

Supersymmetric approach to exact solutions of (1 + 1)-dimensional time-independent Klein-Gordon equation: Application to a position-dependent mass and a \mathcal{PT} -symmetric vector potential

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Abstract. Rigorous use of the SUSYQM approach applied for the Klein-Gordon equation with scalar and vector potentials is discussed. The method is applied to solve exactly, for bound states, two models with position-dependent masses and \mathcal{PT} -symmetric vector potentials, depending on some parameters. The necessary conditions on the parameters to get physical solutions are described. Some special cases are also derived by adjusting the parameters of the models.

1 Introduction

Since the pioneering works of Bender and Boettcher [1,2], it is now recognized that the Hermiticity of the Hamiltonian in Schrödinger equation is not a necessary condition to obtain real eigenvalues for the energy. It has been shown that, one-dimensional stationary Schrödinger equation may exhibit real energy eigenvalues for non-Hermitian potentials provided that the Hamiltonian H has a parity-time reversal symmetry,

$$[\mathcal{PT}, H] = 0, \quad (1)$$

where the action of the space reflection operator \mathcal{P} and the time reversal operator \mathcal{T} on position and momentum operators are given by

$$\mathcal{P} : x \rightarrow -x; \quad p \rightarrow -p \quad \text{and} \quad \mathcal{T} : x \rightarrow x; \quad p \rightarrow -p; \quad i \rightarrow -i. \quad (2)$$

Indeed, the action of \mathcal{PT} on both sides of the Schrödinger equation

$$H\psi_n(x) = E_n\psi_n(x), \quad (3)$$

combined with the properties (1) and (2) leads to

$$H[\mathcal{PT}\psi_n(x)] = E_n^*[\mathcal{PT}\psi_n(x)], \quad (4)$$

where E_n^* denotes the complex conjugate of E_n . Thus, if $\psi_n(x)$ is an eigenfunction of H with eigenvalue E_n , then $\mathcal{PT}\psi_n(x) = \psi_n^*(-x)$ is also an eigenfunction of H with eigenvalue E_n^* . Consequently, for eigenfunctions satisfying $\mathcal{PT}\psi_n(x) = \lambda_n\psi_n(x)$, necessarily $E_n = E_n^*$ and vice versa, since there is no degeneracy in one dimension. In this case, \mathcal{PT} -symmetry is said non-broken, otherwise the \mathcal{PT} -symmetry is said broken and the eigenvalues come in complex conjugate pairs. Furthermore, since $(\mathcal{PT})^2 = 1$, λ_n are phase factors that can be absorbed in the eigenfunctions. Hence, in the case of non-broken \mathcal{PT} -symmetry, the eigenfunctions may be normalized in such a way that $\mathcal{PT}\psi_n(x) = \psi_n(x)$.

The normalization condition in non-broken \mathcal{PT} -symmetric theory is then given by [3,4]

$$\int [\mathcal{PT}\psi_n(x)] \psi_n(x) dx = \int \psi_n^2(x) dx = (-1)^n. \quad (5)$$

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During the last two decades, solvable \mathcal{PT} -symmetric potentials have been extensively studied both in relativistic and non-relativistic quantum mechanics by using different techniques [5–20]. Moreover, some authors have investigated the solutions of Schrödinger equation [21–23] and Dirac equation [24–29] for certain \mathcal{PT} -symmetric potential models with position-dependent mass. Also, problems with position-dependent mass in the context of Klein-Gordon and Dirac equations with Hermitian potentials have been discussed in several works [30–34]. However, in our knowledge, \mathcal{PT} -symmetric potentials have not been studied in the context of Klein-Gordon equation with position-dependent mass. The aim of this work is to fill this gap and solve exactly the (1 + 1)-dimensional time-independent Klein-Gordon equation with position-dependent mass for bound states in the framework of \mathcal{PT} -symmetry. In sect. 2, a summary of the approach of supersymmetric quantum mechanics (SUSYQM) [35, 36] is outlined for \mathcal{PT} -symmetric potentials. In sect. 3, we show how to map Klein-Gordon equation for position-dependent mass with mixing scalar and vector potentials into a Schrödinger-like equation with constant mass and energy-dependent effective potential, suitable for processing by the SUSYQM approach. Section 4 is devoted to applications, where we solve exactly two problems with suitable mass distribution-functions in the presence of \mathcal{PT} -symmetric vector potentials and null scalar potentials, by the approach of SUSYQM.

2 Basic concepts of SUSYQM approach with \mathcal{PT} -symmetric Hamiltonian

In connection to the formalism of SUSYQM for Hermitian Hamiltonians [35, 36], bound-state eigenvalues and corresponding eigenfunctions of a \mathcal{PT} -symmetric one-dimensional Hamiltonian, $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$, with non-broken \mathcal{PT} -symmetry (real eigenvalues), may be obtained in the same way. The partner Hamiltonians $H^{(-)}$ and $H^{(+)}$ associated to H are defined as

$$H^{(-)} = H - E_0 = BA \quad (6)$$

and

$$H^{(+)} = AB, \quad (7)$$

where E_0 is the ground-state energy of the Hamiltonian H , with

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad B = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x), \quad (8)$$

and the superpotential $W(x)$ is a complex function.

Hence, according to eq. (6), H and $H^{(-)}$ have the same eigenfunctions ($\psi_n(x) \sim \psi_n^{(-)}(x)$) and the eigenvalues $\{E_n^{(-)} = E_n - E_0\}$ corresponding to $H^{(-)}$ are semi-positive definite,

$$H^{(-)}\psi_n^{(-)}(x) = BA\psi_n^{(-)}(x) = E_n^{(-)}\psi_n^{(-)}(x), \quad (9)$$

with

$$E_0^{(-)} = 0 \quad \text{and} \quad E_n^{(-)} > 0 \quad \text{for } n = 1, 2, \dots \quad (10)$$

Assuming that the ground-state eigenfunction $\psi_0^{(-)}(x)$ satisfies $A\psi_0^{(-)}(x) = 0$, it is given by

$$\psi_0^{(-)}(x) = N_0 \exp\left(-\frac{\sqrt{2m}}{\hbar} \int^x W(y)dy\right), \quad (11)$$

where N_0 is a normalization constant, such that $\psi_0^{(-)}(x)$ is square integrable in the sense of (5).

The action of the operator A on both sides of eq. (9) leads to

$$H^{(+)}\left(A\psi_n^{(-)}(x)\right) = E_n^{(-)}\left(A\psi_n^{(-)}(x)\right), \quad (12)$$

such that the eigenvalues $E_n^{(+)}$ of $H^{(+)}$ and the normalized corresponding eigenfunctions $\psi_n^{(+)}(x)$ for $n = 0, 1, 2, \dots$, are related to those of $H^{(-)}$ by [35, 36]

$$E_n^{(+)} = E_{n+1}^{(-)}, \quad (13)$$

and

$$\psi_n^{(+)}(x) = \frac{1}{\sqrt{E_{n+1}^{(-)}}} A\psi_n^{(-)}(x). \quad (14)$$

Explicitly, the partner Hamiltonians reads

$$H^{(\mp)} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V^{(\mp)}(x), \tag{15}$$

where the partner potentials, $V^{(\mp)}(x)$, are given by

$$V^{(\mp)}(x) = W^2(x) \mp \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx}. \tag{16}$$

The partner potentials are said shape-invariant potentials [37] if they satisfy

$$V^{(+)}(x; \{a_1\}) = V^{(-)}(x; \{a_2\}) + R(\{a_1\}), \tag{17}$$

where $\{a_1\}$ and $\{a_2\}$ are two sets of real parameters related by a certain function ($\{a_2\} = f(\{a_1\})$) and the remainder $R(\{a_1\})$ is independent of x .

If the requirement (17) is satisfied, one can show [35, 36] that the energy spectrum of $H^{(-)}$ can be deduced algebraically and is given by

$$E_0^{(-)} = 0, \quad E_n^{(-)} = \sum_{k=1}^n R(\{a_k\}) \quad \text{for } n = 1, 2, \dots, \tag{18}$$

with $\{a_k\} = \underbrace{f \circ f \circ \dots \circ f}_{(k-1) \text{ times}}(\{a_1\})$.

The spectrum of the Hamiltonian H is then given by

$$E_n = E_n^{(-)} + E_0. \tag{19}$$

The unnormalized eigenfunctions of the excited states are given by the recurrence formula [35, 36, 38, 39]

$$\psi_n(x; \{a_1\}) = B(\{a_1\})\psi_{n-1}(x; \{a_2\}), \quad \text{for } n \geq 1, \tag{20}$$

which leads to the general formula

$$\psi_n(x; \{a_1\}) = \left[\prod_{i=1}^n B(\{a_i\}) \right] \psi_0(x, \{a_{n+1}\}) \quad \text{for } n \geq 1. \tag{21}$$

3 (1 + 1)-Dimensional time-independent Klein-Gordon equation with position-dependent mass and mixing scalar and vector potentials

The one-dimensional time-independent Klein-Gordon equation for a spinless particle with position-dependent mass $M(x)$, subjected to mixing vector and scalar potentials $V(x)$ and $S(x)$, reads (\hbar is the Plank constant, c is the speed of light)

$$\left(-\hbar^2 \frac{d^2}{dx^2} + \frac{1}{c^2} \left[(M(x)c^2 + S(x))^2 - (E - V(x))^2 \right] \right) \varphi(x) = 0, \tag{22}$$

where E is the energy of the particle and $\varphi(x)$ its corresponding wave function. Since eq. (22) is not an eigenvalues equation, like Schrödinger equation, it is not easy to use SUSYQM approach to solve it and obtain the energy spectrum algebraically. To overcome this difficulty, eq. (22) is often written, in the literature, as an eigenvalues equation in the form

$$\left(-\hbar^2 \frac{d^2}{dx^2} + V_{\text{eff}}(x) \right) \varphi(x) = \tilde{E} \varphi(x), \tag{23}$$

with

$$V_{\text{eff}}(x) = \frac{1}{c^2} \left[(M(x)c^2 + S(x))^2 - V^2(x) + 2EV(x) \right], \tag{24}$$

and

$$\tilde{E} = \frac{E^2}{c^2}. \tag{25}$$

The disadvantage in doing so is that E appears in both the effective potential $V_{\text{eff}}(x)$ and the eigenvalue \tilde{E} . When using SUSYQM approach, the energy E in the hierarchical partner potentials is considered as a parameter that remains unchanged but it changes in the hierarchical eigenvalues, which leads to confusion. In this work, we will follow a different approach that removes the ambiguity.

First, remark that eq. (22) may be seen as a zero energy Schrödinger-like equation for a particle with constant mass ($m = 1/2$), subjected to the E -dependent potential

$$V_E(x) = \frac{1}{c^2} \left[(M(x)c^2 + S(x))^2 - (E - V(x))^2 \right]. \quad (26)$$

Thus, the problem reduces to search the solution of a zero-energy Schrödinger-like equation with a conditional parameter in potential. To solve a typical equation for bound states, *i.e.* discrete real energies $E \equiv E_n$ and normalized wave functions $\varphi(x) \equiv \varphi_n(x)$, using SUSYQM, we consider instead the Schrödinger equation ($2m = 1$)

$$\left(-\hbar^2 \frac{d^2}{dx^2} + V_E(x) \right) \Phi_n(x) = \epsilon_n \Phi_n(x) \quad \text{with } n = 0, 1, 2, \dots, \quad (27)$$

where E is considered as a real parameter in the potential $V_E(x)$. Now, SUSYQM approach can be applied to solve equation (27) without any confusion. When $V_E(x)$ is Hermitian or \mathcal{PT} -symmetric and the \mathcal{PT} -symmetry is not spontaneously broken, the eigenvalues ϵ_n are real functions of the parameter E . Hence, once the eigenvalues ϵ_n and the corresponding eigenfunctions $\Phi_n(x)$ for eq. (27) are obtained, the energies E_n of the original problem (eq. (22)) are given by the real solutions of the equation

$$\epsilon_n(E) = 0, \quad (28)$$

and the wave functions $\varphi_n(x)$ can be deduced by

$$\varphi_n(x) = \Phi_n(x)|_{\epsilon_n(E)=0}. \quad (29)$$

4 Applications

We are interested in this paper to solve exactly eq. (27) for two models with \mathcal{PT} -symmetric E -dependent potential, resulting from complex \mathcal{PT} -symmetric vector potential and null scalar potential. For each model, the mass distribution is suitably chosen such that to obtain an E -dependent potential that is exactly solvable for bound states.

4.1 Model with asymptotically unbounded mass, coupled to a linear \mathcal{PT} -symmetric vector potential

Consider a relativistic position-dependent spinless particle moving on the whole X -axis and subjected to a linear \mathcal{PT} -symmetric vector potential and null scalar potential. The mass distribution and the vector potential are taken, respectively, as

$$M(x) = \sqrt{\mu^2 + \left(\frac{\lambda}{c}\right)^2 x^2}, \quad (30)$$

and

$$V(x) = ic\eta x, \quad (31)$$

where μ is the value of the mass at the origin of the coordinate, λ and η are real parameters with dimension MT^{-1} , and without loss of generality λ is assumed to be positive. Note that the speed of light c is explicitly included in the expressions of $M(x)$ and $V(x)$ only by convenience of calculations. However, λ and η may be seen of order 0 and order 1 compared to c^{-1} respectively, *i.e.*,

$$\lambda = \lambda_0 + O(c^{-1}) \quad \text{and} \quad \eta = \eta_0 c^{-1} + O(c^{-2}). \quad (32)$$

Substituting eqs. (30) and (31) into (26), the E -dependent potential reads

$$V_E(x) = (\lambda^2 + \eta^2) x^2 + i \frac{2\eta E}{c} x + \frac{\mu^2 c^4 - E^2}{c^2}, \quad (33)$$

which is a \mathcal{PT} -symmetric function ($V_E(x) = V_E^*(-x)$) for real values of the energy E .

To solve eq. (27) with the potential (33), by using SUSYQM, we choose the superpotential in the form

$$W_E(x) = Ax + i\frac{\eta E}{cA}, \tag{34}$$

where A is a real parameter.

In order to fix the parameter A and obtain the ground-state eigenvalue $\epsilon_0 \equiv \epsilon_0(E)$, we have to solve the identity

$$V_E(x) - \epsilon_0 = W_E^2(x) - \hbar W_E'(x). \tag{35}$$

Substituting (33) and (34) into (35) and identifying the coefficients of terms in power of x , leads to

$$A^2 = \lambda^2 + \eta^2, \tag{36}$$

and

$$\epsilon_0 = \mu^2 c^2 + \hbar A - \frac{\lambda^2 E^2}{c^2 A^2}. \tag{37}$$

Note that ϵ_0 is real for real values of E . However, using eq. (11), the unnormalized ground-state eigenfunction may be put in the form

$$\Phi_0(x) \sim e^{-\frac{A}{2\hbar}(x+i\frac{\eta E}{cA^2})^2}. \tag{38}$$

By demanding that $\Phi_0(x)$ is normalizable on the real axis in the sense of eq. (5) requires that A is positive, such that the acceptable solution of (36) is

$$A = \sqrt{\lambda^2 + \eta^2}. \tag{39}$$

The supersymmetric partner potentials $V_E^{(\mp)}(x) = W_E^2(x) \mp \hbar W_E'(x)$ are explicitly given by

$$V_E^{(-)}(x) = A^2 x^2 + i\frac{2\eta E}{c}x - \frac{\eta^2 E^2}{c^2 A^2} - \hbar A, \tag{40}$$

$$V_E^{(+)}(x) = A^2 x^2 + i\frac{2\eta E}{c}x - \frac{\eta^2 E^2}{c^2 A^2} + \hbar A. \tag{41}$$

They satisfy the shape invariance condition (17), which reads

$$V_E^{(+)}(x, a_1) = V_E^{(-)}(x, a_2) + R(a_1), \tag{42}$$

with

$$a_1 = A, \quad a_2 = f(a_1) = a_1, \tag{43}$$

and

$$R(a_1) = 2\hbar a_1 = 2\hbar A. \tag{44}$$

Using (19), the energy eigenvalues corresponding to the potential $V_E(x)$ are given by

$$\epsilon_n = \epsilon_n^{(-)} + \epsilon_0, \tag{45}$$

where $\epsilon_n^{(-)}$ are the eigenvalues of the partner $V_E^{(-)}(x)$, which are expressed in terms of the remainder function R as

$$\epsilon_n^{(-)} = \sum_{k=1}^n R(a_k) = 2n\hbar A, \tag{46}$$

and we have used the fact that

$$a_k = \underbrace{f \circ f \circ \dots \circ f}_{(k-1) \text{ times}}(a_1) = a_1. \tag{47}$$

Since $a_k = A > 0$ for all $k = 1, 2, \dots$, the number of eigenvalues is unlimited. Thus, substituting (37) and (46) into (45), the energy eigenvalues ϵ_n are given by

$$\epsilon_n = \mu^2 c^2 + (2n + 1)\hbar A - \frac{\lambda^2 E^2}{c^2 A^2}. \tag{48}$$

By virtue of (28), the generating formula for allowed energy values of the original problem, E_n , may be put in the form

$$E_n = \pm \frac{A}{\lambda} \sqrt{\mu^2 c^4 + (2n+1) \hbar c^2 A}, \quad n = 0, 1, 2, \dots \quad (49)$$

Thus, all the energy values are real, independently of the parameters μ , λ and η , and consequently the \mathcal{PT} -symmetry is always not broken.

To determine the wave functions $\psi_n(x)$ of the original problem ($\psi_n(x) = \Phi_n(x)|_{\epsilon_n(E)=0}$), let us write eq. (27) for $\epsilon_n = 0$ in the form

$$\left(-\hbar^2 \frac{d^2}{dx^2} + A^2 \left(x + i \frac{\eta E_n}{cA^2} \right)^2 \right) \psi_n(x) = (2n+1) \hbar A \psi_n(x), \quad (50)$$

where we made use of (29), (33) and (48).

By defining a new function $\varphi_n(z)$ by

$$\psi_n(x) = e^{-\frac{z^2}{2}} \varphi_n(z), \quad (51)$$

with $z = \sqrt{\frac{A}{\hbar}} \left(x + i \frac{\eta E_n}{cA^2} \right)$ and substituting into (50), it is easily seen that $\varphi_n(z)$ satisfies the Hermite equation

$$\varphi_n''(z) - 2z\varphi_n'(z) + 2n\varphi_n(z) = 0. \quad (52)$$

Hence, the wave functions may be written in the form

$$\psi_n(x) = |N_n| e^{i\frac{n\pi}{2}} e^{-\frac{A}{2\hbar} \left(x + i \frac{\eta E_n}{cA^2} \right)^2} H_n \left(\sqrt{\frac{A}{\hbar}} \left(x + i \frac{\eta E_n}{cA^2} \right) \right), \quad (53)$$

where $H_n(z)$ is the Hermite polynomial and $|N_n|$ is a real normalizing factor. The phase factor $e^{i\frac{n\pi}{2}}$ is introduced explicitly in order to make the wave function $\psi_n(x)$ also eigenfunction of the \mathcal{PT} operator with eigenvalue equal to 1.

Normalizing $\psi_n(x)$ in the sense of eq. (5) allows to fix the normalization factor in the form

$$|N_n| = \left(\frac{A}{\pi \hbar} \right)^{\frac{1}{4}} \left(\frac{1}{2^n n!} \right)^{\frac{1}{2}}. \quad (54)$$

In conclusion, it is obvious from eq. (49) that we have to consider $\lambda \neq 0$ with arbitrary η . This means that the position dependence of the mass which is responsible of the existence of bound states. The vector potential only contributes to the magnification of the energy values. In other words, for a fixed value of λ , the energy values are amplified with increasing values of $|\eta|$. However, in the Hermitian version of the problem, with $V(x) = \eta cx$, the parameter η is to be replaced by $-i\eta$ in eq. (49) so that the role of the vector potential is inverted. Indeed, in this case, for a fixed value λ , energy values decrease with increasing values of $|\eta|$ and bound states exist only if $|\eta| < \lambda$.

4.1.1 Special cases

- Setting $\mu = 0$ in (30), the problem reduces to a particle with a mass distribution as a linear function of the position, given by

$$M(x) = \frac{\lambda}{c} |x|, \quad (55)$$

subjected to the \mathcal{PT} -symmetric vector potential (31). This special case may also be seen as the problem of massless particle subjected to the \mathcal{PT} -symmetric vector potential (31), combined with a real linear scalar potential, $S(x) = \pm \frac{\lambda}{c} x$. The energy values and wave functions reduce to

$$E_n|_{\mu=0} = \pm \frac{c}{\lambda} \sqrt{(2n+1) \hbar A^3}, \quad n = 0, 1, 2, \dots \quad (56)$$

and

$$\psi_n(x)|_{\mu=0} = |N_n| e^{i\frac{n\pi}{2}} e^{-\frac{A}{2\hbar} \left(x + i \frac{\eta E_n|_{\mu=0}}{cA^2} \right)^2} H_n \left(\sqrt{\frac{A}{\hbar}} \left(x + i \frac{\eta E_n|_{\mu=0}}{cA^2} \right) \right), \quad (57)$$

with $|N_n|$ given by (54).

- Setting $\lambda = \mu\omega$, $\eta = \frac{\mu\xi}{c}$ and considering the non-relativistic (NR) limit, by subtracting the rest energy μc^2 from the total positive energy and taking the limit $c \rightarrow \infty$, one gets

$$\lim_{c \rightarrow \infty} (E_n - \mu c^2) = E_n^{NR} \quad \text{and} \quad \lim_{c \rightarrow \infty} \psi_n(x) = \psi_n^{NR}(x),$$

with

$$E_n^{NR} = \left(n + \frac{1}{2}\right) \hbar\omega \tag{58}$$

and

$$\psi_n^{NR}(x) = \left(\frac{\mu\omega}{\hbar\pi}\right)^{\frac{1}{4}} \left(\frac{1}{2^n n!}\right)^{\frac{1}{2}} e^{i\frac{n\pi}{2}} e^{-\frac{\mu\omega}{2\hbar}(x+i\frac{\xi}{\omega^2})^2} H_n\left(\sqrt{\frac{\mu\omega}{\hbar}}\left(x+i\frac{\xi}{\omega^2}\right)\right). \tag{59}$$

We see that in this limit the energy values are those of an harmonic oscillator and the effect of the vector potential appears only in the wave functions. In the absence of the vector potential, $\eta = 0$ (or $\xi = 0$), the wave functions also reduce to those of the harmonic oscillator

$$\psi_n^{NR}(x)|_{\eta=0} = \left(\frac{\mu\omega}{\pi\hbar}\right)^{\frac{1}{4}} \left(\frac{1}{2^n n!}\right)^{\frac{1}{2}} e^{-\frac{\mu\omega}{2\hbar}x^2} H_n\left(\sqrt{\frac{\mu\omega}{\hbar}}x\right). \tag{60}$$

Thus, one can say that this model may be seen as the extension of the one-dimensional non-relativistic harmonic oscillator to the relativistic Klein-Gordon harmonic oscillator. Indeed, by setting $\eta = 0$, $\lambda = \mu\omega$ and taking the non-relativistic limit in the Klein-Gordon equation eq. (22), it can be seen that it reduces to the Schrödinger equation for the harmonic oscillator potential.

4.2 Model with asymptotically bounded mass coupled to a \mathcal{PT} -symmetric hyperbolic vector potential

In this model, we take the mass distribution and the potential functions in the forms

$$M(x) = \sqrt{\mu^2 + \left(\frac{\lambda}{\alpha c}\right)^2 \tanh^2 \alpha x}, \tag{61}$$

and

$$V(x) = i\frac{c\eta}{\alpha} \tanh \alpha x, \tag{62}$$

where μ is the value of the mass at the origin of the coordinate, $\alpha > 0$ and λ, η are real parameters satisfying (32), with $\lambda > 0$.

Substituting eqs. (61) and (62) into (26) and denoting the energy by \mathcal{E} , the effective \mathcal{E} -potential reads

$$V_{\mathcal{E}}(x) = -\frac{\lambda^2 + \eta^2}{\alpha^2 \cosh^2 \alpha x} + i\frac{2\eta\mathcal{E}}{\alpha c} \tanh \alpha x + \frac{\lambda^2 + \eta^2}{\alpha^2} + \frac{\mu^2 c^4 - \mathcal{E}^2}{c^2}, \tag{63}$$

that is, for real values of the energy \mathcal{E} , a shifted \mathcal{PT} -symmetric potential of Rosen-Morse II type.

Choosing the superpotential in the form

$$W_{\mathcal{E}}(x) = \frac{B}{\alpha} \tanh \alpha x + i\frac{\eta\mathcal{E}}{cB}, \tag{64}$$

and using eq. (35), we find that the parameter B and the ground-state energy ϵ_0 are given by

$$B(B + \hbar\alpha^2) = \lambda^2 + \eta^2, \tag{65}$$

and

$$\epsilon_0 = -\left(\frac{B^2}{\alpha^2} - \frac{\eta^2\mathcal{E}^2}{c^2 B^2}\right) + \frac{\lambda^2 + \eta^2}{\alpha^2} + \frac{\mu^2 c^4 - \mathcal{E}^2}{c^2}. \tag{66}$$

The unnormalized ground state eigenfunction reads

$$\Phi_0(x) \sim \exp\left(-\frac{1}{\hbar} \int^x W_{\mathcal{E}}(y) dy\right) = e^{-i\frac{\eta\mathcal{E}}{\hbar c B} x} (\cosh \alpha x)^{-\frac{B}{\hbar\alpha^2}}. \tag{67}$$

Demanding that $\Psi_0(x)$ satisfy the normalization condition in the sense of eq. (5), the parameter B must be positive. Thus, solving eq. (65) with this restriction gives

$$B = \sqrt{\lambda^2 + \eta^2 + \frac{\hbar^2 \alpha^4}{4}} - \frac{\hbar \alpha^2}{2}. \tag{68}$$

The supersymmetric partner potentials are constructed as

$$V_{\mathcal{E}}^{(-)}(x) = W_{\mathcal{E}}^2(x) - \hbar W'_{\mathcal{E}}(x) = -\frac{B(B + \hbar \alpha^2)}{\alpha^2 \cosh^2 \alpha x} + i \frac{2\eta \mathcal{E}}{\alpha} \tanh \alpha x + \frac{B^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 B^2}, \tag{69}$$

$$V_{\mathcal{E}}^{(+)}(x) = W^2(x) + \hbar W'_{\mathcal{E}}(x) = -\frac{B(B - \hbar \alpha^2)}{\alpha^2 \cosh^2 \alpha x} + i \frac{2\eta \mathcal{E}}{\alpha} \tanh \alpha x + \frac{B^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 B^2}, \tag{70}$$

which satisfy the shape invariance condition (17), with

$$a_1 = B, \quad a_2 = f(a_1) = a_1 - \hbar \alpha^2 \tag{71}$$

and

$$R(a_1) = \left(\frac{a_1^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 a_1^2} \right) - \left(\frac{a_2^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 a_2^2} \right). \tag{72}$$

By virtue of (71), one has

$$a_k = \underbrace{f \circ f \circ \dots \circ f}_{(k-1) \text{ times}}(a_1) = a_1 - (k - 1) \hbar \alpha^2,$$

such that the energy spectra for bound states of $V_{\mathcal{E}}^{(-)}(x)$ are given by

$$\begin{aligned} \epsilon_n^{(-)} &= \sum_{k=1}^n R(a_k) = \left(\frac{a_1^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 a_1^2} \right) - \left(\frac{a_{n+1}^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 a_{n+1}^2} \right) \\ &= \left(\frac{B^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 B^2} \right) - \left(\frac{(B - n \hbar \alpha^2)^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 (B - n \hbar \alpha^2)^2} \right). \end{aligned} \tag{73}$$

Using (45) and substituting eqs. (66) and (73) into (45), we get the energy spectra of $V_{\mathcal{E}}(x)$ in the form

$$\epsilon_n = \frac{B(B + \hbar \alpha^2)}{\alpha^2} + \frac{\mu^2 c^4 - \mathcal{E}^2}{c^2} - \left(\frac{(B - n \hbar \alpha^2)^2}{\alpha^2} - \frac{\eta^2 \mathcal{E}^2}{c^2 (B - n \hbar \alpha^2)^2} \right), \tag{74}$$

where n is limited to positive integer numbers satisfying

$$0 \leq n \leq n_{\max} = \left\lfloor \frac{B}{\hbar \alpha^2} \right\rfloor, \tag{75}$$

and $\{k\}$ denotes the largest integer inferior to k .

Thus, the condition $\lambda > 0$ is sufficient for the existence of at least one bound state for the effective potential (real eigenvalues ϵ_n and normalizable corresponding eigenfunctions). However, this will not be necessarily a sufficient condition for the existence of bound states for the original problem.

Putting $\epsilon_n = 0$ in eq. (74) and solving it, the allowed energy values of the original problem, \mathcal{E}_n , are given by the following generating formula:

$$\mathcal{E}_n = \pm \sqrt{\frac{\mu^2 c^4 + \frac{c^2}{\alpha^2} (B(B + \hbar \alpha^2) - (B - n \hbar \alpha^2)^2)}{1 - \frac{\eta^2}{(B - n \hbar \alpha^2)^2}}}, \tag{76}$$

where now allowed values of n must satisfy (75) and also are such that \mathcal{E}_n are real. It is easy to see that, while the numerator of the expression in the square root is always positive if (75) is satisfied, the positivity of the denominator requires the new condition

$$0 \leq n \leq \bar{n}_{\max} = \left\lfloor \frac{B - \eta}{\hbar \alpha^2} \right\rfloor, \tag{77}$$

which is more restrictive than (75).

Hence, using (68) with (77) we find that the existence of at least one bound state for the original problem requires a new constraint on the parameters λ , η and α , given by

$$\lambda^2 > \hbar\alpha^2\eta. \tag{78}$$

This means that when λ increases there is a tendency to increase the number of bound states, while growth of η tends to decrease this number while magnifying the eigenvalues. In other words, for fixed λ , the number of bound states is maximum for null vector potential ($\eta = 0$) and then decreases with increasing η . Thus, the imaginary vector potential tends to reduce the confinement of the particle that is produced by the variation of its mass. Indeed, in the case of null vector potential, one has

$$\mathcal{E}_n|_{\eta=0} = \pm \sqrt{\mu^2 c^4 + \frac{c^2}{\alpha^2} \left[\lambda^2 - \left(\sqrt{\lambda^2 + \frac{\hbar^2 \alpha^4}{4}} - \left(n + \frac{1}{2} \right) \hbar\alpha^2 \right)^2 \right]}, \tag{79}$$

with

$$0 \leq n \leq \bar{n}_{\max}|_{\eta=0} = \left\{ \sqrt{\frac{\lambda^2}{\hbar^2 \alpha^4} + \frac{1}{4}} - \frac{1}{2} \right\}. \tag{80}$$

It appears that

$$\mathcal{E}_n|_{\eta=0} \leq \mathcal{E}_n \tag{81}$$

and

$$\bar{n}_{\max}|_{\eta=0} \geq \bar{n}_{\max}. \tag{82}$$

To obtain the wave functions $\Psi_n(x)$ of the original problem, ($\Psi_n(x) = \Phi_n(x)|_{\epsilon_n(E)=0}$), we proceed as in the previous model. In this case, we are led to solve the following equation:

$$\left(\hbar^2 \frac{d^2}{dx^2} + \frac{B(B + \hbar\alpha^2)}{\alpha^2 \cosh^2 \alpha x} - i \frac{2\eta\mathcal{E}_n}{\alpha c} \tanh \alpha x + \frac{\eta^2 \mathcal{E}_n^2}{c^2(B - n\hbar\alpha^2)^2} - \frac{(B - n\hbar\alpha^2)^2}{\alpha^2} \right) \Psi_n(x) = 0. \tag{83}$$

By the point transformation, defined by

$$z = \tanh(\alpha x); \quad z \in]-1, 1[\quad \text{and} \quad \Psi_n(x) = (1 - z)^{\frac{a_n}{2}} (1 + z)^{\frac{b_n}{2}} \phi_n(z), \tag{84}$$

with

$$a_n = \frac{B - n\hbar\alpha^2}{\hbar\alpha^2} + i \frac{\eta\mathcal{E}_n}{\alpha\hbar c(B - n\hbar\alpha^2)}, \tag{85a}$$

$$b_n = \frac{B - n\hbar\alpha^2}{\hbar\alpha^2} - i \frac{\eta\mathcal{E}_n}{\alpha\hbar c(B - n\hbar\alpha^2)} = a_n^*, \tag{85b}$$

it is straightforward to show that the new function $\phi_n(z)$ satisfies the differential equation of Jacobi polynomials,

$$(1 - z^2) \frac{d^2 \phi_n(z)}{dz^2} + [b_n - a_n - (a_n + b_n + 2)z] \frac{d\phi_n(z)}{dz} + n(n + a_n + b_n + 1) \phi_n(z) = 0. \tag{86}$$

Knowing that a_n and b_n are the complex conjugates of each other and taking account of the following symmetry relation of Jacobi polynomials [40]

$$P_n^{(a_n, b_n)}(-z) = (-1)^n P_n^{(b_n, a_n)}(z), \tag{87}$$

the wave functions $\Psi_n(x)$ may be put in a \mathcal{PT} -symmetric form as follows

$$\Psi_n(x) = |\mathcal{N}_n| e^{i \frac{n\pi}{2}} (1 - \tanh \alpha x)^{\frac{a_n}{2}} (1 + \tanh \alpha x)^{\frac{b_n}{2}} P_n^{(a_n, b_n)}(\tanh \alpha x), \tag{88}$$

where the normalization constant $|\mathcal{N}_n|$ is given by

$$|\mathcal{N}_n| = \sqrt{\frac{2\alpha n! a_n b_n \Gamma(a_n + b_n + n + 1)}{2^{a_n + b_n} (a_n + b_n) \Gamma(a_n + n + 1) \Gamma(b_n + n + 1)}} = \frac{|a_n| \sqrt{\alpha n! \Gamma(2 \operatorname{Re} a_n + n + 1)}}{2^{\operatorname{Re} a_n} \sqrt{\operatorname{Re} a_n} |\Gamma(a_n + n + 1)|}. \tag{89}$$

4.2.1 Special case

- Taking the limit $\alpha \rightarrow 0$ in eqs. (61) and (62), the mass distribution and the vector potential coincide exactly with those of the first model, given respectively by (30) and (31). We will see that, (76), (88) and (89) reduce also to (49), (53) and (54), respectively. Indeed, one has

$$\frac{1}{\alpha^2} \left(B(B + \hbar\alpha^2) - (B - n\hbar\alpha^2)^2 \right) = B(2n + 1)\hbar - n^2\hbar^2\alpha^2 \xrightarrow{\alpha \rightarrow 0} \hbar\sqrt{\lambda^2 + \eta^2} (2n + 1) \tag{90}$$

and

$$1 - \frac{\eta^2}{(B - n\hbar\alpha^2)^2} \xrightarrow{\alpha \rightarrow 0} \frac{\lambda^2}{\lambda^2 + \eta^2}, \tag{91}$$

such that

$$\mathcal{E}_n \xrightarrow{\alpha \rightarrow 0} E_n = \pm \frac{\sqrt{\lambda^2 + \eta^2}}{\lambda} \sqrt{\mu^2 c^4 + (2n + 1)\hbar c^2 \sqrt{(\lambda^2 + \eta^2)}}, \tag{92}$$

which is exactly the relation (49). The number of energy levels is now unlimited, *i.e.* $n = 0, 1, 2, \dots$, as it can be verified by taking the limit $\alpha \rightarrow 0$ in (77). In addition, in the limit $\alpha \rightarrow 0$, the constraint (78) reduces to $\lambda > 0$ as it should be.

As regards the wave functions, keeping only leading terms in the limit $\alpha \rightarrow 0$ leads to

$$\text{Re } a_n \xrightarrow{\alpha \rightarrow 0} \frac{\sqrt{\lambda^2 + \eta^2}}{\hbar\alpha^2} - \left(n + \frac{1}{2} \right) \quad \text{and} \quad \text{Im } a_n \xrightarrow{\alpha \rightarrow 0} \frac{\eta E_n}{\hbar c \alpha \sqrt{\lambda^2 + \eta^2}}, \tag{93}$$

such that a straightforward calculation gives

$$(1 - \tanh \alpha x)^{\frac{\alpha n}{2}} (1 + \tanh \alpha x)^{\frac{\alpha n}{2}} \xrightarrow{\alpha \rightarrow 0} \exp \left(-\frac{\sqrt{\lambda^2 + \eta^2}}{2\hbar} x^2 - i \frac{\eta \bar{E}_n}{\hbar c \sqrt{\lambda^2 + \eta^2}} x \right), \tag{94}$$

and (see definitions (8.960.1, page 999) and (8.950.1, page 996) in ref. [41])

$$P_n^{(a_n, b_n)}(\tanh \alpha x) \xrightarrow{\alpha \rightarrow 0} \frac{1}{2^n n!} \left(\frac{\sqrt{\lambda^2 + \eta^2}}{\alpha^2 \hbar} \right)^{\frac{n}{2}} H_n \left(\left(\frac{\sqrt{\lambda^2 + \eta^2}}{\hbar} \right)^{\frac{1}{2}} \left(x + i \frac{\eta E_n}{c(\lambda^2 + \eta^2)} \right) \right). \tag{95}$$

Otherwise, using Stirling formula

$$\Gamma(X) = \sqrt{2\pi} X^{X-\frac{1}{2}} e^{-X}, \tag{96}$$

that is valid for large X , a straightforward calculation leads to

$$\Gamma(2 \text{Re } a_n + n + 1) \xrightarrow{\alpha \rightarrow 0} \sqrt{2\pi} \left(2 \frac{\sqrt{\lambda^2 + \eta^2}}{\alpha^2 \hbar} \right)^{\frac{2\sqrt{\lambda^2 + \eta^2}}{\alpha^2 \hbar} - n - \frac{1}{2}} e^{-2 \frac{\sqrt{\lambda^2 + \eta^2}}{\alpha^2 \hbar}}, \tag{97}$$

and

$$|\Gamma(a_n + n + 1)| \xrightarrow{\alpha \rightarrow 0} \sqrt{2\pi} \left(\frac{\sqrt{\lambda^2 + \eta^2}}{\alpha^2 \hbar} \right)^{\left(\frac{\sqrt{\lambda^2 + \eta^2}}{\alpha^2 \hbar} \right)} \exp \left(-\frac{\eta^2 E_n^2}{2\hbar c^2 (\lambda^2 + \eta^2)^{\frac{3}{2}}} \right) e^{-\frac{\sqrt{\lambda^2 + \eta^2}}{\hbar \alpha^2}}. \tag{98}$$

Using (93), (97) and (98), we easily verify that, in the leading order of α , the normalization constant $|\mathcal{N}_n|$ reduces to

$$|\mathcal{N}_n| \xrightarrow{\alpha \rightarrow 0} |\bar{\mathcal{N}}_n| = \sqrt{2^n n!} \left(\frac{\sqrt{\lambda^2 + \eta^2}}{\pi \hbar} \right)^{\frac{1}{4}} \left(\frac{\sqrt{\lambda^2 + \eta^2}}{\alpha^2 \hbar} \right)^{-\frac{n}{2}} \exp \left(\frac{\eta^2 E_n^2}{2\hbar c^2 (\lambda^2 + \eta^2)^{\frac{3}{2}}} \right). \tag{99}$$

Finally, substitution of (94), (95) and (99) into (88) leads to

$$\begin{aligned} \Psi_n(x) \xrightarrow{\alpha \rightarrow 0} \psi_n(x) &= \frac{e^{i \frac{n\pi}{2}}}{\sqrt{2^n n!}} \left(\frac{\sqrt{\lambda^2 + \eta^2}}{\pi \hbar} \right)^{\frac{1}{4}} e^{-\frac{\sqrt{\lambda^2 + \eta^2}}{2\hbar} \left(x + i \frac{\eta E_n}{c(\lambda^2 + \eta^2)} \right)^2} \\ &\times H_n \left(\sqrt{\frac{\sqrt{\lambda^2 + \eta^2}}{\hbar}} \left(x + i \frac{\eta E_n}{c(\lambda^2 + \eta^2)} \right) \right), \end{aligned} \tag{100}$$

that coincide exactly with the wave function of the first model (relations (53) and (54)).

5 Conclusion

In this paper, we have discussed bound state solutions of the $(1 + 1)$ -dimensional stationary Klein-Gordon equation with position-dependent mass and \mathcal{PT} -symmetric vector and scalar potentials by the approach of supersymmetric quantum mechanics. We have shown that, for better use of SUSYQM, the problem can be mapped into a constant mass Schrödinger equation with energy-dependent effective potential. This method is applied to solve exactly two models with null scalar potentials and suitable couples of mass distribution and \mathcal{PT} -symmetric vector potential, that, interestingly, coincide in a limiting case.

In the first model, the vector potential is chosen as a \mathcal{PT} -symmetric linear function of the position, and the mass distribution is the square root of a quadratic form. The problem leads to solve Schrödinger equation with quadratic energy-dependent \mathcal{PT} -symmetric potential. The bound-state energies are exactly obtained by SUSYQM and the wave functions are easily deduced.

In the second model, the \mathcal{PT} -symmetric vector potential is chosen as a hyperbolic tangent function, and the mass distribution is the square root of a quadratic form of a hyperbolic tangent function. The problem is then reduced to solve Schrödinger equation with an energy-dependent \mathcal{PT} -symmetric potential of Rosen-Morse II type. Again, SUSYQM approach has been applied successfully to obtain exactly the bound-state energies and to deduce the corresponding wave functions. In particular, we have discussed the constraints that must be satisfied by the parameters of the problem in order to obtain physical results. Furthermore, we have discussed some special cases of the two models and shown that they coincide in a limiting case.

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