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Numerical solution of fractional sub-diffusion and time-fractional diffusion-wave equations via fractional-order Legendre functions

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Abstract. Fractional calculus has been used to model physical and engineering processes that are best described by fractional differential equations. Therefore designing efficient and reliable techniques for the solution of such equations is an important task. In this paper, we propose an efficient and accurate Galerkin method based on the fractional-order Legendre functions (FLFs) for solving the fractional sub-diffusion equation (FSDE) and the time-fractional diffusion-wave equation (FDWE). The time-fractional derivatives for FSDE are described in the Riemann-Liouville sense, while for FDWE are described in the Caputo sense. To this end, we first derive a new operational matrix of fractional integration (OMFI) in the Riemann-Liouville sense for FLFs. Next, we transform the original FSDE into an equivalent problem with fractional derivatives in the Caputo sense. Then the FLFs and their OMFI together with the Galerkin method are used to transform the problems under consideration into the corresponding linear systems of algebraic equations, which can be simply solved to achieve the numerical solutions of the problems. The proposed method is very convenient for solving such kind of problems, since the initial and boundary conditions are taken into account automatically. Furthermore, the efficiency of the proposed method is shown for some concrete examples. The results reveal that the proposed method is very accurate and efficient.

1 introduction

Fractional differential equations have recently attracted increasing attention [1–4], due to the fact that they have many applications in various fields of science and engineering. For example, they can describe many physical and chemical processes, biological systems, etc. In fact, a realistic model of a physical phenomenon which depends not only on the time instant, but also on the previous time history, can be successfully achieved by using fractional calculus. It is worth noting that analytic solutions of most fractional differential equations cannot be obtained explicitly [5], so that new methods, to find numerical solutions of these equations, have practical importance. Due to this fact, in recent years several numerical methods have been proposed for the solution of fractional differential equations (see, *e.g.*, [6–15]).

Fractional partial differential equations (FPDEs) are generalizations of classical partial differential equations (PDEs). Two important classes of FPDEs, widely studied in recent years, are the fractional diffusion equation (FDE) and the fractional diffusion-wave equation (FDWE). The main physical purpose for investigating FDE is describing the phenomena of anomalous diffusion in transport processes through complex and/or disordered systems including fractal media, and fractional kinetic equations have proved particularly useful in the context of anomalous slow diffusion, see, for instance, the review paper [16].

The FSDE is a class of anomalous diffusive systems which is obtained by replacing the time derivative of the ordinary diffusion, by a fractional derivative of order α, with $0 < \alpha < 1$, and is given [17–19] as

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[\mathcal{K}_\alpha \frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(x,t). \tag{1}$$

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Here ${}_0D_t^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative of order $1-\alpha$ with respect to the variable t, $0 < \alpha < 1$ is the anomalous diffusion exponent, $\mathcal{K}_\alpha$ is the generalized diffusion constant, and f is a given known function.

Equation (1) is the evolution equation for the probability density function that describes particles diffusing with mean square displacement $\langle x^2(t)\rangle \sim t^\alpha$ [19,20]. For the case $0 < \alpha < 1$, the diffusion is anomalously slow (sub-diffusion) compared to the normal diffusion behavior with $\alpha = 1$ [20].

There have been various numerical methods for studying and solving the FSDE (see, *e.g.*, [18–29]). Yuste and Acedo [11] combined the forward-time centered-space method and the Grünwald-Letnikov discretization of the Riemann-Liouville derivative to obtain an explicit scheme for fractional diffusion equations. In [13, 21], Chen, Liu and their coworkers have constructed the difference scheme using the Grünwald-Letnikov formula and also presented the Fourier method to show the stability and convergence of the difference scheme for the fractional sub-diffusion and reaction sub-diffusion equations, respectively. Murio [22] developed an implicit finite-difference method for solving the FSDE and also used Fourier method to show the stability. Langlands and Henry [23] developed an implicit difference scheme with convergence order $O(\tau^{\alpha+1} + h^2)$ based on L_1 approximation and numerically verified the unconditional stability of difference scheme but without global convergence analysis. Zhuang, Liu and their coworkers [9, 24] introduced a new way to solve linear and non-linear FSDEs. They first integrated the original differential operator on both sides, then approximated the obtained identity numerically with the idea of numerical integrals. The stability and convergence analyzed in discrete L_2 norm by using the energy method. Two-dimensional anomalous FSDE was treated numerically in [25], where the two methods (explicit and implicit) were proposed by using the relationships between the fractional Grünwald-Letnikov definition and Riemann-Liouville definition. In [17], Cui considered a high-order finite-difference scheme for solving fractional anomalous sub-diffusion equation. The Grünwald formula was used for direct approximation of the Riemann-Liouville fractional derivative in temporal direction and fourth-order compact difference scheme for the spatial discretization, with the convergence order $O(\tau + h^4)$ in discrete L_2 norm. In [26], Chen *et al.* have studied some high accurate numerical methods. By a similar discretization approach, a scheme with convergence order $O(\tau + h^4)$ in L_2 norm was also obtained for the variable-order anomalous differential equation and was analyzed using Fourier method. Gao and Sun [18] have proposed a compact difference scheme for the time fractional sub-diffusion equation, and proved that the scheme was unconditionally stable and convergent in maximum norm with the convergence order $O(\tau^{2-\alpha} + h^4)$. Zhang *et al.* [27] constructed a Crank-Nicolson–type difference scheme and a compact difference scheme for solving the time FSDE with Riemann-Liouville fractional derivative, respectively. They proved that the two difference schemes were unconditionally stable and the numerical solution was convergent in the maximum norm. Zhang *et al.* [28] and Cui [29] constructed alternating direction implicit scheme and compact alternating direction implicit scheme for solving the two-dimensional time FSDE, respectively. Zhao and Sun [20] proposed a Box-type scheme for solving a class of the FSDE with Neumann boundary conditions. Ren *et al.* [19] have proposed a compact difference scheme for the FSDE with Neumann boundary conditions. We also advise the reader to see recent papers [30–32] about FSDE.

The time FDWE is a mathematical model of a wide class of important physical phenomena. This equation is a linear integro-partial differential equation that is obtained from the classical diffusion-wave equation by replacing the second-order time derivative term by a fractional derivative of order $1 < \alpha \leq 2$ [33], and is given in [34] by

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \lambda\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \tag{2}$$

where the parameter α denotes the order of the fractional derivative in the Caputo sense, which will be described in the next section, $\lambda > 0$ is a constant and f is a given known function.

Several numerical methods have been successfully used in the last few years for solving the FDWE, *e.g.*, [33–44]. In [33], Bhrawy *et al.* have proposed a spectral tau method based on the Jacobi operational matrix for numerical solution of time FDWE. Chen *et al.* [34] used the method of separation of variables and implicit finite difference for solving time FDWE with damping. The fundamental solutions for the fractional diffusion-wave equation are obtained in [35] by Mainardi. Wess [36], and Sun and Wu [39] have proposed a fully discrete difference scheme for a diffusion-wave system. Hu and Zhang [41] proposed a compact finite-difference scheme for the fourth-order fractional diffusion-wave system. In [42], Hu and Zhang proposed and investigated finite-difference methods for fourth-order fractional diffusion-wave and sub-diffusion systems. Godinho *et al.* [43] extended the d'Alembert method to space-time modified Riemann-Liouville fractional wave equations. The authors of [44] proposed Sumudu transform method for solving fractional differential equations and fractional diffusion-wave equation.

An usual way to solve functional equations is to express the solution of the problem, under study, as a linear combination of the so-called basis functions. These basis functions can be orthogonal or non-orthogonal. Approximation by orthogonal families of basis functions has found wide applications in sciences and engineering [45]. The main idea of using an orthogonal basis is that the problem under consideration reduces to a system of linear or non-linear algebraic equations [45], which can be simply solved to achieve an approximate solution of the problem. This can be done by truncated series of orthogonal basis functions for the solution of the problem and by using the operational matrices of these basis functions [45].

Depending on their structure, the orthogonal functions may be mainly classified into three families [46]. The first one includes sets of piecewise constant orthogonal functions such as the Walsh functions, block pulse functions, etc. The second one consists of sets of orthogonal polynomials such as Laguerre, Legendre, Chebyshev, etc. It is well known that we can approximate any smooth function by the eigenfunctions of certain singular Sturm-Liouville problems such as Legendre or Chebyshev orthogonal polynomials. In this way, the truncation error approaches zero faster than any negative power of the number of basis functions used in the approximation [46]. This phenomenon is usually referred to as "The spectral accuracy" [46]. The third is the widely used sets of sine-cosine functions in Fourier series.

The derivatives of the basis functions, and their evaluation at some collocation points, give rise to the so-called operational matrices. The operational matrices of fractional derivative and integration have widely been used to solve various types of fractional differential equations in the last decade. In [47], Saadatmandi and Dehghan have proposed a numerical method based on the fractional Caputo operational matrix of Legendre polynomials to numerically solve multi-term FDEs. Doha *et al.* [48] derived the Jacobi operational matrix of the Caputo fractional derivative for solving linear multi-term FDEs. The Chebyshev [49] and Legendre [47] operational matrices were obtained as special cases of Jacobi operational matrix. The operational matrices of fractional derivatives and fractional integrals for generalized Laguerre polynomials were used to solve multi-term FDEs on a semi-infinite interval in [50]. Heydari *et al.* [51] proposed an efficient computational method based on the operational matrix of fractional derivatives of the shifted Chebyshev polynomials to solve fractional biharmonic equation. Bhrawy *et al.* [52] have proposed a space-time Legendre spectral tau method for the two-sided space-time Caputo fractional diffusion-wave equation. Their proposed method is based on shifted Legendre tau procedure in conjunction with the shifted Legendre operational matrices of Riemann-Liouville fractional integral, left-sided and right-sided fractional derivatives. In [53], the authors proposed a numerical technique based on the shifted Legendre polynomials for solving the time-fractional coupled KdV equations. The author of [53] proposed a new spectral collocation algorithm for solving time-space fractional partial differential equations with sub-diffusion and super-diffusion. In their algorithm, the shifted Legendre Gauss-Lobatto collocation scheme and the shifted Chebyshev Gauss-Radau collocation approximations are used for spatial and temporal discretizations, respectively.

In [45], Heydari *et al.* derived a new operational matrix of fractional derivative for the Legendre wavelets and employed it to obtain a numerical solution for the fractional Poisson equation with Dirichlet boundary conditions. The authors of [54] proposed a computational method based on the operational matrices of fractional integration and derivative of the Legendre wavelets for solving fractional partial differential equations with Dirichlet boundary conditions. In [55], the authors used both of the operational matrices of fractional integration and derivative to get a numerical solution for the time-fractional telegraph equation. In [56], the authors proposed a numerical method based on the Legendre wavelets with their operational matrix of fractional integration to solve the time FDWE. The authors of [57] developed the Chebyshev wavelets to solve systems of non-linear singular fractional Volterra integro-differential equations. Recently, Heydari *et al.* [58] have proposed an accurate numerical method based on the Legendre wavelets and their operational matrix of fractional integration for solving fractional optimal control problems. Complex wavelets and operational matrices for such kind of wavelets have been proposed by Cattani in [59, 60]. In [61], this operational matrix was used to solve some integral equations. Other kinds of continously differentiable wavelets, also known as Shannon wavelets, were considered by Cattani in [62, 63] and their operational matrix was used in [64, 65] to solve some integro-differential equation and fractional differential equations. It is worth noting that solutions of fractional equations can contain some fractional-power terms that the classical orthogonal polynomials and the above mentioned classical bases functions can not cope with. Also, the fractional derivatives of a classical polynomial are not polynomials. So, in these cases the rate of convergence of the numerical approximation is not reasonable when the classical polynomial bases and the above-mentioned bases are used. Therefore, we think that a new family of basis functions should be used to eliminate such difficulties in solving fractional functional equations.

In this paper, our main purpose is to apply the FLFs as a generalization of the Legendre polynomials for solving the FSDE (31) and time FDWE (2). In particular, we first derive a new operational matrix of fractional integration in the Riemann-Liouville sense for these basis functions. Next a Galerkin method based on these bases functions together with their operational matrix of fractional integration is proposed to transform the problems under consideration into the corresponding linear systems of algebraic equations, which can be simply solved for achieving the solutions of these problems.

The paper is organized as follows: In sect. 2, some necessary definitions and mathematical preliminaries of the fractional calculus are reviewed. In sect. 3, the FLFs and some of their properties are investigated. In sect. 4, the proposed method is described for solving the FSDE (1). The proposed method is described for solving the time FDWE (2) in sect. 5. In sect. 6, some numerical examples are provided to show the efficiency and accuracy of the proposed method. The conclusion is drawn in sect. 7.

2 Preliminaries and notations

In this section, we give some necessary definitions and mathematical preliminaries of the fractional calculus which are required for establishing our results.

Definition 1. *A real function* $u(x)$*,* $x > 0$*, is said to be in the space* C_υ*,* $\upsilon \in \mathbb{R}$ *if there exists a real number* p $(> \upsilon)$ *such that* $u(x) = x^p u_1(x)$*, where* $u_1(x) \in C[0,\infty]$ *and it is said to be in the space* C_υ^n *if* $u^{(n)} \in C_\upsilon$*,* $n \in \mathbb{N}$*.*

Definition 2. *The Riemann-Liouville fractional integration operator of order* $\alpha \geq 0$ *of a function* $u \in C_\upsilon$*,* $\upsilon \geq -1$*, is defined as* [1]

$$(I^\alpha u)(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-t)^{\alpha-1} u(t)\mathrm{d}t, & \alpha > 0, \\ u(x), & \alpha = 0. \end{cases} \tag{3}$$

It has the following properties [1]*:*

$$\left(I^\alpha I^\beta u\right)(x) = \left(I^{\alpha+\beta} u\right)(x), \qquad I^\alpha x^\nu = \frac{\Gamma(\nu+1)}{\Gamma(\alpha+\nu+1)} x^{\alpha+\nu}, \tag{4}$$

where $\alpha, \beta \geq 0$ *and* $\nu > -1$*.*

Definition 3. *The fractional derivative operator of order* $\alpha > 0$ *in the Caputo sense is defined as* [1]

$$(D_*^\alpha u)(x) = \begin{cases} \dfrac{\mathrm{d}^n u(x)}{\mathrm{d}x^n}, & \alpha = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x (x-t)^{n-\alpha-1} u^{(n)}(t)\mathrm{d}t, & n-1 < \alpha < n, \end{cases} \tag{5}$$

where n *is an integer,* $x > 0$ *and* $u \in C_1^n$*.*

The useful relation between the Riemann-Liouville operator and Caputo operator is given by the following expression [1]*:*

$$(I^\alpha D_*^\alpha u)(x) = u(x) - \sum_{k=0}^{n-1} u^{(k)}(0^+) \frac{x^k}{k!}, \quad x > 0, \ n-1 < \alpha \leq n, \tag{6}$$

where n *is an integer,* $x > 0$ *and* $u \in C_1^n$*.*

For more details about fractional calculus see [1]*.*

3 The FLFs and their properties

The fractional Legendre functions (FLFs) $FL_n^\vartheta(x)$, are a practical solution of the normalized eigenfunctions of the following singular Sturm-Liouville problem [66]:

$$\left(\left(x - x^{1+\vartheta}\right) FL_n'^\vartheta(x)\right)' + \vartheta^2 n(n+1) x^{\vartheta-1} FL_n^\vartheta(x) = 0, \quad \vartheta > 0. \tag{7}$$

The set $\{FL_n^\vartheta(x)\}_{n=0}^\infty$ forms a complete set of orthogonal functions with respect to the weight function $w^\vartheta(x) = x^{\vartheta-1}$ on the interval $[0,1]$. They can be computed with the aid of the following recurrence relation [66]:

$$(n+1) FL_{n+1}^\vartheta(x) = (2n+1)(2x^\vartheta - 1) FL_n^\vartheta(x) - n FL_{n-1}^\vartheta(x), \quad n \in \mathbb{N}, \tag{8}$$

where $FL_0^\vartheta(x) = 1$ and $FL_1^\vartheta(x) = 2x^\vartheta - 1$.

It is worth nothing that, for $\vartheta = 1$, the FLFs will reduce to the shifted Legendre polynomials on the interval $[0,1]$.

The orthogonality condition for these basis functions is given in [66] by

$$\int_0^1 FL_m^\vartheta(x) FL_n^\vartheta(x) w^\vartheta(x)\mathrm{d}x = \frac{1}{(2n+1)\vartheta} \delta_{mn}, \tag{9}$$

where δ_{mn} is the Kroneker delta.

The analytic form of the $FL_n^\vartheta(x)$ of degree $n\vartheta$ is given by

$$FL_n^\vartheta(x) = \sum_{i=0}^n b_{ni}\, x^{i\vartheta}, \quad n = 0, 1, 2, \ldots, \tag{10}$$

where

$$b_{ni} = \frac{(-1)^{n+i}(n+i)!}{(n-i)!(i!)^2}. \tag{11}$$

Any arbitrary function $u(x)$ defined over $[0,1]$ may be expanded by the FLFs as

$$u(x) = \sum_{n=0}^{\infty} c_n FL_n^{\vartheta}(x), \tag{12}$$

where the coefficients c_n are given by

$$c_n = (2n+1)\vartheta \int_0^1 u(x) FL_n^{\vartheta}(x) w^{\vartheta}(x)\mathrm{d}x. \tag{13}$$

If the infinite series in eq. (12) is truncated, then it can be written as

$$u(x) \simeq \sum_{n=0}^{N} c_n FL_n^{\vartheta}(x) = C^T \Psi^{\vartheta}(x), \tag{14}$$

where T indicates transposition, C and $\Psi^{\vartheta}(x)$ are $(N+1)$ column vectors given by

$$\begin{aligned} C &\triangleq [c_0, c_1, \ldots, c_N]^T, \\ \Psi^{\vartheta}(x) &\triangleq \left[FL_0^{\vartheta}(x), FL_1^{\vartheta}(x), \ldots, FL_N^{\vartheta}(x)\right]^T. \end{aligned} \tag{15}$$

Similarly, an arbitrary function of two variables $u(x,t)$ defined over $[0,1] \times [0,1]$, may be expanded by the FLFs as follows:

$$u(x,t) \simeq \sum_{n=0}^{N}\sum_{m=0}^{M} u_{nm} FL_n^{\vartheta}(x) FL_m^{\mu}(t) = \Psi^{\vartheta}(x)^T U \Psi^{\mu}(t), \quad \vartheta,\ \mu > 0, \tag{16}$$

where $\Psi^{\vartheta}(x)$ and $\Psi^{\mu}(t)$ are $(N+1)$- and $(M+1)$-dimensional FLF vectors, respectively, and $U = [u_{nm}]$ is an $(N+1) \times (M+1)$ matrix with entries

$$u_{nm} = (2n+1)(2m+1)\vartheta\mu \int_0^1 \int_0^1 u(x,t) FL_n^{\vartheta}(x) FL_m^{\mu}(t) w^{\vartheta}(x) w^{\mu}(t)\mathrm{d}x\,\mathrm{d}t. \tag{17}$$

Theorem 1. *Let $\Psi^{\vartheta}(x)$ be the FLFs vector defined in eq. (15) and β be a positive constant. The fractional integration of order β in the Riemann-Liouville sense of the vector $\Psi^{\vartheta}(x)$ can be expressed as*

$$(I^{\beta}\Psi^{\vartheta})(x) \simeq {}_xP_{\vartheta}^{\beta} \Psi^{\vartheta}(x), \tag{18}$$

where ${}_xP_{\vartheta}^{\beta}$ is the $(N+1) \times (N+1)$ OMFI of order β for FLFs, and is given as follows:

$${}_xP_{\vartheta}^{\beta} = \begin{pmatrix} \Omega_{\vartheta}^{\beta}(0,0) & \Omega_{\vartheta}^{\beta}(0,1) & \ldots & \Omega_{\vartheta}^{\beta}(0,N) \\ \Omega_{\vartheta}^{\beta}(1,0) & \Omega_{\vartheta}^{\beta}(1,1) & \ldots & \Omega_{\vartheta}^{\beta}(1,N) \\ \vdots & \vdots & \ldots & \vdots \\ \Omega_{\vartheta}^{\beta}(n,0) & \Omega_{\vartheta}^{\beta}(n,1) & \ldots & \Omega_{\vartheta}^{\beta}(n,N) \\ \vdots & \vdots & \vdots & \vdots \\ \Omega_{\vartheta}^{\beta}(N,0) & \Omega_{\vartheta}^{\beta}(N,1) & \ldots & \Omega_{\vartheta}^{\beta}(N,N) \end{pmatrix}, \tag{19}$$

where

$$\Omega_{\vartheta}^{\beta}(n,j) = \sum_{i=0}^{n} \frac{(-1)^{n+i+j}\vartheta(2j+1)(n+i)!\,\Gamma(i\vartheta+1)\,\Gamma\left(\frac{(i+2)\vartheta+\beta}{\vartheta}\right)\Gamma\left(\frac{(j-i)\vartheta-\beta}{\vartheta}\right)}{((i+1)\vartheta+\beta)\,(n-i)!(i!)^2\,\Gamma(i\vartheta+\beta+1)\,\Gamma\left(-\frac{i\vartheta+\beta}{\vartheta}\right)\Gamma\left(\frac{(i+j+2)\vartheta+\beta}{\vartheta}\right)}, \quad n = 0,1,\ldots,N.$$

Proof. From the analytic form of the FLFs expressed in eq. (10), using eq. (4) and since the Riemann-Liouville's fractional integration is a linear operation, we have

$$\left(I^{\beta}FL_n^{\vartheta}\right)(x)=\sum_{i=0}^{n}b_{ni}I^{\beta}x^{i\vartheta}=\sum_{i=0}^{n}\frac{b_{ni}\Gamma(i\vartheta+1)}{\Gamma(i\vartheta+\beta+1)}x^{i\vartheta+\beta}. \tag{20}$$

Now, by expressing $x^{i\vartheta+\beta}$ in terms of the FLFs, we obtain

$$x^{i\vartheta+\beta}\simeq\sum_{j=0}^{N}c_{ij}FL_j^{\vartheta}(x), \tag{21}$$

where c_{ij} is given by eq. (13) with $u(x)=x^{i\vartheta+\beta}$, that is

$$c_{ij}=(2j+1)\vartheta\int_0^1 x^{i\vartheta+\beta}FL_j^{\vartheta}(x)w^{\vartheta}(x)\mathrm{d}x.$$

By using the expansion of the $FL_j^{\vartheta}(x)$, we have

$$c_{ij}=(2j+1)\vartheta\sum_{r=0}^{j}b_{jr}\int_0^1 x^{(i+r)\vartheta+\beta}w^{\vartheta}(x)\mathrm{d}x=(2j+1)\vartheta\sum_{r=0}^{j}\frac{b_{jr}}{(i+r+1)\vartheta+\beta}. \tag{22}$$

Now, by substituting eqs. (21) and (22) into eq. (20), we get

$$\left(I^{\beta}FL_n^{\vartheta}\right)(x)\simeq\sum_{j=0}^{N}\Omega_{\vartheta}^{\beta}(n,j)FL_j^{\vartheta}(x), \tag{23}$$

where $\Omega_{\vartheta}^{\beta}(n,j)=\sum_{i=0}^{n}\Theta_{nji}$, and

$$\Theta_{nji}=\frac{b_{ni}\Gamma(i\vartheta+1)}{\Gamma(i\vartheta+\beta+1)}\times\vartheta(2j+1)\sum_{r=0}^{j}\frac{b_{jr}}{(i+r+1)\vartheta+\beta}. \tag{24}$$

After some simplifications Θ_{nji} can be expressed in the following form:

$$\Theta_{nji}=\frac{(-1)^{n+i+j}\vartheta(2j+1)(n+i)!\Gamma(i\vartheta+1)\Gamma\left(\frac{(i+2)\vartheta+\beta}{\vartheta}\right)\Gamma\left(\frac{(j-i)\vartheta-\beta}{\vartheta}\right)}{((i+1)\vartheta+\beta)(n-i)!(i!)^2\Gamma(i\vartheta+\beta+1)\Gamma\left(-\frac{i\vartheta+\beta}{\vartheta}\right)\Gamma\left(\frac{(i+j+2)\vartheta+\beta}{\vartheta}\right)},\quad j=0,1,\ldots,N.$$

Therefore, eq. (23) can be written as

$$\left(I^{\beta}FL_n^{\vartheta}\right)(x)\simeq\left[\Omega_{\vartheta}^{\beta}(n,0),\Omega_{\vartheta}^{\beta}(n,1),\ldots,\Omega_{\vartheta}^{\beta}(n,N)\right]\Psi^{\vartheta}(x),\quad n=0,1,\ldots,N,$$

which completes the proof.

Theorem 2. *Let $\Psi^{\mu}(t)$ be the FLFs vector defined in eq. (16) and α be a positive constant. The fractional integration of order α in the Riemann-Liouville sense of the vector $\Psi^{\mu}(t)$ can be expressed as follows:*

$$(I^{\alpha}\Psi^{\mu})(t)\simeq{}_tP_{\mu}^{\alpha}\Psi^{\mu}(t), \tag{25}$$

where ${}_tP_{\mu}^{\alpha}$ is the $(M+1)\times(M+1)$ OMFI of order α for FLFs, and is given by:

$${}_tP_{\mu}^{\alpha}=\begin{pmatrix}\Omega_{\mu}^{\alpha}(0,0) & \Omega_{\mu}^{\alpha}(0,1) & \ldots & \Omega_{\mu}^{\alpha}(0,M)\\ \Omega_{\mu}^{\alpha}(1,0) & \Omega_{\mu}^{\alpha}(1,1) & \ldots & \Omega_{\mu}^{\alpha}(1,M)\\ \vdots & \vdots & \ldots & \vdots\\ \Omega_{\mu}^{\alpha}(m,0) & \Omega_{\mu}^{\alpha}(m,1) & \ldots & \Omega_{\mu}^{\alpha}(m,M)\\ \vdots & \vdots & \vdots & \vdots\\ \Omega_{\mu}^{\alpha}(M,0) & \Omega_{\mu}^{\alpha}(M,1) & \ldots & \Omega_{\mu}^{\alpha}(M,M)\end{pmatrix}, \tag{26}$$

where

$$\Omega_{\mu}^{\alpha}(m,j)=\sum_{i=0}^{m}\frac{(-1)^{m+i+j}\mu(2j+1)(m+i)!\Gamma(i\mu+1)\Gamma\left(\frac{(i+2)\mu+\alpha}{\mu}\right)\Gamma\left(\frac{(j-i)\mu-\alpha}{\mu}\right)}{((i+1)\mu+\alpha)\,(m-i)!(i!)^2\Gamma(i\mu+\alpha+1)\Gamma\left(-\frac{i\mu+\alpha}{\mu}\right)\Gamma\left(\frac{(i+j+2)\mu+\alpha}{\mu}\right)},\quad m=0,1,\ldots,M.$$

Proof. The proof is similar to theorem 1.

4 The proposed method for the FSDE

In this section, the FLFs expansion together with their OMFI are used to solve the FSDE given by

$$\frac{\partial u(x,t)}{\partial t}={}_0D_t^{1-\alpha}\left[\mathcal{K}_\alpha\frac{\partial^2 u(x,t)}{\partial x^2}\right]+f(x,t),\quad (x,t)\in[0,1]\times[0,1], \tag{27}$$

subject to the initial condition,

$$u(x,0)=s(x),\quad x\in[0,1], \tag{28}$$

and boundary conditions,

$$u(0,t)=g_0(t),\quad u(1,t)=g_1(t),\quad t\in[0,1], \tag{29}$$

where ${}_0D_t^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative of order $1-\alpha$ with respect to the variable t, $0<\alpha<1$ is the anomalous diffusion exponent, $\mathcal{K}_\alpha$ is the generalized diffusion constant, s, g_0 and g_1 are given functions in $L^2[0,1]$, and f is a given function in $L^2([0,1]\times[0,1])$.

Equation (31) can be written as [18–20] in the following equivalent form:

$${}_0^cD_t^{\alpha}u(x,t)=\mathcal{K}_\alpha\frac{\partial^2 u(x,t)}{\partial x^2}+\left(I_t^{1-\alpha}f\right)(x,t), \tag{30}$$

where ${}_0^cD_t^{\alpha}$ denotes the Caputo fractional derivative of order α with respect to the variable t and $I_t^{1-\alpha}$ denotes the Riemann-Liouville fractional integral of order $1-\alpha$ with respect to the variable t.

Then, we just need to investigate the following problem:

$${}_0^cD_t^{\alpha}u(x,t)=\mathcal{K}_\alpha\frac{\partial^2 u(x,t)}{\partial x^2}+\left(I_t^{1-\alpha}f\right)(x,t),\quad (x,t)\in[0,1]\times[0,1], \tag{31}$$

subject to the initial condition,

$$u(x,0)=s(x),\quad x\in[0,1], \tag{32}$$

and boundary conditions,

$$u(0,t)=g_0(t),\quad u(1,t)=g_1(t),\quad t\in[0,1]. \tag{33}$$

For solving this problem, by applying the Riemann-Liouville fractional integration of order α with respect to t on both sides of eq. (31) and using the initial condition in eq. (32), we obtain

$$u(x,t)-s(x)=\mathcal{K}_\alpha\left(I_t^{\alpha}\frac{\partial^2 u}{\partial x^2}\right)(x,t)+f(x,t), \tag{34}$$

where $q(x,t)=(I_tf)(x,t)$.

Now, we expand $\frac{\partial^2 u(x,t)}{\partial x^2}$ by the FLFs as follows:

$$\frac{\partial^2 u(x,t)}{\partial x^2}\simeq\Psi^{\vartheta}(x)^TU\Psi^{\mu}(t), \tag{35}$$

where $U=[u_{ij}]$ is an $(N+1)\times(M+1)$ unknown matrix, which should be computed and $\Psi^{\vartheta}(x)$ and $\Psi^{\mu}(t)$ are the FLFs vectors defined in eq. (16).

By integrating eq. (35) two times with respect to x, we have

$$u(x,t)\simeq u(0,t)+x\left(\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}\right)+\Psi^{\vartheta}(x)^T\left({}_xP_{\vartheta}^T\right)^2U\Psi^{\mu}(t), \tag{36}$$

and, by putting $x=1$ in eq. (36), and considering the boundary conditions in eq. (33), we obtain

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} \simeq g_1(t) - g_0(t) - \Psi^{\vartheta}(1)^T \left({}_xP_{\vartheta}^T\right)^2 U\Psi^{\mu}(t). \tag{37}$$

We also expand $g_0(t)$ and $g_1(t)$ by the FLFs as follows:

$$g_0(t) \simeq G_0^T\Psi^{\mu}(t), \qquad g_1(t) \simeq G_1^T\Psi^{\mu}(t), \tag{38}$$

where G_0 and G_1 are the FLFs coefficient vectors.

By substituting eq. (38) into eq. (37), we obtain

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} \simeq \left(G_1^T - G_0^T - \Psi^{\vartheta}(1)^T \left({}_xP_{\vartheta}^T\right)^2 U\right)\Psi^{\mu}(t) \triangleq \widetilde{U}^T\Psi^{\mu}(t). \tag{39}$$

Moreover, we expand x and the unit function by the FLFs as follows:

$$x \simeq \Psi^{\vartheta}(x)^T X, \qquad 1 = \Psi^{\vartheta}(x)^T E, \qquad 1 = \hat{E}^T\Psi^{\mu}(t), \tag{40}$$

where X, E and $\hat{E}$ are the FLFs coefficient vectors.

Now by substituting eq. (39) into eq. (36) and using eq. (40), we have

$$u(x,t) \simeq \Psi^{\vartheta}(x)^T \left[EG_0^T + X\widetilde{U}^T + \left({}_xP_{\vartheta}^T\right)^2 U\right]\Psi^{\mu}(t) \triangleq \Psi^{\vartheta}(x)^T \Lambda\Psi^{\mu}(t). \tag{41}$$

Furthermore, we expand $s(x)$ and $q(x,t)$ by the FLFs as follows:

$$s(x) \simeq \Psi^{\vartheta}(x)^T S, \qquad q(x,t) \simeq \Psi^{\vartheta}(x)^T Q\Psi^{\mu}(t), \tag{42}$$

where S is the known FLFs coefficients vector for $s(x)$, and Q is the known FLFs coefficients matrix for $q(x,t)$. Then, by substituting eqs. (35), (41) and (42) into eq. (34) and using OMFI of FLFs, we write the residual function $R(x,t)$ for eq. (31) as follows:

$$R(x,t) = \Psi^{\vartheta}(x)^T \left[\Lambda - \mathcal{K}_{\alpha}U\,{}_tP_{\mu}^{\alpha} - S\hat{E}^T - Q\right]\Psi^{\mu}(t). \tag{43}$$

As in a typical Galerkin method [46], we generate $(N+1)\times(M+1)$ linear algebraic equations with the unknown expansion coefficients, u_{ij}, $i=1,2,\ldots,(N+1)$; $j=1,2,\ldots,(M+1)$ as

$$\int_0^1\int_0^1 R(x,t)Fl_i^{\vartheta}(x)FL_j^{\mu}(t)w^{\vartheta}(x)w^{\mu}(t)\mathrm{d}x\,\mathrm{d}t = 0, \quad i=1,2,\ldots,(N+1),\ j=1,2,\ldots,(M+1). \tag{44}$$

Finally, by solving this system for the unknown matrix U, we obtain an approximate solution for the problem by substituting U in eq. (41).

The algorithm of the proposed method is presented as follows:

Algorithm 1

Input: $N, M \in \mathbb{N}$, $\mu, \vartheta \in \mathbb{R}^+$; $\mathcal{K}_{\alpha} \in \mathbb{R}^+$, $0<\alpha<1$; the functions s, g_0, g_1 and f.
Step 1: Define the FLFs $FL_n^{\vartheta}(x)$ and $FL_m^{\mu}(t)$ by eq. (10), and the weight functions $w^{\vartheta}(x)$ and $w^{\mu}(t)$.
Step 2: Construct the FLFs vectors $\Psi^{\vartheta}(x)$ and $\Psi^{\mu}(t)$ from eqs. (15) and (16).
Step 3: Compute the operational matrices ${}_xP_{\vartheta}^2$ and ${}_tP_{\mu}^{\alpha}$ using eqs. (19) and (26).
Step 4: Define the $(N+1)\times(M+1)$ unknown matrix $U=[u_{ij}]$.
Step 5: Compute the vectors G_0, G_1, X, E and S in eqs. (38), (40) and (42).
Step 6: Compute the vector $\widetilde{U}$ by eq. (39).
Step 7: Compute the matrix Λ using eq. (41).
Step 8: Compute the matrix Q by eq. (42).
Step 9: Compute the residual function $R(x,t)$ using eq.(43).
Step 10: Put $\int_0^1\int_0^1 R(x,t)Fl_i^{\vartheta}(x)FL_j^{\mu}(t)w^{\vartheta}(x)w^{\mu}(t)\mathrm{d}x\,\mathrm{d}t=0$, $i=1,2,\ldots,(N+1)$, $j=1,2,\ldots,(M+1)$.
Step 11: Solve the linear system of algebraic equations in Step 10 for the unknown matrix U.
Output: The approximate solution: $u(x,t)\simeq\Psi(x)^T\Lambda\Psi(t)$.

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5 The proposed method for the time FDWE

In this section, we use the OMFI of the FLFs to solve the time FDWE given by:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\lambda\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t),\quad (x,t)\in[0,1]\times[0,1],\quad 1<\alpha\le 2, \tag{45}$$

subject to the initial conditions,

$$u(x,0)=f_0(x),\quad u_t(x,0)=f_1(x),\quad x\in[0,1], \tag{46}$$

and boundary conditions,

$$u(0,t)=g_0(t),\quad u(1,t)=g_1(t),\quad t\in[0,1], \tag{47}$$

where $\lambda>0$ is a constant, f_0, f_1, g_0 and g_1 are given functions in $L^2[0,1]$, and q is a given function in $L^2([0,1]\times[0,1])$.

To solve eq. (45), by applying the Riemann-Liouville fractional integration of order α with respect to t on both sides of eq. (45) and considering the initial conditions in eq. (46), we obtain

$$u(x,t)-v(x,t)+\lambda\left(I_t^{\alpha-1}u\right)(x,t)=\left(I_t^\alpha\frac{\partial^2 u}{\partial x^2}\right)(x,t)+\left(I_t^\alpha q\right)(x,t), \tag{48}$$

where $v(x,t)=f_0(x)+tf_1(x)-\frac{\lambda t^{\alpha-1}}{\Gamma(\alpha)}f_0(x)$.

Now we approximate $\frac{\partial^2 u(x,t)}{\partial x^2}$ by the FLFs as follows:

$$\frac{\partial^2 u(x,t)}{\partial x^2}\simeq\Psi^\vartheta(x)^T U\Psi^\mu(t), \tag{49}$$

where $U=[u_{ij}]$ is an $(N+1)\times(M+1)$ unknown matrix which should be found.

Moreover, by integrating eq. (49) two times with respect to x, we have

$$u(x,t)\simeq u(0,t)+x\left(\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}\right)+\Psi^\vartheta(x)^T\left({}_xP_\vartheta^T\right)^2U\Psi^\mu(t), \tag{50}$$

and, by putting $x=1$ in eq. (50), and considering the boundary conditions in eq. (47), we obtain

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}\simeq g_1(t)-g_0(t)-\Psi^\vartheta(1)^T\left({}_xP_\vartheta^T\right)^2U\Psi^\mu(t). \tag{51}$$

We also expand $g_0(t)$ and $g_1(t)$ by the FLFs as follows:

$$g_0(t)\simeq G_0^T\Psi^\mu(t),\qquad g_1(t)\simeq G_1^T\Psi^\mu(t), \tag{52}$$

where G_0 and G_1 are the FLFs coefficient vectors.

By substituting eq. (52) into eq. (51), we obtain

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0}\simeq\left(G_1^T-G_0^T-\Psi^\vartheta(1)^T\left({}_xP_\vartheta^T\right)^2U\right)\Psi^\mu(t)\triangleq\widetilde{U}^T\Psi^\mu(t). \tag{53}$$

Moreover, we expand x and the unit function by the FLFs as follows:

$$x\simeq\Psi^\vartheta(x)^T X,\qquad 1=\Psi^\vartheta(x)^T E, \tag{54}$$

where X and E are the FLFs coefficient vectors.

Now, by substituting eq. (53) into eq. (50) and using eq. (54), we have

$$u(x,t)\simeq\Psi^\vartheta(x)^T\left[EG_0^T+X\widetilde{U}^T+\left({}_xP_\vartheta^T\right)^2U\right]\Psi^\mu(t)\triangleq\Psi^\vartheta(x)^T\Lambda\Psi^\mu(t). \tag{55}$$

Furthermore, we expand $v(x,t)$ and $f(x,t)$ by the FLFs as follows:

$$v(x,t)\simeq\Psi^\vartheta(x)^T V\Psi^\mu(t),\qquad f(x,t)\simeq\Psi^\vartheta(x)^T F\Psi^\mu(t), \tag{56}$$

where V and F are the known FLFs coefficient matrices for $v(x,t)$ and $f(x,t)$, respectively.

Then, by substituting eqs. (49), (55) and (56) into eq. (48) and using OMFI for FLFs, we write the residual function $R(x,t)$ for eq. (45) as follows:

$$R(x,t) = \Psi^{\vartheta}(x)^T \left[\Lambda + \lambda \Lambda \, {}_tP_{\mu}^{(\alpha-1)} - U \, {}_tP_{\mu}^{\alpha} - V - F \, {}_tP_{\mu}^{\alpha} \right] \Psi^{\mu}(t). \tag{57}$$

As in a typical Galerkin method [46], we generate $(N+1) \times (M+1)$ linear algebraic equations with the unknown expansion coefficients, u_{ij}, $i = 1, 2, \ldots, (N+1)$; $j = 1, 2, \ldots, (M+1)$ as

$$\int_0^1 \int_0^1 R(x,t) Fl_i^{\vartheta}(x) FL_j^{\mu}(t) w^{\vartheta}(x) w^{\mu}(t) \mathrm{d}x \, \mathrm{d}t = 0, \quad i = 1, 2, \ldots, (N+1), \; j = 1, 2, \ldots, (M+1). \tag{58}$$

Finally, by solving this system for the unknown matrix U, we obtain an approximate solution for the problem by substituting U in eq. (55).

The algorithm of the proposed method is presented as follows:

Algorithm 2

Input: $N, M \in \mathbb{N}$, $\mu, \vartheta \in \mathbb{R}^+$; $\lambda \in \mathbb{R}^+$, $1 < \alpha \leq 2$; the functions f_0, f_1, g_0, g_1 and f.
Step 1: Define the FLFs $FL_n^{\vartheta}(x)$ and $FL_m^{\mu}(t)$ by eq. (10), and the weight functions $w^{\vartheta}(x)$ and $w^{\mu}(t)$.
Step 2: Construct the FLFs vectors $\Psi^{\vartheta}(x)$ and $\Psi^{\mu}(t)$ from eqs. (15) and (16).
Step 3: Compute the operational matrices ${}_xP_{\vartheta}^2$, ${}_tP_{\mu}^{(\alpha-1)}$ and ${}_tP_{\mu}^{\alpha}$ using eqs. (19) and (26).
Step 4: Define the $(N+1) \times (M+1)$ unknown matrix $U = [u_{ij}]$.
Step 5: Compute the vectors G_0, G_1, X and E in eqs. (52) and (54).
Step 6: Compute the vector $\widetilde{U}$ using eq. (53).
Step 7: Compute the matrix Λ by eq. (55).
Step 8: Compute the matrices V and F using eq. (56).
Step 9: Compute the residual function $R(x,t)$ by eq. (57).
Step 10: Put $\int_0^1 \int_0^1 R(x,t) Fl_i^{\vartheta}(x) FL_j^{\mu}(t) w^{\vartheta}(x) w^{\mu}(t) \mathrm{d}x \, \mathrm{d}t = 0$, $i = 1, 2, \ldots, (N+1)$, $j = 1, 2, \ldots, (M+1)$.
Step 11: Solve the linear system of algebraic equations in Step 10 for the unknown matrix U.
Output: The approximate solution: $u(x,t) \simeq \Psi(x)^T \Lambda \Psi(t)$.

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6 Illustrative test problems

In this section, some numerical examples are provided to demonstrate the efficiency and reliability of the proposed method. Also, the absolute errors of the proposed method in some different points $(x_i, t_i) \in [0,1] \times [0,1]$ are reported as follows:

$$|e(x_i, t_i)| = \left| \Psi^{\vartheta}(x_i)^T \Lambda \Psi^{\mu}(t_i) - u(x_i, t_i) \right|.$$

It is worth to mention that all numeric computations are performed via MAPLE 17 with 50 decimal digits.

Example 1

Consider the following time FSDE [17,18]:

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\alpha} \left[\frac{\partial^2 u(x,t)}{\partial x^2} \right] + e^x \left((1+\alpha) t^{\alpha} - \frac{\Gamma(2+\alpha)}{\Gamma(1+2\alpha)} t^{2\alpha} \right),$$

subject to the initial and boundary conditions,

$$u(x,0) = 0, \qquad u(0,t) = t^{1+\alpha}, \qquad u(1,t) = e t^{1+\alpha},$$

and the exact solution $u(x,t) = e^x t^{1+\alpha}$.

This problem is also solved by the proposed method for $N = M = 10$. The graphs of the absolute errors for $(\vartheta = 1, \mu = \frac{1}{4})$ and $\alpha = 0.25$ at $t = 0.5$ and $x = 0.5$ are shown in fig. 1. The space-time graphs of the approximate solution and absolute error for $\alpha = 0.25$ are shown in fig. 2. From figs. 1 and 2, it can be seen that the proposed

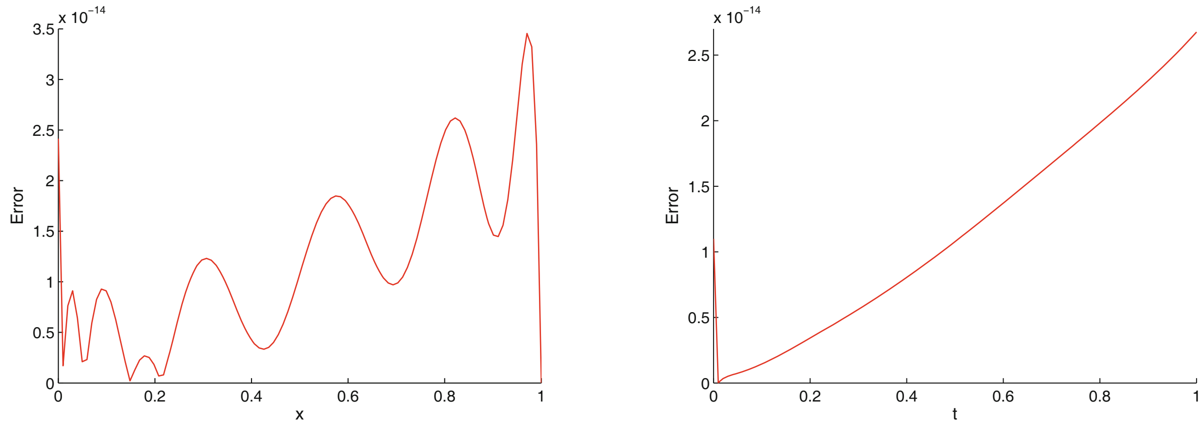

Fig. 1. The graphs of the absolute errors at $t = 0.5$ (left) and $x = 0.5$ (right) for $\alpha = 0.25$ with $(\vartheta = 1, \mu = \frac{1}{4})$ for example 1.

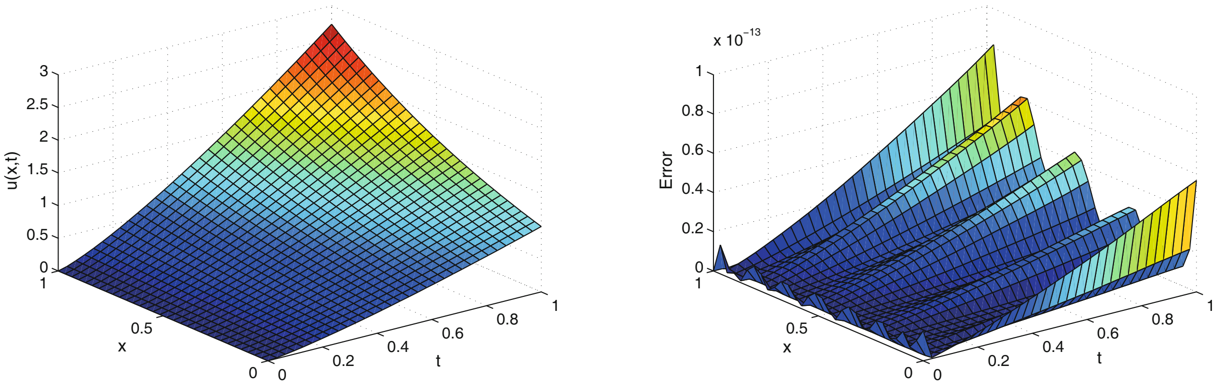

Fig. 2. The space-time graphs of the approximate solution (left) and absolute error (right) for $\alpha = 0.25$ with $(\vartheta = 1, \mu = \frac{1}{4})$ for example 1.

Table 1. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points (x_i, t_i) for example 1.

	$(\vartheta = 1, \mu = \frac{1}{4})$			$(\vartheta = 1, \mu = \frac{1}{2})$		
(x, t)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
$(0.1, 0.1)$	1.1653E-15	5.4630E-16	8.9967E-17	5.4027E-07	5.4630E-16	6.6203E-08
$(0.2, 0.2)$	4.8299E-16	3.4482E-16	3.7702E-17	5.6531E-07	3.4481E-16	6.8081E-08
$(0.3, 0.3)$	6.3654E-15	4.3503E-15	2.7852E-15	9.1462E-07	4.3503E-15	9.9607E-08
$(0.4, 0.4)$	3.1785E-15	2.0544E-15	1.0999E-15	3.5436E-07	2.0544E-15	3.0814E-08
$(0.5, 0.5)$	1.2673E-14	1.0547E-14	8.5076E-15	7.6824E-07	1.0547E-14	9.0129E-08
$(0.6, 0.6)$	2.2290E-14	1.9065E-14	1.6046E-14	7.7987E-07	1.9065E-14	7.9461E-08
$(0.7, 0.7)$	1.5053E-14	1.3220E-14	1.1282E-14	7.8472E-07	1.3220E-14	9.3327E-08
$(0.8, 0.8)$	4.4996E-14	4.2386E-14	3.9374E-14	1.4603E-06	4.2386E-14	1.5929E-07
$(0.9, 0.9)$	3.0894E-14	2.9780E-14	2.8455E-14	7.8523E-07	2.9780E-14	8.1111E-08

method is very efficient and accurate for solving this problem. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points $(x_i, t_i) \in [0, 1] \times [0, 1]$ for $\vartheta = 1$ and some values for μ are shown in tables 1 and 2. From the numerical results reported in tables 1 and 2, it can be observed that the best choice for (ϑ, μ) to obtain accurate numerical solutions for all chosen $0 < \alpha < 1$ is $(\vartheta = 1, \mu = \frac{1}{4})$. In general, it can be concluded that by choosing suitable values for (ϑ, μ) only a small number of the FLFs is needed to obtain satisfactory results. In [17], Cui applied the high-order compact finite-difference method for solving this problem for two cases $\alpha = 0.25$ and $\alpha = 0.75$. The maximum absolute errors obtained in [17] for the cases $\alpha = 0.25$ and $\alpha = 0.75$ are 1.8928×10^{-5} and 5.6363×10^{-5}, respectively. In [18], Gao and Sun proposed a compact finite-difference scheme for this problem,

Table 2. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points (x_i, t_i) for example 1.

	$(\vartheta = 1, \mu = \frac{3}{4})$			$(\vartheta = 1, \mu = 1)$		
(x, t)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
$(0.1, 0.1)$	1.0970E-05	5.4625E-16	1.5824E-06	2.7476E-05	1.7375E-05	5.7483E-06
$(0.2, 0.2)$	8.3776E-06	3.4481E-16	1.2396E-06	1.5595E-05	1.0133E-05	3.4165E-06
$(0.3, 0.3)$	9.7453E-06	4.3502E-15	1.3699E-06	2.9960E-05	1.8121E-05	5.7788E-06
$(0.4, 0.4)$	5.7152E-06	2.0544E-15	7.2884E-07	3.6442E-05	2.1315E-05	6.6293E-06
$(0.5, 0.5)$	2.4679E-06	8.9159E-15	7.0020E-07	2.0080E-05	1.1169E-05	3.3581E-06
$(0.6, 0.6)$	1.0663E-06	1.9065E-14	1.5126E-06	1.1095E-05	7.2043E-06	2.3820E-06
$(0.7, 0.7)$	5.8631E-06	1.3221E-14	7.5966E-07	3.5053E-05	2.0899E-05	6.5923E-06
$(0.8, 0.8)$	4.3308E-06	4.2386E-14	6.9124E-07	4.3419E-05	2.5446E-05	7.9587E-06
$(0.9, 0.9)$	1.0588E-05	2.9780E-14	1.5640E-06	5.2516E-05	3.0834E-05	9.6672E-06

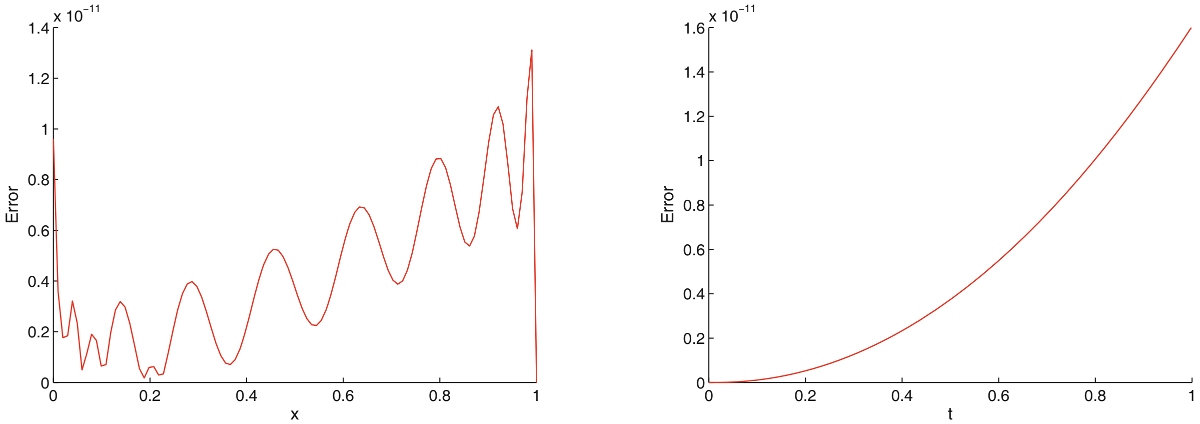

Fig. 3. The graphs of the absolute errors at $t = 0.5$ (left) and $x = 0.5$ (right) for $\alpha = 0.50$ with $(\vartheta = 1, \mu = \frac{1}{2})$ for example 2.

and compared the numerical solutions obtained for the cases $\alpha = 0.25$ and $\alpha = 0.75$ with numerical results obtained in [17]. They showed that their numerical results are apparently more accurate than numerical results via difference scheme in [17]. It is also worth mentioning that the maximum absolute errors obtained in [18] for the cases $\alpha = 0.25$ and $\alpha = 0.75$ are 1.4338×10^{-8} and 7.1085×10^{-6}, respectively. From table 1, it can be seen that for $(\vartheta = 1, \mu = \frac{1}{4})$ our results are more accurate than results in [17, 18]. It is worth noting that the implementation of our proposed method is very simple in comparison with the above mentioned methods.

Example 2

Consider the following time FSDE [17]:

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + \left(2t + \frac{8\pi^2 t^{\alpha+1}}{\Gamma(2+\alpha)}\right)\sin(2\pi x),$$

with the homogeneous initial and boundary conditions, and the exact solution $u(x,t) = t^2\sin(2\pi x)$.

This problem is solved by the proposed method for $N = M = 15$. The graphs of the absolute errors for $(\vartheta = 1, \mu = \frac{1}{2})$ and $\alpha = 0.50$ at $t = 0.5$ and $x = 0.5$ are shown in fig. 3. The space-time graphs of the approximate solution and absolute error for $\alpha = 0.50$ are shown in fig. 4. From figs. 3 and 4, one can see that the proposed method provides numerical solutions with high accuracy for the problem. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points $(x_i, t_i) \in [0,1] \times [0,1]$ for $\vartheta = 1$ and some values for μ are shown in tables 3 and 4. From the numerical results reported in tables 3 and 4, it can be observed that the best choices for values (ϑ, μ) to obtain accurate numerical solutions for all chosen $0 < \alpha < 1$ are $(\vartheta = 1, \mu = \frac{1}{4}, \frac{1}{2}, 1)$. Furthermore, it can be seen that by choosing suitable (ϑ, μ) only a small number of the FLFs is needed to obtain satisfactory results.

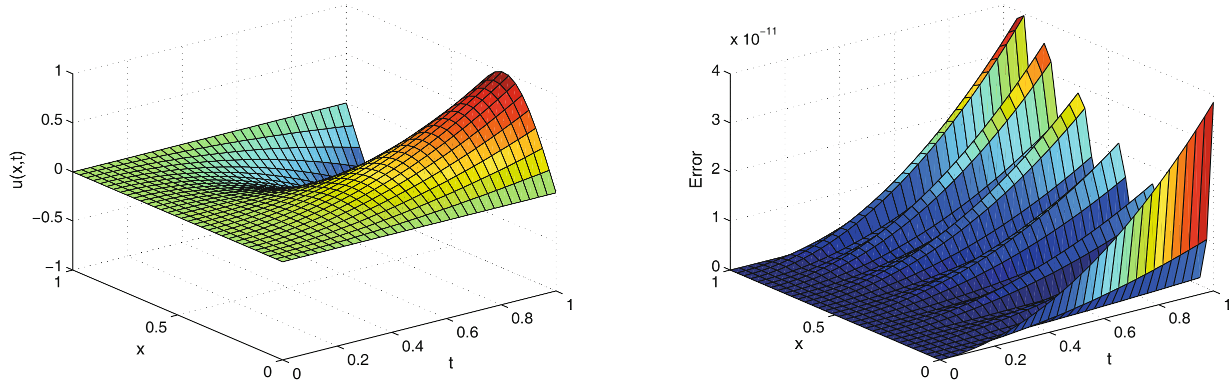

Fig. 4. The space-time graphs of the approximate solution (left) and absolute error (right) for $\alpha = 0.50$ with $(\vartheta = 1, \mu = \frac{1}{2})$ for example 2.

Table 3. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points (x_i, t_i) for example 2.

	$(\vartheta = 1, \mu = \frac{1}{4})$			$(\vartheta = 1, \mu = \frac{1}{2})$		
(x, t)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
$(0.1, 0.1)$	2.0893E-14	2.9398E-14	3.9670E-14	2.0893E-14	2.9398E-14	3.9670E-14
$(0.2, 0.2)$	9.0586E-14	1.3648E-13	1.9747E-13	9.0586E-14	1.3648E-13	1.9747E-13
$(0.3, 0.3)$	1.3784E-12	1.2651E-12	1.1144E-12	1.3784E-14	1.2651E-12	1.1144E-12
$(0.4, 0.4)$	1.5418E-12	1.3410E-12	1.0796E-12	1.5418E-12	1.3410E-12	1.0796E-12
$(0.5, 0.5)$	3.6180E-12	3.3177E-12	2.9391E-12	3.6180E-12	3.3177E-12	2.9391E-12
$(0.6, 0.6)$	8.2284E-12	7.8632E-12	7.4165E-12	8.2284E-12	7.8632E-12	7.4165E-12
$(0.7, 0.7)$	8.7568E-12	8.3601E-12	7.8917E-12	8.7568E-12	8.3601E-12	7.8917E-12
$(0.8, 0.8)$	2.3438E-11	2.3076E-11	2.2666E-11	2.3438E-11	2.3076E-11	2.2666E-11
$(0.9, 0.9)$	3.0885E-11	3.0649E-11	3.0392E-11	3.0885E-11	3.0649E-11	3.0392E-11

Table 4. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points (x_i, t_i) for example 2.

	$(\vartheta = 1, \mu = \frac{3}{4})$			$(\vartheta = 1, \mu = 1)$		
(x, t)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
$(0.1, 0.1)$	4.8506E-09	5.5714E-09	7.3549E-09	2.0893E-14	2.9399E-14	3.9671E-14
$(0.2, 0.2)$	3.2969E-08	3.0956E-08	3.0936E-08	9.0586E-14	1.3648E-13	1.9747E-13
$(0.3, 0.3)$	2.1475E-08	1.8831E-08	1.7510E-08	1.3784E-12	1.2651E-12	1.1144E-12
$(0.4, 0.4)$	1.1284E-08	9.5736E-09	8.6436E-08	1.5418E-12	1.3410E-12	1.0796E-12
$(0.5, 0.5)$	4.0329E-12	3.7410E-12	3.3720E-12	4.0329E-12	3.7410E-12	3.3720E-12
$(0.6, 0.6)$	1.3401E-08	1.2229E-08	1.2005E-08	8.2284E-12	7.8632E-12	7.4165E-12
$(0.7, 0.7)$	2.3177E-08	2.1914E-08	2.2290E-08	8.7568E-12	8.3601E-12	7.8917E-12
$(0.8, 0.8)$	2.2582E-08	2.0672E-08	2.0685E-08	2.3438E-11	2.3076E-11	2.2666E-11
$(0.9, 0.9)$	1.3519E-08	3.8082E-09	7.5906E-09	3.0885E-11	3.0649E-11	3.0392E-11

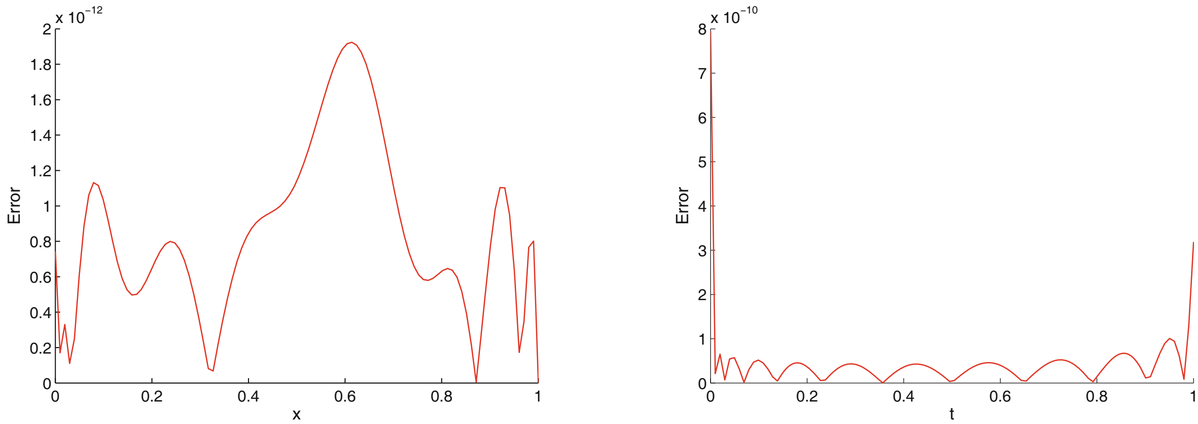

Fig. 5. The graphs of the absolute errors at $t = 0.5$ (left) and $x = 0.5$ (right) for $\alpha = 0.75$ with ($\vartheta = 1, \mu = \frac{1}{2}$) for example 3.

In [17], Cui applied high-order compact finite-difference method for solving this problem and investigated the numerical solutions for $\alpha = 0.50$. The maximum absolute error obtained in [17] is 5.0200×10^{-4}. It is also expressed that the coefficient matrix of the unknowns in his method is tridiagonal and the scheme can be solved by the Thomas algorithm, but since all time history must be in memory, the memory requirement are costly, and the computer memory will limit the step sizes. To avoid of this fact the "Short Memory" principle has been used, but it is worth noting that the numerical experiments are not very satisfactory unless sufficiently many previous time steps have been included. However, we believe that our method is more efficient and accurate in comparison with the above mentioned method. Furthermore, the implementation of our proposed method is very simple in comparison with the mentioned method.

Example 3

Consider the following time FSDE [18,19]:

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\alpha}\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] + f(x,t),$$

with the homogenous initial and boundary conditions, and

$$f(x,t) = (2+\alpha)e^x x^2(1-x)^2 t^{\alpha+1} - \frac{\Gamma(3+\alpha)}{\Gamma(2\alpha+2)}t^{2\alpha+1}e^x\left(2 - 8x + x^2 + 6x^3 + x^4\right).$$

The exact solution for this problem is given in [18,19] as $u(x,t) = e^x x^2(1-x)^2 t^{\alpha+2}$.

This problem is also solved by the proposed method for $N = M = 12$. The graphs of the absolute errors for ($\vartheta = 1, \mu = \frac{1}{2}$) and $\alpha = 0.75$ at $t = 0.5$ and $x = 0.5$ are shown in fig. 5. The space-time graphs of the approximate solution and absolute error for $\alpha = 0.75$ are shown in fig. 6. From figs. 5 and 6, it can be observed that the proposed method is very accurate for solving this problem. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points $(x_i, t_i) \in [0,1] \times [0,1]$ for $\vartheta = 1$ and some values for μ are shown in tables 5 and 6. From tables 5 and 6, one can see that the best choice for values (ϑ, μ) to obtain accurate numerical solutions for all chosen $0 < \alpha < 1$ is ($\vartheta = 1, \mu = \frac{1}{4}$). Furthermore, it can be seen that by choosing suitable (ϑ, μ), only a few number of the FLFs is needed to obtain satisfactory results. In [18], Zhao and Sun, and in [19] Ren *et al.* have proposed a box-type scheme and a compact difference scheme, respectively for solving this problem with Neumann boundary conditions.

Example 4

Consider the following time FDWE [34]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 1 < \alpha \leq 2,$$

with the homogenous initial and boundary conditions, and

$$f(x,t) = \frac{2x(1-x)}{\Gamma(3-\alpha)}t^{2-\alpha} + 2tx(1-x) + 2t^2.$$

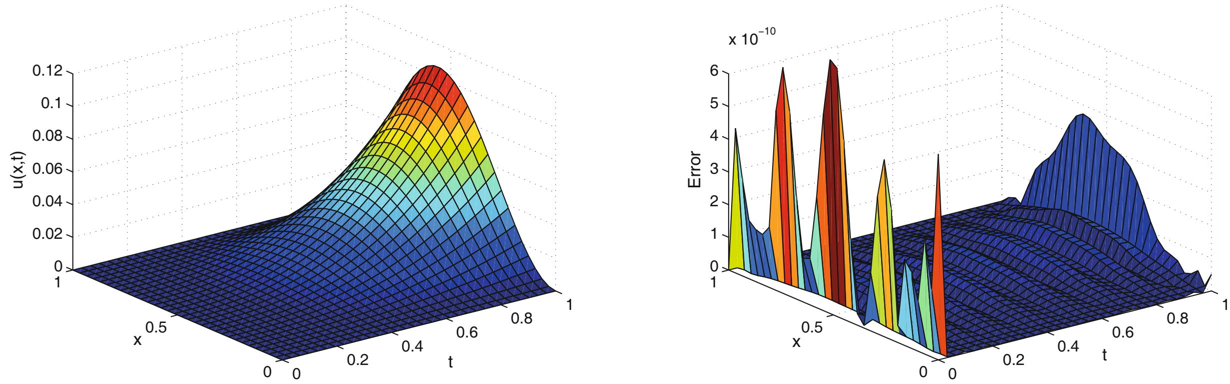

Fig. 6. The space-time graphs of the approximate solution (left) and absolute error (right) for $\alpha = 0.75$ with $(\vartheta = 1, \mu = \frac{1}{2})$ for example 3.

Table 5. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points (x_i, t_i) for example 3.

	$(\vartheta = 1, \mu = \frac{1}{4})$			$(\vartheta = 1, \mu = \frac{1}{2})$		
(x,t)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
$(0.1, 0.1)$	1.0591E-14	1.6530E-13	1.0008E-12	2.4706E-11	7.2647E-14	5.3118E-12
$(0.2, 0.2)$	2.7800E-15	1.0559E-13	4.4145E-13	5.6743E-11	7.4827E-14	1.2859E-11
$(0.3, 0.3)$	4.2109E-14	3.8730E-14	2.0540E-13	1.1572E-10	9.6851E-14	2.5763E-11
$(0.4, 0.4)$	3.6027E-13	3.4266E-13	9.3389E-13	1.2821E-10	1.3099E-13	3.2749E-11
$(0.5, 0.5)$	1.2908E-13	8.4652E-14	8.1460E-14	1.7255E-11	4.2559E-13	1.1406E-12
$(0.6, 0.6)$	6.7786E-13	3.3773E-13	1.6059E-12	1.8323E-10	1.4608E-12	4.1547E-11
$(0.7, 0.7)$	4.9752E-13	2.3678E-13	8.1717E-13	1.4779E-10	1.0059E-12	3.9874E-11
$(0.8, 0.8)$	5.5828E-13	3.1919E-13	1.9012E-12	1.2709E-11	1.0278E-12	7.8293E-12
$(0.9, 0.9)$	9.7112E-13	8.9222E-13	6.1592E-12	1.7107E-11	1.9818E-12	7.3444E-12

Table 6. The absolute errors of the approximate solutions for some values of $0 < \alpha < 1$ at some different points (x_i, t_i) for example 3.

	$(\vartheta = 1, \mu = \frac{3}{4})$			$(\vartheta = 1, \mu = 1)$		
(x,t)	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$
$(0.1, 0.1)$	3.6692E-14	2.0811E-10	9.6903E-11	5.0957E-09	3.2396E-09	1.0391E-09
$(0.2, 0.2)$	1.5601E-15	8.4011E-10	4.0451E-10	1.7284E-08	1.1483E-08	4.0656E-09
$(0.3, 0.3)$	3.1166E-14	1.3156E-09	6.3452E-10	2.6434E-08	1.7734E-08	6.2650E-09
$(0.4, 0.4)$	1.6146E-13	1.2714E-09	6.2185E-10	3.0799E-08	2.1441E-08	7.5830E-09
$(0.5, 0.5)$	1.5324E-14	2.4291E-10	1.3709E-10	2.4346E-08	1.8432E-08	6.6094E-09
$(0.6, 0.6)$	3.5489E-13	1.1940E-09	5.7066E-10	9.3755E-09	9.1548E-09	3.3786E-09
$(0.7, 0.7)$	3.7479E-13	1.8083E-09	8.9997E-10	2.5547E-09	6.9719E-10	2.5998E-10
$(0.8, 0.8)$	4.6625E-13	1.3820E-09	6.9911E-10	3.1730E-09	7.3498E-10	5.2779E-10
$(0.9, 0.9)$	9.5645E-13	6.1495E-10	3.1335E-10	3.3165E-09	2.4949E-09	2.1812E-10

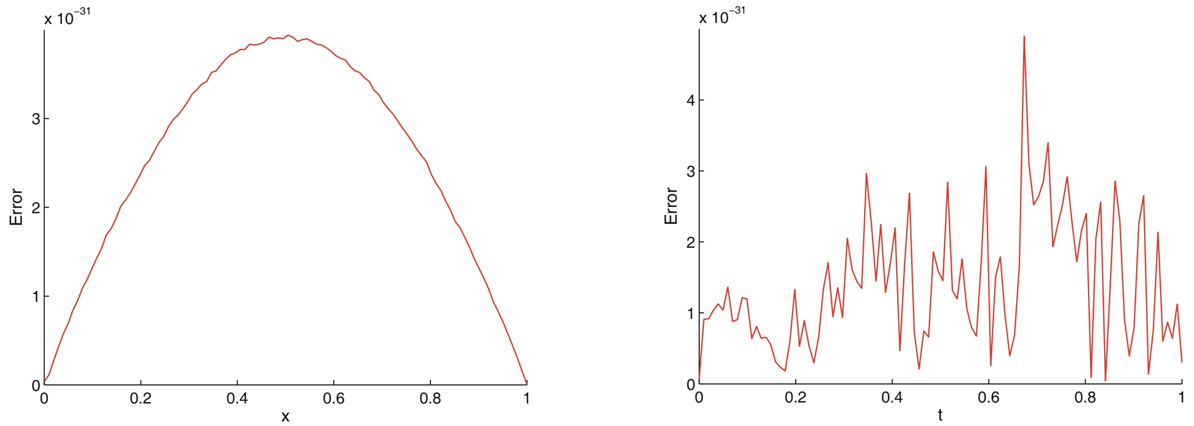

Fig. 7. The graphs of the absolute errors at $t = 0.5$ (left) and $x = 0.5$ (right) for $\alpha = 1.50$ with $(\vartheta = 1, \mu = \frac{1}{4})$ for example 4.

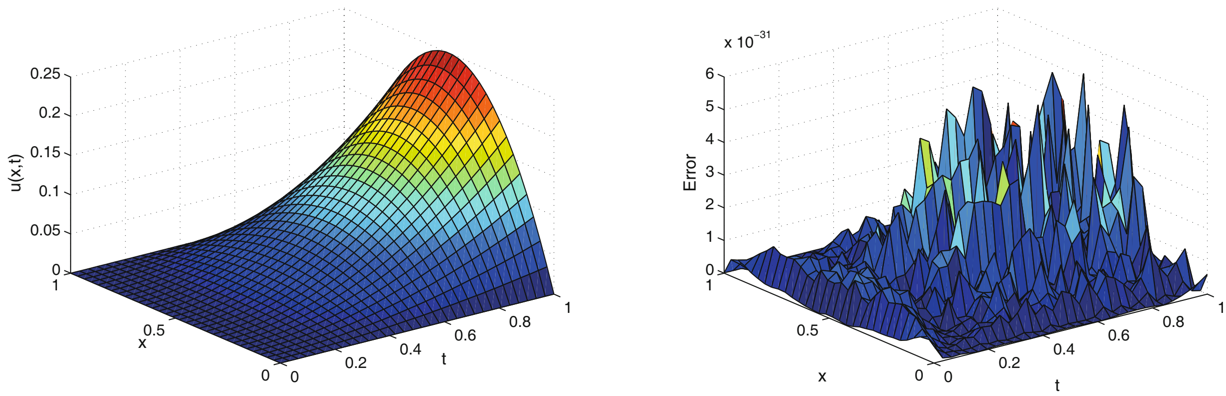

Fig. 8. The space-time graphs of the approximate solution (left) and absolute error (right) for $\alpha = 1.50$ with $(\vartheta = 1, \mu = \frac{1}{4})$ for example 4.

Table 7. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points (x_i, t_i) for example 4.

	$(\vartheta = 1, \mu = \frac{1}{4})$			$(\vartheta = 1, \mu = \frac{1}{2})$		
(x, t)	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$
$(0.1, 0.1)$	1.3500E-33	2.0760E-32	1.5710E-32	3.5748E-08	9.8100E-33	7.8522E-07
$(0.2, 0.2)$	5.0000E-32	6.3400E-32	1.4700E-32	1.0457E-07	6.2000E-33	1.6394E-06
$(0.3, 0.3)$	2.6000E-32	9.3000E-32	6.5000E-32	1.0680E-08	7.0000E-33	1.1692E-06
$(0.4, 0.4)$	1.7000E-32	1.1900E-31	1.3700E-31	2.0266E-07	2.0000E-32	2.5900E-06
$(0.5, 0.5)$	2.2400E-31	3.9800E-31	9.3000E-32	1.1693E-07	6.0000E-33	3.1399E-06
$(0.6, 0.6)$	3.2100E-31	4.7800E-31	2.2700E-31	1.7656E-07	8.0000E-33	6.3779E-07
$(0.7, 0.7)$	3.3000E-31	2.0000E-31	3.9000E-31	2.2697E-07	0.0000	3.3699E-06
$(0.8, 0.8)$	5.0000E-31	1.3000E-31	0.0000	5.7136E-08	1.0000E-32	1.0418E-06
$(0.9, 0.9)$	8.2000E-31	5.7000E-32	8.6000E-32	1.3647E-07	5.0000E-33	1.4630E-06

The exact solution for this problem is given in [34] as $u(x,t) = t^2 x(1-x)$.

This problem is also solved by the proposed method for $N = M = 8$. The graphs of the absolute errors for $(\vartheta = 1, \mu = \frac{1}{4})$ and $\alpha = 1.50$ at $t = 0.5$ and $x = 0.5$ are shown in fig. 7. The space-time graphs of the approximate solution and absolute error for $\alpha = 1.50$ are shown in fig. 8. By figs. 7 and 8, it can be seen that the proposed method provides numerical solutions with high accuracy for the problem. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points $(x_i, t_i) \in [0,1] \times [0,1]$ for $\vartheta = 1$ and some values for μ are shown in tables 7 and 8. From the numerical results reported in tables 7 and 8, one can see that the best choice for (ϑ, μ) to obtain accurate numerical solutions for all chosen $1 < \alpha < 2$ is $(\vartheta = 1, \mu = \frac{1}{4})$.

Table 8. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points (x_i, t_i) for example 4.

(x,t)	$(\vartheta = 1, \mu = \frac{3}{4})$ $\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$(\vartheta = 1, \mu = 1)$ $\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$
$(0.1, 0.1)$	7.7163E-08	3.0712E-07	7.4457E-07	4.3944E-10	1.6842E-07	5.0477E-07
$(0.2, 0.2)$	9.6531E-08	1.6941E-06	4.9945E-06	8.7663E-07	2.6294E-06	3.4478E-06
$(0.3, 0.3)$	3.0379E-07	8.2147E-08	2.8548E-06	1.6197E-06	3.9313E-06	3.7111E-06
$(0.4, 0.4)$	4.2050E-07	1.3287E-06	1.3821E-06	1.9740E-06	1.3696E-06	2.6916E-06
$(0.5, 0.5)$	1.8205E-07	8.0071E-07	1.9800E-06	2.0156E-06	4.8060E-06	4.3428E-06
$(0.6, 0.6)$	5.4480E-07	2.0092E-06	5.9535E-06	3.3103E-07	3.1998E-07	1.8222E-06
$(0.7, 0.7)$	1.0100E-07	1.0678E-07	2.2874E-06	1.6515E-06	4.0298E-06	3.6475E-06
$(0.8, 0.8)$	4.5130E-07	1.2586E-06	2.4733E-06	1.0386E-06	1.6663E-06	2.4491E-07
$(0.9, 0.9)$	1.9383E-07	5.1424E-07	3.4417E-07	9.8661E-08	2.4378E-07	8.7888E-07

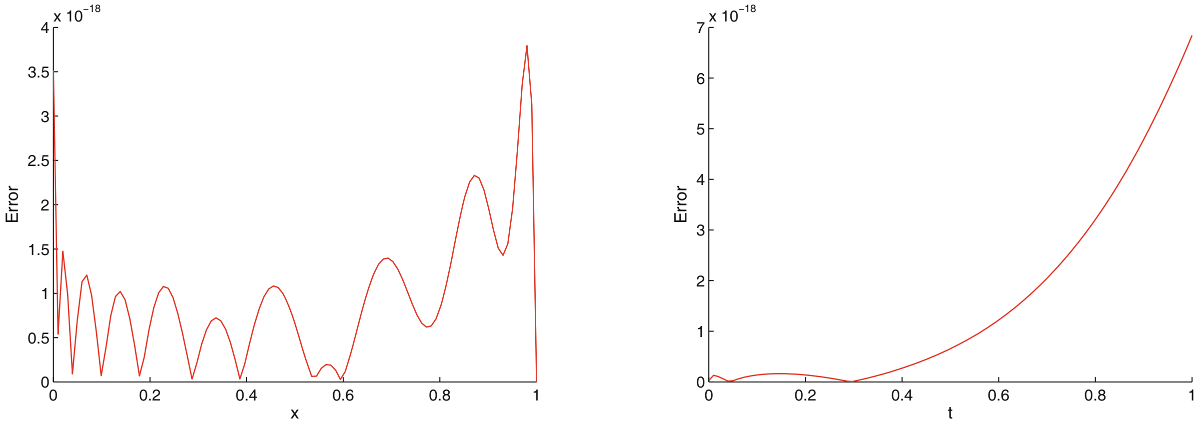

Fig. 9. The graphs of the absolute errors at $t = 0.5$ (left) and $x = 0.5$ (right) for $\alpha = 0.25$ with $(\vartheta = 1, \mu = \frac{1}{4})$ for example 5.

Example 5

Consider the following time FDWE [34, 40]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 1 < \alpha \leq 2,$$

subject to the initial and boundary conditions,

$$u(x,0) = 0, \qquad u_t(x,0) = 0, \qquad u(0,t) = t^3, \qquad u(1,t) = et^3,$$

and

$$f(x,t) = \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} e^x + 3t^2 e^x - t^3 e^x.$$

The exact solution for this problem is given in [34, 40] as $u(x,t) = e^x t^3$.

We solved this problem by the proposed method for $N = M = 12$. The graphs of the absolute errors for $(\vartheta = 1, \mu = \frac{1}{4})$ and $\alpha = 1.25$ at $t = 0.5$ and $x = 0.5$ are shown in fig. 9. The space-time graphs of the approximate solution and absolute error for $\alpha = 1.25$ are shown in fig. 10. By figs. 9 and 10, it can be seen that the proposed method is very accurate for solving the problem. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points $(x_i, t_i) \in [0,1] \times [0,1]$ for $\vartheta = 1$ and some values for μ are shown in tables 9 and 10. From the numerical results reported in tables 9 and 10, one can see that the best choice for (ϑ, μ) to obtain accurate numerical solutions for all chosen $1 < \alpha < 2$ is $(\vartheta = 1, \mu = \frac{1}{4})$. In [67], Liu *et al.* proposed the fractional predictor-corrector method (FPCM) to solve this problem for the case $\alpha = 1.85$. The maximum absolute error obtained in [67] is 1.6341×10^{-3}. However, it can be seen that our numerical results are more accurate in comparison with those based on the FPCM.

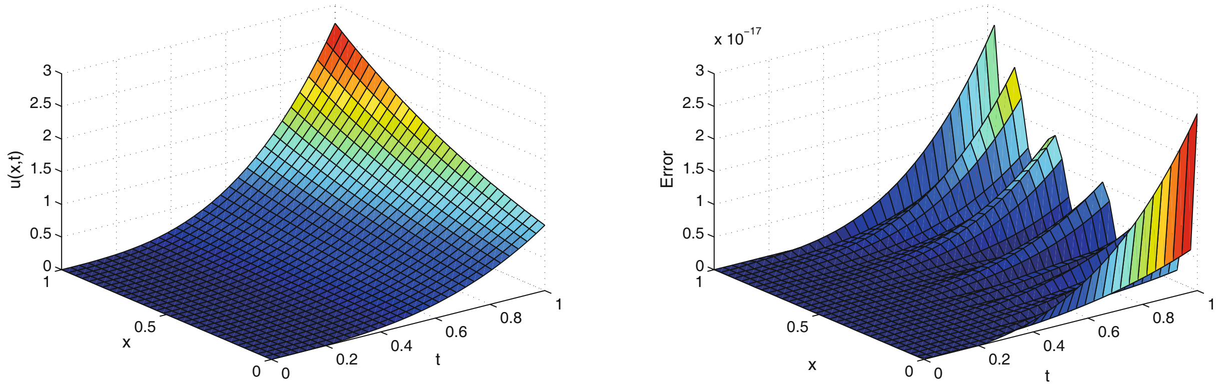

Fig. 10. The space-time graphs of the approximate solution (left) and absolute error (right) for $\alpha = 1.25$ with $(\vartheta = 1, \mu = \frac{1}{4})$ for example 5.

Table 9. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points (x_i, t_i) for example 5.

	$(\vartheta = 1, \mu = \frac{1}{4})$			$(\vartheta = 1, \mu = \frac{1}{2})$		
(x, t)	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$
$(0.1, 0.1)$	1.3169E-20	2.6021E-21	9.3498E-21	1.1088E-10	2.4947E-21	1.4311E-09
$(0.2, 0.2)$	3.2588E-20	6.9387E-20	5.9722E-20	4.0681E-10	6.9410E-20	1.9862E-09
$(0.3, 0.3)$	1.2798E-19	5.1705E-20	4.4853E-20	1.1970E-10	5.1537E-20	1.0093E-09
$(0.4, 0.4)$	1.2678E-19	2.5709E-19	3.3075E-19	6.9363E-10	2.5787E-19	4.3781E-09
$(0.5, 0.5)$	8.1870E-19	8.3886E-19	9.2864E-19	9.0826E-10	8.3880E-19	2.4925E-09
$(0.6, 0.6)$	3.1825E-19	7.5240E-19	1.2362E-18	2.9761E-10	7.5231E-19	4.3308E-09
$(0.7, 0.7)$	3.9360E-18	3.4125E-18	2.8110E-18	1.5694E-09	3.4120E-18	5.5774E-09
$(0.8, 0.8)$	3.9710E-18	3.4186E-18	2.6320E-18	1.4391E-09	3.4193E-18	1.6162E-09
$(0.9, 0.9)$	1.2346E-17	1.1996E-17	1.1088E-17	8.8595E-10	1.1997E-17	2.1890E-09

Table 10. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points (x_i, t_i) for example 5.

	$(\vartheta = 1, \mu = \frac{3}{4})$			$(\vartheta = 1, \mu = 1)$		
(x, t)	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$
$(0.1, 0.1)$	5.2702E-09	1.0422E-09	3.2897E-08	2.7359E-08	8.5538E-08	1.2025E-07
$(0.2, 0.2)$	3.3186E-10	1.9765E-10	3.0303E-09	1.0747E-08	4.6804E-08	8.5811E-08
$(0.3, 0.3)$	1.0661E-08	1.9756E-09	6.0188E-08	3.3736E-08	1.0006E-07	1.1897E-07
$(0.4, 0.4)$	1.7464E-08	1.9134E-09	1.1131E-08	6.9245E-08	1.5694E-07	1.5560E-07
$(0.5, 0.5)$	1.8382E-08	1.8851E-09	5.1733E-08	8.7506E-08	1.6409E-07	1.5362E-07
$(0.6, 0.6)$	6.2774E-10	9.8058E-10	3.7139E-08	7.5321E-08	1.0155E-07	7.2003E-08
$(0.7, 0.7)$	2.0593E-08	3.3893E-09	4.1812E-08	4.7613E-08	3.3217E-09	5.8660E-08
$(0.8, 0.8)$	3.6073E-08	5.3958E-09	6.5686E-08	4.8316E-08	3.9548E-08	2.4194E-07
$(0.9, 0.9)$	3.6955E-08	6.7146E-09	1.5555E-08	1.5717E-07	3.5338E-07	6.0079E-08

Example 6

Consider the following time FDWD:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 1 < \alpha \le 2,$$

subject to the homogenous initial and boundary conditions, and

$$f(x,t) = \left((\alpha+1)t^\alpha - \alpha\, t^{\alpha-1} + \Gamma(\alpha+2)t - \Gamma(\alpha+1)\right)\left(x^{\frac{5}{2}} - x\right) - \frac{15}{4}\sqrt{x}\left(t^{\alpha+1} - t^\alpha\right).$$

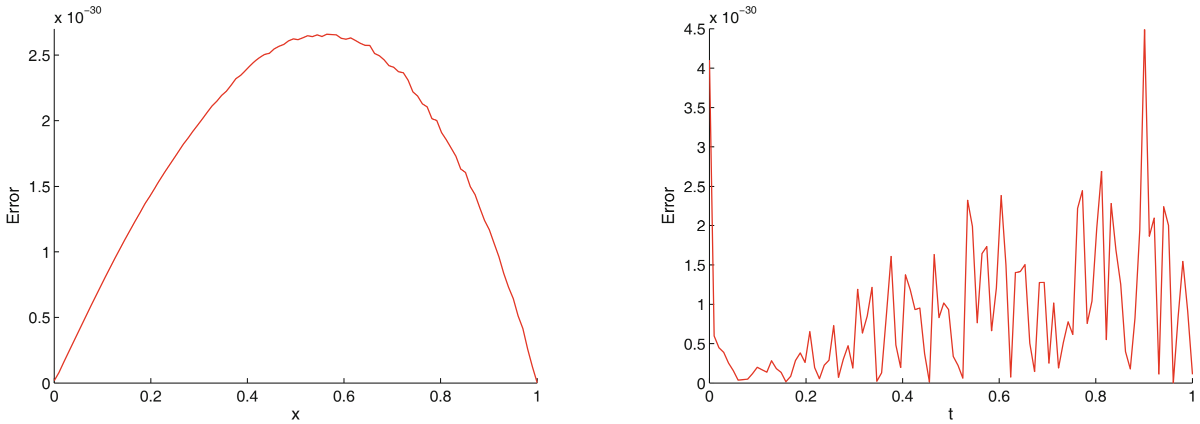

Fig. 11. The graphs of the absolute errors at $t = 0.5$ (left) and $x = 0.5$ (right) for $\alpha = 1.75$ with $(\vartheta = \frac{1}{2}, \mu = \frac{1}{4})$ for example 6.

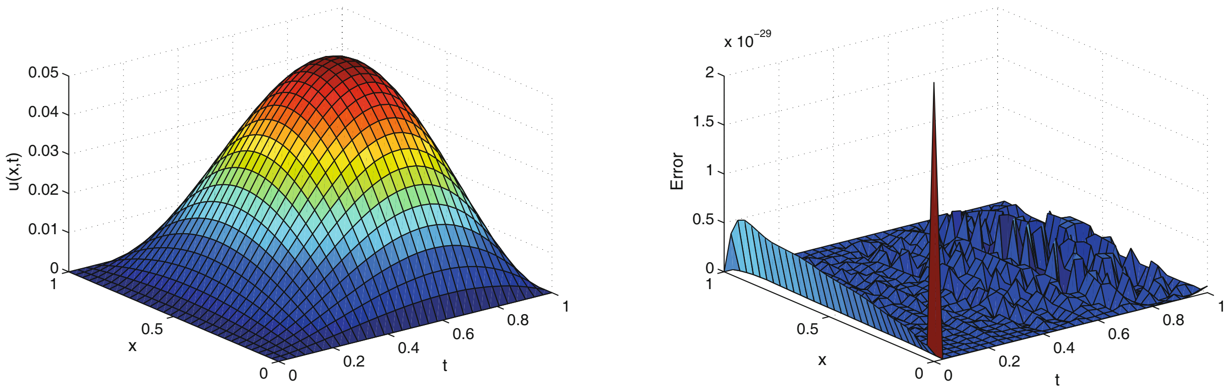

Fig. 12. The space-time graphs of the approximate solution (left) and absolute error (right) in the case $\alpha = 1.75$ with $(\vartheta = \frac{1}{2}, \mu = \frac{1}{4})$ for example 6.

The exact solution of this problem is $u(x,t) = (t^{\alpha+1} - t^{\alpha})(x^{\frac{5}{2}} - x)$.

We solved this problem by the proposed method for $N = M = 11$. The graphs of the absolute errors for $(\vartheta = \frac{1}{2}, \mu = \frac{1}{4})$ and $\alpha = 1.75$ at $t = 0.5$ and $x = 0.5$ are shown in fig. 11. The space-time graphs of the approximate solution and absolute error for $\alpha = 1.75$ are shown in fig. 12. By figs. 11 and 12, it can be seen that the proposed method provides numerical solutions with high accuracy for the problem. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points $(x_i, t_i) \in [0,1] \times [0,1]$ for $\vartheta = \frac{1}{2}, 1$ and some values for μ are shown in tables 11–14. From tables 11–14, it can be observed that the best choice for (ϑ, μ) to obtain accurate numerical solutions for all chosen $1 < \alpha < 2$ is $(\vartheta = \frac{1}{2}, \mu = \frac{1}{4})$.

7 Conclusion

In this paper, an efficient and accurate Galerkin method using fractional Legendre Functions (FLFs) together with their operational matrix of fractional integration (OMFI) was proposed to obtain approximate solutions for the fractional sub-diffusion equation (FSDE) and time-fractional diffusion-wave equation (FDWE). Also, a new OMFI in the Riemann-Liouville sense for FLFs was derived. The FLFs and their OMFI have been used to convert the problems under consideration into some corresponding linear systems of algebraic equations to achieve approximate solutions for the problems. The method discussed in this paper is very convenient for solving such kind of problems, since the initial and boundary conditions are taken into account automatically. The efficiency of the propoesed method have been tested on several non-trivial examples, where the numerical solutiona have been explicitly computed and graphically represented. The applicability and accuracy was also carefully checked for these numerical examples. The obtained results were in a good agreement with the exact solutions. We have also shown that only a small number of FLFs are needed to obtain satisfactory results. Taking into account the efficiency of this method we believe that it can be also used for the numerical solution of other kinds of fractional partial differential equations such as space fractional diffusion-wave equation and it can be easily applied also to the analysis of fourth-order FDWE.

Table 11. The absolute errors of the approximate solution for some values of $1 < \alpha < 2$ at some different points (x_i, t_i) for example 6.

	$(\vartheta = \frac{1}{2}, \mu = \frac{1}{4})$			$(\vartheta = \frac{1}{2}, \mu = \frac{1}{2})$		
(x, t)	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$
$(0.1, 0.1)$	6.3200E-32	8.6500E-32	8.6000E-33	1.2569E-07	6.2000E-33	3.5023E-10
$(0.2, 0.2)$	2.5000E-32	1.8800E-31	4.2160E-31	3.7701E-07	3.0000E-33	1.4116E-09
$(0.3, 0.3)$	4.9000E-32	3.7500E-31	4.2600E-31	1.4160E-06	1.0000E-32	9.9022E-09
$(0.4, 0.4)$	7.6000E-32	6.8200E-31	7.7900E-31	1.0872E-06	2.1000E-32	7.9740E-09
$(0.5, 0.5)$	7.8000E-32	2.1600E-31	3.3400E-31	1.3095E-06	1.5000E-32	5.6182E-09
$(0.6, 0.6)$	5.8400E-31	8.8100E-31	2.6000E-32	1.9804E-06	3.8000E-32	6.4448E-09
$(0.7, 0.7)$	6.1600E-31	5.3500E-31	2.6400E-31	1.0948E-06	6.0000E-33	5.0656E-09
$(0.8, 0.8)$	4.4000E-32	3.2300E-31	6.8500E-31	2.1323E-06	6.0000E-33	3.0504E-09
$(0.9, 0.9)$	2.4600E-31	7.3000E-31	1.0540E-30	7.9575E-07	1.0000E-32	1.3878E-09

Table 12. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points (x_i, t_i) for example 6.

	$(\vartheta = \frac{1}{2}, \mu = \frac{3}{4})$			$(\vartheta = \frac{1}{2}, \mu = 1)$		
(x, t)	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$
$(0.1, 0.1)$	1.3812E-06	2.4916E-07	4.0603E-08	2.9244E-06	9.7970E-07	1.6120E-07
$(0.2, 0.2)$	1.8276E-06	2.1906E-07	1.2339E-08	6.0657E-06	2.4510E-06	5.4776E-07
$(0.3, 0.3)$	3.3628E-06	5.7925E-07	7.7451E-08	6.4277E-06	2.1519E-06	3.8354E-07
$(0.4, 0.4)$	3.8603E-08	4.3200E-08	1.0323E-07	2.7923E-06	1.1673E-07	1.9105E-07
$(0.5, 0.5)$	3.8652E-06	4.1543E-07	1.4404E-07	4.1092E-06	2.6654E-06	8.1157E-07
$(0.6, 0.6)$	5.2350E-06	6.4437E-07	2.1652E-08	9.1482E-06	4.0138E-06	9.8223E-07
$(0.7, 0.7)$	1.0579E-06	3.1710E-07	4.3693E-08	9.0008E-06	3.2715E-06	6.9048E-07
$(0.8, 0.8)$	2.4499E-06	3.3965E-07	8.2515E-08	6.2824E-06	1.9940E-06	3.7061E-07
$(0.9, 0.9)$	2.3150E-06	2.7708E-07	5.7861E-08	4.7581E-06	1.6187E-06	3.2544E-07

Table 13. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points (x_i, t_i) for example 6.

	$(\vartheta = 1, \mu = \frac{1}{4})$			$(\vartheta = 1, \mu = \frac{1}{2})$		
(x, t)	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$
$(0.1, 0.1)$	1.3139E-07	7.2598E-08	3.8468E-08	5.7255E-09	7.2646E-08	3.8887E-08
$(0.2, 0.2)$	2.0302E-07	1.3417E-07	8.9142E-08	1.7392E-07	1.3403E-07	9.0606E-08
$(0.3, 0.3)$	1.9110E-07	1.5239E-07	1.1404E-07	1.2247E-06	1.5228E-07	1.2356E-07
$(0.4, 0.4)$	9.8715E-08	4.8248E-07	1.9372E-08	9.8850E-07	4.8220E-08	2.7166E-08
$(0.5, 0.5)$	3.3980E-07	2.4799E-07	1.6580E-07	9.6975E-07	2.4815E-07	1.7143E-07
$(0.6, 0.6)$	6.6207E-08	2.5887E-08	1.9787E-08	2.0467E-06	2.5683E-08	2.5959E-08
$(0.7, 0.7)$	4.0262E-07	3.5436E-07	3.0420E-07	1.4975E-06	3.5461E-07	2.9868E-07
$(0.8, 0.8)$	4.8309E-07	4.6348E-07	4.4429E-07	1.6490E-06	4.6340E-07	4.4737E-07
$(0.9, 0.9)$	2.2936E-07	2.3739E-07	2.4840E-07	1.0242E-06	2.3729E-07	2.4991E-07

Table 14. The absolute errors of the approximate solutions for some values of $1 < \alpha < 2$ at some different points (x_i, t_i) for example 6.

	$(\vartheta = 1, \mu = \frac{3}{4})$			$(\vartheta = 1, \mu = 1)$		
(x, t)	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$	$\alpha = 1.25$	$\alpha = 1.50$	$\alpha = 1.75$
$(0.1, 0.1)$	1.5126E-06	3.2178E-07	7.9234E-08	2.7929E-06	9.0706E-07	1.2276E-07
$(0.2, 0.2)$	2.0307E-06	3.5307E-07	7.6859E-08	5.8627E-06	2.3170E-06	4.5845E-07
$(0.3, 0.3)$	3.5541E-06	3.5307E-07	1.9111E-07	6.2364E-06	1.9997E-06	2.7000E-07
$(0.4, 0.4)$	6.0037E-08	4.9557E-09	1.2239E-07	2.6933E-06	6.8240E-08	2.1030E-07
$(0.5, 0.5)$	3.4945E-06	5.7933E-07	1.4622E-08	1.5074E-06	1.5620E-06	4.5899E-07
$(0.6, 0.6)$	5.1687E-06	6.1874E-07	1.9675E-09	9.2141E-06	4.0393E-06	9.6261E-07
$(0.7, 0.7)$	1.4600E-06	3.8607E-07	3.4772E-07	8.5976E-06	2.9167E-06	3.8622E-07
$(0.8, 0.8)$	2.9331E-06	8.0289E-07	3.6172E-07	6.7645E-06	2.4573E-06	8.1527E-07
$(0.9, 0.9)$	2.0860E-06	3.8854E-08	3.0635E-07	4.5252E-06	1.3782E-06	7.9251E-08

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