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New solutions for conformable fractional Boussinesq and combined KdV-mKdV equations using Jacobi elliptic function expansion method

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Abstract. In this paper, the Jacobi elliptic function expansion method is proposed for the first time to construct the exact solutions of the time conformable fractional two-dimensional Boussinesq equation and the combined KdV-mKdV equation. New exact solutions are found. This method is based on Jacobi elliptic functions. The results obtained confirm that the proposed method is an efficient technique for analytic treatment of a wide variety of nonlinear conformable time-fractional partial differential equations.

1 Introduction

Since it has been understood that the fractional differential equations are valuable tools in the modeling of many phenomena in biology, chemistry, economy, engineering, physics and other areas of applications, they have drawn much attention by scientists [1–4]. Due to this, scientists have been giving different definitions to these fractional derivatives such as Grünwald-Letnikov, Riemann-Liouville and Caputo's fractional derivatives [5–7]. Within these definitions of fractional derivatives, the most popular ones are the following:

1) The Riemann-Liouville Fractional Derivative Definition: If n is a positive integer and $\alpha \in [n-1, n)$, the α derivative of a function f is given by [7]

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} \mathrm{d}x.$$

2) The Caputo Fractional Derivative Definition: If n is a positive integer and $\alpha \in [n-1, n)$, the α derivative of a function f is given by [7]

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} \mathrm{d}x.$$

Although the above definitions are used in most articles, there are some flaws in these definitions, such as:

1) The Riemann-Liouville derivative does not satisfy $D_a^\alpha 1 = 0$ (while the Caputo derivative satisfies this condition), if α is not a natural number.
2) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:

$$D_a^\alpha(fg) = gD_a^\alpha(f) + fD_a^\alpha(g)\,.$$

3) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^\alpha\left(\frac{f}{g}\right) = \frac{fD_a^\alpha(f) - gD_a^\alpha(g)}{g^2}\,.$$

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4) All fractional derivatives do not satisfy the chain rule:

$$D_a^\alpha(fog)(t) = f^\alpha(g(t))g^\alpha(t).$$

5) All fractional derivatives do not satisfy $D^\alpha D^\beta = D^{\alpha+\beta}$ in general.
6) The Caputo definition assumes that the function f is differentiable.

To overcome these difficulties, recently, an interesting and helpful work on the theory of derivatives and integrals of fractional order was done by Khalil *et al.* [8], which is the simplest definition to recognize the fractional derivative and integral.

Definition. Let $f : [0, \infty) \to \mathbb{R}$ be a function. The α-th order *conformable fractional derivative* of f is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $a > 0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then we define $f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$, and the *conformable fractional integral* of a function f starting from $a \geq 0$ is defined as

$$I_\alpha^a(f)(t) = \int_a^t \frac{f(x)}{x^{1-\alpha}} \mathrm{d}x,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$. The following theorem points out some properties which are satisfied by the conformable fractional derivative [8].

Theorem 1. *Let $\alpha \in (0, 1]$ and suppose f, g are α-differentiable at point $t > 0$. Then*

1) $T_\alpha(cf + dg) = cT_\alpha(f) + cT_\alpha(g)$ *for all* $a, b \in \mathbb{R}$.
2) $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.
3) $T_\alpha(\lambda) = 0$ *for all constant functions* $f(t) = \lambda$.
4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
5) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
6) *If, in addition to f differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{\mathrm{d}f}{\mathrm{d}t}$.

This new fractional definition has some physical applications. For instance, Hammad and Khalil [9] gave the solution for the conformable fractional heat equation. Chung [10] used the conformable fractional derivative and integral to discuss fractional Newtonian mechanics; Kurt *et al.* [11] expressed the exact and approximate solutions of the time conformable fractional Burgers' equation, which arises in numerous physical areas, such as gas dynamics, heat conduction, elasticity theory, turbulence theory, shock wave theory, fluid mechanics, termaviscous fluids, hydrodynamic waves and elastic waves. It is clearly seen that further studies and explanations can be made regarding the physical meaning and physical applications of this new subject area.

Looking for the exact solutions of nonlinear wave equations is very important. Up to now, many powerful methods have been developed, such as the exp-function method [12], the hyperbolic function method [13], the first integral method [14], and so on. Recently, the Jacobi elliptic function expansion method [15] has been proposed to construct exact solutions to nonlinear wave equations. In this paper, we obtain the exact solutions of the time-fractional combined KdV-mKdV equation and Boussinesq equation by means of the conformable fractional derivative by the Jacobi elliptic function expansion method.

2 The Jacobi elliptic function expansion method

Now let us simply describe the procedure of the Jacobi elliptic function expansion method. Let us consider a given nonlinear wave equation, say, in two variables:

$$F\left(u, \frac{\partial^\alpha u}{\partial t^\alpha}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^{2\alpha} u}{\partial t^{2\alpha}}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \dots\right) = 0. \tag{1}$$

Transforming eq. (1), applying the chain rule [16],

$$u = u(\xi), \quad \xi = m\frac{t^\alpha}{\alpha} + nx + sy, \tag{2}$$

Table 1. Solutions of $F(\xi)$ in eq. (5) for the special chosen values of P, Q and R [17,18].

	P	Q	R	F
1	r^2	$-(1+r^2)$	1	sn, cd
2	$-r^2$	$2r^2-1$	$1-r^2$	cn
3	-1	$2-r^2$	r^2-1	dn
4	1	$-(1+r^2)$	r^2	ns, dc
5	$1-r^2$	$2r^2-1$	$-r^2$	nc
6	r^2-1	$2-r^2$	-1	nd
7	$1-r^2$	$2-r^2$	1	sc
8	$-r^2(1-r^2)$	$2r^2-1$	1	sd
9	1	$2-r^2$	$1-r^2$	cs
10	1	$2r^2-1$	$-r^2(1-r^2)$	ds
11	$\frac{-1}{4}$	$\frac{r^2+1}{2}$	$-\frac{(1-r^2)^2}{4}$	$r\,cn \mp \mathrm{dn}$
12	$\frac{1}{4}$	$\frac{-2r^2+1}{2}$	$\frac{1}{4}$	$n\mathrm{s} \mp \mathrm{cs}$
13	$\frac{1-r^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1-r^2}{4}$	$\mathrm{nc} \mp \mathrm{sc}$
14	$\frac{1}{4}$	$\frac{r^2-2}{2}$	$\frac{r^4}{4}$	$n\mathrm{s} \mp \mathrm{ds}$
15	$\frac{r^2}{4}$	$\frac{r^2-2}{2}$	$\frac{r^2}{4}$	$\mathrm{sn} \mp \mathrm{i}\,cn$, $\frac{\mathrm{dn}}{\sqrt{1-r^2}\,\mathrm{sn} \mp cn}$
16	$\frac{1}{4}$	$\frac{1-2r^2}{2}$	$\frac{1}{4}$	$r\,cn \mp \mathrm{i}\,\mathrm{dn}$, $\frac{\mathrm{sn}}{1 \mp cn}$
17	$\frac{r^2}{4}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}}{1 \mp \mathrm{dn}}$
18	$\frac{r^2-1}{4}$	$\frac{r^2+1}{2}$	$\frac{r^2-1}{4}$	$\frac{\mathrm{dn}}{1 \mp r\,\mathrm{sn}}$
19	$\frac{1-r^2}{4}$	$\frac{r^2+1}{2}$	$\frac{-r^2+1}{4}$	$\frac{cn}{1 \mp \mathrm{sn}}$
20	$\frac{(1-r^2)^2}{4}$	$\frac{r^2+1}{2}$	$\frac{1}{4}$	$\frac{\mathrm{sn}}{\mathrm{dn} \mp cn}$
21	$\frac{r^4}{4}$	$\frac{r^2-2}{2}$	$\frac{1}{4}$	$\frac{cn}{\sqrt{1-r^2} \mp \mathrm{dn}}$

where m, n and s are the arbitrary constants,

$$\frac{\partial^\alpha(\cdot)}{\partial t^\alpha} = m\frac{\mathrm{d}(\cdot)}{\mathrm{d}\xi}, \qquad \frac{\partial(\cdot)}{\partial x} = n\frac{\mathrm{d}(\cdot)}{\mathrm{d}\xi}\,, \qquad \frac{\partial(\cdot)}{\partial y} = s\frac{\mathrm{d}(\cdot)}{\mathrm{d}\xi}, \ldots, \tag{3}$$

yields an ordinary differential equation (ODE) for $u(\xi)$,

$$O(u, u', u'', u''', \ldots). \tag{4}$$

The main idea in this generalized indirect method is to use the opportunity of the solutions of an auxiliary ordinary differential equation (the first kind of Jacobian equation containing three parameters) to build abundant families of Jacobian elliptic solutions for the above-mentioned equation. The auxiliary equation can be shown

$$(F')^2(\xi) = PF^4(\xi) + QF^2(\xi) + R, \tag{5}$$

where $F' = \mathrm{d}F/\mathrm{d}\xi$, $\xi = \xi(x,t,y)$ and P, Q, R are constants. Equation (5) has its solutions in table 1 where $i^2 = -1$. The Jacobian elliptic functions $\mathrm{sn}\,\xi = \mathrm{sn}(\xi;r)$, $cn\,\xi = cn(\xi;r)$ and $\mathrm{dn}\,\xi = \mathrm{dn}(\xi;r)$, where $r(0<r<1)$ is the modulus

Table 2. Jacobi elliptic functions for $r \to 0$ and $r \to 1$ [17,18].

		$r \to 0$	$r \to 1$			$r \to 0$	$r \to 1$
1	$\operatorname{sn} u$	$\sin u$	$\tanh u$	7	$\operatorname{dc} u$	$\sec u$	1
2	$cn\, u$	$\cos u$	$\operatorname{sech} u$	8	$\operatorname{nc} u$	$\sec u$	$\cosh u$
3	$\operatorname{dn} u$	1	$\operatorname{sech} u$	9	$\operatorname{sc} u$	$\tan u$	$\sinh u$
4	$\operatorname{cd} u$	$\cos u$	1	10	$ns\, u$	$\csc u$	$\coth u$
5	$\operatorname{sd} u$	$\sin u$	$\sinh u$	11	$\operatorname{ds} u$	$\csc u$	$\operatorname{csch} u$
6	$\operatorname{nd} u$	1	$\cosh u$	12	$\operatorname{cs} u$	$\cot u$	$\operatorname{csch} u$

of the elliptic function, are double periodic and satisfy the following properties:

$$\begin{aligned}
\operatorname{sn}^2 \xi + cn^2 \xi &= 1, \\
\operatorname{dn}^2 \xi + r^2 \operatorname{sn}^2 \xi &= 1, \\
\frac{\mathrm{d}}{\mathrm{d}\xi} \operatorname{sn} \xi &= cn\, \xi \operatorname{dn} \xi, \\
\frac{\mathrm{d}}{\mathrm{d}\xi}\, cn\, \xi &= -\operatorname{sn} \xi \operatorname{dn} \xi, \\
\frac{\mathrm{d}}{\mathrm{d}\xi} \operatorname{dn} \xi &= -r^2\, cn\, \xi \operatorname{sn} \xi.
\end{aligned}$$

Since $r \to 0$ and $r \to 1$, Jacobi elliptic functions, which are listed in table 2, turn into trigonometric and hyperbolic functions, hence we obtain the trigonometric function solutions and solitonic solutions of the considered equation.

By the Jacobi elliptic function expansion method, $u(\xi)$ can be expressed as a finite series of Jacobi elliptic functions,

$$u(\xi) = \sum_{i=0}^{k} a_i F^i(\xi), \tag{6}$$

where $F(\xi)$ is the solution of the nonlinear ordinary equation (5) and k, a_i $(i = 0, 1, 2, \ldots, k)$ are constants to be determined later. The integer k in (6) can be determined by balancing the highest-order linear term,

$$O\left(\frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\right) = k + p, \quad p = 0, 1, 2, \ldots, \tag{7}$$

and the highest-order nonlinear term,

$$O\left(u^q \frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\right) = (q+1)k + p, \quad p = 0, 1, \ldots,\ q = 0, 1, \ldots, \tag{8}$$

in (4).

Substituting (6) and setting all the coefficients of powers F to be zero, then a system of nonlinear algebraic equations for a_i $(i = 0, 1, 2, \ldots, n)$ is derived, by solving this system with the aid of Mathematica and using all the values for P, Q, R (5) in table 1.

After all this procedure combining the values with eq. (6) and the auxiliary equation we choose, we can get exact solutions for eq. (1).

3 Exact traveling wave solutions to the time-fractional two-dimensional Boussinesq equation

Let us consider the conformable time-fractional Boussinesq equation,

$$\frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 (u^2)}{\partial x^2} - \frac{\partial^4 u}{\partial x^4} = 0, \tag{9}$$

where $\alpha \in (0,1)$ and $\frac{\partial^\alpha u}{\partial t^\alpha}$ means conformable fractional derivative of function $u(x,y,t)$. This equation is used in the analysis of long waves in shallow water. It is also used in the analysis of many other physical applications, such as the percolation of water in a porous subsurface of a horizontal layer of material. Applying the wave transformation (2) to eq. (9) and then integrating twice yields

$$(m^2 - n^2 - s^2)u - n^2u^2 - n^4u_{\xi\xi} = 0. \tag{10}$$

By balancing the highest-order linear term and the highest-order nonlinear term we obtain $k = 2$, thus the solution of eq. (9) can be expressed as

$$u(\xi) = a_0 + a_1F(\xi) + a_2F^2(\xi). \tag{11}$$

So, differentiating eq. (11) twice,

$$u_{\xi\xi} = a_1F''(\xi) + 2a_2((F'(\xi))^2 + F(\xi)F''(\xi)), \tag{12}$$

then by using (5),

$$F'' = 2PF^3 + QF. \tag{13}$$

Inserting (13) and (5) into (12)

$$u_{\xi\xi} = a_1\left(2PF^3 + QF\right) + 2a_2\left(PF^4 + QF^2 + R\right) + 2a_2F\left(2PF^3 + QF\right). \tag{14}$$

Substituting eq. (11) and eq. (14) in eq. (10) and setting each coefficient of F to zero, yields an equations system,

$$\begin{aligned}
&a_0m^2 - a_0^2n^2 - a_0s^2 - a_0n^2 - 2a_2n^4R = 0,\\
&a_1n^2 - a_1m^2 + 2a_0a_1n^2 + a_1n^4Q + a_1s^2 = 0,\\
&a_2n^2 - a_2m^2 + 2a_0a_2n^2 + a_1^2n^2 + a_2s^2 + 4a_2n^4Q = 0,\\
&2a_1a_2n^2 + 2a_1n^4P = 0,\\
&a_2^2n^2 - 6a_2n^4P = 0.
\end{aligned}$$

By solving this system with the aid of Mathematica, two cases arises:

1) Case 1:

$$a_1 = 0, \qquad a_0 = -2n^2Q - 2n^2\sqrt{Q^2 - 3PR}, \qquad a_2 = -6n^2P, \qquad m = \mp\sqrt{n^2 + s^2 - 4n^4\sqrt{Q^2 - 3PR}}.$$

2) Case 2:

$$a_1 = 0, \qquad a_0 = -2n^2Q + 2n^2\sqrt{Q^2 - 3PR}, \qquad a_2 = -6n^2P, \qquad m = \mp\sqrt{n^2 + s^2 + 4n^4\sqrt{Q^2 - 3PR}}.$$

Pursuing the solution procedure mentioned above and combining, respectively, the values with eq. (11), we can get exact solutions of eq. (9) for all cases as follows.

Solutions for Case 1

When $P = r^2$, $Q = -(1 + r^2)$, $R = 1$ are chosen, $F = \mathrm{sn}$ from table 1, thus

$$u(x, y, t) = 2n^2\left(1 + r^2 - \sqrt{1 - r^2 + r^4}\right) - 6n^2r^2\,\mathrm{sn}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right),$$

and considering $r \to 1$, from table 2, the solitary wave solution can be obtained as

$$u_{1,1}(x, y, t) = 2n^2 - 6n^2\tanh^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right), \tag{15}$$

where $m = \mp\sqrt{n^2 - 4n^4 + s^2}$.

Choosing $P = -r^2$, $Q = 2r^2 - 1$, $R = 1 - r^2$, it can be denoted, from table 1 $F = cn$, hence

$$u(x, y, t) = 2n^2\left(1 - 2r2 - \sqrt{1 - r2 + r^4}\right) + 6n2r^2\,cn^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right). \tag{16}$$

For $r \to 1$ from table 2, the solitary wave solution is expressed as

$$u_{1,2}(x,y,t) = -4n^2 + 6n^2 \operatorname{sech}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right),$$

where $m = \mp\sqrt{n^2 - 4n^4 + s^2}$. It is clearly seen that this solution is the same as that of (15).

While $P = -1$, $Q = 2 - r^2$, $R = r^2 - 1$, it can be deducted from table 1 $F = \mathrm{dn}$, then the solution can be evaluated as

$$u(x,y,t) = 2n^2\left(-2 + r^2 - \sqrt{1 - r^2 + r^4}\right) + 6n^2 \operatorname{dn}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

For $r \to 1$, it is clearly seen that this solution is the same solution as (15).

Setting $P = 1$, $Q = -(1 + r^2)$, $R = r^2$, from table 1 $F = n\mathrm{s}$, due to this settings

$$u(x,y,t) = 2n^2\left(1 + r^2 - \sqrt{1 - r^2 + r^4}\right) - 6n^2 \operatorname{ns}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

Furthermore, if $r \to 1$ from table 2 the solitary wave solution of eq. (9) is as follows:

$$u_{1,3}(x,y,t) = 2n^2 - 6n^2 \coth^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)$$

and, if $r \to 0$, from table 2 the periodic solution of eq. (9) can be derived as

$$u_{1,4}(x,y,t) = -6n^2 \csc^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right), \tag{17}$$

where $m = \mp\sqrt{n^2 - 4n^4 + s^2}$.

Supposing $P = 1$, $Q = -(1 + r^2)$, $R = r^2$, from table 1 this choices correspond to $F = \mathrm{dc}$, so

$$U(x,y,t) = 2n^2\left(1 + r^2 - \sqrt{1 - r^2 + r^4}\right) - 6n^2 \operatorname{dc}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)$$

is found and, for $r \to 0$ from table 2, the periodic solution can be obtained as

$$u_{1,5}(x,y,t) = -6n^2 \sec^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right), \tag{18}$$

where $m = \mp\sqrt{n^2 - 4n^4 + s^2}$.

Considering $P = 1$, $Q = 2 - r^2$, $R = 1 - r^2$ and, from table 1, $F = \mathrm{cs}$, the solution can be acquired as

$$u(x,y,t) = 2n^2\left(-2 + r^2 - \sqrt{1 - r^2 + r^4}\right) - 6n^2 \operatorname{cs}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

As $r \to 0$, from table 2, we get the periodic solution as

$$u_{1,6}(x,y,t) = -6n^2 - 6n^2 \cot^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)$$

and, as $r \to 1$, from table 2 we get the solitary wave solution as from table 2,

$$u_{1,7}(x,y,t) = -4n^2 - 6n^2 \operatorname{csch}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right), \tag{19}$$

where $m = \mp\sqrt{n^2 - 4n^4 + s^2}$. The first solution is the same as that of (17).

Also regarding $P = 1 - r^2$, $Q = 2 - r^2$, $R = 1$ and, from table 1, $F = \mathrm{sc}$, the solution can be evaluated as

$$u(x, y, t) = 2n^2 \left(-2 + r^2 - \sqrt{1 - r^2 + r^4}\right) - 6n^2(1 - r^2)\,\mathrm{sc}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

In addition, for $r \to 0$, from table 2 the periodic solution can be stated as

$$u_{1,8}(x, y, t) = -6n^2 - 6n^2 \tan^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right),$$

where $m = \mp\sqrt{n^2 - 4n^4 + s^2}$. The obtained solution is the same as that of (18).

Also assigning $P = 1$, $Q = 2r^2 - 1$, $R = r^4 - r^2$ and $F = \mathrm{ds}$ from table 1, the solution is found as

$$u(x, y, t) = 2n^2 \left(1 - 2r^2 - \sqrt{1 - r^2 + r^4}\right) - 6n^2\,\mathrm{ds}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

In this case for $r \to 0$, from table 2 the solution obtained is the same as that of (17) and, for $r \to 1$, the solution obtained is the same as that of (19).

When P, Q, R are chosen as $P = -\frac{1}{4}$, $Q = \frac{r^2+1}{2}$, $R = -\frac{(1-r^2)^2}{4}$, F is going to be as $F = r\,cn \mp \mathrm{dn}$ from table 1, so the solution can be obtained as

$$u(x, y, t) = -n^2\left(1 + r^2 + \frac{\sqrt{1 + 14r^2 + r^4}}{2}\right) + \frac{3n^2}{2}\left(r\,cn\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \mathrm{dn}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2.$$

Furthermore, for $r \to 1$, the solution is found to be the same of (15) using table 1.

If we choose P, Q, R as $P = \frac{1}{4}$, $Q = \frac{-2r^2+1}{2}$, $R = \frac{1}{4}$, from table 1 F is obtained as $F = ns \mp \mathrm{cs}$, in this way the solution can be expressed as

$$u(x, y, t) = -n^2\left(1 - 2r^2 + \frac{\sqrt{1 - 16r^2 + 16r^4}}{2}\right) - \frac{3n^2}{2}\left(ns\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \mathrm{cs}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2.$$

Moreover, for $r \to 0$, from table 2 the periodic solution can be stated as

$$u_{1,9}(x, y, t) = -\frac{3n^2}{2} - \frac{3n^2}{2}\left(\csc\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \cot\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2,$$

and for $r \to 1$ the solitary solution can be found as

$$u_{1,10}(x, y, t) = \frac{n^2}{2} - \frac{3n^2}{2}\left(\coth\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \mathrm{csch}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2,$$

where $m = \mp\sqrt{n^2 - n^4 + s^2}$.

For choices $P = \frac{1-r^2}{4}$, $Q = \frac{r^2+1}{2}$, $R = \frac{1-r^2}{4}$, from table 1 F can be written as $F = \mathrm{nc} \mp \mathrm{sc}$, thus the solution is found as

$$u(x, y, t) = -n^2\left(1 + r^2 + \frac{\sqrt{1 + 14r^2 + r^4}}{2}\right) - \frac{3n^2}{2}(1 - r^2)\left(\mathrm{nc}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \mathrm{sc}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2.$$

For $r \to 0$, with the help of table 2, the periodic solution can be acquired as

$$u_{1,11}(x, y, t) = -\frac{3n^2}{2} - \frac{3n^2}{2}\left(\sec\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \tan\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2,$$

where $m = \mp\sqrt{n^2 - n^4 + s^2}$.

Setting $P = \frac{r^2}{4}$, $Q = \frac{r^2-2}{2}$, $R = \frac{r^2}{4}$, from table 1 $F = \mathrm{sn} \mp i\, cn$, due to this settings,

$$u(x,y,t) = n^2\left(2 - r^2 - \frac{\sqrt{16-16r^2+r^4}}{2}\right) - \frac{3n^2}{2}r^2\left(\mathrm{sn}\left(m\frac{t^\alpha}{\alpha}+nx+sy\right) \mp i\, cn\left(m\frac{t^\alpha}{\alpha}+nx+sy\right)\right)^2.$$

For $r \to 1$, by using table 2, the solitary solution can be evaluated as

$$u_{1,12}(x,y,t) = \frac{n^2}{2} - \frac{3n^2}{2}\left(\tanh\left(m\frac{t^\alpha}{\alpha}+nx+sy\right) \mp i\,\mathrm{sech}\left(m\frac{t^\alpha}{\alpha}+nx+sy\right)\right)^2, \tag{20}$$

where $m = \mp\sqrt{n^2 - n^4 + s^2}$.

Regarding $P = \frac{1}{4}$, $Q = \frac{1-2r^2}{2}$, $R = \frac{1}{4}$, from table 1, it is obtained that $F = r\,\mathrm{sn} \mp i\,\mathrm{dn}$ and the solution can be expressed as

$$u(x,y,t) = -n^2\left(1 - 2r^2 + \frac{\sqrt{1-16r^2+16r^4}}{2}\right) - \frac{3n^2}{2}\left(r\,\mathrm{sn}\left(m\frac{t^\alpha}{\alpha}+nx+sy\right) \mp i\,\mathrm{dn}\left(m\frac{t^\alpha}{\alpha}+nx+sy\right)\right)^2.$$

For $r \to 1$, the solution is obtained as that of (20).

Considering $P = \frac{1}{4}$, $Q = \frac{1-2r^2}{2}$, $R = \frac{1}{4}$, from table 1, F can be expressed as $F = \frac{\mathrm{sn}}{1 \mp cn}$, so the solution can be found as

$$u(x,y,t) = -n^2\left(1 - 2r^2 + \frac{\sqrt{1-16r^2+16r^4}}{2}\right) - \frac{3n^2}{2}\left(\frac{\mathrm{sn}(m\frac{t^\alpha}{\alpha}+nx+sy)}{1 \mp cn(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2.$$

Also, for $r \to 0$, by use of table 2, the periodic solution can be acquired as

$$u_{1,13}(x,y,t) = -\frac{3n^2}{2} - \frac{3n^2}{2}\left(\frac{\sin(m\frac{t^\alpha}{\alpha}+nx+sy)}{1 \mp \cos(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2 \tag{21}$$

and, for $r \to 1$, the solitary wave solution can be stated as

$$u_{1,14}(x,y,t) = \frac{n^2}{2} - \frac{3n^2}{2}\left(\frac{\tanh(m\frac{t^\alpha}{\alpha}+nx+sy)}{1 \mp \mathrm{sech}(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2, \tag{22}$$

where $m = \mp\sqrt{n^2 - n^4 + s^2}$.

Supposing $P = \frac{r^2}{4}$, $Q = \frac{r^2-2}{2}$, $R = \frac{1}{4}$, it can be deducted from table 1 $F = \frac{\mathrm{sn}}{1\mp\mathrm{dn}}$ and the solution can be evaluated as

$$u(x,y,t) = n^2\left(2 - r^2 - \frac{\sqrt{16-19r^2+4r^4}}{2}\right) - \frac{3n^2}{2}r^2\left(\frac{\mathrm{sn}(m\frac{t^\alpha}{\alpha}+nx+sy)}{1 \mp \mathrm{dn}(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2.$$

For $r \to 1$, the solution is obtained the same as that of (22).

Assigning $P = \frac{1-r^2}{4}$, $Q = \frac{r^2+1}{2}$, $R = \frac{1-r^2}{4}$, from table 1, this assignment corresponds to $F = \frac{cn}{1\mp\mathrm{sn}}$, so the solution can be found as

$$u(x,y,t) = -n^2\left(1 + r^2 + \frac{\sqrt{1+14r^2+r^4}}{2}\right) - \frac{3n^2}{2}(1-r^2)\left(\frac{cn(m\frac{t^\alpha}{\alpha}+nx+sy)}{1 \mp \mathrm{sn}(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2.$$

Considering $r \to 0$, in the light of table 2, the periodic solution can be shown as

$$u_{1,15}(x,y,t) = -\frac{3n^2}{2} - \frac{3n^2}{2}\left(\frac{\cos(m\frac{t^\alpha}{\alpha}+nx+sy)}{1 \mp \sin(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2,$$

where $m = \mp\sqrt{n^2 - n^4 + s^2}$.

Choosing $P = \frac{(1-r^2)^2}{4}$, $Q = \frac{r^2+1}{2}$, $R = \frac{1}{4}$ from table 1, this choice follows $F = \frac{\mathrm{sn}}{\mathrm{dn} \mp cn}$, so that the solution can be obtained as

$$u(x,y,t) = -n^2 \left(1 + r^2 + \frac{\sqrt{1+14r^2+r^4}}{2}\right) - \frac{3n^2}{2}(1-r^2)^2 \left(\frac{\mathrm{sn}(m\frac{t^\alpha}{\alpha} + nx + sy)}{\mathrm{dn}(m\frac{t^\alpha}{\alpha} + nx + sy) \mp cn(m\frac{t^\alpha}{\alpha} + nx + sy)}\right)^2.$$

For $r \to 0$ the solution is obtained as that of (21).

Solutions for Case 2

When $P = r^2$, $Q = -(1+r^2)$, $R = 1$ are chosen, $F = \mathrm{sn}$, from table 1, thus

$$u(x,y,t) = 2n^2\left(1 + r^2 + \sqrt{1-r^2+r^4}\right) - 6n^2r^2\,\mathrm{sn}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)$$

and considering $r \to 1$, from table 2, the solitary wave solution can be obtained as

$$u_{2,1}(x,y,t) = 6n^2 - 6n^2\tanh^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right), \tag{23}$$

where $m = \mp\sqrt{n^2 + 4n^4 + s^2}$.

Choosing $P = -r^2$, $Q = 2r^2 - 1$, $R = 1 - r^2$, it can be denoted, from table 1, $F = cn$, hence

$$u(x,y,t) = 2n^2\left(1 - 2r^2 + \sqrt{1-r^2+r^4}\right) + 6n^2r^2\, cn^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right). \tag{24}$$

For $r \to 1$, from table 2, the solitary wave solution is expressed as

$$uu_{2,2}(x,y,t) = 6n^2\,\mathrm{sech}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right),$$

where $m = \mp\sqrt{n^2 + 4n^4 + s^2}$. It is clearly seen that this solution is the same as that of (23).

While $P = -1$, $Q = 2 - r^2$, $R = r^2 - 1$, it can be deducted, from table 1, $F = \mathrm{dn}$, then the solution can be evaluated as

$$u(x,y,t) = 2n^2\left(-2 + r^2 + \sqrt{1-r^2+r^4}\right) + 6n^2\,\mathrm{dn}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

For $r \to 1$, it is clearly seen that this solution is the same as that of (23).

Setting $P = 1$, $Q = -(1+r^2)$, $R = r^2$, from table 1, $F = ns$, due to this settings

$$u(x,y,t) = 2n^2\left(1 + r^2 + \sqrt{1-r^2+r^4}\right) - 6n^2\,\mathrm{ns}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

Furthermore, if $r \to 1$ from table 2, the solitary wave solution of eq. (9) is as follows:

$$u_{2,3}(x,y,t) = 6n^2 - 6n^2\coth^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)$$

and, if $r \to 0$, from table 2 the periodic solution of eq. (9) can be written as

$$u_{2,4}(x,y,t) = 4n^2 - 6n^2\csc^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right), \tag{25}$$

where $m = \mp\sqrt{n^2 + 4n^4 + s^2}$.

Supposing $P = 1$, $Q = -(1+r^2)$, $R = r^2$, from table 1, this choice corresponds to $F = \mathrm{dc}$, so

$$u(x,y,t) = 2m^2\left(1 + r^2 + \sqrt{1-r^2+r^4}\right) - 6m^2\,\mathrm{dc}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)$$

is found and, for $r \to 0$, from table 2, the periodic solution can be obtained as

$$u_{2,5}(x,y,t) = 4n^2 - 6n^2 \sec^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right), \tag{26}$$

where $m = \mp\sqrt{n^2 + 4n^4 + s^2}$.

Considering $P = 1$, $Q = 2 - r^2$, $R = 1 - r^2$ and from table 1, $F = \mathrm{cs}$, the solution can be acquired as

$$u(x,y,t) = 2n^2\left(-2 + r^2 + \sqrt{1 - r^2 + r^4}\right) - 6n^2 \,\mathrm{cs}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

As $r \to 0$, from table 2 we get the periodic solution as

$$uu_{2,6}(x,y,t) = -2n^2 - 6n^2 \cot^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)$$

and, as $r \to 1$, from table 2 we get the solitary wave solution

$$u_{2,7}(x,y,t) = -6n^2 \,\mathrm{csch}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right), \tag{27}$$

where $m = \mp\sqrt{n^2 + 4n^4 + s^2}$. The first solution is the same as that of (25).

Also, regarding $P = 1 - r^2$, $Q = 2 - r^2$, $R = 1$ and, from table 1, $F = \mathrm{sc}$, the solution can be evaluated as

$$u(x,y,t) = 2n^2\left(-2 + r^2 + \sqrt{1 - r^2 + r^4}\right) - 6n^2(1 - r^2)\,\mathrm{sc}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

In addition, for $r \to 0$, from table 2 the periodic solution can be stated as

$$uu_{2,8}(x,y,t) = -2n^2 - 6n^2 \tan^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right),$$

where $m = \mp\sqrt{n^2 + 4n^4 + s^2}$. The obtained solution is the same as that of (26).

Also assigning $P = 1$, $Q = 2r^2 - 1$, $R = r^4 - r^2$ and $F = \mathrm{ds}$, from table 1, the solution is found as

$$u(x,y,t) = 2n^2\left(1 - 2r^2 + \sqrt{1 - r^2 + r^4}\right) - 6n^2 \,\mathrm{ds}^2\left(m\frac{t^\alpha}{\alpha} + nx + sy\right).$$

In this case, for $r \to 0$, from table 2 the solution is obtained as that of (25) and, for $r \to 1$, the solution is obtained as that of (27). When P, Q, R are chosen as $P = -\frac{1}{4}$, $Q = \frac{r^2+1}{2}$, $R = -\frac{(1-r^2)^2}{4}$, F is going to be as $F = r\,cn \mp \mathrm{dn}$ from table 1, so the solution can be obtained as

$$\begin{aligned} u(x,y,t) = {} & -n^2\left(1 + r^2 + \frac{\sqrt{1 + 14r^2 + r^4}}{2}\right) \\ & + \frac{3n^2}{2}\left(r\,cn\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \mathrm{dn}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2. \end{aligned}$$

Furthermore, for $r \to 1$, the solution is found as that of (23) using table 1.

If we choose P, Q, R as $P = \frac{1}{4}$, $Q = \frac{-2r^2+1}{2}$, $R = \frac{1}{4}$, from table 1, F is obtained as $F = ns \mp \mathrm{cs}$, in this way the solution can be expressed as

$$u(x,y,t) = -n^2\left(1 - 2r^2 - \frac{\sqrt{1 - 16r^2 + 16r^4}}{2}\right) - \frac{3n^2}{2}\left(ns\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \mathrm{cs}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2.$$

Moreover, for $r \to 0$, from table 2 the periodic solution can be stated as

$$u_{2,9}(x,y,t) = -\frac{n^2}{2} - \frac{3n^2}{2}\left(\csc\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \cot\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2$$

and, for $r \to 1$, the solitary solution can be found as

$$u_{2,10}(x,y,t) = \frac{3n^2}{2} - \frac{3n^2}{2}\left(\coth\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \operatorname{csch}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2,$$

where $m = \mp\sqrt{n^2 + n^4 + s^2}$.

For choices $P = \frac{1-r^2}{4}$, $Q = \frac{r^2+1}{2}$, $R = \frac{1-r^2}{4}$, from table 1, F can be written as $F = \mathrm{nc} \mp \mathrm{sc}$, thus solution is found as

$$u(x,y,t) = -n^2\left(1 + r^2 - \frac{\sqrt{1+14r^2+r^4}}{2}\right) - \frac{3n^2}{2}(1-r^2)\left(\mathrm{nc}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \mathrm{sc}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2.$$

For $r \to 0$, with the help of table 2, the periodic solution can be acquired as

$$u_{2,11}(x,y,t) = -\frac{n^2}{2} - \frac{3n^2}{2}\left(\sec\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp \tan\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2,$$

where $m = \mp\sqrt{n^2 + n^4 + s^2}$.

Setting $P = \frac{r^2}{4}$, $Q = \frac{r^2-2}{2}$, $R = \frac{r^2}{4}$, from table 1, $F = \mathrm{sn} \mp i\, cn$, due to this settings

$$u(x,y,t) = n^2\left(2 - r^2 + \frac{\sqrt{16-16r^2+r^4}}{2}\right) - \frac{n^2}{2}r^2\left(\mathrm{sn}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp i\, cn\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2.$$

For $r \to 1$, by using table 2, the solitary solution can be evaluated as

$$u_{2,12}(x,y,t) = \frac{3n^2}{2} - \frac{3n^2}{2}\left(\tanh\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp i\operatorname{sech}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2, \tag{28}$$

where $m = \mp\sqrt{n^2 + n^4 + \alpha_2^2}$.

Regarding $P = \frac{1}{4}$, $Q = \frac{1-2r^2}{2}$, $R = \frac{1}{4}$, from table 1, one obtains that $F = r\,\mathrm{sn} \mp i\,\mathrm{dn}$ and the solution can be expressed as

$$u(x,y,t) = -n^2\left(1 - 2r^2 - \frac{\sqrt{1-16r^2+16r^4}}{2}\right) - \frac{3n^2}{2}\left(r\,\mathrm{sn}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right) \mp i\,\mathrm{dn}\left(m\frac{t^\alpha}{\alpha} + nx + sy\right)\right)^2.$$

For $r \to 1$, the solution is obtained as that of (28).

Considering $P = \frac{1}{4}$, $Q = \frac{1-2r^2}{2}$, $R = \frac{1}{4}$, from table 1, F can be expressed as $F = \frac{\mathrm{sn}}{1 \mp cn}$, so the solution can be found as

$$u(x,y,t) = -n^2\left(1 - 2r^2 - \frac{\sqrt{1-16r^2+16r^4}}{2}\right) - \frac{3n^2}{2}\left(\frac{\mathrm{sn}(m\frac{t^\alpha}{\alpha} + nx + sy)}{1 \mp cn(m\frac{t^\alpha}{\alpha} + nx + sy)}\right)^2.$$

Also for $r \to 0$, by way of table 2, the periodic solution can be acquired as

$$u_{2,13}(x,y,t) = -\frac{n^2}{2} - \frac{3n^2}{2}\left(\frac{\sin(m\frac{t^\alpha}{\alpha} + nx + sy)}{1 \mp \cos(m\frac{t^\alpha}{\alpha} + nx + sy)}\right)^2, \tag{29}$$

and, for $r \to 1$, the solitary wave solution can be stated as

$$u_{2,14}(x,y,t) = \frac{3n^2}{2} - \frac{3n^2}{2}\left(\frac{\tanh(m\frac{t^\alpha}{\alpha} + nx + sy)}{1 \mp \operatorname{sech}(m\frac{t^\alpha}{\alpha} + nx + sy)}\right)^2, \tag{30}$$

where $m = \mp\sqrt{n^2 + n^4 + s^2}$.

Supposing $P=\frac{r^2}{4}$, $Q=\frac{r^2-2}{2}$, $R=\frac{1}{4}$, it can be deduced, from table 1, $F=\frac{\mathrm{sn}}{1\mp\mathrm{dn}}$ and the solution can be evaluated as

$$u(x,y,t)=n^2\left(2-r^2+\frac{\sqrt{16-19r^2+4r^4}}{2}\right)-\frac{3n^2}{2}r^2\left(\frac{\mathrm{sn}(m\frac{t^\alpha}{\alpha}+nx+sy)}{1\mp\mathrm{dn}(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2.$$

For $r\to 1$, the solution is obtained as that of (30).

Assigning $P=\frac{1-r^2}{4}$, $Q=\frac{r^2+1}{2}$, $R=\frac{1-r^2}{4}$, from table 1, this assignment corresponds to $F=\frac{cn}{1\mp\mathrm{sn}}$, so the solution can be found as

$$u(x,y,t)=-n^2\left(1+r^2-\frac{\sqrt{1+14r^2+r^4}}{2}\right)-\frac{3n^2}{2}(1-r^2)\left(\frac{cn(m\frac{t^\alpha}{\alpha}+nx+sy)}{1\mp\mathrm{sn}(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2.$$

Considering $r\to 0$, in the light of table 2, the periodic solution can be shown as

$$u_{2,15}(x,y,t)=-\frac{n^2}{2}-\frac{3n^2}{2}\left(\frac{\cos(m\frac{t^\alpha}{\alpha}+nx+sy)}{1\mp\sin(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2,$$

where $m=\mp\sqrt{n^2+n^4+s^2}$.

Choosing $P=\frac{(1-r^2)^2}{4}$, $Q=\frac{r^2+1}{2}$, $R=\frac{1}{4}$ from table 1, this choice follows $F=\frac{\mathrm{sn}}{\mathrm{dn}\mp cn}$, so that the solution can be obtained as

$$u(x,y,t)=-n^2\left(1+r^2-\frac{\sqrt{1+14r^2+r^4}}{2}\right)$$
$$-\frac{3n^2}{2}(1-r^2)^2\left(\frac{\mathrm{sn}(m\frac{t^\alpha}{\alpha}+nx+sy)}{\mathrm{dn}(m\frac{t^\alpha}{\alpha}+nx+sy)\mp cn(m\frac{t^\alpha}{\alpha}+nx+sy)}\right)^2.$$

For $r\to 0$ the solution is obtained as that of (29).

4 Exact traveling wave solutions to the time-fractional combined KdV-mKdV equation

The KdV and mKdV equations are most popular soliton equations and have been comprehensively investigated. But nonlinear terms of the KdV and mKdV equations often simultaneuosly exist in some problems, such as fluid physics and quantum field theory, and form the combined KdV-mKdV equation. In this paper, we restrict our attention to the study of the conformable time-fractional combined KdV-mKdV equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha}+mu\frac{\partial u}{\partial x}+nu^2\frac{\partial u}{\partial x}-s\frac{\partial^3 u}{\partial x^3}=0, \tag{31}$$

where $\alpha\in(0,1)$ and $\frac{\partial^\alpha u}{\partial t^\alpha}$ means conformable fractional derivative of function $u(x,t)$. Applying the wave transformation,

$$u=u(\xi),\qquad \xi=w\frac{t^\alpha}{\alpha}+cx, \tag{32}$$

eq. (31) turns into the following form:

$$wu_\xi+cmuu_\xi+cnu^2u_\xi-\mathrm{sc}^3\,u_{\xi\xi\xi}=0. \tag{33}$$

Balancing the highest-order linear term and the highest-order nonlinear term, we obtain $k=1$, thus the solution of eq. (31) can be stated as

$$u=a_0+a_1F(\xi). \tag{34}$$

Thus from eq. (34) and eq. (5),

$$u_\xi=a_1F'(\xi) \tag{35}$$

and

$$u_{\xi\xi\xi}=a_1F'''(\xi)=a_1(6PF^2F'+QF'), \tag{36}$$

where $F'''(\xi) = 6PF^2F' + QF'$ is obtained by differentiating (5) two times. Substituting (35) and (36) into eq. (33) and setting each coefficient of F to be zero, an equation system arises

$$a_1(a_0cm + a_0^2\, cn - c^3Qs + w) = 0$$
$$a_1(a_1^2\, cn - 6c^3Ps) = 0$$
$$a_1(a_1cm + 2a_0a_1\, cn) = 0.$$

Solving this system by using Mathematica yields

$$a_0 = -\frac{m}{2n}, \qquad a_1 = \mp\frac{c\sqrt{6Ps}}{\sqrt{n}}, \qquad w = \frac{c(m^2 + 4c^2nQs)}{4n}.$$

Following the same solution procedure, which is mentioned in sect. 2, respectively, using table 1 and table 2 and combining the values with eq. (34), we can get exact solutions of eq. (31) as follows:

$$u_1(x,t) = -\frac{m \mp 2c\sqrt{6\,ns}\tanh(cx + \frac{ct^\alpha(m^2-8c^2\,ns)}{4n\alpha})}{2n}$$
$$u_2(x,t) = -\frac{m \mp 2c\sqrt{6\,ns}\coth(cx + \frac{ct^\alpha(m^2-8c^2\,ns)}{4n\alpha})}{2n}$$
$$u_3(x,t) = -\frac{m \mp 2c\sqrt{6\,ns}\csc(cx + \frac{ct^\alpha(m^2-4c^2\,ns)}{4n\alpha})}{2n}$$
$$u_4(x,t) = -\frac{m \mp 2c\sqrt{6\,ns}\sec(cx + \frac{ct^\alpha(m^2-4c^2\,ns)}{4n\alpha})}{2n}$$
$$u_5(x,t) = -\frac{m \mp 2c\sqrt{6\,ns}\tan(cx + \frac{ct^\alpha(m^2+8c^2\,ns)}{4n\alpha})}{2n}$$
$$u_6(x,t) = -\frac{m \mp 2c\sqrt{6\,ns}\cot(cx + \frac{ct^\alpha(m^2+8c^2\,ns)}{4n\alpha})}{2n}$$
$$u_7(x,t) = -\frac{m \mp 2c\sqrt{6\,ns}\,\mathrm{csch}(cx + \frac{ct^\alpha(m^2+4c^2\,ns)}{4n\alpha})}{2n}$$
$$u_8(x,t) = -\frac{m \mp c\sqrt{6\,ns}(\sec(cx + \frac{ct^\alpha(m^2+2c^2\,ns)}{4n\alpha}) \mp \tan(cx + \frac{ct^\alpha(m^2+2c^2\,ns)}{4n\alpha}))}{2n}$$
$$u_9(x,t) = -\frac{m \mp c\sqrt{6\,ns}(\coth(cx + \frac{ct^\alpha(m^2-2c^2\,ns)}{4n\alpha}) \mp \mathrm{csch}(cx + \frac{ct^\alpha(m^2-2c^2\,ns)}{4n\alpha}))}{2n}$$
$$u_{10}(x,t) = -\frac{m \mp \frac{c\sqrt{6\,ns}\tanh(cx+\frac{ct^\alpha(m^2-2c^2\,ns)}{4n\alpha})}{1\mp\mathrm{sech}(cx+\frac{ct^\alpha(m^2-2c^2\,ns)}{4n\alpha})}}{2n}$$
$$u_{11}(x,t) = -\frac{m \mp \frac{c\sqrt{6\,ns}\cos(cx+\frac{ct^\alpha(m^2+2c^2\,ns)}{4n\alpha})}{1\mp\sin(cx+\frac{ct^\alpha(m^2+2c^2\,ns)}{4n\alpha})}}{2n}$$
$$u_{11}(x,t) = -\frac{m \mp \frac{c\sqrt{6\,ns}\sin(cx+\frac{ct^\alpha(m^2+2c^2\,ns)}{4n\alpha})}{1\mp\cos(cx+\frac{ct^\alpha(m^2+2c^2\,ns)}{4n\alpha})}}{2n}$$
$$u_{12}(x,t) = -\frac{m \mp 2ic\sqrt{6\,ns}\,\mathrm{sech}(cx + \frac{ct^\alpha(m^2+4c^2\,ns)}{4n\alpha})}{2n}$$
$$u_{13}(x,t) = -\frac{m \mp c\sqrt{6\,ns}(\tanh(cx + \frac{ct^\alpha(m^2-2c^2\,ns)}{4n\alpha}) + i\,\mathrm{sech}(cx + \frac{ct^\alpha(m^2-2c^2\,ns)}{4n\alpha}))}{2n}.$$

5 Conclusions

In this paper, the Jacobi elliptic function expansion method is used to obtain the exact solutions of some conformable nonlinear time-fractional wave equations. The Jacobi elliptic function expansion method has several advantages according to other traditional methods. This method obtains the results directly, quickly and needs simple algorithms in programming. Additionally by using conformable fractional derivative definition, fractional wave equations can be solved easily. Consequently, it is easily seen that many different type of fractional PDEs are suitable to solve by given solution procedure.

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