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The anti-de Sitter spacetime as a time machine

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Abstract. We construct an axially symmetric spacetime admitting, after a certain instant, closed timelike curves (CTCs) indicating that it is a time-machine spacetime. The spacetime, which is locally anti-de Sitter, is a four-dimensional extension of the Misner space with identical causality-violating properties. In this spacetime, CTCs evolve from a casually well-behaved initial hypersurface.

1 Introduction

The anti-de Sitter (AdS) spacetime has been an important topic of study in recent times on account of the celebrated AdS/CFT correspondence [1] which provides a connection between a quantum theory of gravity on an asymptotically AdS spacetime and a lower-dimensional conformal field theory (CFT) on the boundary of the spacetime. Thus, the study of causality violation in AdS space is of relevance. It is well known that the AdS is intrinsically causality-violating on account of the periodicity of the time coordinate. However, generally, one works in the universal covering space of the AdS, where the time coordinate is unwrapped to take values along the whole real line and, consequently, free from CTCs. The anti-de Sitter one is a conformally flat space of constant four-dimensional curvature satisfying the field equations for an Einstein space,

$$R_{\mu\nu} = \Lambda\, g_{\mu\nu}, \tag{1}$$

with $\Lambda < 0$. The anti-de Sitter space is a maximally symmetric space, consequently a space of constant curvature, locally characterised by

$$R_{\mu\nu\sigma\rho} = K\left(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}\right). \tag{2}$$

The Minkowski space, the de Sitter space and the anti-de Sitter space are the only spaces of constant curvature satisfying the vacuum equation (1) and, in addition, are conformally flat, *i.e.* the Weyl tensor

$$C_{\mu\nu\sigma\rho} = 0. \tag{3}$$

Equations (1), (2) and (3) uniquely determine these three simplest exact solutions, with $K < 0$, for the anti-de Sitter space. Additionally, we note that any spacetime with a curvature tensor expressible in the form (2) is asymptotically (anti-)de Sitter.

In this paper, we attempt to write down an axially symmetric metric, locally isometric to the anti-de Sitter space, where closed timelike curves (CTCs) appear after a certain instant —a time-machine spacetime. In this context we distinguish between eternal time-machine spacetimes, such as that of Gödel [2], or the van Stockum metric [3], where CTCs always exist, and true time-machine spacetimes, typically the one discussed in [4], where CTCs appear after a certain instant. Another time-machine spacetime of great interest is the Misner space [5]. It is characterized by a two-dimensional line element,

$$\mathrm{d}s^2 = -\mathrm{d}T\,\mathrm{d}X - T\,\mathrm{d}X^2, \tag{4}$$

where $-\infty < T < \infty$, but the coordinate X is periodic. The metric (4) is regular everwhere, as $\det g = -1$, including at $T = 0$. The curves $T = T_0$, where T_0 is a constant, are closed since X is periodic. The curves $T < 0$ are spacelike, $T > 0$ are timelike, indicating the formation of CTCs at $T = 0$.

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The axially symmetric form of the anti-de Sitter (AdS) spacetime discussed in this paper reduces to the Misner space in two dimensions. Here the time-machine behaviour of the Misner space is carried over to the AdS space in four dimensions. We note that there has been an earlier attempt at constructing a Misner-like AdS spacetime by Li [6] with the line element

$$\mathrm{d}s^2 = -(\mathrm{d}t - t\,\mathrm{d}\psi)^2 + \alpha^2\,\mathrm{d}\psi^2 + (\mathrm{d}y - y\,\mathrm{d}\psi)^2 + (\mathrm{d}z - z\,\mathrm{d}\psi)^2. \tag{5}$$

In (5), there are no CTCs when $t^2 < \alpha^2 + y^2 + z^2$, but CTCs appear in the region with $t^2 > \alpha^2 + y^2 + z^2$. However, this spacetime, as in the case of the Misner space, is multiply connected, and we attempt to overcome this problem by constructing a Misner-like, AdS spacetime which is simply connected.

2 Axially symmetric AdS spacetime

Consider the line element

$$\mathrm{d}s^2 = \coth^2(\alpha\,r)\,\mathrm{d}r^2 + 2\sinh^2(\alpha\,r)\left(\mathrm{d}z^2 + \sqrt{2}\cosh t\,\mathrm{d}t\,\mathrm{d}\phi - \sinh t\,\mathrm{d}\phi^2\right) - \frac{1}{2}\sinh^2(\alpha\,r)\cosh^2 t\,\mathrm{csch}\,t\,\mathrm{d}t^2, \tag{6}$$

where the constant α is real and positive. Here ϕ is periodic $\phi \sim \phi + 2\pi$, while for t, z, we have $-\infty < z,\, t < \infty$ and for the radial coordinate $0 \leq r < \infty$. Note that (6) reduces to $2D$ Misner space for constant r and z indicating that it is a $4D$ extension of it. The metric above is Lorentzian with the metric tensor having determinant $\det g = -2\cosh^2(\alpha\,r)\cosh^2 t\sinh^4(\alpha\,r)$ and the spacetime signature is $(+,+,+,-)$.

The metric (6) is conformally flat, its Weyl tensor satisfying (3). It is also an Einstein space with

$$\Lambda = -3\,\alpha^2, \tag{7}$$

i.e., a vacuum spacetime with negative cosmological constant. Moreover, by explicit calculation, we have determined that (2) is satisfied with $K = -\alpha^2$. The curvature scalar is calculated to have the value $R = -12\,\alpha^2$. Hence, we conclude that (6) is an anti-de Sitter space. We note that, as required, we have

$$\Lambda = \frac{1}{4}\,R \quad \text{and} \quad K = \frac{1}{12}\,R. \tag{8}$$

We have also calculated some of the curvature invariants and find that they are constant everywhere. For instance,

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 24\,\alpha^4 \tag{9}$$

and

$$R_{\mu\nu\rho\sigma;\lambda}R^{\mu\nu\rho\sigma;\lambda} = 0, \tag{10}$$

indicating that there are no curvature singularities in the spacetime represented by (6).

Closed timelike curves are generated in the spacetime represented by (6). The orbits of constant r, z and t are closed since ϕ is periodic. These orbits are spacelike for $t < 0$, null curves at $t = 0$ and timelike at $t > 0$, which indicate that this spacetime gives rise to CTCs when $t > 0$ in a manner identical to the Misner space. We also note that one can also ensure that the CTCs arise from an initially spacelike hypersurface. A $t = \text{const.}$ hypersurface is spacelike if the norm of the gradient of t, $\nabla_\mu t\nabla^\mu t < 0$, null if $\nabla_\mu t\nabla^\mu t = 0$, and timelike when $\nabla_\mu t\nabla^\mu t > 0$. For the spacetime considered here,

$$\nabla_\mu t\nabla^\mu t = 2\,\mathrm{csch}^2(\alpha\,r)\,\mathrm{sech}^2\,t\sinh t. \tag{11}$$

It is clear from the above that (for $r \neq 0$) a $t = \text{const.}$ hypersurface will remain spacelike for $t < 0$. For $t = 0$, the hypersurface is null and timelike for $t > 0$.

Static, cylindrically symmetric AdS spacetimes are characterised by the presence of deficit angles, which may be interpreted as infinite rods and cylinders characterised by a lack of a elementary flatness at the symmetry axis. In fact, the conformally flat de Sitter and anti-de Sitter spaces are not compatible with static cylindrical symmetry [7]. In our case, however, the metric functions are not all independent of time and satisfy the requirements for axial symmetry. In what follows, this is explicitly shown.

The spacetime represented by (6) admits the Killing vector $\boldsymbol{\eta} = \partial_\theta$, having the normal form

$$\eta^\mu = (0, 1, 0, 0). \tag{12}$$

Its covector is

$$\eta_\mu = \sinh^2(\alpha\,r)\,\left(0, -2\sinh t, 0, \sqrt{2}\cosh t\right). \tag{13}$$

The vector (12) satisfies the Killing equation $\eta_{\mu;\,\nu} + \eta_{\nu;\,\mu} = 0$. A cyclically symmetric spacetime admits a Killing vector with spacelike, closed orbits. If the norm of this vanishes at the origin, it indicates the presence of a non-empty axis of symmetry and the spacetime is called axially symmetric (see refs. [8, 9] and references therein). The Killing vector above η^μ has the norm

$$\eta_\mu \eta^\mu = -2\sinh^2(\alpha\, r)\sinh t. \tag{14}$$

Clearly, closed orbits of the above are spacelike for $t < 0$ and the norm vanishes at $r = 0$ (since $\alpha > 0$). The spacetime is regular on the axis as the condition for elementary flatness, namely, that in the limit of the symmetry axis one must have [8–10]

$$\frac{(\nabla_a(\eta_\mu \eta^\mu))(\nabla^a(\eta_\mu \eta^\mu))}{4\,\eta_\mu \eta^\mu} \to 1, \tag{15}$$

which also holds for $t < 0$. This indicates that the ϕ co-ordinate has period $2\,\pi$ and r is a radial coordinate. Moreover, since the spacetime contains the symmetry axis, it is simply connected.

3 Conclusion

In this paper, we have attempted to construct an axial symmetric anti-de Sitter space which is a four-dimensional extension of the Misner space and possesses its time-machine properties. We note that although there are a considerable number of known metrics that admit CTCs that always exist, *i.e.* eternal time machine, only a handful of true time-machine spacetimes, where CTCs appear at a specific instant, have been studied. Many of these possess the additional disadvantage of not satisfying the energy conditions and hence cannot be considered valid spacetimes.

We have determined that (6) represents the AdS space by showing that this metric satisfies the properties, which uniquely define it. It is also possible to transform this line element via a series of transformations to the form

$$\mathrm{d}s^2 = \mathrm{d}x^2 + e^{\beta\, x}\left(-\mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{16}$$

which is one of the known forms [11] of the anti-de Sitter space.

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