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An improved iterative technique for solving nonlinear doubly singular two-point boundary value problems

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Abstract. This paper presents a new iterative technique for solving nonlinear singular two-point boundary value problems with Neumann and Robin boundary conditions. The method is based on the homotopy perturbation method and the integral equation formalism in which a recursive scheme is established for the components of the approximate series solution. This method does not involve solution of a sequence of nonlinear algebraic or transcendental equations for the unknown coefficients as in some other iterative techniques developed for singular boundary value problems. The convergence result for the proposed method is established in the paper. The method is illustrated by four numerical examples, two of which have physical significance: The first problem is an application of the reaction-diffusion process in a porous spherical catalyst and the second problem arises in the study of steady-state oxygen-diffusion in a spherical cell with Michaelis-Menten uptake kinetics.

1 Introduction

We consider the following doubly singular two-point boundary value problems:

$$(g(x)y')' = g(x)p(x)f(x,y), \quad 0 < x \leq 1, \tag{1}$$

subject to the boundary conditions

$$y'(0) = 0, \qquad \mu y(1) + \sigma y'(1) = B. \tag{2}$$

Here, $\mu > 0$, $\sigma \geq 0$, and B is finite constant. The condition $g(0) = 0$ states that the problem (1) is singular. Further, if $p(x)$ is allowed to be discontinuous at $x = 0$, the problem (1) and (2) is called doubly singular [1].

The following conditions have been imposed on the functions $g(x)$, $p(x)$ and $f(x,y)$:

$C1$: $f(x,y)$ is continuous for all $(x,y) \in \{[0,1] \times R\}$,

$C2$: $\partial f(x,y)/\partial y$ exists and is continuous for all $(x,y) \in \{[0,1] \times R\}$

$C3$: $\partial f(x,y)/\partial y \geq 0$,

$C4$: $g(x) \geq 0$, $g(0) = 0$,

$C5$: $g(x) \in C^1(0,1]$,

$C6$: $1/g(x) \in L^1(0,1]$,

$C7$: $p(x) > 0$ in $(0,1]$, $p(x) \in L^1(0,1]$,

$C8$: $\displaystyle\int_0^1 \frac{1}{g(\eta)} \int_0^\eta g(t)p(t)\,\mathrm{d}t\,\mathrm{d}\eta < \infty$,

$C9$: $f(x,y)$ satisfies the Lipschitz condition $|f(x,y_1) - f(x,y_2)| \leq \sigma^*|y_1 - y_2|$,

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where σ^* is Lipschitz's constant and moreover it is required that $\sigma^* M < 1$, where M is a bound that depends on the different functions and boundary conditions that define the problem, see Lemma 1.

Pandey [2–4] has established the existence and uniqueness of the solution to problem (1) with $p(x) = 1$ and with boundary conditions $y(0) = 0$ (or $y'(0) = 0$) and $y(1) = A$. Further, Pandey and Verma [5,6] have established the existence and uniqueness results for the solution of problem (1) with boundary conditions $y(0) = 0$, $\mu y(1) + \sigma y'(1) = B$.

Singular boundary value problems frequently arise in the modeling of many problems in physical, biological, chemical and engineering sciences [7–18]. In particular, the problem (1) and (2) with $g(x) = x^2$, $p(x) = 1$ and $f(x,y) = \delta y/(y+\lambda)$, $\delta > 0$, $\lambda > 0$ arises in the study of steady-state oxygen-diffusion in a spherical cell with Michaelis-Menten uptake kinetics [7,8]. The problem (1) and (2) with $g(x) = x$, $p(x) = 1$ arises in the study of thermal explosions for $f(x,y) = -\nu e^y$ [11]. Further, the problem (1) and (2) arises in the study of the equilibrium of isothermal gas sphere when $g(x) = x^2$, $p(x) = 1$ and $f(x,y) = -y^5$, [12]. Another case of physical significance is when $g(x) = x^2$, $p(x) = 1$ and $f(x,y) = \theta^2 y^m$, which arises in the formulation of reaction-diffusion process in a porous spherical catalyst [13]. Here θ^2 denotes the Thiele modulus.

In recent years, the development of new numerical and analytical tools for solving singular boundary value problems has emerged as an area of great promise. The solution of the nonlinear singular boundary value problems arising in various physical models discussed above is numerically challenging because of singularity behavior at the origin. Several numerical and analytical techniques have been proposed by various authors for solving the singular differential equation (1) with $g(x) = x^\alpha$ (or $g(x) = x^\alpha s(x)$), ($\alpha > 0$, $s(x) \geq 0$) and $p(x) = 1$ and with boundary conditions $y(0) = 0$ (or $y'(0) = 0$) and $y(1) = A$ (or $\mu y(1) + \sigma y'(1) = B$), such as finite difference method [19–25], spline method [26–28], finite element method [29] and decomposition method [30]. Asaithambi *et al.* [31] presented a numerical technique for obtaining pointwise bounds for the solution of a class of nonlinear singular boundary-value problems arising in physiology. Moreover, El-Gebeily and Abu-Zaid [32] have developed a finite difference method based on a uniform mesh for the solution of the following linear singular boundary value problem:

$$(g(x)y')' = w(x)(f(x) - q(x)y(x)), \quad 0 < x \leq 1,$$

with boundary conditions

$$y'(0) = 0, \qquad y(1) = 0.$$

However, no numerical or analytical method has yet been developed to approximate the solution of nonlinear doubly singular boundary value problem (1) with Neumann and Robin boundary conditions (2). Although some recursive methods [33–35] have been developed for solving the problem (1) and (2) with $g(x) = x^\alpha$, $\alpha > 0$ and $p(x) = 1$ and however, these methods require the computation of an undetermined coefficient because the explicit expression for the components of the approximate series solution contains an unknown constant. Such constant is determined by solving a sequence of nonlinear algebraic or transcendental equations of higher order, which increases the computational cost. In some cases, the undetermined coefficient may not be obtained uniquely.

The aim of this study is to introduce a new algorithm to approximate the solution of nonlinear doubly singular boundary value problem (1)-(2). There are three major steps occurring in this algorithm:

i) Convert the original problem (1)-(2) into an equivalent nonlinear integral equation.
ii) Employ boundary conditions (2) to eliminate the undetermined coefficients associated with the resulting integral equation.
iii) Implement the homotopy perturbation method [36] to the integral equation without undeter-mined coefficients for the solution of the considered problem.

In addition to the development of the method, we establish its convergence result. The method is illustrated by one nonlinear doubly singular problem with known exact solution. The method is also implemented to obtain approximate solution of two singular boundary value problems- first one arises in the study of steady-state oxygen diffusion in a spherical cell [7], and second one arises in the study of reaction-diffusion process in a porous spherical catalyst [13]. Numerical results obtained by the proposed method are compared with that obtained using the finite difference method [23], B-Spline method [27] and the method in [31].

This article is organized as follows. In the following section we will discuss basic principles of the standard homotopy-perturbation method. We derive a new recursive scheme to obtain solution of singular boundary value problem (1)-(2) in sect. 3. Section 4 contains the convergence analysis of the proposed method. In sect. 5, the method is applied on four numerical examples taken from the literature and comparison is made between the present method with some existing methods. Finally, conclusions are drawn in sect. 6.

2 Review of the standard homotopy-perturbation method

In this section, we give a brief description of the standard homotopy perturbation method (HPM). HPM is a combination of the classical perturbation method and the homotopy concept as used in topology, originally introduced by J.-H. He [36]. This method has been effectively applied to a wide class of functional equations [37–43].

We consider the following nonlinear differential equation:

$$G(y) + f(r) = 0, \tag{3}$$

with boundary conditions

$$B^*\left(y, \frac{\partial y}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{4}$$

where G is a differential operator, $f(r)$ denotes an analytical function, y is an unknown function, Γ is the boundary of the domain Ω and B^* is a boundary operator. The operator G can be divided into two parts L and N, where L is linear and N is nonlinear. Equation (3) can therefore be written as

$$L(y) + N(y) + f(r) = 0. \tag{5}$$

Basing on the homotopy idea, one can construct a homotopy equation $u(r,p) : \Omega \times [0,1] \to R$ for eq. (3) which satisfies

$$H(u,p) = (1-p)[L(u) - L(u_0)] + p[G(u) + f(r)] = 0, \tag{6}$$

or

$$H(u,p) = L(u) - L(u_0) + pL(u_0) + p[N(u) + f(r)] = 0, \tag{7}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of eq. (3), which satisfies the boundary condition of eq. (4).

It is clear that, for $H(u,p) = 0$, we have

$$H(u,0) = L(u) - L(u_0) = 0, \tag{8}$$

$$H(u,1) = G(u) + f(r) = 0. \tag{9}$$

When the homotopy parameter p varies through 0 to 1, $u(r,p)$ varies correspondingly through $y_0(r,p)$ to $y(r,p)$. In topology, this is called deformation and $H(u,0)$ and $H(u,1)$ are called homotopic.

By means of homotopy perturbation method, p is implemented as a small parameter, the solution u of eq. (6) can be expressed as a power series in p:

$$u = \sum_{i=0}^{\infty} u_i p^i. \tag{10}$$

Setting $p = 1$ in (10), one can obtain the approximate solution of eq. (3) as

$$y = \sum_{i=0}^{\infty} u_i(x). \tag{11}$$

3 Derivation of an improved iterative technique

Here we derive a new recursive scheme for solving nonlinear doubly singular differential equation (1) with Neumann and Robin boundary conditions (2). For the purpose, we set $z(x) = g(x)y'$ in eq. (1), then integrating eq. (1) from 0 to x, we get

$$z(x) = z(0) + \int_0^x g(t)p(t)f(t,y)\mathrm{d}t. \tag{12}$$

Imposing the boundary condition at $x = 0$ in eq. (12), it follows that

$$y'(x) = \frac{1}{g(x)} \int_0^x g(t)p(t)f(t,y)\mathrm{d}t. \tag{13}$$

Again, integrating eq. (13) from x to 1, yields

$$y(x) = y(1) - \int_x^1 \frac{1}{g(\eta)} \left(\int_0^\eta g(t)p(t)f(t,y)\mathrm{d}t\right) \mathrm{d}\eta. \tag{14}$$

We set $y(1) = C^*$, where C^* is not known. To determine the value of C^* we apply the right end boundary condition to eq. (14).

Thus, using the boundary condition at $x = 1$ we obtain as follows:

$$y(1) = C^* = \frac{B}{\mu} - \frac{\sigma}{\mu g(1)} \int_0^1 g(t)p(t)f(t,y)\mathrm{d}t. \tag{15}$$

Substituting (15) into eq. (14) yields

$$y(x) = \frac{B}{\mu} - \frac{\sigma}{\mu g(1)} \int_0^1 g(t)p(t)f(t,y)\mathrm{d}t - \int_x^1 \frac{1}{g(\eta)} \left(\int_0^\eta g(t)p(t)f(t,y)\mathrm{d}t \right) \mathrm{d}\eta. \tag{16}$$

Interchanging the order of integration in (16), we get

$$\begin{aligned} y(x) = \frac{B}{\mu} - \frac{\sigma}{\mu g(1)} \int_0^1 g(t)p(t)f(t,y)\mathrm{d}t - \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta \right) g(t)p(t)f(t,y)\mathrm{d}t \\ - \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta \right) g(t)p(t)f(t,y)\mathrm{d}t. \end{aligned} \tag{17}$$

Now the nonlinear integral equation (17) without any undetermined coefficients is treated by using the HPM to establish a recursive scheme for the solution of the original problem (1)-(2). In view of this, we consider eq. (17) as

$$\begin{aligned} L(y) = y(x) - \frac{B}{\mu} + \frac{\sigma}{\mu g(1)} \int_0^1 g(t)p(t)f(t,y)\mathrm{d}t + \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta \right) g(t)p(t)f(t,y)\mathrm{d}t \\ + \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta \right) g(t)p(t)f(t,y)\mathrm{d}t = 0. \end{aligned} \tag{18}$$

By means of the homotopy perturbation method, we construct the following homotopy:

$$H(y,p) = (1-p)F(y) + pL(y), \tag{19}$$

where $F(y) = y(x) - \frac{B}{\mu}$.

If $p = 0$, then eq. (19) becomes

$$H(y,0) = F(y) \tag{20}$$

and, when $p = 1$, eq. (19) turns out to be the original equation, *i.e.*

$$\begin{aligned} H(y,1) = y(x) - \frac{B}{\mu} + \frac{\sigma}{\mu g(1)} \int_0^1 g(t)p(t)f(t,y)\mathrm{d}t + \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta \right) g(t)p(t)f(t,y)\mathrm{d}t \\ + \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta \right) g(t)p(t)f(t,y)\mathrm{d}t. \end{aligned} \tag{21}$$

The solution of (21) and the nonlinear function $f(x,y)$ in (21) are decomposed by an infinite series of embedded parameter p as

$$y = \sum_{i=0}^{\infty} y_i p^i$$

and

$$f(x,y) = \sum_{n=0}^{\infty} H_n(y_0, y_1, \ldots, y_n)p^n, \tag{22}$$

where the H_n is the so-called He's polynomial, which can be calculated by using the formula

$$H_n(y_0, y_1, \ldots, y_n) = \frac{1}{n!} \left(\frac{\mathrm{d}^n f(x,y)}{\mathrm{d}p^n} \right)_{p=0}. \tag{23}$$

Inserting eq. (10) into eq. (21) and then equating the identical powers of p, we obtain a set of nonlinear integral equations:

$$\begin{aligned}
p^0 &: y_0(t) = \frac{B}{\mu},\\
p^1 &: y_1(t) = -\frac{\sigma}{\mu g(1)}\int_0^1 g(t)p(t)H_0(t,y)\mathrm{d}t - \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_0(t,y)\mathrm{d}t\\
&\quad -\int_x^1\left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_0(t,y)\mathrm{d}t,\\
p^2 &: y_2(t) = -\frac{\sigma}{\mu g(1)}\int_0^1 g(t)p(t)H_1(t,y)\mathrm{d}t - \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_1(t,y)\mathrm{d}t\\
&\quad -\int_x^1\left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_1(t,y)\mathrm{d}t,\\
p^3 &: y_3(t) = -\frac{\sigma}{\mu g(1)}\int_0^1 g(t)p(t)H_2(t,y)\mathrm{d}t - \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_2(t,y)\mathrm{d}t\\
&\quad -\int_x^1\left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_2(t,y)\mathrm{d}t,\\
p^4 &: y_4(t) = -\frac{\sigma}{\mu g(1)}\int_0^1 g(t)p(t)H_3(t,y)\mathrm{d}t - \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_3(t,y)\mathrm{d}t\\
&\quad -\int_x^1\left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_3(t,y)\mathrm{d}t,
\end{aligned}\tag{24}$$

and so on.

The present method can be defined by the following recurrence relation:

$$\begin{aligned}
y_0(t) &= \frac{B}{\mu},\\
y_i(t) &= -\frac{\sigma}{\mu g(1)}\int_0^1 g(t)p(t)H_{i-1}(t,y)\mathrm{d}t - \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_{i-1}(t,y)\mathrm{d}t\\
&\quad -\int_x^1\left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)H_{i-1}(t,y)\mathrm{d}t, \quad i\geq 1.
\end{aligned}\tag{25}$$

Hence, the n-th order approximation to the doubly singular boundary value problem (1) with boundary condition (2) can be obtained as

$$\phi_n = y_0 + y_1 + y_2 + \ldots + y_n. \tag{26}$$

We note that the recursive scheme defined in (25) does not require the computation of unknown constants for solving the singular boundary value problem (1)-(2). Hence, there is a huge gain in efficiency since the computationally expensive unknown constants evaluation is not performed.

4 Convergence of the method

In this section, we establish the convergence result of the proposed method (25) developed for solving the nonlinear doubly singular boundary value problem (1) and (2). For this, we write the equation (21) in operator form as

$$y = C + N(y), \tag{27}$$

where $C = \frac{B}{\mu}$ and

$$N(y) = -\frac{\sigma}{\mu g(1)}\int_0^1 g(t)p(t)f(t,y)\mathrm{d}t - \int_0^x\left(\int_x^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)f(t,y)\mathrm{d}t - \int_x^1\left(\int_t^1 \frac{1}{g(\eta)}\mathrm{d}\eta\right) g(t)p(t)f(t,y)\mathrm{d}t,$$

or

$$N(y) = -\int_0^1 k(x,t)g(t)p(t)f(t,y)\mathrm{d}t, \tag{28}$$

where $k(x,t)$ is the kernel and is given by

$$k(x,t) = \begin{cases} \displaystyle\int_x^1 \frac{1}{g(\eta)}\,\mathrm{d}\eta + \frac{\sigma}{\mu g(1)}\,, & t \le x, \\ \displaystyle\int_t^1 \frac{1}{g(\eta)}\,\mathrm{d}\eta + \frac{\sigma}{\mu g(1)}\,, & t > x. \end{cases}$$

Lemma 1. *Let g satisfy $C5$ and $C6$ and p satisfies $C7$. Then there exists a constant M such that*

$$\max_{0\le x\le 1}\left|\int_0^1 k(x,t)g(t)p(t)\mathrm{d}t\right| = M < \infty.$$

Proof. The proof of the Lemma is quite straightforward. In view of the conditions defined in $C5$–$C8$ and the kernel $k(x,t)$ as defined in (28) we must have

$$\max_{0\le x\le 1}\left|\int_0^1 k(x,t)g(t)p(t)\mathrm{d}t\right| = M < \infty.$$

Lemma 2. *Let $\{S_n\}$ be a sequence of partial sum of the series solution $\sum_{i=0}^{\infty} y_i$ with components $y_i, i \ge 0$, defined by eq. (25) and N be a nonlinear operator defined in (28). Then the sequence $\{S_n\}$ can be written in operator form as $S_n = C + N(S_{n-1})$, $n \ge 1$, with $C = \frac{B}{\mu}$.*

Proof. Let $S_n = \sum_{i=0}^{n} y_i$ be the n-th partial sum of the series solution $\sum_{i=0}^{\infty} y_i$.

With the help of the solution components defined in (25), one can easily obtain

$$\begin{aligned} S_n &= \sum_{i=0}^{n} y_i \\ &= \frac{B}{\mu} - \frac{\sigma}{\mu g(1)}\int_0^1 g(t)p(t)\sum_{i=1}^{n} H_{i-1}(t)p^{i-1}\,\mathrm{d}t - \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}\,\mathrm{d}\eta\right) g(t)p(t)\sum_{i=1}^{n} H_{i-1}(t)p^{i-1}\,\mathrm{d}t \\ &\quad - \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}\,\mathrm{d}\eta\right) g(t)p(t)\sum_{i=1}^{n} H_{i-1}(t)p^{i-1}\,\mathrm{d}t. \end{aligned} \tag{29}$$

Inserting eq. (22) in (29) we obtain

$$\begin{aligned} S_n = \frac{B}{\mu} &- \frac{\sigma}{\mu g(1)}\int_0^1 g(t)p(t)f(t,S_{n-1})\mathrm{d}t - \int_0^x \left(\int_x^1 \frac{1}{g(\eta)}\,\mathrm{d}\eta\right) g(t)p(t)f(t,S_{n-1})\mathrm{d}t \\ &- \int_x^1 \left(\int_t^1 \frac{1}{g(\eta)}\,\mathrm{d}\eta\right) g(t)p(t)f(t,S_{n-1})\mathrm{d}t. \end{aligned} \tag{30}$$

Hence the sequence generated by the method can be written as

$$S_n = C + N(S_{n-1}), \quad n \ge 1, \quad \text{with} \quad C = \frac{B}{\mu} = y_0.$$

This completes the proof of Lemma 2.

Theorem 1. *Suppose that $X = C[0,1]$ be a Banach space with the norm $\|z\| = \max_{x\in[0,1]}|z(x)|$, $x \in X$. Let $N : X \to X$ be a nonlinear mapping defined by (28). Also assume that $f(x,y)$ satisfies the condition $C9$. Let $\beta = \sigma^* M$, where the constant M defined in Lemma 1. Then eq. (27) has a unique solution in X.*

Proof. In view of Lemma 1 and using the Lipschitz condition on $f(x,y)$, we have for any $y_1, y_2 \in X$:

$$\begin{aligned}
|N(y_1) - N(y_2)| &= \left| \int_0^1 k(x,t)g(t)p(t)f(t,y_1)\mathrm{d}t - \int_0^1 k(x,t)g(t)p(t)f(t,y_2)\mathrm{d}t \right| \\
&= \left| \int_0^1 k(x,t)g(t)p(t)[f(t,y_1) - f(t,y_2)]\mathrm{d}t \right| \\
&\le \max_{x\in[0,1]} \left| \int_0^1 k(x,t)g(t)p(t)\mathrm{d}t \right| \times \max_{t\in[0,1]} |f(t,y_1) - f(t,y_2)| \\
&= M \times \max_{t\in[0,1]} |f(t,y_1) - f(t,y_2)| \\
&\le M \times \max_{t\in[0,1]} \sigma^* |y_1 - y_2| \\
&= M\sigma^* \|y_1 - y_2\|.
\end{aligned}$$

Setting $\beta = \sigma^* M$, we have

$$|N(y_1) - N(y_2)| \le \beta \|y_1 - y_2\|.$$

If $\beta < 1$, then the nonlinear mapping N is contraction.

Hence by the Banach contraction mapping theorem, eq. (27) has a unique solution in X.

Theorem 2. *Suppose that $X = C[0,1]$ be a Banach space with the norm $\|z\| = \max_{x\in[0,1]} |z(x)|$, $x \in X$. Let $N : X \to X$ is the nonlinear mapping defined by (28) which satisfies the Lipschitz condition $\|N(x_1) - N(x_2)\| \le \beta\|x_1 - x_2\|$, for all $x_1, x_2 \in X$, with $0 \le \beta < 1$. If we assume that $\|y_0\| < \infty$, then the sequence $S_n = C + N(S_{n-1})$, converges to the exact solution y.*

Proof. Let S_n be the n-th partial sum of the series $\sum_{i=0}^{\infty} y_i$, as defined by $S_n = C + N(S_{n-1})$.

The convergence of the sequence S_n is equivalent to the convergence of the series $\sum_{i=0}^{\infty} y_i$.

We complete the proof showing that

i)
$$\|S_{n+1} - S_n\| \le \beta^n \|y_0\|, \quad \forall n; \tag{31}$$

ii) the sequence S_n is a Cauchy sequence in $X = C[0,1]$.

The proof of the assertion (31) follows by the method of induction. Using the definition of nonlinear operator N, for $n = 1$ we have

$$\|S_2 - S_1\| = \|N(S_1) - N(S_0)\| \le \beta\|S_1 - S_0\| = \beta\|y_0\|.$$

So the result is true for $n = 1$.

Assume that the assertion (31) is valid for $n = k$, *i.e.*

$$\|S_{k+1} - S_k\| = \|N(S_k) - N(S_{k-1})\| \le \beta\|S_k - S_{k-1}\| = \beta^k\|y_0\|$$

Now we have to prove that the result is true for $n = k+1$:

$$\|S_{k+2} - S_{k+1}\| = \|N(S_{k+1}) - N(S_k)\| \le \beta\|S_{k+1} - S_k\| \le \beta^{k+1}\|S_1 - S_0\| = \beta^{k+1}\|y_0\|.$$

Hence the result is true for all values of n.

We next show that S_n is a Cauchy sequence on the Banach space X.

For every $m, n \in N$, $m \le n$,

$$\begin{aligned}
\|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+2} - S_{m+1}) + (S_{m+1} - S_m)\| \\
&\le \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| + \ldots + \|S_{m+2} - S_{m+1}\| + \|S_{m+1} - S_m\|.
\end{aligned}$$

Now using (31) we have

$$\begin{aligned}
\|S_n - S_m\| &\le \beta^{n-1}\|y_0\| + \beta^{n-2}\|y_0\| + \ldots + \beta^{m+1}\|y_0\| + \beta^m\|y_0\| \\
&\le \|y_0\|\beta^m(1 + \beta + \beta^2 + \ldots + \beta^{n-1-m}) \\
&\le \|y_0\|\beta^m \left(\frac{1 - \beta^{n-m}}{1 - \beta} \right),
\end{aligned}$$

with $0 < \beta < 1$, and $\|y_0\| < \infty$, by assumption, the immediate consequence of above inequality is that

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0.$$

Therefore, S_n is a Cauchy sequence in the Banach space X. This implies that the series solution $\sum_{i=0}^{\infty} y_i$ converges to the exact solution y.

5 Numerical experiments

In this section, we demonstrate the applicability, flexibility and accuracy of the present scheme by applying it to several nonlinear singular boundary value problems. We verify whether the numerical examples do satisfy the assumption of the convergence theorem before applying the described method to obtain their solution. In addition, numerical results are compared with that of [23,27,31]. All the numerical computations were done using symbolic computation software package Maple.

Example 1

Consider the following nonlinear doubly singular boundary value problem:

$$(x^{\alpha}y')' = \rho x^{\alpha+\rho-2}\frac{(\rho x^{\rho}e^{y} - (\alpha+\rho-1))}{4+x^{\rho}}, \tag{32}$$

$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5\ln(1/5) - 1. \tag{33}$$

The exact solution is given by $Y(x) = -\ln(4 + x^{\rho})$.

We solve the problem (32)-(33) for $\alpha = 0.5$ and $\rho = 1$. This problem corresponds to eqs. (1) and (2) with $g(x) = x^{0.5}$, $p(x) = x^{-0.5}$, $f(x, y) = x^{-0.5}e^{y}(xe^{y} - 0.5)$, $\mu = 5$, $\sigma = 1$ and $B = 5\ln(1/5) - 1$,

We first verify whether the particular condition of the convergence theorem is met or not. For this purpose, we therefore need to check that the singular boundary value problem (32)-(33) does satisfy the condition $\sigma^{*}M < 1$.

In the considered example we have $M = \max_{0\le x\le 1}\left|\int_0^1 k(x,t)g(t)p(t)\mathrm{d}t\right| = 0.87$.

However, the function $f(x, y) = x^{-0.5}e^{y}(xe^{y} - 0.5)$ does not satisfy the Lipschitz's condition over the domain $\{[0, 1] \times R\}$, which means that our method does not satisfy the required condition.

Next we check whether our method for this example is convergent.

Using eq. (23), we compute homotopy polynomials $H_n(x)$:

$$\begin{aligned}
H_0(x) &= c_1[c_2e^{2y_0} - c_3e^{y_0}]\\
H_1(x) &= c_1[2c_2y_1e^{2y_0} - c_3y_1e^{y_0}]\\
H_2(x) &= 0.5c_1(4c_2y_2e^{2y_0} + 4c_2y_1^2e^{2y_0} - 2c_3y_2e^{y_0} - c_3y_1^2e^{y_0})\\
H_3(x) &= \frac{1}{6}c_1(12c_2y_3e^{2y_0} + 24c_2y_2y_1e^{2y_0} + 8c_2y_1^3e^{2y_0} - 6c_3y_3e^{y_0} - 6c_3y_2y_1e^{y_0} - c_3y_1^3e^{y_0})\\
&\ \vdots\ ,
\end{aligned} \tag{34}$$

with $c_1 = x^{-0.5}$, $c_2 = x$, $c_3 = 0.5$.

Inserting the homotopy polynomials (34) in eq. (25), we can obtain solution components $y_n(x)$ of approximate series solution $y(x)$.

Hence the third-order term of the approximate solution of (32)-(33) is as follows:

$$\begin{aligned}
\phi_3(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) &= 0.1052631579(1 - x^{\cdot 0.5})(15.55588431\\
&- 1.340640092x^{5})x^{(9/2)} - \ln(5) + 2.101421926x^{5}\\
&- 2.334913251x^{(9/2)} - 0.4554272101x^{10} + 0.4793970636x^{(19/2)} + 0.007582152671^{(39/2)}\\
&- 0.07502591587x^{(29/2)} + 1.645499640 * 10^{(-10)}x^{14} - 4.253111820 * 10^{(-10)}x^{9} - 0.007392598834x^{20}\\
&+ 0.07252505186x^{15} - 2.0(1 - x^{\cdot 0.5})(0.0004851054948x^{(39/2)} - 0.01693248240x^{(29/2)} + 0.1067324924x^{(19/2)}\\
&- 1.627892791 * 10^{(-10)}x^{9} - 0.2507971246 * x^{(9/2)}) + 0.000006729778544x^{(59/2)} - 0.0003912137400x^{(49/2)}\\
&- 2.000000000(1 - x^{0.5})(0.000003364889279x^{(59/2)} - 0.02058047546x^{(29/2)} + 0.06240603435 * x^{(19/2)}\\
&+ 0.003305970830x^{(39/2)} - 0.0001956068698x^{(49/2)} - 0.09792874783x^{(9/2)} + 0.0003833894648x^{25}\\
&- 0.000006617615582x^{30} + 0.2041749375.
\end{aligned}$$

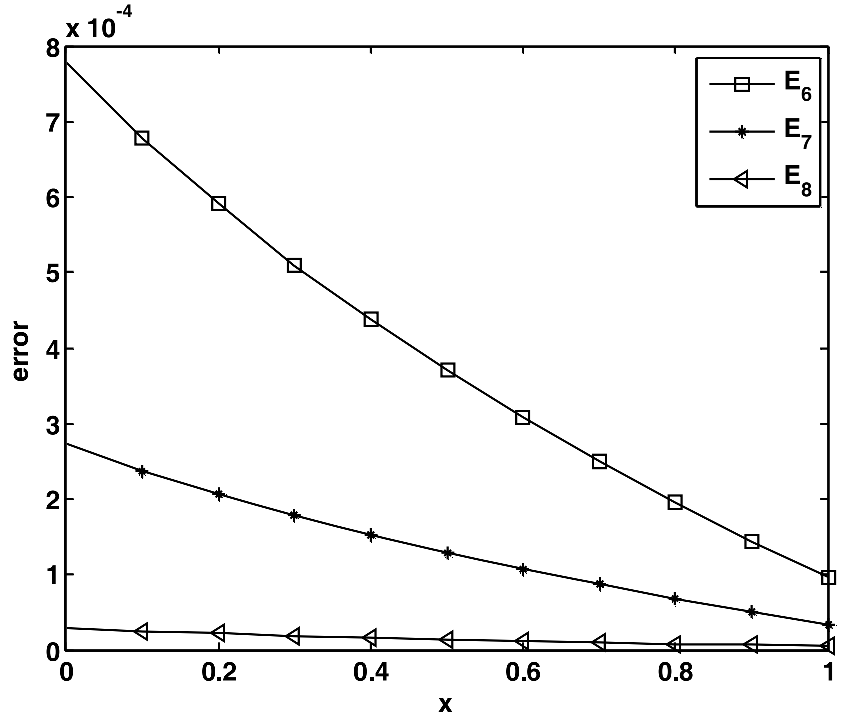

Fig. 1. Numerical results of absolute error for example 1.

Table 1. Maximum absolute error for example 1.

n (No. of terms)	Maximum absolute error
6	7.77×10^{-4}
7	2.73×10^{-4}
8	2.89×10^{-5}

To determine the accuracy of the proposed method against the exact solution, we compute the maximum absolute error, as defined by

$$E_n(x) = \mathrm{Max}_{x\in[0,1]}|\phi_n(x) - Y(x)|.$$

Here, $Y(x)$ is the exact solution of the problem and $\phi_n(x)$ is the truncated n-terms approximate series solution.

The results of the maximum absolute errors of the problem (32)-(33) for $n = 6, 7, 8$, are presented in fig. 1 and table 1. Numerical results reveal that the present method with few solution components approximates the exact solution very well and the method for the problem (32)-(33) is convergent. The accuracy of the solution increases as the number of the solution components in the approximate series solution $y(x)$ increases.

Example 2

Consider the nonlinear singular boundary value problem

$$(x^2 y'(x))' = x^2 \left(\frac{ky(x)}{y(x)+\eta}\right), \tag{35}$$

$$y'(0) = 0, \qquad 5y(1) + y'(1) = 5. \tag{36}$$

This problem arises in the study of steady-state oxygen diffusion in a spherical cell [7]. Here, k and η are finite positive constants and represent the reaction rate and the Michaelis constant, respectively. We take $k = 0.76129$ and $\eta = 0.03119$.

This problem corresponds to eqs. (1) and (2) with $g(x) = x^2$, $p(x) = 1$, $\mu = 5$, $\sigma = 1$, $B = 5$ and $f(x, y) = \frac{0.76129y(x)}{y(x)+0.03119}$.

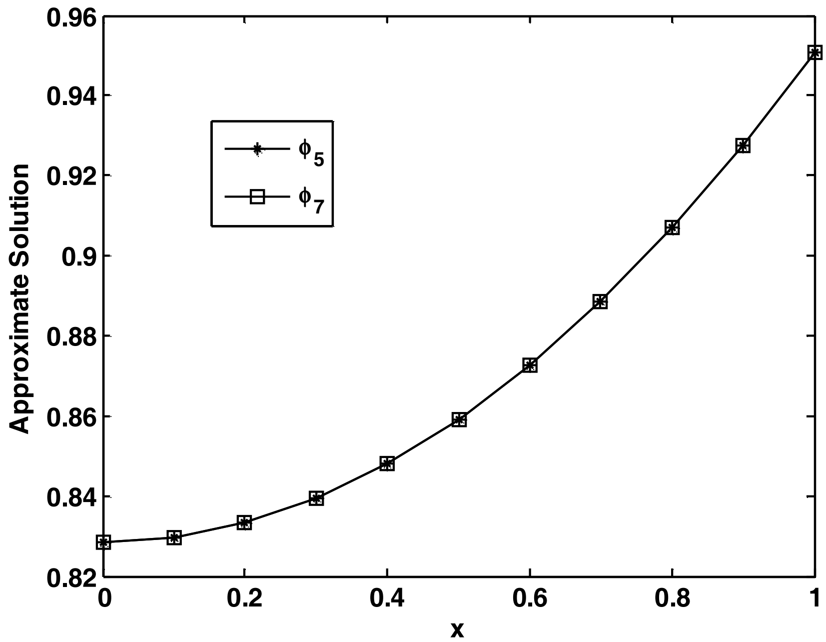

Fig. 2. Approximate solution of example 2.

We first verify that the problem (35)-(36) does satisfy the condition $\sigma^* M < 1$.

In the considered problem we have $M = \max_{0\le x\le 1} \left| \int_0^1 k(x,t)g(t)p(t)\mathrm{d}t \right| = 0.23$ and $\sigma^* = 0.033$. Hence

$$\sigma^* M = 0.0076 < 1,$$

which means that the problem (35)-(36) does satisfy the assumption of the convergence theorem.

We solve the problem (35)-(36) using the proposed technique described by eq. (25). Using eq. (23), we compute polynomials $H_n(x)$:

$$\begin{aligned} &H_0(x) = 0.7382635596\\ &H_1(x) = 0.0223299687y_1.\\ &H_2(x) = -0.02165456250y_1^2 + 0.02232996850y_2.\\ &H_3(x) = 0.0209995865y_1^3 - 0.0433091253y_1y_2 + 0.02232996850y_3,\\ &\quad\vdots \end{aligned} \tag{37}$$

Inserting the polynomials (37) in eq. (25), we can obtain solution components $y_n(x)$ of approximate series solution $y(x)$.

The fourth-order term of the approximate solution of (35)-(36) is as follows:

$$\begin{aligned} \phi_4(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) &= 0.8284816685 + 0.1207562728x^2\\ &- (0.0005495134060 * x^4 - 0.001282197947x^2)\\ &+ 0.0009676639911 * x^4 - (-0.00004639689451x^6 + 0.0001807305688x^4 - 0.0002093525882x^2)\\ &- 0.00008060486614x^6 - (0.000004246062983x^8 - 0.00002269299075x^6 + 0.00004388721798x^4\\ &- 0.00003355199008x^2) + 0.000004776820856x^8. \end{aligned}$$

The approximate solution of the problem (35)-(36) for $n = 5, 7$ is depicted in fig. 2. Comparison of the numerical results obtained by our method, the finite difference method [23], B-Spline method [27] and a numerical method [31] derived using the direct integration method as opposed to the commonly used finite differences is presented in table 2. It is clearly evident that the results obtained by the proposed method with few solution components are in very good agreement with that of [23,27,31], and the proposed method for this problem is convergent.

Table 2. Approximate solution of example 2.

x	Present method ϕ_5	Present method ϕ_7	FDM solution [23]	Pointwise bound solution [31]	B-Spline solution [27]
0.0	0.8284830753	0.8284832866	0.8284831497	0.8284752	0.82848327295802
0.1	0.8297058822	0.8297060888	0.8297060742	0.8296982	0.82970607521884
0.2	0.8333745371	0.8333747303	0.8333747157	0.8333673	0.83337471691089
0.3	0.8394897383	0.8394899111	0.8394898966	0.8394831	0.83948989814383
0.4	0.8480526347	0.8480527826	0.8480527684	0.8480467	0.84805277036165
0.5	0.8590648040	0.8590649253	0.8590649116	0.8590596	0.85906491397434
0.6	0.8725282230	0.8725283186	0.8725283056	0.8725237	0.87252830841853
0.7	0.8884452322	0.8884453046	0.8884452928	0.8884408	0.88844529589927
0.8	0.9068184947	0.9068185474	0.9068185369	0.9068145	0.90681854026297
0.9	0.9276509514	0.9276509879	0.9276509791	0.9276474	0.92765098252660
1.0	0.9509457747	0.9509457981	0.9509457914	0.9509432	0.95094579461056

Example 3

Consider the nonlinear singular boundary value problem

$$(x^2 y'(x))' = x^2\theta^2 y^m, \tag{38}$$

$$y'(0) = 0, \quad y(1) = 1. \tag{39}$$

This problem is an application of reaction-diffusion process in a porous spherical catalyst [13].

The problem (38)-(39), in dimensionless form, is derived from the following singular boundary value problems by introducing dimensionless variables $y = C_A/C_{AS}$, $x = r/R$, $\theta = \sqrt{R^2 k_n C_{AS}^{n-1}/D_e}$:

$$(r^2 C_A(r))' = r^2 \frac{K_n C_A(r)^m}{D_e}\,, \tag{40}$$

subject to the boundary conditions

$$C_A(R) = C_{AS}, \text{ (at catalyst surface)}, \tag{41}$$

$$C_A'(0) = 0, \text{ (at the centre of the catalyst)}. \tag{42}$$

The details of the mathematical formulation of the above reaction-diffusion problem can be found in [13].

This problem corresponds to eqs. (1) and (2) with $g(x) = x^2$, $p(x) = 1$ and $f(x, y) = \theta^2 y^m$.

Here θ^2 describes a ratio of chemical reaction rate at the catalyst surface in the absence of mass transfer limitation to the rate of diffusion through the catalyst.

We solve the problem (38)-(39) for $\theta = 2$ and $m = 0.5$,

First, we corroborate that the considered problem with $\theta = 2$ and $m = 0.5$ does satisfy the condition $\sigma^* M < 1$.

Here we have $M = \max_{0\le x\le 1} \left|\int_0^1 k(x,t)g(t)p(t)\mathrm{d}t\right| = 0.17$ and $\sigma^* = 2.9$.

Hence

$$\sigma^* M = 0.493 < 1,$$

which means that the present method defined by eq. (25) can be certainly used to solve the problem (38)-(39).

The sixth-order term of the approximate solution of (38)-(39) is given by

$$\begin{aligned}\phi_6(x) = {} & 0.4759232396057792 + 0.4599128498387758x^2 - 1.210189853060109 \times 10^{-16}x^3 \\ & + 0.06666666666666665x^4 + 6.100444223851771 \times 10^{-18}x^5 - 0.003054785700288346x^6 \\ & + 3.794707603699265 \times 10^{-19}x^7 + 0.0006897903194199509x^8 - 3.252606517456513 \times 10^{-19}x^9 \\ & - 0.0001590508257174927 4x^{10} + 1.084202172485504 \times 10^{-19}x^{11} + 0.000021290095364169396x^{12}.\end{aligned}$$

The approximate solutions for different values of n are depicted in fig. 3 and table 3.

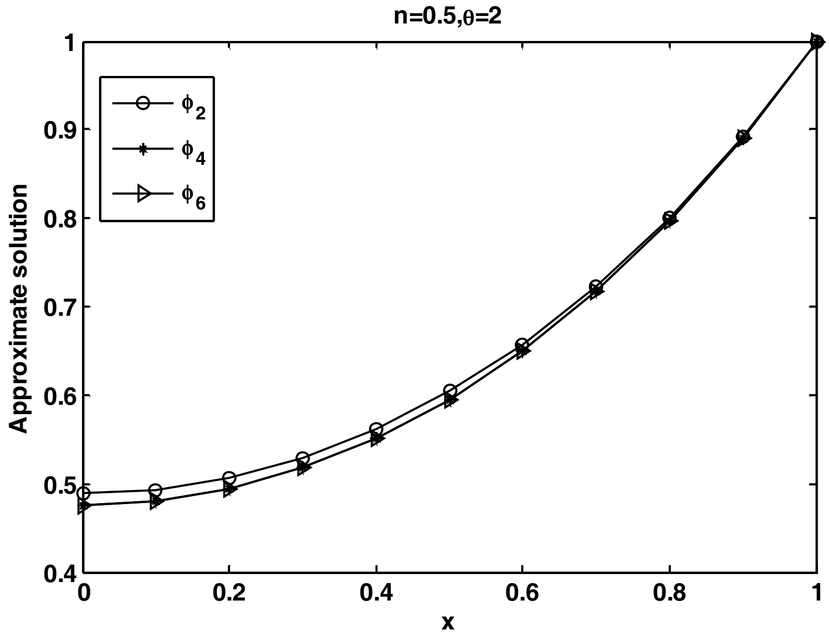

Fig. 3. Approximate solution of example 3.

Table 3. Approximate solution of example 3.

x	ϕ_2	ϕ_4	ϕ_6
0.0	0.488889	0.475838	0.475923
0.1	0.493340	0.480444	0.480529
0.2	0.506773	0.494343	0.494426
0.3	0.529429	0.517773	0.517853
0.4	0.561707	0.551127	0.551204
0.5	0.604167	0.594953	0.595023
0.6	0.657529	0.649939	0.650000
0.7	0.722673	0.716915	0.716963
0.8	0.800640	0.796841	0.796873
0.9	0.892629	0.890801	0.890817
1.0	1	1	1

Example 4

Consider the following singular boundary value problem:

$$(xy')' = -4x\left(1 - \frac{y(x)}{1-y(x)}\right), \quad 0 < x < 1,$$

subject to the boundary conditions

$$y'(0) = 0, \qquad y(1) = 0.$$

The exact solution of the problem is not known. This problem corresponds to eqs. (1) and (2) with $g(x) = x$, $p(x) = 1$, $\mu = 1$, $\sigma = 0$, $B = 0$ and $f(x,y) = -4\big(1 - \frac{y(x)}{1-y(x)}\big)$.

We solve this equation by using our method defined in (25). The above problem does not satisfy the condition of the convergence theorem as $\sigma^* M = 2.48 > 1$. Since the problem does not have the exact solution, we determine the maximum absolute residual error function, as defined by

$$R_n(x) = \mathrm{Max}_{x\in[0,1]}\left|(x\phi_n'(x))' + 4x\left(1 - \frac{\phi_n(x)}{1-\phi_n(x)}\right)\right|.$$

Table 4. Numerical results of maximum absolute residual error of example 4.

n	Maximum absolute residual error
5	9.162
6	1.446
7	5.735
8	1.571
9	227.248
10	22.8366

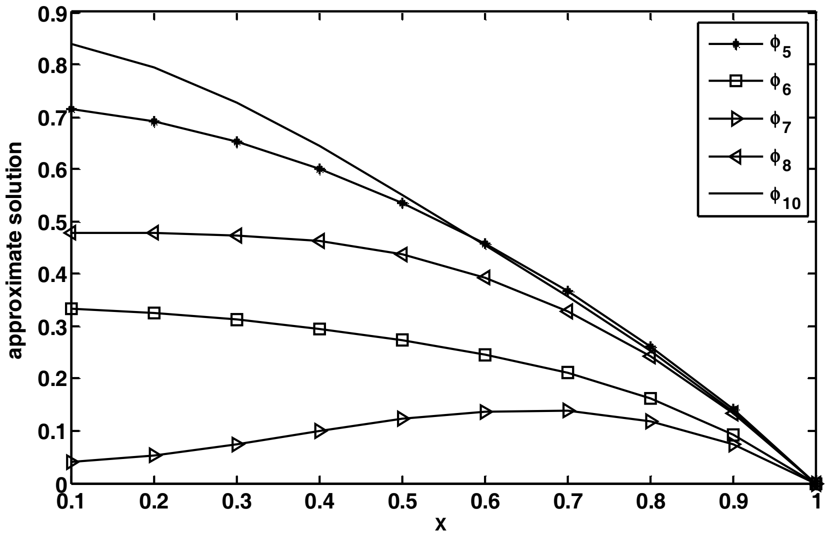

Fig. 4. Approximate solution of example 4.

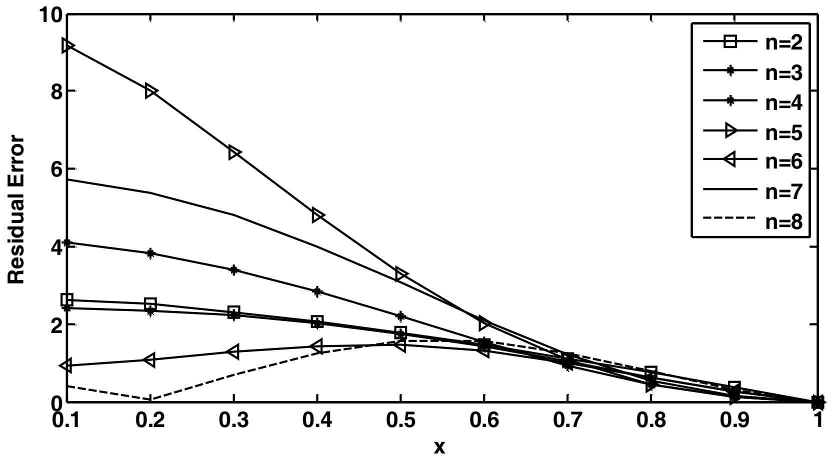

Fig. 5. Residual error of example 4.

The results of the maximum absolute residual errors for different values of n are tabulated in table 4. Further the approximate solution and the residual error for different values n are given in figs. 4 and 5, respectively. It is obvious from the figures and table that the proposed method for this equation is divergent.

6 Conclusions

To conclude, an efficient iterative technique has been developed in this study for the numerical solution of nonlinear doubly singular two-point boundary value problem with Neumann and Robin boundary conditions. The convergence analysis of the method has been discussed in the paper. Four nonlinear examples have been considered to illustrate the applicability of the method. It has been observed that example 1 does not satisfy the assumption of the theorem and converge to the solution. Further, examples 2 and 3 met the conditions required in the convergence proof and converge. However, example 4 does neither satisfy the condition, nor even converge. The reason lies in the fact that the nonlinearity confronted in this example is in the form of a rational function. Because of the strong nonlinearly of the right hand side function $f(x,y)$ of the singular differential equation in example 4, the method yields divergent solution.

In addition, the numerical results obtained by the present method were compared with those obtained by finite difference method, B-spline approach and a numerical method based on the direct integration. The advantage of the present method over these three methods is that the present method does not require any discretization of the variable and the method is computationally very efficient. Another advantage of the present algorithm is that it requires few solution components to produce similar results. Furthermore, the advantage of our method over other existing recursive schemes [33–35] is that it does not require the computation of undetermined coefficients. Hence, the proposed method can then be used for obtaining approximate solutions for nonlinear boundary value problems with mixed type of boundary conditions, but also for nonlinear integral equations of different types.

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