Eur. Phys. J. Plus (2016) **131**: 98
DOI 10.1140/epjp/i2016-16098-2

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

On Lie symmetries, exact solutions and integrability to the KdV-Sawada-Kotera-Ramani equation

Pan-Li Ma[1,2], Shou-Fu Tian[1,2,3,a], Tian-Tian Zhang[1,2,b], and Xing-Yong Zhang[1,2,c]

[1] Department of Mathematics, China University of Mining and Technology, Xuzhou 221116, People's Republic of China
[2] Center of Nonlinear Equations, China University of Mining and Technology, Xuzhou 221116, People's Republic of China
[3] Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK

Received: 24 November 2015 / Revised: 16 January 2016
Published online: 15 April 2016 –

Abstract. In this paper, the KdV-Sawada-Kotera-Ramani equation is investigated, which is used to describe the resonances of solitons in one-dimensional space. By using the Lie symmetry analysis method, the vector field and optimal system of the equation are derived, respectively. The optimal system is further used to study the symmetry reductions and exact solutions. Furthermore, the exact analytic solutions of the equation can be obtained by considering the power series theory. Finally, the complete integrability of the equation is systematically presented by using binary Bell's polynomials, which includes the bilinear representation, bilinear Bäcklund transformation, Lax pair and infinite conservation laws. Based on its bilinear representation, the N-soliton solutions of the equation are also constructed with exact analytic expression.

1 Introduction

It is well known that nonlinear evolution equations (NLEEs) have an important effect on the study of nonlinear physical phenomena. Due to the importance of those NLEEs, it is very significative to study their Lie symmetry analysis, exact solutions, various soliton solutions and completely integrable properties, which includes bilinear form, Lax pairs, infinite symmetries, Hamiltonian structure, bilinear Bäcklund transformation, etc. Nowadays, there are many kinds of methods to construct exact solutions of the NLEEs in soliton theory, such as the inverse scattering transform [1], Lie group [2], Darboux transformation [3], Hirota's bilinear method [4,5], algebro-geometrical approach [6] and Painlevé analysis [7, 8], etc. The Hirota bilinear method developed by Hirota is one of powerful and effective approachs to construct exact solutions of NLEEs. Once the bilinear form of a nonlinear equation is obtained by a dependent variable transformation, it will be easy to get its multi-soliton solutions [9–21]. Moreover, by means of the Lie symmetry analysis and dynamical system method, one can also obtain the symmetries and exact explicit solutions of NLEEs.

In this paper, we will study the following KdV-Sawada-Kotera-Ramani equation [22–26]:

$$u_t + a(3u^2 + u_{xx})_x + b(15u^3 + 15uu_{xx} + u_{xxxx})_x = 0, \tag{1}$$

which was used to describe the resonances of solitons in a one-dimensional space by Hirota and Ito [22]. They found that two solitons near the resonant state exhibit some new phenomena. The existence of conservation law for this equation was further proved by Konno [27]. The KdV-Sawada-Kotera-Ramani equation (1) is a linear combination of the KdV equation and the Sawada-Kotera equation.

When $b = 0$, eq. (1) is reduced to the KdV equation as follows:

$$u_t + a(6uu_x + u_{xxx}) = 0. \tag{2}$$

This fundamental equation describes the weakly nonlinear waves in the one dimensional media with weak dispersion. From [28], one can see that it is the first nonlinear equation integrated by use of the Inverse Scattering Method [29].

[a] e-mail: sftian@cumt.edu.cn (corresponding author)
[b] e-mail: ttzhang@cumt.edu.cn (corresponding author)
[c] e-mail: zhangxingyong163@163.com (corresponding author)

When $a = 0$, eq. (1) is reduced to the Sawada-Kotera equation given by

$$u_t + b(45u^2 u_x + 15u_x u_{xx} + 15uu_{xxx} + u_{xxxxx}) = 0, \tag{3}$$

which belongs to the completely integrable hierarchy of higher-order KdV equations, and has many sets of conservation laws [30].

The main purpose of this paper is to investigate the Lie symmetry analysis, optimal system and exact solution of the KdV-Sawada-Kotera-Ramani equation. In addition, based on binary Bell polynomial [31–36], we will systematically study its bilinear representation, Bäcklund transformation, Lax pair and infinite conservation laws, respectively.

The structure of this paper is as follows. In sect. 2, based on Lie symmetry analysis method, we study the vector field and optimal system of eq. (1). In sect. 3, the similarity reductions and exact solutions of eq. (1) are investigated by means of optimal system. In sect. 4, based on the power series method, the exact analytic solutions of the equation are obtained. The convergence of power series solutions of eq. (1) is also analyzed. In sect. 5, we systematically construct the bilinear representation by using Bell polynomial approach, based on which, its N-soliton solutions are also derived. In sect. 6, we derive the bilinear Bäcklund transformation and Lax pair of KdV-Sawada-Kotera-Ramani equation, respectively. In sect. 7, by virtue of the obtained Lax equation, the infinite conservation laws of the equation are also derived.

2 Lie sysmmetry analysis

In this section, we investigate the Lie symmetry and optimal system of KdV-Sawada-Kotera-Ramani equation. The geometric vector field of eq. (1) is given as follows:

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}\,, \tag{4}$$

where the coefficient functions $\xi(x,t,u), \tau(x,t,u), \phi(x,t,u)$ are to be determined. It is equivalent to a one-parameter Lie group as below

$$\begin{aligned} x^* &= x + \epsilon\xi(x,t,u) + o(\epsilon^2),\\ t^* &= t + \epsilon\tau(x,t,u) + o(\epsilon^2),\\ u^* &= u + \epsilon\phi(x,t,u) + o(\epsilon^2), \end{aligned} \tag{5}$$

where ϵ is a group parameter. If the vector field (4) generates a symmetry of eq. (1), then V should be satisfied the following Lie symmetry condition:

$$pr^{(5)}V(F)|_{F=0} = 0, \tag{6}$$

where $F = u_t + a(6uu_x + u_{xxx}) + b(45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx})$. Furthermore, the prolongation of $pr^{(5)}V$ is of the following form:

$$\begin{aligned} pr^{(5)}V(F) = V &+ \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}}\\ &+ \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xxxxx}\frac{\partial}{\partial u_{xxxxx}}\,, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \phi^x &= D_x\phi - u_xD_x\xi - u_tD_x\tau, \quad \phi^t = D_t\phi - u_xD_t\xi - u_tD_t\tau,\\ \phi^{xx} &= D_x^2\phi_- u_xD_x^2\xi - u_tD_x^2\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau, \end{aligned} \tag{8}$$

and D_x, D_t are total derivative operators as follows:

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \cdots\,,\\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots\,. \end{aligned} \tag{9}$$

Then, when $b \neq 0$, we get the following vector field of the KdV-Sawada-Kotera-Ramani equation (1) by means of Lie symmetry analysis method

$$V_1 = \left(\frac{x}{5} - \frac{4a^2t}{25b}\right)\partial x + t\partial t - \frac{30bu + 2a}{75b}\partial u, \qquad V_2 = \partial t, \qquad V_3 = \partial x. \tag{10}$$

When $b=0$, eq. (1) is reduced to the KdV equation (2). Following the same way as eq. (1), the vector field of eq. (2) is given by

$$V_1=\frac{x}{3}\partial x+t\partial t-\frac{2u}{3}\partial u,\qquad V_2=t\partial x+\frac{1}{6a}\partial u,\qquad V_3=\partial t,\qquad V_4=\partial x. \tag{11}$$

It is necessary to verify that the vector field is closed under the Lie bracket. In view of the vector field of eq. (1), one has

$$\begin{aligned}&[V_1,V_1]=[V_2,V_2]=[V_3,V_3]=0,\qquad [V_2,V_3]=-[V_3,V_2]=0,\\&[V_1,V_2]=-[V_2,V_1]=\frac{4a^2}{25b}V_3-V_2,\qquad [V_1,V_3]=-[V_3,V_1]=-\frac{1}{5}V_3,\end{aligned} \tag{12}$$

where the commutator operators $[V_s,V_t]=V_sV_t-V_tV_s$.

Then by using the following Lie series:

$$Ad(\exp(\varepsilon V_i))V_j=V_j-\varepsilon[V_i,V_j]+\frac{1}{2}\varepsilon^2[V_i,[V_i,V_j]]-\cdots, \tag{13}$$

one can obtain the adjoint representation of the vector field. For the KdV-Sawada-Kotera-Ramani equation (1), we have the adjoint representation of the vector field as follows:

$$Ad(\exp(\varepsilon V_i))V_i=V_i,\quad i=1,2,3, \tag{14}$$

$$\begin{aligned}&Ad(\exp(\varepsilon V_1))V_2=V_2-\frac{4a^2}{25b}\varepsilon V_3+\varepsilon V_2,\quad Ad(\exp(\varepsilon V_1))V_3=V_3+\frac{\varepsilon}{5}V_3,\quad Ad(\exp(\varepsilon V_2))V_1=V_1+\frac{4a^2}{25b}\varepsilon V_3-\varepsilon V_2,\\&Ad(\exp(\varepsilon V_2))V_3=V_3,\quad Ad(\exp(\varepsilon V_3))V_1=V_1-\frac{\varepsilon}{5}V_3,\quad Ad(\exp(\varepsilon V_3))V_2=V_2,\end{aligned} \tag{15}$$

with any $\varepsilon\in R$.

The adjoint representation of the vector field of eq. (2) can be obtained in the similar way. According to the adjoint representation of the vector field, we have the optimal system of the KdV-Sawada-Kotera-Ramani equation as follows:

$$\{V_1,V_2,V_3+rV_2\}, \tag{16}$$

in which r is an arbitrary constant.

Following the same computational procedure, the optimal system of eq. (2) is given by

$$\{V_1,V_2,V_3,V_4,V_2+rV_3\}, \tag{17}$$

in which r is an arbitrary constant.

3 Similarity reductions and exact solutions

In the preceding section, we study the vector fields and the optimal systems of the KdV-Sawada-Kotera-Ramani equation and the KdV equation, respectively. In this section, based on the obtained optimal systems, we will study the similarity reductions and exact solutions of these equations.

3.1 Reductions and exact solutions of eq. (1)

Case I. For the generator V_1, one has

$$u=f(\xi)t^{-\frac{2}{5}}-\frac{a}{15b}, \tag{18}$$

where $\xi=xt^{-\frac{1}{5}}+\frac{a^2}{5b}t^{\frac{4}{5}}$. Combining (18) and (1), we can obtain the following result

$$-\frac{1}{5}f'\xi-\frac{2}{5}f+45bf^2f'+15bf'f''+15bff'''+bf^{(5)}=0, \tag{19}$$

where $f'=\frac{\mathrm{d}f}{\mathrm{d}\xi}$.

Case II. For the generator V_2, one has

$$u=f(\xi), \tag{20}$$

where $\xi = x$. Combining (20) and (1), we can obtain the following ordinary differential equation:

$$a(6ff' + f''') + b(45f^2 f' + 15f'f'' + 15ff''' + f^{(5)}) = 0, \tag{21}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$.

Case III. For the linear combination $V_3 + rV_2$, one has

$$u = f(\xi), \tag{22}$$

where $\xi = -rx + t$. Combining (22) and (1), we can obtain the following ordinary differential equation:

$$f' + a\left(-6rff' - r^3 f'''\right) + b\left(-45rf^2 f' - 15r^3 f'f'' - 15r^3 ff''' - r^5 f^{(5)}\right) = 0, \tag{23}$$

in which $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$.

3.2 Reductions and exact solutions of eq. (2)

Case I. For the generator V_1, one has

$$u = f(\xi)t^{-\frac{2}{3}}, \tag{24}$$

where $\xi = x^3 t^{-1}$. Combining (24) and (2), we obtain the following result:

$$-f'\xi - \frac{2}{3}f + a[18ff'\xi^{\frac{2}{3}} + 27f^{'''}\xi^2 + 36f''\xi + 18f'\xi + 6f'] = 0, \tag{25}$$

in which $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$.

Case II. For the generator V_2, one has

$$u = f(\xi) + \frac{1}{6a}xt^{-1}, \tag{26}$$

where $\xi = t$. Combining (26) and (2), one reduces this equation to the following ordinary differential equation:

$$f' + f\xi^{-1} = 0, \tag{27}$$

in which $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$. Solving eq. (27) yields $f = \frac{c}{t}$, where c is an arbitrary constant. From eq. (26), one has

$$u = \frac{c}{t} + \frac{x}{6at}\,. \tag{28}$$

Case III. For the generator V_3, one has

$$u = f(\xi), \tag{29}$$

where $\xi = x$. Combining (29) and (2), we can obtain the following ordinary differential equation:

$$6ff' + f^{'''} = 0, \tag{30}$$

in which $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$. Solving eq. (30) yields

$$f = -2\wp(\xi + c_1, 0, \xi + c_2), \tag{31}$$

where $\wp(\cdot,\cdot,\cdot)$ is the Weierstrass elliptic function. Considering eqs. (29) and (31), one can obtain the following solution:

$$u = -2\wp(x + c_1, 0, x + c_2). \tag{32}$$

Case IV. For the generator V_4, one obtains the trivial solution of eq. (2) given by $u(x,t) = c$, where c is an arbitrary constant.

Case V. For the linear combination $V_2 + rV_3$, one has

$$u = f(\xi) + \frac{1}{6ar}t, \tag{33}$$

where $\xi = x - \frac{1}{2r}t^2$. Combining (33) and (2), we have the following result

$$\frac{1}{6ar} + 6aff' + af^{'''} = 0, \tag{34}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\xi}$. One can see that eq. (34) is the first Painlevé-like equation.

4 The exact power series solutions

In this section, based on the power series method, we will investigate the exact analytic solutions of the reduced equations. Once the exact analytic solutions of the reduced equations are obtained, we can get the exact power series solutions for the original partial differential equations. In the following, we will take eqs. (19), (21) and (23) as examples.

4.1 Exact analytic solutions of eq. (19)

First of all, for eq. (19), we will construct a solution of it in the following form:

$$f(\xi)=\sum_{n=0}^{\infty}c_n\xi^n. \tag{35}$$

Combining (35) and (19), one obtains the following equation:

$$\begin{aligned}
&-\frac{1}{5}\left[c_1+\sum_{n=1}^{\infty}(n+1)c_{n+1})\xi^n\right]\xi-\frac{2}{5}\left[c_0+\sum_{n=1}^{\infty}c_n\xi^n\right]+45b\left[c_0^2c_1+\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}(n+1-k)c_jc_{k-j}c_{n+1-k}\xi^n\right]\\
&+15b\left[2c_1c_2+\sum_{n=1}^{\infty}\sum_{k=0}^{n}(k+1)(n-k+1)(n-k+2)c_{k+1}c_{n-k+2}\xi^n\right]\\
&+15b\left[6c_0c_3+\sum_{n=1}^{\infty}\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_kc_{n-k+3}\xi^n\right]\\
&+b\left[120c_5+\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)(n+5)c_{n+5}\xi^n\right]=0.
\end{aligned} \tag{36}$$

When $n=0$, comparing coefficients of ξ yields

$$c_5=-\frac{1}{120b}\left(-\frac{2}{5}c_0+45bc_0^2c_1+30bc_1c_2+90bc_0c_3\right). \tag{37}$$

When $n\geq 1$, one has

$$\begin{aligned}
c_{n+5}=&-\frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)}\left[-\frac{n+2}{5}c_n+45\sum_{k=0}^{n}\sum_{j=0}^{k}(n-k+1)c_jc_{k-j}c_{n+1-k}\right.\\
&\left.+15b\sum_{k=0}^{n}(k+1)(n-k+1)(n-k+2)c_{k+1}c_{n-k+2}+15b\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_kc_{n-k+3}\right].
\end{aligned} \tag{38}$$

From (37) and (38), one can obtain all the coefficients $c_n(n\geq 5)$ of the eq. (35), for example

$$c_6=-\frac{1}{720b}\left(-\frac{3}{5}c_1+90bc_0c_2+90bc_0c_1^2+180bc_1c_3+60bc_2^2+360bc_0c_4\right),\ldots. \tag{39}$$

Therefore, for arbitrary chosen constant numbers c_0,c_1,c_2,c_3 and c_4, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined from (37) and (38) in a unique manner. It shows that eq. (19) exists a power series solution (35) with the coefficients given by (37) and (38). Moreover, it is easy to prove the convergence of the power series (35) with the coefficients given by (37) and (38). Thus, we can write the power series solution of eq. (19) in the following form:

$$\begin{aligned} f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + c_5\xi^5 + \sum_{n=1}^{\infty} c_{n+5}\xi^{n+5} \\ &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 - \frac{1}{120b}\left(-\frac{2}{5}c_0 + 45bc_0^2c_1 + 30bc_1c_2 + 90bc_0c_3\right)\xi^5 \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)} \left[-\frac{n+2}{5}c_n + 45\sum_{k=0}^{n}\sum_{j=0}^{k}(n-k+1)c_jc_{k-j}c_{n+1-k} \right. \\ &\quad \left. +15b\sum_{k=0}^{n}(k+1)(n-k+1)(n-k+2)c_{k+1}c_{n-k+2} + 15b\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_kc_{n-k+3}\right]\xi^{n+5}. \end{aligned} \tag{40}$$

Furthermore, we get the exact power series solution of eq. (1) as follows:

$$\begin{aligned} u(x,t) &= c_0 + c_1\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right) + \cdots + c_4\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)^4 + c_5\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)^5 + \sum_{n=1}^{\infty} c_{n+5}\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)^{n+5} \\ &= c_0 + c_1\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right) + \cdots + c_4\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)^4 \\ &\quad - \frac{1}{120b}\left(-\frac{2}{5}c_0 + 45bc_0^2c_1 + 30bc_1c_2 + 90bc_0c_3\right)\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)^5 \\ &\quad - \sum_{n=1}^{\infty} \frac{1}{b(n+1)(n+2)(n+3)(n+4)(n+5)} \left[-\frac{n+2}{5}c_n + 45\sum_{k=0}^{n}\sum_{j=0}^{k}(n-k+1)c_jc_{k-j}c_{n+1-k} \right. \\ &\quad \left. +15b\sum_{k=0}^{n}(k+1)(n-k+1)(n-k+2)c_{k+1}c_{n-k+2} + 15b\sum_{k=0}^{n}(n-k+1)(n-k+2)(n-k+3)c_kc_{n-k+3}\right] \\ &\quad \times \left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)x^{n+5}, \end{aligned} \tag{41}$$

in which c_i, $(i = 0, 1, 2, 3, 4)$ are arbitrary constants, and other coefficients c_n $(n \geq 5)$ can be obtained from (37) and (38).

In physical applications, based on the above calculation, it is more convenient to write the solution of eq. (1) as the following form:

$$\begin{aligned} u(x,t) &= c_0 + c_1\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right) + \cdots + c_4\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)^4 \\ &\quad - \frac{1}{120b}\left(-\frac{2}{5}c_0 + 45bc_0^2c_1 + 30bc_1c_2 + 90bc_0c_3\right)\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)^5 \\ &\quad - \frac{1}{720b}\left(-\frac{3}{5}c_1 + 90bc_0c_2 + 90bc_0c_1^2 + 180bc_1c_3 + 60bc_2^2 + 360bc_0c_4\right)\left(xt^{-\frac{1}{5}} + \frac{a^2}{5b}t^{\frac{4}{5}}\right)^6 \cdots. \end{aligned} \tag{42}$$

4.2 Exact analytic solutions of eq. (21)

For eq. (21), integrating it with respect to x yields

$$3af^2 + af'' + 15bf^3 + 15bff'' + bf^{(4)} + c = 0, \tag{43}$$

in which c is an integration constant. Then we will construct a solution of eq. (43) in a power series of the form (35). Combining (35) and (43), one obtains the following equation

$$3a\left[c_0^2+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}c_kc_{n-k}\right)\xi^n\right]+a\left[2c_2+\sum_{n=1}^{\infty}(n+1)(n+2)c_{n+2}\xi^n\right]+15b\left[c_0^3+\sum_{n=1}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}c_jc_{k-j}c_{n-k}\xi^n\right]$$
$$+15b\left[2c_0c_2+\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}(n-k+2)(n-k+1)c_kc_{n-k+2}\right)\xi^n\right]+b\left[24c_4+\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)c_{n+4}\xi^n\right]$$
$$+c=0. \tag{44}$$

When $n=0$, comparing coefficients of ξ yields

$$c_4=-\frac{1}{24b}\left(3ac_0^2+2ac_2+15bc_0^3+30bc_0c_2+c\right). \tag{45}$$

When $n\geq 1$, one has

$$c_{n+4}=-\frac{1}{b(n+1)(n+2)(n+3)(n+4)}\left(3a\sum_{k=0}^{n}c_kc_{n-k}+a(n+1)(n+2)c_{n+2}+15b\sum_{k=0}^{n}\sum_{j=0}^{k}c_jc_{k-j}c_{n-k}\right.$$
$$\left.+15b\sum_{k=0}^{n}(n-k+2)(n-k+1)c_kc_{n-k+2}\right). \tag{46}$$

From (45) and (46), one can obtain all the coefficients $c_n (n\geq 4)$ of the eq. (35), such as

$$c_5=-\frac{1}{120b}\left(6ac_0c_1+6ac_3+45bc_0^2c_1+90bc_0c_3+30bc_1c_2\right),$$
$$c_6=-\frac{1}{360b}\left(6ac_0c_2+3ac_1^2+12ac_4+45bc_0^2c_2+45bc_0c_1^2+180bc_0c_4+90bc_1c_3+30bc_2^2\right),\ldots. \tag{47}$$

Therefore, for arbitrary chosen constant numbers c_0, c_1, c_2 and c_3, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined from (45) and (46) in a unique manner. The power series solution of eq. (43) can be written as the following form:

$$f(\xi)=c_0+c_1\xi+c_2\xi^2+c_3\xi^3-\frac{1}{24b}\left(3ac_0^2+2ac_2+15bc_0^3+30bc_0c_2+c\right)\xi^4$$
$$-\sum_{n=1}^{\infty}\frac{1}{b(n+1)(n+2)(n+3)(n+4)}\left[3a\sum_{k=0}^{n}c_kc_{n-k}+a(n+1)(n+2)c_{n+2}+15b\sum_{k=0}^{n}\sum_{j=0}^{k}c_jc_{k-j}c_{n-k}\right.$$
$$\left.+15b\sum_{k=0}^{n}(n-k+2)(n-k+1)c_kc_{n-k+2}\right]\xi^{n+4}. \tag{48}$$

Furthermore, we get the following exact power series solution of eq. (1):

$$u(x,t)=c_0+c_1x+c_2x^2+c_3x^3-\frac{1}{24b}\left(3ac_0^2+2ac_2+15bc_0^3+30bc_0c_2+c\right)x^4$$
$$-\sum_{n=1}^{\infty}\frac{1}{b(n+1)(n+2)(n+3)(n+4)}\left[3a\sum_{k=0}^{n}c_kc_{n-k}+a(n+1)(n+2)c_{n+2}+15b\sum_{k=0}^{n}\sum_{j=0}^{k}c_jc_{k-j}c_{n-k}\right.$$
$$\left.+15b\sum_{k=0}^{n}(n-k+2)(n-k+1)c_kc_{n-k+2}\right]x^{n+4}, \tag{49}$$

in which c_i $(i=0,1,2,3)$ are arbitrary constants, and other coefficients c_n $(n\geq 4)$ can be determined from (45) and (46).

4.3 Exact analytic solutions of eq. (23)

In the similar way, we also can construct a solution of eq. (23) in the power series form (35). By combining (35) and (23), and comparing the corresponding coefficients yields

$$
\begin{aligned}
c_4 &= \frac{1}{24br^5}\left(c_0 - 3arc_0^2 - 2ar^3c_2 - 15brc_0^3 - 30br^3c_0c_2 + g\right),\\
c_{n+4} &= -\frac{1}{br^5(n+1)(n+2)(n+3)(n+4)}\left[c_n - 3ar\sum_{k=0}^{n}c_kc_{n-k} - ar^3(n+1)(n+2)c_{n+2} - 15br\sum_{k=0}^{n}\sum_{j=0}^{k}c_jc_{k-j}c_{n-k}\right.\\
&\quad\left. -15br^3\sum_{k=0}^{n}(n-k+2)(n-k+1)c_kc_{n-k+2}\right], \quad n = 1,2,3\cdots. \qquad (50)
\end{aligned}
$$

From (50), we can obtain all the coefficients c_n $(n \geq 4)$ of the power series eq. (35), for example

$$
c_5 = \frac{1}{120br^5}\left(c_1 - 6arc_0c_1 - 6ar^3c_3 - 45brc_0^2c_1 - 90br^3c_0c_3 - 30br^3c_1c_2\right), \dots. \qquad (51)
$$

Therefore, for arbitrary chosen constant numbers c_0, c_1, c_2 and c_3, the other terms of the sequence $\{c_n\}_{n=0}^{\infty}$ can be determined from (50) in a unique manner. It shows that eq. (23) exists a power series solution (35) with the coefficients given by (50). Following the same way, one can also investigate the power series solution of eq. (25) to find the corresponding solutions of the KdV equation.

4.4 Convergence analysis of the power series solutions

In this subsection, we will prove the convergence of power series solution (35) for eq. (19). For (38), we have

$$
|c_{n+5}| \leq M\left[|c_n| + \sum_{k=0}^{n}\sum_{j=0}^{k}|c_j||c_{k-j}||c_{n+1-k}| + \sum_{k=0}^{n}|c_{k+1}||c_{n-k+2}| + \sum_{k=0}^{n}|c_k||c_{n-k+3}|\right], \quad n = 0,1,2,\cdots, \qquad (52)
$$

where $M = \max\{45, 15b\}$. Then, we define a new power series

$$
R = R(\xi) = \sum_{n=0}^{\infty} r_n\xi^n, \qquad (53)
$$

with $r_i = |c_i|$ $(i = 0,1,2,3,4)$ and $r_{n+5} = M[r_n + \sum_{k=0}^{n}\sum_{j=0}^{k} r_jr_{k-j}r_{n-k+1} + \sum_{k=0}^{n} r_{k+1}r_{n-k+2} + \sum_{k=0}^{n} r_kr_{n-k+3}]$, $(n = 0,1,2,\cdots)$. It is clearly show that

$$
|c_n| \leq r_n, \quad n = 0,1,2,\dots. \qquad (54)
$$

In other words, the series $R = R(\xi) = \sum_{n=0}^{\infty} r_n\xi^n$ is a majorant series of eq. (35).

Then, we will show that the series $R = R(\xi)$ has positive radius of convergence. Actually, we can write $R(\xi)$ in the following form:

$$
\begin{aligned}
R(\xi) &= r_0 + r_1\xi + r_2\xi^2 + r_3\xi^3 + r_4\xi^4 + \sum_{n=0}^{\infty} r_{n+5}\xi^{n+5}\\
&= r_0 + r_1\xi + r_2\xi^2 + r_3\xi^3 + r_4\xi^4 + M\left[\sum_{n=0}^{\infty} r_n\xi^{n+5} + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{k} r_jr_{k-j}r_{n-k+1}\xi^{n+5} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} r_{k+1}r_{n-k+2}\xi^{n+5}\right.\\
&\quad\left. + \sum_{n=0}^{\infty}\sum_{k=0}^{n} r_kr_{n-k+3}\xi^{n+5}\right]\\
&= r_0 + r_1\xi + r_2\xi^2 + r_3\xi^3 + r_4\xi^4 + M\left[R(\xi)\xi^5 + R^3(\xi)\xi^4 - r_0R^2(\xi)\xi^4 + (R - r_0)(R - r_0 - r_1\xi)\xi^2\right.\\
&\quad\left. + R(R - r_0 - r_1\xi - r_2\xi^2)\xi^2\right]. \qquad (55)
\end{aligned}
$$

Consider the functional equation about the independent variable ξ

$$F(\xi,R)=R-r_0-r_1\xi-r_2\xi^2-r_3\xi^3-r_4\xi^4-M\left[R(\xi)\xi^5+R^3(\xi)\xi^4-r_0R^2(\xi)\xi^4+(R-r_0)(R-r_0-r_1\xi)\xi^2\right.$$
$$\left.+R(R-r_0-r_1\xi-r_2\xi^2)\xi^2\right]. \tag{56}$$

From the above formula, we see F is analytic in the neighborhood of $(0,r_0)$ and

$$F(0,r_0)=0,\qquad F'_R(0,r_0)=1\neq 0. \tag{57}$$

Based on the following theorem, we see that the $R=R(\xi)$ is analytic in a neighborhood of the point $(0,r_0)$ and with the positive radius. It shows that the power series (35) is convergent in a neighborhood of the point $(0,r_0)$.

Theorem [37]. Let f be a $\mathscr{C}'$-mapping of an open set $E\subset R^{n+m}$ into R^n, such that $f(a,b)=0$ for some point $(a,b)\in E$. Assume that $A=f'(a,b)$ and A_x is invertible. Then the following properties hold in the open sets $U\subset R^{n+m}$ and $W\subset R^m$ with $(a,b)\in U$ and $b\in W$.
i) For each $y\in W$, there exist a unique x such that

$$(x,y)\in U\ \text{ and }\ f(x,y)=0. \tag{58}$$

ii) If x is defined to be $g(y)$, then

$$\begin{aligned} g(b)&=a,\\ f(g(y),y)&=0\ \ (y\in W),\\ g'(b)&=-(A_x)^{-1}A_y, \end{aligned} \tag{59}$$

where g is a $\mathscr{C}'$-mapping of W into R^n.
iii) The function g is "implicitly" defined by (59).

5 Bilinear representation and N-soliton solutions

In order to explore the existence of linearizable representation of eq. (1), one introduces a potential field

$$u=h(t)q_{2x}, \tag{60}$$

where $q=q(x,t)$ and $h=h(t)$ are two free functions to be the suitable choice such that eq. (1) connects with $\mathcal{P}$-polynomials. Combining transformation (60) and (1) yields

$$h_t(t)q_{2x}+h(t)q_{2x,t}+a(6h(t)^2q_{2x}q_{3x}+h(t)q_{5x})+b(45h^3(t)q_{2x}^2q_{3x}+15h^2(t)q_{3x}q_{4x}+15h^2(t)q_{2x}q_{5x}+h(t)q_{7x})=0. \tag{61}$$

Integrating eq. (61) with respect to x and taking the function $h(t)=1$, we obtain the result as follows:

$$E(q)=q_{x,t}+a(q_{4x}+3q_{2x}^2)+b(15q_{2x}^3+15q_{2x}q_{4x}+q_{6x})+c=0, \tag{62}$$

where $c=c(t)$ is an integration constant. By utilizing the formula (A.6) in the appendix, eq. (62) can be rewritten in a combination form of $\mathcal{P}$-polynomials

$$E(q)\equiv P_{x,t}(q)+aP_{4x}(q)+bP_{6x}(q)+c=0. \tag{63}$$

Considering the property of multi-dimensional Bell polynomials and using the following change:

$$q=2(\ln f)\Longleftrightarrow u=h(t)q_{2x}=2(\ln f)_{xx}, \tag{64}$$

we can obtain the bilinear representation of KdV-Sawada-Kotera-Ramani equation

$$\mathscr{D}(D_t,D_x)\equiv(D_xD_t+aD_x^4+bD_x^6+c)f\cdot f=0. \tag{65}$$

According to the obtained bilinear representation, we obtain the one-soliton solution of the KdV-Sawada-Kotera-Ramani equation

$$u=2\partial_x^2\ln(1+e^\eta),\qquad \eta=\mu x-(a\mu^3+b\mu^5)t+\delta, \tag{66}$$

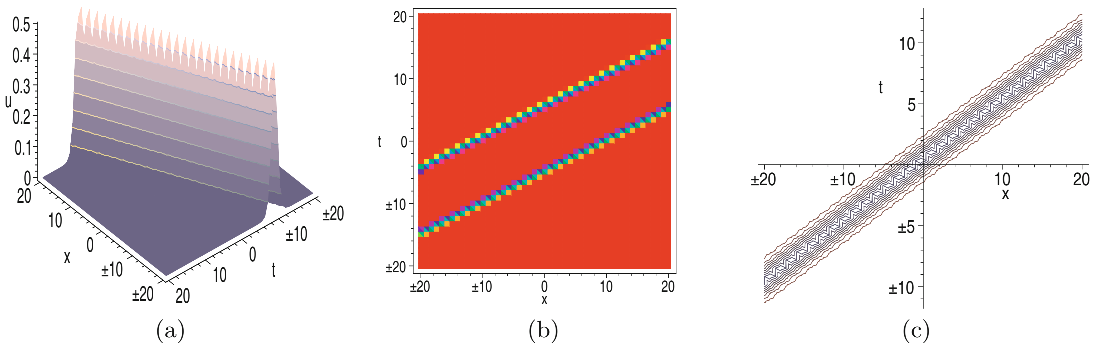


Fig. 1. Propagation situations of the one solitary wave for the KdV-Sawada-Kotera-Ramani equation (1) via expression (66) with $a = 1$, $b = 1$, $\mu = 1$, $\delta = 1$. (a) The perspective view of the wave. (b) The overhead view of the wave. (c) The corresponding contour plot.

by taking $c = 0$, where $\mu \neq 0$, and δ is an arbitrary constant. In a similar way, we obtain the two-soliton solution of eq. (1)

$$\begin{aligned} u &= 2\partial_x^2 \ln(1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_1+\eta_2+A_{12}}), \qquad \eta_i = \mu_i x - (a\mu_i^3 + b\mu_i^5)t + \delta_i, \\ e^{A_{12}} &= -\frac{(\mu_1 - \mu_2)(\nu_1 - \nu_2) + a(\mu_1 - \mu_2)^4 + b(\mu_1 - \mu_2)^6}{(\mu_1 + \mu_2)(\nu_1 + \nu_2) + a(\mu_1 + \mu_2)^4 + b(\mu_1 + \mu_2)^6}, \\ \nu_i &= -a\mu_i^3 - b\mu_i^5, \end{aligned} \tag{67}$$

in which $\mu_i \neq 0$, and $\nu_i, \delta_i,\ (i = 1, 2)$ are arbitrary constants. More generally, when $c = 0$, the KdV-Sawada-Kotera-Ramani equation admits the following N-soliton solution:

$$u = 2(\ln f)_{xx}, \quad f = \sum_{\rho=0,1} \exp\left(\sum_{j=1}^{N} \rho_j \eta_j + \sum_{1\le j<i\le N}^{N} \rho_i \rho_j A_{ij}\right), \tag{68}$$

with

$$\begin{aligned} \eta_i &= \mu_i x - (a\mu_i^3 + b\mu_i^5)t + \delta_i, \\ e^{A_{ij}} &= -\frac{(\mu_i - \mu_j)(\nu_i - \nu_j) + a(\mu_i - \mu_j)^4 + b(\mu_i - \mu_j)^6}{(\mu_i + \mu_j)(\nu_i + \nu_j) + a(\mu_i + \mu_j)^4 + b(\mu_i + \mu_j)^6}, \\ \nu_i &= -a\mu_i^3 - b\mu_i^5, \quad (1 \le j < i \le N) \end{aligned} \tag{69}$$

in which μ_j, ν_j are parameters characterizing the jth soliton, $\sum_{1\le j<i\le N}^{N}$ is the summation of all possible pairs taken from N elements with the condition $1 \le j < i \le N$, and $\sum_{\rho=0,1}$ represents the summation over all possible combinations of $\rho_i, \rho_j = 0, 1$ $(i, j = 1, 2, \ldots, N)$.

To further investigate the properties of the soliton solutions of KdV-Sawada-Kotera-Ramani equation, we present some figures to describe the propagation situations of the solitary waves based on the above obtained soliton solutions. Figure 1 show the propagation situations of the one solitary waves with appropriate parameters in eq. (66). Figure 2 show the propagation situations of the two solitary waves with appropriate parameters in eq. (67).

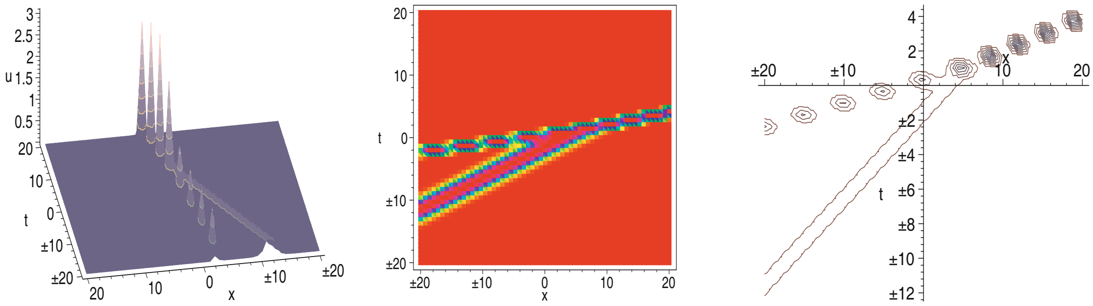
6 The bilinear Bäcklund transformation and associated Lax pair

To obtain the bilinear Bäcklund transformation of the KdV-Sawada-Kotera-Ramani equation (1), we introduce two different solutions of eq. (62)

$$q = 2\ln g, \qquad q' = 2\ln f. \tag{70}$$

Combining (70) and the two-field condition from (62) yields

$$E(q') - E(q) = (q'-q)_{x,t} + a[(q'-q)_{4x} + 3(q'-q)_{2x}(q'+q)_{2x}] + b[(q'-q)_{6x} + 15(q'_{2x}q'_{4x} - q_{2x}q_{4x}) + 15(q'^3_{2x} - q^3_{2x})] = 0. \tag{71}$$



(a) (b) (c)

Fig. 2. Propagation situations of the two solitary waves for the KdV-Sawada-Kotera-Ramani equation (1) via expression (67) with $a = 1$, $b = 1$, $\mu_1 = 1$, $\mu_2 = -1.5$, $\nu_1 = -2$, $\nu_2 = 10.96875$, $\delta_1 = 1$, $\delta_2 = 0$. (a) The perspective view of the wave. (b) The overhead view of the wave. (c) The corresponding contour plot.

This two-field condition can be regarded as an ansatz for a bilinear Bäcklund transformation, which may produce the required transformation under applicable constraints. In order to find such constraints, two new variables are introduced by

$$v = (q' - q)/2 = \ln(f/g), \qquad \omega = (q' + q)/2 = \ln(fg). \tag{72}$$

By using the new variables, condition (71) can be rewritten in the following form:

$$\begin{aligned} E(q') - E(q) = E(\omega + v) - E(\omega - v) &= 2v_{x,t} + a[2v_{4x} + 12v_{2x}\omega_{2x}] + b[2v_{6x} + 30v_{4x}\omega_{2x} + 30v_{2x}\omega_{4x} + 90v_{2x}\omega_{2x}^2 + 30v_{2x}^3] \\ &= \partial_x[2\mathscr{Y}_t(v) - 3b\mathscr{Y}_{5x}(v,\omega) + 2a\mathscr{Y}_{3x}(v,\omega)] + \mathscr{R}(v,\omega) = 0, \end{aligned} \tag{73}$$

where

$$\begin{aligned} \mathscr{R}(v,\omega) = b[&5v_{6x} + 60v_{4x}\omega_{2x} + 30v_{3x}\omega_{3x} + 45v_{2x}\omega_{4x} + 15v_x\omega_{5x} + 60v_xv_{2x}v_{3x} + 30v_x^2v_{4x} \\ &+ 135v_{2x}\omega_{2x}^2 + 90v_x\omega_{2x}\omega_{3x} + 90v_x^2v_{2x}\omega_{2x} + 30v_x^3\omega_{3x} + 15v_x^4v_{2x} + 30v_{2x}^3] + a[6v_{2x}\omega_{2x} - 6v_x\omega_{3x} - 6v_x^2v_{2x}]. \end{aligned} \tag{74}$$

By introducing the following constraints:

$$\begin{aligned} \mathscr{Y}_{3x}(v,\omega) &= \lambda, \\ \mathscr{Y}_{2x}(v,\omega) + \alpha\mathscr{Y}_x(v,\omega) &= \beta, \end{aligned} \tag{75}$$

$\mathscr{R}(v,\omega)$ is obtained as

$$\begin{aligned} \mathscr{R}(v,\omega) = b\,&\left[-30v_xv_{2x}v_{3x} - 15v_x^2v_{4x} - 180v_x^2v_{2x}\omega_{2x} - 60v_x^3\omega_{3x} - 75v_x^4v_{2x} - 15v_{4x}\omega_{2x} - 15v_{3x}\omega_{3x}\right. \\ &\left.-45v_{2x}\omega_{2x}^2 - 90v_x\omega_{2x}\omega_{3x}\right] + a\left[6v_{2x}\omega_{2x} - 6v_x\omega_{3x} - 6v_x^2v_{2x}\right], \end{aligned} \tag{76}$$

i.e.

$$\mathscr{R}(v,\omega) = \partial_x[-15b\lambda\mathscr{Y}_{2x}(v,\omega) + 6a\beta\mathscr{Y}_x(v,\omega)], \tag{77}$$

where $\lambda,\ \alpha,\ \beta$ are arbitrary parameters.

Combining relations (75) and (77), we can obtain a coupled system of $\mathscr{Y}$-polynomials

$$\begin{aligned} &\mathscr{Y}_{3x}(v,\omega) = \lambda, \\ &\mathscr{Y}_{2x}(v,\omega) + \alpha\mathscr{Y}_x(v,\omega) = \beta, \\ &\partial_x\left[2\mathscr{Y}_t(v,\omega) + 2a\left(\mathscr{Y}_{3x}(v,\omega) + 3\beta\mathscr{Y}_x(v,\omega)\right) - b\left(3\mathscr{Y}_{5x}(v,\omega) + 15\lambda\mathscr{Y}_{2x}(v,\omega)\right)\right] = 0. \end{aligned} \tag{78}$$

By means of the identity (A.5), we obtain the following bilinear Bäcklund transformation:

$$\begin{aligned} &(D_x^3 - \lambda)f\cdot g = 0, \\ &(D_x^2 + \alpha D_x - \beta)f\cdot g = 0, \\ &(2D_t + 2aD_x^3 + 6a\beta D_x - 3bD_x^5 - 15b\lambda D_x^2 + \gamma)f\cdot g = 0, \end{aligned} \tag{79}$$

where λ, α, β are arbitrary parameters, $\gamma = \gamma(t)$ is an arbitrary function.

Next, based on the system (78), we will construct the Lax pair of eq. (1). By using the Hopf-Cole transformation $v = \ln\psi$, the Bell system (78) is linearized into the following system:

$$(\mathscr{L}_1)\psi \equiv \psi_{3x} + 3u\psi_x - \lambda\psi = 0, \tag{80a}$$
$$(\mathscr{L}_2)\psi \equiv u\psi + \psi_x + \alpha\psi_x - \beta\psi = 0, \tag{80b}$$
$$(\mathscr{L}_3 + 2\partial_t)\psi = 2\psi_t + 2a(3u\psi_x + \psi_{3x}) + 6a\beta\psi_x - 3b(\psi_{5x} + 10u\psi_{3x} + 5u_{2x}\psi_x + 15u^2\psi_x) - 15b\lambda(u\psi + \psi_{2x}) = 0, \tag{80c}$$

where $\lambda, \alpha,\ \beta$ are arbitrary parameters and u is a solution of eq. (1). Under the condition eq. (80b), the expression $\psi_{3x,t} = \psi_{t,3x}$ yields eq. (1). Thus, the system (80) can be considered as the Lax pair of eq. (1).

7 Infinite conservation laws

In order to redecompose the two-field condition (71) into the x- and y-derivative of $\mathscr{Y}$-polynomials, we revisit $\mathscr{R}(v,\omega)$ in the condition (73) and write it in another form

$$\mathscr{R}(v,\omega) = [-15b\lambda(v_x^2 + \omega_{2x}) + 6a\beta v_x]_x + [2v_x]_t = 0. \tag{81}$$

Moreover, the two-field constraint system (78) can be rewritten in the following form:

$$\begin{aligned}
&v_x^2 + \omega_{2x} + \alpha v_x - \beta = 0,\\
&v_{3x} + 3v_x\omega_{2x} + v_x^3 - \lambda = 0,\\
&[2a(v_{3x} + 3v_x\omega_{2x} + v_x^3) + 6a\beta v_x - 3b(v_{5x} + 10v_{3x}\omega_{2x} + 5v_x\omega_{4x} + 10v_x^2 v_{3x} + 15v_x\omega_{2x}^2 + 10v_x^3\omega_{2x} + v_x^5)\\
&-15b\lambda(v_x^2 + \omega_{2x})]_x + [2v_x]_t = 0.
\end{aligned} \tag{82}$$

Introducing the new potential function

$$\eta = (q'_x - q_x)/2, \tag{83}$$

and utilizing relation (72), we obtain the following formula:

$$v_x = \eta, \qquad \omega_x = q_x + \eta. \tag{84}$$

Combining (84) and (82), and taking $\beta = \varepsilon^2,\ \lambda = \varepsilon^3$, we can decompose the two-field condition (73) into two Riccati-type equations

$$\begin{aligned}
&\eta^2 + q_{2x} + \eta_x + \alpha\eta - \varepsilon^2 = 0,\\
&\eta_{2x} + 3\eta(q_{2x} + \eta_x) + \eta^3 - \varepsilon^3 = 0,
\end{aligned} \tag{85}$$

which are new potential functions with respect to q, and a divergence-type equation as follows:

$$\begin{aligned}
&\partial_x[2a\varepsilon^3 + 6a\varepsilon^2\eta - 3b(\eta_{4x} + 10\varepsilon^2\eta_{2x} - 15\alpha\eta\eta_{2x} - 10\eta^2\eta_{2x} - 10\eta\eta_x^2 + 15\varepsilon^4\eta - 30\alpha\varepsilon^2\eta^2 + 15\alpha^2\eta^3 - 20\varepsilon^2\eta^3\\
&+ 20\alpha\eta^4 + 6\eta^5 + 5\varepsilon^5 - 5\alpha\varepsilon^3\eta)] + [2\eta]_t = 0,
\end{aligned} \tag{86}$$

where (86) is obtained by using (85).

Inserting the following expansion:

$$\eta = \varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n(q, q_x, q_{2x}, \ldots)\varepsilon^{-n} \tag{87}$$

into the linear combination as follows:

$$\eta^2 + q_{2x} + \eta_x + \alpha\eta - \varepsilon^2 + r[\eta_{2x} + 3\eta(q_{2x} + \eta_x) + \eta^3 - \varepsilon^3] = 0, \tag{88}$$

and equating the coefficients for ε, where $r \neq 0$, we can obtain the conversed densities $\mathscr{I}_n$'s as follows:

$$\begin{aligned}
&\mathscr{I}_1 = -u - \frac{\alpha}{3r},\\
&\mathscr{I}_2 = u_x + \frac{u}{3r} + \frac{2\alpha}{9r^2}, \ldots.
\end{aligned} \tag{89}$$

Then combining expansion (87) and the divergence-type equation (86) yields

$$\partial_x \left\{ 2a\varepsilon^3 + 6a\varepsilon^2 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right) - 3b \left[\sum_{n=1}^{\infty} \mathscr{I}_{n,4x}\varepsilon^{-n} + 10\varepsilon^2 \sum_{n=1}^{\infty} \mathscr{I}_{n,2x}\varepsilon^{-n} - 15\alpha \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right) \sum_{n=1}^{\infty} \mathscr{I}_{n,2x}\varepsilon^{-n} \right.\right.$$
$$-10 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right)^2 \sum_{n=1}^{\infty} \mathscr{I}_{n,2x}\varepsilon^{-n} - 10 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right) \left(\sum_{n=1}^{\infty} \mathscr{I}_{n,x}\varepsilon^{-n}\right)^2 + 15\varepsilon^4 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right)$$
$$-30\alpha\varepsilon^2 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right)^2 + 15\alpha^2 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right)^3 - 20\varepsilon^2 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right)^3 + 20\alpha \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right)^4$$
$$\left.\left. +6 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right)^5 + 5\varepsilon^5 - 5\alpha\varepsilon^3 \left(\varepsilon + \sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right) \right] \right\} + \partial_t \left(2\varepsilon + 2\sum_{n=1}^{\infty} \mathscr{I}_n \varepsilon^{-n}\right) = 0. \tag{90}$$

From eq. (90), the fluxes $\mathscr{J}_n$'s are given by

$$\mathscr{J}_1 = 6a\mathscr{I}_3 - 3b[\mathscr{I}_{1,4x} - 20\mathscr{I}_1\mathscr{I}_{1,2x} - 10\mathscr{I}_{1,x}^2 - 15\alpha\mathscr{I}_{2,2x} - 15\mathscr{I}_5 + 15\alpha\mathscr{I}_4 + 180\alpha\mathscr{I}_1\mathscr{I}_2 + 45\alpha^2\mathscr{I}_3 + 45\alpha^2\mathscr{I}_1^2],$$
$$\mathscr{J}_2 = 6a\mathscr{I}_4 - 3b[\mathscr{I}_{2,4x} - 20\mathscr{I}_1\mathscr{I}_{2,2x} - 20\mathscr{I}_2\mathscr{I}_{1,2x} - 20\mathscr{I}_{1,x}\mathscr{I}_{2,x} - 15\alpha\mathscr{I}_1\mathscr{I}_{1,2x} - 15\alpha\mathscr{I}_{3,2x} - 15\mathscr{I}_6 + 15\alpha\mathscr{I}_5$$
$$+180\alpha\mathscr{I}_1\mathscr{I}_3 + 90\alpha\mathscr{I}_2^2 + 80\alpha\mathscr{I}_1^3 + 45\alpha^2\mathscr{I}_4 + 90\alpha^2\mathscr{I}_1\mathscr{I}_2 + 120\mathscr{I}_1^2\mathscr{I}_2], \ldots. \tag{91}$$

The conversed densities $\mathscr{I}_n$'s and $\mathscr{J}_n$'s provide us the infinite conservation laws as follows

$$\mathscr{I}_{n,t} + \mathscr{J}_{n,x} = 0, \quad n = 1, 2, \ldots. \tag{92}$$

8 Conclusions and discussions

In this paper, we study the Lie symmetries, exact solutions and integrability of KdV-Sawada-Kotera-Ramani equation. On the one hand, based on Lie symmetry analysis method, the vector field and optimal system are obtained. Then the symmetry reductions and exact solutions are also obtained by employing the optimal system. Moreover, we construct the exact analytic solutions of the equation by utilizing the power series method with the convergence of power series solutions. On the other hand, we systematically investigate the integrability of KdV-Sawada-Kotera-Ramani equation by using Bell polynomial approach, such as N-soliton solutions, Lax pair, Bäcklund transformation and infinite conservation laws. The $\mathcal{P}$-polynomial expression and $\mathscr{Y}$- polynomial of KdV-Sawada-Kotera-Ramani equation are obtained, respectively, which can be cast into the bilinear form and the bilinear Bäcklund transformation. Furthermore, by linearizing the Bell-polynomial-typed Bäcklund transformation, the corresponding Lax pair is also derived. Besides, by using the Hirota bilinear method, the N-soliton solutions of the KdV-Sawada-Kotera-Ramani equation are also obtained. On the basis of binary Bell polynomial form, we found infinite conservation laws of the KdV-Sawada-Kotera-Ramani equation.

In conclusion, we construct the Lie symmetries, exact solutions and integrability to the KdV-Sawada-Kotera-Ramani equation. Based on the Bell's polynomials, a straightforward way is explicitly provided to construct its bilinear equation, N-soliton solutions, Lax pair, Bäcklund transformation and infinite conservation laws for such equation. This method is also suitable for other nonlinear differential equations.

The authors are greatly indebted to the editor and reviewers for their helpful comments and constructive suggestions, and one of the authors Pan-Li Ma is grateful for Li Wang's valuable discussion and help. This work is supported by the Fundamental Research Funds for the Central Universities of China under the Grant No. 2015QNA53.

Appendix A. Multidimensional bell polynomials

In what follows, we simply recall some necessary notations on multidimensional binary Bell polynomials; for example, to Lembert and Gilson's work [38–40]. Let $f = f(x_1, x_2, \ldots, x_n)$ be a $\mathbb{C}^\infty$ function with multi-variables, the polynomial

$$Y_{n_1x_1,\ldots,n_rx_r}(f) \equiv Y_{n_1,\ldots,n_r}(f_{l_1x_1}, \ldots, f_{l_rx_r}) = e^{-f}\partial_{x_1}^{n_1} \cdots \partial_{x_r}^{n_r} e^f \tag{A.1}$$

is called the muliti-dimensional Bell polynomial, where $f_{l_1x_1,\ldots,l_rx_r} = \partial_{x_1}^{l_1}\cdots\partial_{x_r}^{l_r}$ $(0 \le l_i \le n_i, i = 1,2,\ldots,r)$. Taking $n = 1$, the Bell polynomials are presented as follows:

$$Y_{nx}(f) \equiv Y_n(f_1,f_2,\ldots,f_n) = \sum \frac{n!}{s_1!\cdots s_n!(1!)^{s_1}\cdots(n!)^{s_n}} f_1^{s_1}\cdots f_n^{s_n}, \quad n = \sum_{k=1}^{n} ks_k,$$
$$Y_x(f) = f_x, \quad Y_{2x} = f_{2x} + f_x^2, \quad Y_{3x}(f) = f_{3x} + 3f_xf_{2x} + f_x^3, \ldots. \tag{A.2}$$

To build a relationship between the Bell polynomials and the Hirota D-operator, the multidimensional binary polynomials can be defined as follows [39]:

$$\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(v,\omega) = Y_{n_1,\ldots,n_r}(f)\,|_{f_{l_1x_1,\ldots,l_rx_r}=\begin{cases} v_{l_1x_1,\ldots,l_rx_r}, & l_1+\cdots+l_r \quad \text{is odd,}\\ \omega_{l_1x_1,\ldots,l_rx_r}, & l_1+\cdots+l_r \quad \text{is even,}\end{cases}}$$
$$\mathscr{Y}_x(v,\omega) = v_x, \quad \mathscr{Y}_{2x}(v,\omega) = v_x^2 + \omega_{2x}, \quad \mathscr{Y}_{x,t}(v,\omega) = v_xv_t + \omega_{xt},$$
$$\mathscr{Y}_{3x}(v,\omega) = v_{3x} + 3v_x\omega_{2x} + v_x^3, \ldots, \tag{A.3}$$

which inherit the easily recognizable partial structure of the Bell polynomials. The relation between the $\mathscr{Y}$-polynomials and the Hirota bilinear equation $D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G$ can be given by the identity [39]

$$\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\nu = \ln F/G, \omega = \ln FG) = (FG)^{-1}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G, \tag{A.4}$$

where F and G are both the functions about x and t. In particular, taking $F = G$, the formula (A.4) becomes

$$F^{-2}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot F = \mathscr{Y}(0, q = 2\ln F) = \begin{cases} 0, & \text{if} \quad n_1+\cdots+n_r \quad \text{is odd,}\\ P_{n_1x_1,\ldots,n_rx_r}(q), & \text{if} \quad n_1+\ldots+n_r \quad \text{is even,}\end{cases} \tag{A.5}$$

in which the P-polynomials can be replaced by an equally recognizable even-part partitional structure,

$$P_{2x}(q) = q_{2x}, \quad P_{x,t}(q) = q_{xt}, \quad P_{4x}(q) = q_{4x} + 3q_{2x}^2, \quad P_{6x}(q) = q_{6x} + 15q_{2x}q_{4x} + 15q_{2x}^3, \ldots. \tag{A.6}$$

The binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\nu,\omega)$ can be divided into P-polynomials and Y-polynomials

$$(FG)^{-1}D_{x_1}^{n_1},\ldots,D_{x_r}^{n_r}F\cdot G = \mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\nu,\omega)\big|_{\nu=\ln F/G,\omega=\ln FG}$$
$$= \mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\nu,\nu+q)\big|_{\nu=\ln F/G,\omega=\ln FG}$$
$$= \sum_{n_1+n_2+\cdots+n_r=even}\ \sum_{l_1=0}^{n_1}\cdots\sum_{l_r}^{n_r}\prod_{i=0}^{r}\binom{n_i}{l_i}P_{l_1x_1,\ldots,l_rx_r}(q)Y_{(n_1-l_1)x_1,\ldots,(n_r-l_r)x_r}(\nu). \tag{A.7}$$

The multidimensional Bell polynomials admit the following key property:

$$Y_{n_1x_1,\ldots,n_rx_r}(\nu)\big|_{\nu=\ln\psi} = \psi_{n_1x_1,\ldots,n_rx_r}/\psi, \tag{A.8}$$

which means that the binary Bell polynomials $\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\nu,\omega)$ can still be linearized by taking advantage of the Hopf-Cole transformation $\nu = \ln\psi$, *i.e.* $\psi = F/G$. The associated Lax system of the nonlinear equations can be obtained by means of formulas (A.7) and (A.8).

References

1. M.J. Ablowitz, P.A. Clarkson, *Solitons; Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, 1991).
2. G.W. Bluman, S. Kumei, *Symmetries and Differential Equations*, in *Applied Mathematical Sciences*, Book **81** (Springer-Verlag, New York, 1989).
3. V.B. Matveev, M.A. Salle, *Darboux Transformation and Solitons* (Springer, 1991).
4. R. Hirota, *Direct Methods in Soliton Theory* (Springer, 2004).
5. X.B. Hu, C.X. Li, J.J.C. Nimmo, G.F. Yu, J. Phys. A **38**, 195 (2005).
6. E. Belokolos, A. Bobenko, V. Enol'skij, A. Its, V. Matveev, *Algebro-Geometrical Approach to Nonlinear Integrable Equations* (Springer, 1994).
7. J. Weiss, M. Tabor, G. Carnevale, J. Math. Phys. **24**, 522 (1983).

8. S.Y. Lou, Z. Naturforsch. **53a**, 251 (1998).
9. E.G. Fan, Y.C. Hon, Phys. Rev. E **78**, 036607 (2008).
10. E.G. Fan, J. Phys. A **42**, 095206 (2009).
11. W.X. Ma, R. Zhou, L. Gao, Mod. Phys. Lett. A **24**, 1677 (2009).
12. J.M. Tu, S.F. Tian, M.J. Xu, T.T. Zhang, Appl. Math. Comput. **275**, 345 (2016).
13. A.M. Wazwaz, *Partial Differential Equations: Methods and Applications* (Balkema Publishers, The Netherlands, 2002).
14. B. Tian, Y.T. Gao, Eur. Phys. J. B **30**, 97 (1995).
15. S.F. Tian, Y.F. Zhang, B.L. Feng, H.Q. Zhang, Chin. Ann. Math. **36B**, 543 (2015).
16. M. Eslami, A. Neirameh, Eur. Phys. J. Plus **129**, 54 (2014).
17. M.J. Xu, S.F. Tian, J.M. Tu, P.L. Ma, T.T. Zhang, Eur. Phys. J. Plus **130**, 174 (2015).
18. P.L. Ma, S.F. Tian, J.M. Tu, M.J. Xu, Eur. Phys. J. Plus **130**, 98 (2015).
19. P.L. Ma, S.F. Tian, T.T. Zhang, Appl. Math. Lett. **50**, 146 (2015).
20. Y. Amadou, G. Betchewe, Douvagai, M. Justin, S.Y. Doka, K.T. Crepin, Eur. Phys. J. Plus **130**, 13 (2015).
21. S.F. Tian, T.T. Zhang, P.L. Ma, X.Y. Zhang, J. Nonlinear Math. Phys. **22**, 180 (2015).
22. R. Hirota, M. Ito, J. Phys. Soc. Jpn. **52**, 744 (1983).
23. J. Yang, Stud. Appl. Math. **106**, 337 (2001).
24. M. Inc, B. Kilic, Waves Random Complex Media **24**, 393 (2014).
25. W.X. Ma, J.H. Lee, Chaos Solitons Fractals. **42**, 1356 (2009).
26. E.J. Parkes, B.R. Duffy, P.C. Abbott, Phys. Lett. A. **295**, 280 (2002).
27. K. Konno, J. Phys. Soc. Jpn. **61**, 51 (1992).
28. D.J. Korteweg, G.de Vries, Philos. Mag. **39**, 422 (1895).
29. C.S. Gardner, J.M. Greene, M.D. Kruskal, R.M. Miura, Rev. Let. **19**, 1095 (1967).
30. U. Göktas, W. Hereman, J. Symb. Comput. **11**, 1 (2008).
31. S.F. Tian, H.Q. Zhang, J. Phys. A: Math. Theor. **45**, 055203 (2012).
32. S.F. Tian, P.L. Ma, Commun. Theor. Phys. **62**, 245 (2014).
33. S.F. Tian, H.Q. Zhang, J. Math. Anal. Appl. **371**, 585 (2010).
34. S.F. Tian, H.Q. Zhang, Commun. Nonlinear Sci. Numer. Simulat. **16**, 173 (2011).
35. S.F. Tian, H.Q. Zhang, Chaos Solitons Fractals **47**, 27 (2013).
36. S.F. Tian, H.Q. Zhang, Stud. Appl. Math. **132**, 212 (2014).
37. W. Rudin, *Principles of Mathematical Analysis* (China Machine Press, Beijing, 2004).
38. E.T. Bell, Ann. Math. **35**, 258 (1834).
39. C. Gilson, F. Lambert, J. Nimmo, R. Willox, Proc. R. Soc. London A. **452**, 223 (1996).
40. F. Lambert, I. Loris, J. Springael, Inverse Probl. **17**, 1067 (2001).