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On the thermodynamics of the cosmological apparent horizon

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Abstract. It has been shown by Cai *et al.* that the apparent horizon of radius r_0 in the cosmological Friedmann space-time emits radiation at the temperature $T_0 = 1/2\pi r_0$. Here, we derive this result from the Wheeler-DeWitt equation for the wave function of the Universe Ψ, starting from a classical gravitational Lagrangian L that contains a quadratic higher-derivative term $\mathcal{R}^2$, the scalar component of which is non-tachyonic, by application of the horizon hypothesis and definition of the physical three-space on the time-slice $\mathrm{d}x_0 = 0$. We also extend our previous analysis of the Wheeler-DeWitt equation for the wave function Φ of the apparent horizon of the de Sitter space-time to include the case of a more general energy-momentum source, that generates an arbitrary Friedmann space-time, confirming the expression for T_0 after application of the ADM formalism.

1 Introduction

Apparent horizons in space-time lead to thermal effects which have been intensively investigated and continue to be the subject of much research, involving both classical thermodynamics and quantum field theory, since the original discovery by Hawking [1] that a Schwarzschild [2] black hole of mass M emits radiation at a temperature given by the formula

$$T_{\mathrm{H}} = \frac{1}{8\pi M} = \frac{1}{4\pi r_0}\,, \tag{1}$$

in natural units where $c = \hbar = k_{\mathrm{B}} = G_{\mathrm{N}} = 1$ and $r_0 \equiv 2M$ is the horizon radius.

The Schwarzschild space-time is most simply defined in static coordinates $(\bar{t}, r, \theta, \varphi)$, the line element being

$$\mathrm{d}s^2 = (1 - 2Mr^{-1})\mathrm{d}\bar{t}^2 - (1 - 2Mr^{-1})^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2, \tag{2}$$

where $\mathrm{d}\Omega^2 \equiv \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$ specifies the unit two-sphere. More recently, however, the analysis of radiative processes has been extended to dynamical space-times, in particular the cosmological Friedmann space-time

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\mathrm{d}\boldsymbol{x}^2, \tag{3}$$

where $t \equiv x^0$ is comoving time and $a(t) \equiv \mathrm{e}^{\alpha(t)}$ is the radius function of the three-space

$$\mathrm{d}\boldsymbol{x}^2 = \frac{\mathrm{d}\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2\mathrm{d}\Omega^2, \tag{4}$$

of curvature $k = 0, \pm 1$. Irrespective of the value of k, Cai *et al.* [3] find that the apparent horizon exhibits thermal characteristics, at the temperature

$$T_0 = \frac{1}{2\pi r_0}\,, \tag{5}$$

a result confirmed by Li *et al.* [4] for the emission of fermionic particles, and which we have also derived via quantum cosmology [5].

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Expressed in geometrical terms, the formulae (1) and (5) differ by a factor of 2, raising the question why this is so. Let us first consider the problem from the classical viewpoint. In order to clarify the rôle played by the apparent horizon, we have to recast the line element (3) into the form analogous to eq. (2), by introducing the new radial coordinate $r \equiv a\tilde{r}$, which yields

$$\mathrm{d}s^2 = \left(\frac{1 - r^2/r_0^2}{1 - kr^2/a^2}\right)\mathrm{d}t^2 + \left(\frac{2\xi r}{1 - kr^2/a^2}\right)\mathrm{d}t\,\mathrm{d}r - \left(\frac{\mathrm{d}r^2}{1 - kr^2/a^2} + r^2\mathrm{d}\Omega^2\right), \tag{6}$$

where the Hubble parameter is defined as $\xi = \dot{a}/a = \dot{\alpha}$ and $^{\bullet} \equiv \mathrm{d}/\mathrm{d}t$.

Equation (6) can be written in the local form due to von Weyssenhoff [6],

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}\tau^2 - \mathrm{d}\sigma^2 = g_{00}^{-1}\mathrm{d}x_0^2 - \gamma_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta, \tag{7}$$

where $\mathrm{d}\tau \equiv \mathrm{d}x_0/\sqrt{g_{00}}$ is the proper time[1] and

$$\mathrm{d}\sigma^2 \equiv \gamma_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta = \left(-g_{\alpha\beta} + g_{0\alpha}g_{0\beta}/g_{00}\right)\mathrm{d}x^\alpha\mathrm{d}x^\beta \tag{8}$$

is the proper three-space measured by a local observer. We have

$$\mathrm{d}s^2 = \left(\frac{1 - kr^2/a^2}{1 - r^2/r_0^2}\right)\mathrm{d}x_0^2 - \left(\frac{\mathrm{d}r^2}{1 - r^2/r_0^2} + r^2\mathrm{d}\Omega^2\right), \tag{9}$$

remembering [7] the definition of the cosmological apparent horizon as the boundary hypersurface of an anti-trapped region with topology S^2, given by the equation

$$(\nabla r)^2 = 0, \tag{10}$$

the solution to which is

$$r_0 = 1/\sqrt{\xi^2 + k/a^2}\,. \tag{11}$$

On Einstein-shell, the Friedmann equation

$$\xi^2 = \frac{8\pi}{3}\rho - \frac{k}{a^2} \tag{12}$$

enables us to re-express eq. (11) as $r_0^2 = 3/8\pi\rho(t)$, while the inverse metric (6) is

$$g^{00} = 1, \qquad g^{01} = \xi r, \qquad g^{11} = -(1 - r^2/r_0^2), \qquad g^{22} = -r^{-2}, \qquad g^{33} = -r^{-2}\,\mathrm{cosec}^2\,\theta, \tag{13}$$

for all values of k.

The total proper energy contained within the apparent horizon is

$$M(t) = \int \sqrt{-g}\rho(t)\mathrm{d}^3x = 4\pi\int_0^{r_0}\frac{r^2\rho(t)\mathrm{d}r}{\sqrt{1 - kr^2/a^2}}\,, \tag{14}$$

where $\sqrt{-g} = \sqrt{g_{00}}\sqrt{\gamma}$ and $\gamma = \det\gamma_{\alpha\beta}$. When $k = 0$, we have $M(t) = 4\pi\rho(t)r_0^3/3$. The quantity $M(t)$ is in fact the Misner-Sharp [8] energy, which Kodama [9] has used to construct a conserved energy flux. The line element (9) can be re-expressed in the canonical double-null form [9], which in the signature $(+\,-\,-\,-)$ is

$$\mathrm{d}s^2 = \frac{4r_{,u}r_{,v}\mathrm{d}u\,\mathrm{d}v}{g^{11}} - r^2\mathrm{d}\Omega^2, \tag{15}$$

where

$$g^{11} = g_{11}^{-1} = -\left(1 - r^2/r_0^2\right) \tag{16}$$

and the advanced and retarded null coordinates are defined as

$$\mathrm{d}v = \sqrt{1 - kr^2/a^2}\mathrm{d}x_0 + \mathrm{d}r, \qquad \mathrm{d}u = \sqrt{1 - kr^2/a^2}\mathrm{d}x_0 - \mathrm{d}r = \mathrm{d}v - 2\mathrm{d}r, \tag{17}$$

[1] The holonomic time coordinate t' corresponding to x_0 is obtained via the integrating factor $f(t,r) \equiv \mathrm{d}t'/\mathrm{d}x_0$, so defined that $\partial(fg_{00})/\partial r = \partial(fg_{01})/\partial t$. From eq. (6), we write $f(t,r) = (1 - kr^2/a^2)h(t)$, where $h(t)$ satisfies the differential equation $\mathrm{d}(\ln h)/\mathrm{d}t = -(\dot{\xi} + 2/r_0^2)/\xi$. For example, when $k = 0$ and for the perfect-fluid source defined by eq. (28), we have $f(t,r) \equiv f(t) = h(t)$, $\xi = 2/3\gamma' t$, $\dot{\xi} = -\xi/t$ and $1/r_0^2 = \xi^2$, yielding the solution $f(t) = t^{1-4/3\gamma'}$ and hence $t' = t^{-4/3\gamma'}[t^2/(2 - 4/3\gamma') + r^2/3\gamma']$ if $\gamma' \neq 0, 2/3$. Note, for a radiative universe where $\gamma' = 4/3$, that $f = 1$ and therefore $x_0 \equiv t' = t + r^2/4t$ is holonomic in this case.

respectively. From expression (17), we construct the derivatives

$$\frac{\partial}{\partial v} = \frac{1}{2}\left(\frac{1}{\sqrt{1-kr^2/a^2}}\frac{\partial}{\partial x_0} + \frac{\partial}{\partial r}\right), \qquad \frac{\partial}{\partial u} = \frac{1}{2}\left(\frac{1}{\sqrt{1-kr^2/a^2}}\frac{\partial}{\partial x_0} - \frac{\partial}{\partial r}\right), \tag{18}$$

while $\partial r/\partial v = -\partial r/\partial u = 1/2$, and therefore the Kodama vector is

$$K \equiv -\frac{1}{2}g^{11}\left[\left(\frac{1}{\partial r/\partial v}\right)\frac{\partial}{\partial v} - \left(\frac{1}{\partial r/\partial u}\right)\frac{\partial}{\partial u}\right] = \frac{1-r^2/r_0^2}{\sqrt{1-kr^2/a^2}}\frac{\partial}{\partial x_0} \tag{19}$$

referred to the local basis vectors, or in component form referred to the metric (6),

$$K_i = \frac{1}{\sqrt{1-kr^2/a^2}}\left(1-r^2/r_0^2, \xi r, 0, 0\right) \qquad K^i = \left(\sqrt{1-kr^2/a^2}, 0, 0, 0\right). \tag{20}$$

The scalar invariant is independent of k,

$$K^2 = K_i K^i = 1 - r^2/r_0^2, \tag{21}$$

showing that K_i is time-like, null or space-like for $r < r_0$, $r = r_0$ or $r > r_0$, respectively, as noted in ref. [3].

2 Classical thermodynamics

An important quantity characterizing a thermodynamical system is the Helmholtz free energy $F \equiv fV$, defined by the equation

$$F = U - TS \tag{22}$$

in terms of the temperature T, which is an intensive variable, and the internal energy $U \equiv \rho V$ and entropy $S \equiv sV$, which are extensive variables, where V is the volume, f, ρ and s being the corresponding densities. After applying the fundamental differential relationship

$$\mathrm{d}U = T\mathrm{d}S - p\mathrm{d}V, \tag{23}$$

that is, the first law of thermodynamics referred to a fixed amount of matter, we obtain the equation

$$(f+p)\mathrm{d}\ln V = T\mathrm{d}s - \mathrm{d}\rho, \tag{24}$$

where, from eq. (22),

$$f = \rho - Ts. \tag{25}$$

In flat space-time, V is arbitrary, and can be varied independently of T, as a consequence of which eq. (24) splits up into the two separate equations

$$f = -p, \qquad T\mathrm{d}s = \mathrm{d}\rho, \tag{26}$$

the second of which is the Gibbs-Duhem relation. When the space-time is curved, however, V may be temperature-dependent, in which case eqs. (26) no longer necessarily hold.

In ref. [5], hereafter called paper I, we analyzed the spatially flat Friedmann space-time generated by dust, described by the metric (3) with $k = 0$, that is a Universe composed of pressure-free matter which expands adiabatically doing no $p\mathrm{d}V$ work against the boundary, the total mass-energy content being constant. We have

$$M_{\mathrm{U}} = \rho(t)a^3(t) = \rho_0 a_0^3, \tag{27}$$

since $\rho(t) = \rho_0 t^{-2}$ and $a(t) = a_0 t^{2/3}$, assuming a perfect fluid with pressure

$$p = (\gamma' - 1)\,\rho \tag{28}$$

and setting the adiabatic index $\gamma' = 1$.

The radius function $a(t)$ is arbitrary, *a priori*, and can be normalized such that the volume is $V = a^3$ by setting the fundamental fiducial three-volume equal to unity via eq. (I22),

$$V_3 = \int \mathrm{d}^3x\sqrt{-\tilde{g}} = 1. \tag{29}$$

This arbitrariness in $a(t)$ means that V is independent of T. From eqs. (26), it therefore follows that the free-energy density vanishes, this being explicable by the vanishing of the total energy density, matter plus gravitational, as expressed by the Friedmann equation (12) written in the form

$$\rho - 3\xi^2/8\pi = 0, \tag{30}$$

when $k = 0$. Equation (22) can then be understood in the present application, where $F = 0$, if the gravitational energy $(-3\xi^2 V/8\pi)$ is reinterpreted as the term $(-TS)$.

To make this hypothesis precise, we consider not the whole Universe contained within the volume $a^3(t)$, but rather the causally connected region contained within the apparent horizon of an observer at $r = 0$, the proper volume of which is

$$V_0 = 4\pi r_0^3/3, \tag{31}$$

so that

$$3\xi^2 V_0/8\pi \equiv r_0/2 = T_0 S. \tag{32}$$

It then follows from eqs. (5), (22) and (32) that the entropy is given by the formula

$$S = U/T_0 = \pi r_0^2 = A/4, \tag{33}$$

where $A \equiv 4\pi r_0^2$ is the area of the apparent horizon, which is the result obtained by Bekenstein [10] and Hawking [11] for the entropy of the black hole.

If we now take the differential of eq. (22) setting $F = 0$, in place of eq. (23) we obtain

$$\mathrm{d}U = T_0\mathrm{d}S + S\mathrm{d}T_0 = T_0\mathrm{d}S/2. \tag{34}$$

Even though $p = 0$, the first law of thermodynamics does not hold, due to the additional term $S\mathrm{d}T_0 \equiv -T_0\mathrm{d}S/2$, which is explained by the fact that the Kodama observer detects an amount of matter which is not fixed. For although the total proper mass-energy of the Universe, given by eq. (27), is constant, the proper mass-energy contained within the apparent horizon is not constant, being

$$U = 4\pi r_0^3\rho/3 = r_0/2. \tag{35}$$

By contrast, in the case of the Schwarzschild black hole, the stationary observer at spatial infinity does detect a fixed amount of matter M, the sum of the residual mass of the hole itself plus the matter-energy contained in the emitted radiation. Therefore, the first law of thermodynamics has to hold in its normal form: writing

$$S = \pi r_0^2 = U/2T_{\mathrm{H}}, \tag{36}$$

we find that

$$\mathrm{d}U = T_{\mathrm{H}}\mathrm{d}S \tag{37}$$

after setting $U = M$ and substituting from eq. (1). The volume of the black hole changes during the radiation process, so that ρ, s and T_{H} are not independent of V, and although eq. (24) still holds, eqs. (26) do not. There is no "pressure" as such, but the free energy density does not vanish, as noted expressly by Padmanabhan [12].

We now see that the difference by a factor of 2 between eqs. (1) and (5) is explained by the factor of 2 difference between eq. (37) and eq. (34), or equivalently between eq. (36) and eq. (33), when referred to the ratio U/TS.

3 The Wheeler-DeWitt equation in the Friedmann space-time

Previously, in paper I, we derived the temperature of the cosmological apparent horizon, given by eq. (5), from the Wheeler-DeWitt equation for the wave function Ψ of a spatially flat, Friedmann dust Universe, after Euclideanization of the time coordinate t. Without repeating all the details, we focus here on certain general features of the analysis.

The starting point is the effective action including quadratic higher-derivative gravitational terms $\mathcal{R}^2 \equiv \mu R^2 - \nu R_{ij}R^{ij}$,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(-\frac{R}{2\kappa^2} + \mathcal{R}^2 + L_{\mathrm{m}}\right), \tag{38}$$

where $\kappa^2 \equiv 8\pi G_{\mathrm{N}} = 8\pi$ is the gravitational coupling, R the Ricci scalar, R_{ij} the Ricci tensor and L_{m} the matter Lagrangian. In the space-time (3), setting $k = 0$ and remembering eq. (29), expression (38) takes the form

$$S = \int \mathrm{d}t\left(-\frac{3\dot{\alpha}^2}{\kappa^2} + \beta\ddot{\alpha}^2 + L_{\mathrm{m}}\right)\mathrm{e}^{3\alpha}, \tag{39}$$

where we have introduced the parameter $\beta \equiv 3/\kappa^2 M_0^2 = 12(3\mu - \nu)$, M_0 and M_2 being the spin-0 and spin-2 particle masses, defined by

$$M_0^2 = 1/4\,(3\mu - \nu)\,\kappa^2, \qquad M_2^2 = 1/2\nu\kappa^2. \tag{40}$$

The spin-2 particle is absent, due to conformal invariance of the metric (3).

Note that the Lagrangian (39) contains no quartic term $\dot{\alpha}^4$ (which is only true in space-time dimensionality $\mathcal{D} = 4$), but that it contains a quadratic term in second derivatives $\ddot{\alpha}^2$. To deal with this, we apply the method of Ostrogradsky [13], defining the auxiliary coordinate

$$q = -\partial\mathcal{L}/\partial\ddot{\alpha} = -2\beta\ddot{\alpha}\mathrm{e}^{3\alpha}, \tag{41}$$

in terms of which the Lagrangian (39) can be rewritten, dropping all divergences, as

$$\mathcal{L} = \dot{\alpha}\dot{q} - \left(\frac{3\dot{\alpha}^2}{\kappa^2} + \beta\ddot{\alpha}^2\right)\mathrm{e}^{3\alpha} + \mathcal{L}_\mathrm{m} \equiv \mathcal{L}_\mathrm{g} + \mathcal{L}_\mathrm{m}. \tag{42}$$

The Hamiltonian is therefore

$$\mathcal{H} = \pi_\alpha\dot{\alpha} + \pi_\xi\dot{\xi} - \mathcal{L}_\mathrm{g} + \mathcal{H}_\mathrm{m}, \tag{43}$$

where the canonical momenta are

$$\pi_\alpha = \partial\mathcal{L}/\partial\dot{\alpha} = \dot{q} - 6\dot{\alpha}/\kappa^2, \qquad \pi_q = \partial\mathcal{L}/\partial\dot{q} = \dot{\alpha} \tag{44}$$

and $\mathcal{H}_\mathrm{m}$ is the matter Hamiltonian density.

We have interchanged the coordinate q and momentum π_q according to

$$q = -\pi_\xi, \qquad \pi_q = \xi = \dot{\alpha}, \tag{45}$$

as explained in ref. [14], with particular reference to the dimensionally reduced heterotic superstring theory of Gross *et al.* [15–17], for which $\mu = \nu = B$, $\beta = 24B$, where B is given by the integral over the internal space

$$B = \frac{1}{8}\zeta(3)\kappa^6 A_\mathrm{r} B_\mathrm{r}^{-2} \int \mathrm{d}^6y\sqrt{\bar{g}}\bar{R}_{\mu\nu\xi o}\bar{R}^{\mu\nu\xi o} \Big/ \int \mathrm{d}^6y\sqrt{\bar{g}}, \tag{46}$$

$\zeta(3) = 1.202$, A_r and B_r are moduli and $\bar{R}_{\mu\nu\xi o}$ is the Riemann-Christoffel tensor of the space $\bar{g}_{\mu\nu}$ —see also Horowitz [18].

In the semi-classical approximation for the wave function,

$$\Psi \approx \exp(\mathrm{i}S), \tag{47}$$

the canonical momenta are replaced by their operator equivalents

$$\pi_\alpha \to -\mathrm{i}\partial_\alpha, \qquad \pi_\xi \to -\mathrm{i}\partial_\xi. \tag{48}$$

The action integral (39) is rendered finite by means of the horizon hypothesis [19–21], whereby the spatial integral is bounded by the causal horizon at the distance ξ^{-1}. The Wheeler [22]-DeWitt [23] equation, obtained by promoting the Hamiltonian constraint $\mathcal{H} = 0$ into the operator equation $\widehat{\mathcal{H}}\Psi = 0$ after substituting from the replacements (48), can be written in the form of a Schrödinger equation,

$$\frac{\mathrm{i}\partial\Psi}{\partial t} = \left[-\frac{\mathrm{e}^{-3\alpha}}{4\beta}\frac{\partial^2}{\partial\xi^2} + \left(\frac{3\xi^2\mathrm{e}^{3\alpha}}{\kappa^2} + \mathcal{H}_\mathrm{m}\right)\right]\Psi \equiv \widetilde{\mathcal{H}}\Psi, \tag{49}$$

where $\widetilde{\mathcal{H}}$ is the pseudo-Hamiltonian.

Operator-ordering ambiguities have been ignored in the double derivative $\partial^2/\partial\xi^2$, which is anyway negligible far from the Planck era at times $t \gg t_\mathrm{P} \equiv G_\mathrm{N}^{1/2}$, where classically $(1/4\beta a^3)\pi_\xi^2 = (\beta/t^2)\xi^2 a^3 \ll (3/8\pi t_\mathrm{P}^2)\xi^2 a^3$. Thus, in the semi-classical approximation, from eqs. (12) and (27) we have

$$\widetilde{\mathcal{H}} \approx \frac{3\xi^2\mathrm{e}^{3\alpha}}{\kappa^2} + \mathcal{H}_\mathrm{m} \approx 2M_\mathrm{U}. \tag{50}$$

It is important to note that the application of this method results in an effective doubling of the matter Hamiltonian. Although the gravitational energy itself is negative, the classical Friedmann equation (12) reading equivalently

$\mathcal{H}_\mathrm{g} \equiv -3\xi^2 \mathrm{e}^{3\alpha}/\kappa^2 = -\mathcal{H}_\mathrm{m}$, the presence of the higher-derivative terms $\mathcal{R}^2$ has the effect of reversing the sign of $\mathcal{H}_\mathrm{g}$ in expression (50) for $\widetilde{\mathcal{H}}$. That is to say, the pseudo-Hamiltonian is positive semi-definite, even though it is independent of β in this approximation. (In fact $\widetilde{\mathcal{H}} \approx 2M_\mathrm{U}$ for all spatial curvatures $k = 0, \pm 1$.)

For the dust Universe defined by eq. (27), M_U, and hence $\widetilde{\mathcal{H}}$, is constant, so that either directly from the ansatz (47) or by integration of eq. (49), we find that the wave function is

$$\Psi \approx \Psi_0 \exp(-\mathrm{i}\widetilde{\mathcal{H}}t). \tag{51}$$

After performing the Wick rotation $t \to -\mathrm{i}\tilde{t}$, we rewrite eq. (51) in the form

$$\Psi \approx \Psi_0 \exp(-\widetilde{\mathcal{H}}t), \tag{52}$$

which can be interpreted as a Boltzmann distribution at the temperature given by

$$T = 1/2\lambda\tilde{t}. \tag{53}$$

The factor of 2 in the denominator of eq. (53) is the factor of 2 in eq. (50), while the parameter λ is the "form factor" introduced in I. To obtain the temperature measured by the Kodama observer, we have to transform from comoving time $t \equiv x^0$ to the time coordinate x_0. From eq. (9), we see that the local, physical three-dimensional subspace $\mathrm{d}\sigma^2$ defined on the time-slice $\mathrm{d}x_0 = 0$ has compact topology with volume $\tilde{V}_3 = 2\pi^2 r_0^3$ for all values of k. In deriving eq. (53), we have assumed a cut-off at $a = r_0$ corresponding to the flat topology with volume a^3. Conversion to the physical coordinate system requires transformation of the flat volume a^3 to the compact volume $[2\pi^2/(4\pi/3)]a^3 = (3\pi/2)a^3$, implying that $\lambda = 3\pi/2$, as indicated in I, and resulting in eq. (I(E) 35), which is eq. (5).

Thus, if we define $M_\mathrm{U} = \lambda\widetilde{M}_\mathrm{U}$, the exponent in eq. (52) can be rewritten as

$$\widetilde{\mathcal{H}}\tilde{t} = 2\lambda\widetilde{M}_\mathrm{U}\tilde{t} = \widetilde{M}_\mathrm{U}/T_0 = \widetilde{S}, \tag{54}$$

where $\widetilde{S}$ is the entropy referred to the volume $\widetilde{V}_3$, in agreement with eq. (22), setting $\widetilde{F} = 0$, and with the probabilistic interpretation of Ψ.

4 The comoving observer

Central to our discussion is the notion of a comoving observer. Pauli [24] noted, in the context of special relativity, that the three-surfaces of simultaneity for an observer moving with the volume element are given by the equation

$$u_i \mathrm{d}x^i = 0, \tag{55}$$

where u_i is the unit four-velocity of the observer and $u_i u^i = 1$. This idea carries over to general relativity, when the metric g_{ij} typically has non-vanishing off-diagonal space-time components $g_{0\alpha}$, $g^{0\alpha}$, as a consequence of which eq. (55) admits two natural solutions: we can choose

$$u_i = {}_*u_i = (1,0,0,0)/\sqrt{g^{00}} \quad \text{or} \quad u^i = {}^*u^i = (1,0,0,0)/\sqrt{g_{00}}\,, \tag{56}$$

corresponding to the coordinate basis vectors

$${}_*u = {}_*u_i \partial/\partial x_i = \partial/\sqrt{g^{00}}\partial x_0 \quad \text{or} \quad {}^*u = {}^*u^i \partial/\partial x^i = \partial/\sqrt{g_{00}}\partial x^0, \tag{57}$$

respectively.

Note that $g_{00} = g^{00} = 1$ in the comoving coordinate system $(t, \tilde{r}, \theta, \varphi)$ defining the line element (3), for which ${}_*u_i = {}^*u^i = (1,0,0,0)$, while from eqs. (13), $g^{00} = 1$ in the coordinate system (t, r, θ, φ), when ${}_*u_i = (1,0,0,0)$. By comparison with eq. (19), we see that the Kodama vector can be written as [9]

$$K = \left(1 - r^2/r_0^2\right)^{1/2} {}^*u. \tag{58}$$

A separate formalism can be constructed around each choice defined by eqs. (56). In the case $u^i = {}^*u^i$, eq. (55) yields $\mathrm{d}x_0 = 0$ and the spatial three-volume is the physical volume $V_3 = \int \sqrt{\gamma}\mathrm{d}^3x$ measured by a local observer, defined by eqs. (7), (8) —see Landau and Lifschitz [25].

The division into space and time can be put on a covariant basis via the tensor h_{ij}, defined by [26,27]

$$h_i^j = \delta_i^j - u_i u^j, \tag{59}$$

which has the property of projecting all four-tensors into the three-space orthogonal to the time lines u^i, since $h^i_i = 3$ and $h_{ij}u^j = 0$. Indices are raised and lowered by means of the four-metric g_{ij}, g^{ij}, the line element taking the form quite generally, provided that u^i can be defined,

$$\mathrm{d}s^2 = \left(u_i \mathrm{d}x^i\right)^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{60}$$

When $u^i = {}^*u^i$, we find that

$$h_{00} = h_{0\alpha} = 0, \qquad h^{00} = -\gamma_\alpha\gamma^\alpha, \qquad h^{0\alpha} = -\gamma^\alpha \qquad h_{\alpha\beta} = -\gamma_{\alpha\beta}, \qquad h^{\alpha\beta} = -\gamma^{\alpha\beta}, \tag{61}$$

where the three-vector $\boldsymbol{\gamma}$ is defined as $\gamma_\alpha = -g_{0\alpha}/g_{00}$, $\gamma^\alpha = -g^{0\alpha}$, indices α, β being raised and lowered by the three-metric $\gamma_{\alpha\beta}$, $\gamma^{\alpha\beta} \equiv -g^{\alpha\beta}$.

In the alternative case $u_i = {}_*u_i$, the quantity $u_i\mathrm{d}x^i = \mathrm{d}x^0/\sqrt{g^{00}}$, and we obtain the formalism due to Arnowitt, Deser and Misner [28] (ADM) which is based on the holonomic time coordinate x^0. Thus,

$$h^{00} = h^{0\alpha} = 0, \qquad h_{00} = -N_\alpha N^\alpha, \qquad h_{0\alpha} = -N_\alpha \qquad h_{\alpha\beta} = -{}^3g_{\alpha\beta}, \qquad h^{\alpha\beta} = -{}^3g^{\alpha\beta}, \tag{62}$$

where the three-vector $\boldsymbol{N}$ is defined as $N_\alpha = -g_{0\alpha}$, $N^\alpha = -g^{0\alpha}/g^{00}$, indices α, β being raised and lowered by the three-metric ${}^3g_{\alpha\beta} \equiv -g_{\alpha\beta}$, ${}^3g^{\alpha\beta} \equiv -g^{\alpha\beta} + g^{0\alpha}g^{0\beta}/g^{00}$, $g^{00} = 1/N^2$ and $g_{00} = N^2 - N_\alpha N^\alpha$.

Correspondingly, the line element (60) can be written as eq. (7),

$$\mathrm{d}s^2 = g_{00}^{-1}\left(\mathrm{d}x_0\right)^2 - \gamma_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta, \tag{63}$$

or as

$$\mathrm{d}s^2 = N^2\left(\mathrm{d}x^0\right)^2 - {}^3g^{\alpha\beta}\mathrm{d}x_\alpha\mathrm{d}x_\beta. \tag{64}$$

The local physical three-space $\mathrm{d}\sigma^2$ defined by eq. (8) is the chronometrically invariant subspace in the sense of Zel'manov [29], being invariant on the time-slice $\mathrm{d}x_0 = 0$ under the group of coordinate transformations

$$x^0 \to x'^0(x^i), \qquad x^\alpha \to x'^\alpha(x^\beta). \tag{65}$$

While the ADM 3-space ${}^3g^{\alpha\beta}$ is defined on the time-slice $\mathrm{d}x^0 = 0$.

5 The Wheeler-DeWitt equation on the apparent horizon

The derivation of the temperature T_0 of the apparent horizon given by eq. (5), in paper I for the heterotic superstring theory and more generally in sect. 3 above, by means of the quantum-cosmological Wheeler-DeWitt equation for the wave function Ψ of the Friedmann Universe generated by a dust source, is based upon the line element (3) expressed in the coordinates $(t, \tilde{r}, \theta, \varphi)$, involving a transformation to coordinates $(t, r \equiv a\tilde{r}, \theta, \varphi)$ at the end of the calculation to realize the horizon hypothesis.

It is also possible, however, to formulate the Wheeler-DeWitt equation directly on the apparent horizon, starting from the line element (6) expressed in the coordinates (t, r, θ, φ) (in which the metric is non-singular on the horizon, since $1 - kr^2/a^2 = 1 - r^2/r_0^2 + \xi^2r^2 > 0\,\forall\, r \le r_0$), as we have shown in refs. [30,31] regarding the Schwarzschild black hole and in ref. [31], hereafter called paper II, vis-à-vis the cosmological metric (6).

The line element is written in the general form

$$\mathrm{d}s^2 = \tilde{h}_{ab}\mathrm{d}x^a\mathrm{d}x^b - \phi^2\mathrm{d}\Omega^2, \tag{66}$$

where $a, b = 0, 1$ and the two-metric $\tilde{h}_{ab}$ is parametrized in the ADM formalism due to Hajicek [32], which in the signature $(+---)$ reads (see eq. (II18))

$$\tilde{h}_{ab} = \begin{pmatrix} \tilde{\alpha}^2 - \tilde{\beta}^2/\tilde{\gamma} & -\tilde{\beta} \\ -\tilde{\beta} & -\tilde{\gamma} \end{pmatrix}, \qquad \tilde{h}^{ab} = \begin{pmatrix} 1/\tilde{\alpha}^2 & -\tilde{\beta}/\tilde{\alpha}^2\tilde{\gamma} \\ -\tilde{\beta}/\tilde{\alpha}^2\tilde{\gamma} & -1/\tilde{\gamma} + \tilde{\beta}^2/\tilde{\alpha}^2\tilde{\gamma}^2 \end{pmatrix}, \tag{67}$$

while $\phi \equiv r$. The quantities N, N_1, N^1 defined from eqs. (62) are

$$N = \tilde{\alpha}, \qquad N_1 = \tilde{\beta}, \qquad N^1 = \tilde{\beta}/\tilde{\gamma}. \tag{68}$$

Applied to the problem of the radiating Schwarzschild space-time, the metric defined by eqs. (67) is that of Vaidya [33,34] (see also Lindquist *et al.* [35]), where

$$\tilde{\alpha} = (1 + 2M/r)^{-1/2}, \qquad \tilde{\beta} = 2M/r, \qquad \tilde{\gamma} = 1/\tilde{\alpha}^2 = 1 + 2M/r. \tag{69}$$

In the cosmological case, the parameters $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are given by eqs. (II59),

$$\tilde{\alpha} = 1, \qquad \tilde{\beta} = -\xi r/(1 - kr^2/a^2), \qquad \tilde{\gamma} = 1/(1 - kr^2/a^2). \tag{70}$$

Full details of the analysis are contained in paper II, the chief result of which, following the method of Tomimatsu [36], is the Wheeler-DeWitt equation for the wave function Φ on the apparent horizon, given by eq. (II93), correcting a minus sign,

$$\mathrm{i}\frac{\partial \Phi}{\partial \phi} = \left(-\frac{1}{4\pi\phi^2}\frac{\partial^2}{\partial \zeta^2} + V \right)\Phi. \tag{71}$$

Here, ζ is a matter field and the potential $V \equiv V^0 \equiv V^1$ is defined, for the de Sitter space-time generated by a cosmological constant Λ, by eqs. (II88), correcting the sign of $\tilde{\lambda}'$, where $\tilde{\lambda} \equiv \tilde{\beta}/\tilde{\alpha}\sqrt{\tilde{\gamma}} = -\xi r/(1 - kr^2/a^2)^{1/2}$, as

$$V^0 = 1 + \sqrt{\tilde{\gamma}}\left(\frac{1}{\sqrt{\tilde{\gamma}}}\right)'\phi + \frac{1}{2}\tilde{\gamma}\left(\Lambda\phi^2 - 1\right), \qquad V^1 = 1 + \left[\sqrt{\tilde{\gamma}}\left(\frac{1}{\sqrt{\tilde{\gamma}}}\right)' - \tilde{\lambda}'\right]\phi, \tag{72}$$

and where $' \equiv \mathrm{d}/\mathrm{d}r$.

Setting $\Lambda = 3/r_0^2$, it is straightforward to show from eqs. (70) that

$$V^0 = V^1 = 2 \tag{73}$$

on the apparent horizon for all spatial curvatures $k = 0, \pm 1$. Now the potential V^1, defined as part of the momentum $\mathcal{H}^1$, does not contain Λ, suggesting that eqs. (73) remain true for *any* Friedmann space-time. To prove this we need to establish two results: firstly, that Λ can be replaced by $3/r_0^2$ for any perfect-fluid source density in the potential V^0, which forms part of the Hamiltonian $\mathcal{H}^0$; and secondly, that V^1 is then unchanged.

In fact, although Λ originates in the Lagrangian of eq. (II53) as a cosmological constant, it occurs in eqs. (72) as a multiple of the energy density, $8\pi\rho$, and from eqs. (11), (12), we see indeed that $8\pi\rho = 3/r_0^2$ for any perfect fluid. While the momentum constraint $\mathcal{H}^1 = 0$, of which V^1 forms a part, is classically the $\binom{0}{1}$ component of the Einstein equations —see p. 236 of ref. [28]. And since the unit four-velocity is given by the first of eqs. (56), $u_i = {}_*u_i$, in the ADM space-time convention, then $u_1 = 0$, implying that V^1 is defined purely geometrically, irrespective of the matter source, for which

$$T^0_1 \equiv (\rho + p)u_1 u^0 = 0. \tag{74}$$

(See also sect. 7 below.)

Thus, when the matter field is ignorable, eq. (71) can be written in the approximate form

$$\mathrm{i}\frac{\partial \Phi}{\partial \phi} \approx V\Phi, \tag{75}$$

where the potential V is constant on the apparent horizon. The Euclideanization argument of paper II, sect. 7, then implies that the temperature obtained previously as eq. (II101) for the de Sitter space-time, namely

$$T = 1/2\pi r_0, \tag{76}$$

is valid for any perfect-fluid source, including in particular the cosmic dust considered in paper I, and for all spatial curvatures $k = 0, \pm 1$, in agreement with the result found in refs. [3,4] as eq. (5).

6 Phenomenology

At the present epoch, the cosmic temperature defined by eq. (5) is far below the limit of observational detection, for numerically

$$T_0 = 3.940 \times 10^{-30} h\,\mathrm{K}, \tag{77}$$

where the Hubble parameter today is $\xi_0 = 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, some thirty orders of magnitude less than the temperature $T_c \approx 2.73\,\mathrm{K}$ of the cosmic microwave background radiation. The possibility of experimental verification would improve for a hypothetical observer travelling backwards in time, since $T_0 \sim \xi \sim t^{-1}$ while $T_c \sim a^{-1} \sim t^{-1/2}$ in a radiation-dominated universe, but even then the two effects do not become comparable until the Planck era at $t \sim 10^{-43}$ s.

In energetic terms, $k_B T_0 \sim 2\pi\hbar\nu_0$, where the frequency $\nu_0 = c/r_0$, that is to say, the thermal fluctuations resulting from the presence of the apparent horizon occur at approximately the same level as the zero-point quantum fluctuations of a harmonic oscillator with wavelength $\lambda \sim r_0$.

The origin of the horizon thermal effect is quite different, however, as explained by Mamaev *et al.* [37] in their analysis of the Friedmann space-time in the vacuum limit $\xi = 0$, $k = +1$, for which, from eq. (11), $r_0 = a$. It was found in ref. [37] not only that created pairs of virtual bosonic particles are characterized by temperature $T = 1/2\pi a$, in accordance with eq. (5), but also that the spectrum is exactly Planckian for massless particles, or that of a relativistic Bose gas with zero chemical potential for massive particles, the effect being due to the non-Euclidean spatial sections associated with the finite distance r_0 to the apparent horizon.

In the vacuum theory, r_0 is only real and finite if $k = 1$. The spatial curvature is produced by matter with the stringy equation of state $\gamma' = 2/3$ and density $\rho = 3/8\pi a^2$, following from eq. (12).

In the non-vacuum case $\xi^2 \neq 0$ considered in the present paper, however, the matter source is a perfect fluid with arbitrary γ'. The horizon distance r_0 can then be real and finite for all three spatial curvatures $k = 0, \pm 1$, giving rise to the non-zero temperature (5), essentially because of the closed topology of the three-dimensional physical subspace $d\sigma^2$ defined by eq. (9) on the slice $dx_0 = 0$.

Although unobservable at the present day, the result of a non-vanishing temperature (77), given fundamentally by eq. (5), is clearly of theoretical importance in linking thermal and gravitational cosmic phenomena. The derivation of eq. (5) in sect. 5 via the Wheeler-DeWitt equation formulated on the apparent horizon proceeds from the Lagrangian for the Einstein-Hilbert theory alone coupled to matter,

$$\mathcal{L} = \sqrt{-g}\left(-\frac{R}{2\kappa^2} + L_m\right), \tag{78}$$

without the inclusion of any higher-derivative gravitational terms $\mathcal{R}^n, n \geq 2$. While the previous derivation in sect. 3 from the Wheeler-DeWitt equation for the Friedmann dust Universe requires the additional existence of a quadratic higher-derivative term, defined by eqs. (38) and (39) as

$$\mathcal{L}^{(2)} = \sqrt{-g}\mathcal{R}^2 = \beta\ddot{\alpha}^2 e^{3\alpha}, \tag{79}$$

this being necessary to obtain the wave equation in the Schrödinger form (49).

The potential from which T_0 is calculated, however, is independent of β, which should be positive for reasons of causality, but occurs only in the kinetic term $-(1/4\beta e^{3\alpha})\partial^2\Psi/\partial\xi^2$ in eq. (49), that can be ignored at sufficiently large radius a (that is, sufficiently large time t). In this sense, it is the Einstein-Hilbert term $-R/2\kappa^2$ which actually determines T in both derivations.

7 The matter Lagrangian

Let us note that the matter source is generally specified through its constituent fields, and is then particularly suited to description by the ADM space-time decomposition, in a background Friedmann space-time. Consider first the dust Universe assumed in sect. 3. In this case L_m can be defined via a scalar dust field χ of mass m, as shown by Salopek *et al.* [38]. For a perfect fluid, L_m is given by the pressure p, provided that $p \neq 0$. Dust is characterized by vanishing pressure, however, and therefore it is necessary to specify the source by the addition of a constraint, enforced by a Lagrange multiplier l, which is just the dust density ρ, up to a numerical factor of $(-1/2)$.

Thus, in the metric signature $(+---)$, we find that [38]

$$L_{\text{dust}} = l\left(1 - g^{ij}\chi_{,i}\chi_{,j}/m^2\right), \tag{80}$$

which yields the constraint equation

$$\frac{\partial\mathcal{L}_{\text{dust}}}{\partial l} = \sqrt{-g}\left(1 - g^{ij}\chi_{,i}\chi_{,j}/m^2\right) = 0. \tag{81}$$

It is therefore possible to define a unit time-like vector

$$u_i = \chi_{,i}/m, \tag{82}$$

satisfying $u_i u^i = 1$. The energy-momentum tensor is

$$T_{ij} = \rho u_i u_j, \tag{83}$$

and taking into account the constraint (81), the Hamiltonian is

$$H = \dot{\chi}\frac{\partial L_{\rm dust}}{\partial \dot{\chi}} - L_{\rm dust} = \rho u_0 u^0 = T_0^0, \tag{84}$$

since $L_{\rm dust} = 0$.

Now if χ is generating the Friedmann space-time (3), it can only depend upon the time coordinate t, assuming that $k = 0$, in which case $\chi = \chi(t)$ and from eq. (82), $u_i = {}_*u_i$, in the notation of eqs. (56), corresponding to the ADM convention. Thus, $u_\alpha = 0$ and $T_\alpha^0 = 0$, as required by eq. (74).

When the pressure is non-vanishing, it is also possible to describe the matter by means of a scalar field, as we have shown [39], although the resulting theory is non-linear unless $\gamma' = 2$. The matter Lagrangian is defined in terms of a scalar field η by eq. (76) of ref. [39],

$$L_{\rm m} = \frac{1}{2}(\gamma' - 1)\left[(\nabla\eta)^2\right]^{\gamma'/2(\gamma'-1)} \equiv (\gamma' - 1)\,\rho = p, \tag{85}$$

from which we derive the energy-momentum tensor

$$T_{ij} = \frac{1}{2}\gamma'\left[(\nabla\eta)^2\right]^{[\gamma'/2(\gamma'-1)]}\left[\gamma' u_i u_j - (\gamma' - 1) g_{ij}\right], \tag{86}$$

where the unit time-like four-velocity is

$$u_i = \eta_{,i}/\sqrt{(\nabla\eta)^2}\,. \tag{87}$$

If the matter Lagrangian (85) is the source of the Friedmann space-time (3), then $\eta = \eta(t)$, so that $u_i = {}_*u_i$ and consequently again

$$T_\alpha^0 = \frac{1}{2}\gamma'\left[(\nabla\eta)^2\right]^{[-(\gamma'-2)/2(\gamma'-1)]}\eta_{,\alpha}\eta_{,j}g^{0j} = 0, \tag{88}$$

as required by eq. (74), while the Hamiltonian is

$$T_0^0 = \frac{1}{2}\left[g^{00}\dot{\eta}^2\right]^{\gamma'/2(\gamma'-1)} = \rho. \tag{89}$$

It therefore apppears that the matter part of the momentum constraint vanishes for any perfect fluid,

$$\mathcal{H}^\alpha_{\rm matter} \equiv T_\alpha^0 = 0, \tag{90}$$

as stated previously in sect. 5.

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