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Dynamics of solitary-wave structures in one-dimensional Gross-Pitaevskii equation with distributed coefficients

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The first author, Emmanuel Kengne, dedicates this work to Mr. Tchango Bonaventure

Abstract. Motivated by recent experimental investigations in the context of matter wave solitons in Bose-Einstein condensates (BECs), we consider the 1+1 Gross-Pitaevskii equation with complex time-varying harmonic potential, and time-varying cubic and quintic nonlinearities. By performing a modified lens-type transformation for the one-dimensional GP equation, we present one and/or two parameter exact analytical solutions which describe the propagation of bright, kink, and dark solitary waves on the vanishing continuous wave (cw) background. Based on exact analytical solutions of the GP equation, we investigate analytically the dynamics of matter-wave solitons in the BEC systems. Our studies show that the solitons' amplitude depends on both the scattering length and the feeding/loss term of the potential while their motion depends on the external trapping potential and solution parameters.

1 Introduction

First realized experimentally in 1995 for rubidium [1], lithium [2, 3], and sodium [4], Bose-Einstein condensates provide unique opportunities for exploring quantum phenomena on a macroscopic scale. At absolute zero temperature, the properties of a condensate are usually described by the time-dependent, nonlinear, mean-field Gross-Pitaevskii equation [5] with nonlinear terms that describe the interatomic interactions. The s-wave scattering length, $a_s(t) = a[1+\Delta/(B_0-B(t))]$ (where $B(t)$ is the time-dependent externally applied magnetic field, Δ is the width of resonance and B_0 is the resonant value of the magnetic field), plays an important role in the description of an atom-atom interaction at ultralow temperatures. The magnitude and sign of the s-wave scattering length, can be tuned to any value, small or large, negative or positive by applying an external magnetic field. The presence of an attractive interaction ($a_s(t) < 0$) between the atoms has a profound effect on the stability of a BEC, since a large enough attractive interaction will cause the BEC to become unstable and collapse in some way. The two-body interaction, corresponding to a cubic nonlinear term in the GP equation, has been reported to be generally the dominant one [6] and can be described by a single parameter (scattering length) where the effects of the three-body interaction are negligible. However, the three-body interaction (quintic nonlinear term of the GP equation) can start to play an important role if the atom density is considerably high [7–9]. In the simple case of the cigar-shaped trapping potential, the GP equation can be integrated out, leading to the quasi one-dimensional (1-D) dimensionless equation

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}+g\left|\psi\right|^2\psi+\chi_0\left|\psi\right|^4\psi-V(x,t)\psi=0, \tag{1}$$

where t and x are the temporal and spatial coordinates measured in harmonic-oscillator units $1/\omega_\perp$ and $a_\perp=\sqrt{\hbar/m\omega_\perp}$, respectively. Here, $a_\perp$ and $a_0=\sqrt{\hbar/m\omega_0}$ are linear oscillator lengths in the transverse and cigar-axis directions, respectively. $\omega_\perp$ and ω_0 are corresponding harmonic oscillator frequencies, and m is the atomic mass. Parameters g and χ_0 are the strengths of time-dependent two-body and three-body interatomic interactions, respectively. These parameters are negative for repulsive interatomic interactions (or defocusing nonlinearities) and positive for attractive

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ones (focusing nonlinearities). The parameter of the cubic nonlinearities is given by $g = -\frac{2a_s}{3a_B}$, where a_B is the Bohr radius [10–12]. Even though the strength χ_0 of the three-body interaction could be a complex quantity, one can safely neglect the imaginary part which represents the three-body loss when the density is not too high or the experimental period is not too long. Moreover, the three-body loss would be more pronounced in higher-dimensional BECs as compared to quasi 1-D BECs. Because the three-body losses are dissipative terms, we must neglect them to conserve the energy and the total number of atoms from the viewpoint of integrability of the dynamical system [13,14]. Usually the strength χ_0 of the three-body interaction is very small when compared with strength g of the two-body interaction, as pointed out by Gammal [14].

In our studies, we consider BECs with time-dependent scattering length in a parabolic background with a complex potential. For a more general study, the two terms are taken to be time dependent. Hence the potential is taken in the form [15,16]

$$V(x,t) = k(t)x^2 + i\gamma_0(t), \tag{2}$$

where $k(t)$ is the strength of the magnetic trap, can be positive (confining potential) or negative (repulsive potential), and $\gamma_0(t)$ is a small parameter related to the feeding ($\gamma_0 > 0$) or loss ($\gamma_0 < 0$) of atoms in the condensate resulting from the contact with the thermal cloud and three-body recombination [16–18]. Because $|k| = \frac{\omega_0^2}{2\omega_\perp^2} \ll 1$, parameter k expresses the trapping frequency in the x-direction [19,20]. It should be pointed out that the s-wave scattering length a_s as well as the trapping frequency in the elongated axis ω_0 can be functions of time t (while in deriving eq. (1) $\omega_\perp$ is kept constant) [21,22]. When $\gamma_0(t) \neq 0$ (presence of the effect of gain/loss of atoms), the GP equation (1) becomes a nonconservative system and hence there is no soliton in the conventional sense. However, as we will see, one can still look for nonautonomous solitons by suitably tailoring the gain/loss of atoms.

The purpose of this work is to use eq. (1) for investigating the dynamics of matter-wave solitons of BECs with time-dependent scattering length in a parabolic background with a complex potential. Eventhough the generation of matter-wave solitons of BECs with both two- and three-body interactions in a complex potential has been investigated [23], the integrability of the associated model has not been spelt out in detail. To obtain a flexible solution which is capable of solving the above problem is very desirable. Traditionally, people relied too much on numerical approaches for investigating the dynamics of matter-wave solitons of BECs. This may be necessary when the analytical solutions were not available. But if both analytical and numerical solutions can be obtained for the same issue, the analytical one is often preferred. Except for its simplicity being used to compile computer codes, the analytical solution is very attractive since its efficiency depends weakly on the dimensions of the problem, in contrast to the numerical methods. In the absence of the potential term, *i.e.*, when $k(t) = 0$ and $\gamma_0(t) = 0$, eq. (1) becomes a cubic-quintic nonlinear Schrödinger equation and methods of finding its special solitonlike solutions for constant a g can be found in refs. [24–27]. In the absence of the effects of gain/loss of atoms and for time-independent scattering length ($g(t)$ = constant) and strength of the magnetic trap ($k(t)$ = constant), Kumar *et al.* [28] derived the associated Lax-pair and generate the bright soliton solutions of eq. (1). In the absence of the three-body interaction, and for time-independent scattering length ($g(t)$ = constant) and strength of the magnetic trap ($k(t)$ = constant), Kengne and Talla [17] used the Darboux transformation to derive exact bright soliton solution of eq. (1). The dynamics of matter-wave solitons in Bose-Einstein condensates with time modulated nonlinearities or/and controlling potential in the absence of the effect of gain/loss of atoms ($\gamma_0(t) = 0$) have been intensively investigated and several analytical procedures have been developed to derive nonautonomous solitons admitting different density profiles [29–37]. In the presence of the effect of gain/loss of atoms ($\gamma_0(t) \neq 0$), similarity transformation mapping method have been used to derive special soliton-like solutions of eq. (1) with potential (2) [38,39].

In this work, we aim to present in the conventional sense the explicit analytical solitonlike solutions of eq. (1) with potential (2) that may describe the dynamics of matter-wave solitary-waves of BECs with time-dependent s-wave scattering length in time-dependent harmonic trapping potential with feeding/loss parameter. The rest of the paper is organized as follows. In sect. 2, we use a modified lens-type transformation to transform the GP equation (1) with potential (2) into an elliptic ordinary differential equation (EODE) and present analytical soliton solutions for eq. (1). Based on the exact analytical solitary wave-like solutions, we show in sect. 3 that the method used in our work is an experimental technique for the generation of soliton patterns in BECs. Finally, we summarize the main results in sect. 4.

2 Solitons of BECs BECs with time-dependent scattering length in time-dependent complex potential with both two- and three-body interactions

In this section, we derive exact analytical soliton solutions of the GP eq. (1) with the complex potential (2). We begin with the following modified lens-type transformation [22]:

$$\psi(x,t) = \frac{1}{\ell(t)} a(T) u(X) \exp\left[\eta(t) + i\left(f(t)x^2 + \frac{K}{\ell(t)}x - \varphi(T)\right)\right], \tag{3}$$

where $\ell(t)$, $T = T(t)$, $\eta(t)$, $f(t)$ are functions of time t, $a(T) \neq 0$ and $\varphi(T)$ are functions of T, $u(X)$ is a function of $X = X(x,t) = x/\ell(t) - KT + K_0$, K and K_0 being two real constants. Inserting eq. (3) into eq. (1) and demanding that

$$\frac{\mathrm{d}f}{\mathrm{d}t} + 2f^2 + k = 0, \tag{4a}$$

$$\frac{1}{\ell}\frac{\mathrm{d}\ell}{\mathrm{d}t} - 2f = 0, \tag{4b}$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{1}{\ell^2} = 0, \tag{4c}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}t} - \gamma_0 = 0, \tag{4d}$$

$$\frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} - f = 0, \tag{4e}$$

yield

$$\left(\frac{\mathrm{d}u}{\mathrm{d}X}\right)^2 = -\frac{2a^4\chi_0}{3\ell^2}\exp[4\eta]u^6 + 2\left(\frac{1}{2}K^2 - \frac{\mathrm{d}\varphi}{\mathrm{d}T}\right)u^2 - a^2 g\exp[2\eta]u^4 + \delta, \tag{5}$$

where $\delta = \delta(t)$ is an arbitrary function of time t. Introducing the new unknown variable $\rho = u^2(X)$ reduces eq. (5) in the following form:

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}X}\right)^2 = \alpha\rho^4 + 4\beta\rho^3 + 6\gamma\rho^2 + 4\delta\rho \overset{\text{def.}}{=} R(\rho), \tag{6a}$$

$$\alpha = -\frac{8a^4\chi_0}{3\ell^2}\exp[4\eta], \qquad \beta = -a^2 g\exp[2\eta], \qquad \gamma = \frac{2}{3}\left(K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}\right) = \frac{2}{3}\left(K^2 - 2\ell^2\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right). \tag{6b}$$

Equation (6a) is a special case of the EODE. We distinguish two cases, the case where $\alpha = 0$ corresponding to BECs with two-body interatomic interactions and the case where $\alpha \neq 0$ associated with BECs with both two- and three-body interactions.

2.1 Case $\alpha = 0$: One-dimensional Gross-Pitaevskii equation with cubic nonlinearity

When $\alpha = 0$, *i.e.*, $\chi_0 = 0$, eq. (5) becomes

$$\left(\frac{\mathrm{d}u}{\mathrm{d}X}\right)^2 = -a^2 g\exp[2\eta]u^4 + \left(K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}\right)u^2 + \delta. \tag{7}$$

Equation (7) is known as the EODE of the first kind. In some special cases, we can derive its exact analytical solitary wave-like solutions.

2.1.1 First class of exact analytical solitary wave-like solutions of eq. (7)

The first class of exact analytical solitary wave-like solutions of eq. (7) is obtained by asking that functions $u = 1/\cosh[\mu X]$, $u = 1/(A + B\cosh^2[\mu X])$ and $u = \tanh[\mu X]$ satisfy eq. (7). This leads to the following bright soliton, solitary wave-like, and kink soliton solutions of eq. (7):

$$u(X) = \frac{1}{\cosh[a\sqrt{g}\exp[\eta]X]}, \qquad 2\frac{\mathrm{d}\varphi}{\mathrm{d}T} + a^2 g\exp[2\eta] - K^2 = 0, \quad g > 0, \tag{8a}$$

$$u(X) = \pm\frac{\exp[-\eta]}{a}\sqrt{\frac{K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}}{g}}\,\frac{1}{1 - 2\cosh^2\left[\frac{1}{2}\sqrt{K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}}X\right]}, \quad g > 0 \quad \text{and} \quad K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T} > 0, \tag{8b}$$

$$u(X) = \pm\tanh[a\sqrt{-g}\exp[\eta]X], \quad g < 0, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}T} + a^2 g\exp[2\eta] - \frac{1}{2}K^2 = 0, \tag{8c}$$

respectively. Solutions (8a) and (8b) are obtained with $\delta = 0$, while solution (8c) is obtained with $\delta \neq 0$.

2.1.2 Second class of exact analytical solitary wave-like solutions of eq. (7)

The second class of exact analytical solitary wave-like solutions of eq. (7) is obtained by imposing to eq. (7) to have a special form. If $g(t) < 0$ and $\gamma_0(t)$ is so that function $f = -\frac{1}{2}\frac{2g\gamma_0 + \mathrm{d}g/\mathrm{d}t}{g}$ satisfies the Riccati equation (4a), then by choosing $a(t)$ and $\varphi(t)$ from conditions $4ga^2 + \exp[-2\eta] = 0$ and $2\frac{\mathrm{d}\varphi}{\mathrm{d}T} - K^2 - \frac{1}{2} = 0$ (*i.e.*, $2\ell^2\frac{\mathrm{d}\varphi}{\mathrm{d}t} - K^2 - \frac{1}{2} = 0$), then eq. (7) for $\delta = \frac{1}{4}$ admits the kink solitary wave-like solution

$$u(X) = \frac{\sinh[X]}{1 + \cosh[X]}\,. \tag{8d}$$

2.1.3 Third class of exact analytical solitary wave-like solutions of eq. (7)

To obtain the third class of exact analytical solitary wave-like solutions of eq. (7), we use the Weierstrass' elliptic function method [40, 41] (see the below appendix) and, respectively, obtain the following bright, kink, and dark solitary wave-like solutions:

$$u(X) = \pm\frac{1}{a}\sqrt{\frac{K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}}{g}}\exp[-\eta]\left(1 - \frac{12}{1 + 6\cosh^2\left[\frac{1}{2}\sqrt{K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}}X\right]}\right), \quad g > 0 \quad \text{and} \quad K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T} > 0, \tag{9a}$$

$$u(X) = \frac{1}{2a}\sqrt{\frac{K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}}{2g}}\exp[-\eta]\frac{4 + 3\sinh\left[2\sqrt{\frac{\mathrm{d}\varphi}{\mathrm{d}T} - PK^2}X\right]}{5 + 3\cosh\left[2\sqrt{\frac{\mathrm{d}\varphi}{\mathrm{d}T} - PK^2}X\right]}, \quad g < 0 \quad \text{and} \quad \frac{\mathrm{d}\varphi}{\mathrm{d}T} - PK^2 > 0, \tag{9b}$$

and

$$u(X) = \pm\frac{\exp[-\eta]}{2}\sqrt{\frac{K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}}{2g}}\frac{\sqrt{2 + \sinh^2[\sqrt{3e_1}X] + \sinh^4[\sqrt{3e_1}X] - \sinh[2\sqrt{3e_1}X]}}{2 + \sinh^2[\sqrt{3e_1}X]}, \quad g < 0 \quad \text{and} \quad \frac{\mathrm{d}\varphi}{\mathrm{d}T} - \frac{K^2}{2} > 0, \tag{9c}$$

where $e_1 = (2\frac{\mathrm{d}\varphi}{\mathrm{d}T} - K^2)/6$. Each of solutions (9a), (9b), and (9c) contains the real constant K and the functional real parameter $\varphi(T)$. It should be noted that solution (9c) is obtained through solving the EODE (6a) with $\alpha = 0$.

2.2 Case $\alpha \neq 0$: One-dimensional Gross-Pitaevskii equation with cubic-quintic nonlinearity

A large set of nonnegative exact analytical soliton solutions of eq. (6a) can be found by choosing either $\gamma = 0$ or $\delta(t) = 0$. If we choose $\gamma = 0$ (*i.e.*, if $\varphi(T)$ is taken from condition $\frac{\mathrm{d}\varphi}{\mathrm{d}T} - \frac{K^2}{2} = 0$), then the conditions for soliton solutions read $\alpha > 0$ (meaning that $\chi_0 < 0$) and $\delta = -(\frac{8}{\alpha})^2(\frac{\beta}{3})^3 \neq 0$. Therefore, $\rho_0 = 0$ is a simple root of polynomial $R(\rho)$. To this simple root corresponds the following nonnegative bright solitary wave-like solution of eq. (6a) (see the below appendix and refs. [40, 41])

$$\rho(X) = -\frac{\ell^2 g}{a^2\chi_0}\frac{\exp[-2\eta]}{\left[1 + 3\cosh^2\left[\frac{g\ell}{\sqrt{-\chi_0}}X\right]\right]}, \quad \chi_0 < 0, \quad g > 0. \tag{10a}$$

Now, let us consider the situation when $\delta(t) = 0$ and $\gamma \neq 0$. In this case, polynomial $R(\rho)$ admits two simple roots if $\ell^2 g^2 + 4\chi_0\gamma > 0$. According to refs. [40, 41] (see also the below appendix), exact nonnegative bright solitary wave-like solution of (6a) under condition $\delta(t) = 0$ can be given by

$$\rho(X) = \frac{3\gamma\rho_0\cosh^2\left[\sqrt{\frac{3}{2}\gamma}X\right]}{6\gamma + 2\beta\rho_0 + 3\gamma\cosh^2\left[\sqrt{\frac{3}{2}\gamma}X\right]}, \quad \text{if} \quad \gamma > 0 \quad \text{and} \quad \ell^2 g^2 + 4\chi_0\gamma > 0, \tag{10b}$$

where $\rho_0 = -\frac{6\ell(\ell g \pm \sqrt{\ell^2 g^2 + 4\chi_0\gamma})\exp[-2\eta]}{8a^2\chi_0}$. For $\rho(X)$ to be nonnegative, it is necessary and sufficient that $9\gamma + 2\beta\rho_0 > 0$ and $\rho_0 > 0$. Condition $\rho_0 > 0$ is satisfied if and only, if either $\chi_0 < 0$, $g > 0$ and $\rho_0 = -\frac{3\ell(\ell g \pm \sqrt{\ell^2 g^2 + 4\chi_0\gamma})\exp[-2\eta]}{4a^2\chi_0}$

or $\chi_0 > 0$ and $\rho_0 = -\frac{3\ell(\ell g-\sqrt{\ell^2 g^2+4\chi_0\gamma})\exp[-2\eta]}{4a^2\chi_0}$. Therefore parameter K and the functional parameter $\varphi(T)$ must be chosen from

$$K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T} > 0 \qquad \text{and} \qquad 4\chi_0\left(K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}\right) + \ell g\left(\ell g \pm \sqrt{\ell^2 g^2 + 4\chi_0\gamma}\right) < 0,$$

if $\chi_0 < 0$ and $g > 0$, and

$$K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T} > 0 \qquad \text{and} \qquad 4\chi_0\left(K^2 - 2\frac{\mathrm{d}\varphi}{\mathrm{d}T}\right) + \ell g\left(\ell g - \sqrt{\ell^2 g^2 + 4\chi_0\gamma}\right) > 0,$$

if $\chi_0 > 0$.

3 Discussions and results

In this section, we use the exact analytical solution of the GP equation (1) with potential (2) to investigate analytically the dynamics of matter-wave solitons in the one-dimensional BEC system. In our discussions, we mainly use the following three strength of the magnetic trap: i) the constant strength of the magnetic trap $k = -2\lambda^2$ ($\lambda \simeq 0.05$) [22,42], ii) the temporal periodic modulation of strength of the magnetic trap $k(t) = -\frac{m\omega^2}{2}\frac{[m+\sin[\omega t]+m\cos^2[\omega t]]}{(1+m\sin[\omega t])^2}$ with $0 < m < 1$ [43], and the interesting case of time-dependent potential which corresponds to the strength of the magnetic trap $k(t) = (t+t_0)^{-2}/8$ [16, 22], where t_0 is any real constant whose sign is related to the sign of $f(t)$; $t_0 f(t) > 0$ and which essentially determines the width of the trap at time $t = t_0$ according to $k(t)$. Case $t_0 < 0$ describes a BEC in a shrinking trap while case $t_0 > 0$ corresponds to a broadening condensate. Inserting these expressions for $k(t)$ into the Riccati equation (4a) leads to the following particular solutions $f(t) = \pm\lambda$ for constant k, $f(t) = -\frac{m\omega}{2}\frac{\cos[\omega t]}{1+m\sin[\omega t]}$ for the temporal periodic $k(t)$, and $f(t) = (t+t_0)^{-1}/4$ for the last case of k.

3.1 Dynamics of matter-wave solitons in 1-D Gross-Pitaevskii equation with cubic nonlinearity

We start the discussions with the case of BEC systems with two-body interaction. In our discussions, we will distinguish the case when the BECs density does not contain the functional parameter $\varphi(t)$ and the case which $\varphi(t)$ appears explicitly in the BECs density. We limit ourselves to the investigation of dynamics of bright solitons BECs described by the GP equation (1) with external potential (2). Without loss of generality, we focus on the following two cases.

3.1.1 Use of function $u(X)$ given by eq. (8a) (a case of bright soliton with vanishing boundary conditions)

Let $g > 0$ and φ be a function satisfying equation $2\frac{\mathrm{d}\varphi}{\mathrm{d}T} + a^2 g\exp[2\eta] - K^2 = 0$, *i.e.*,

$$2\ell^2\frac{\mathrm{d}\varphi}{\mathrm{d}t} + a^2 g\exp[2\eta] - K^2 = 0. \tag{11}$$

Then, eq. (8a) is a solution of eq. (7) leading to the following solution of the GP equation (1) with potential (2):

$$\psi(x,t) = \frac{\ell_0^{-1}a(t)\exp\left[\eta_0 + \int_0^t[\gamma_0(\tau) - 2f(\tau)]\mathrm{d}\tau\right]\exp[i\theta]}{\cosh\left[\frac{a(t)\sqrt{g}\exp[\eta_0+\int_0^t[\gamma_0(\tau)-2f(\tau)]\mathrm{d}\tau]}{\ell_0}\left(x - \frac{\exp[\int_0^t 2f(\tau)\mathrm{d}\tau]}{\ell_0}\left(K\int_0^t\exp\left[-4\int_0^t f(\tau)\mathrm{d}\tau\right]\mathrm{d}\tau + K_0\right)\right)\right]}, \tag{12}$$

where $\theta(x,t) = f(t)x^2 + \frac{K}{\ell(t)}x - \varphi(t)$, $a(t) = \widetilde{a}_0\exp\left[\int_0^t f(\tau)\mathrm{d}\tau\right]$, $\widetilde{a}_0$, ℓ_0, η_0, K, and K_0 are real constants, and $f(t)$ is a solution of the Riccati equation (4a). Equation (12) is just the bright one-soliton solution for eq. (1). It follows from solution (12) that:

i) The amplitude $\ell_0^{-1}a(t)$ of the bright soliton is time-dependent and is proportional to $\exp[\eta_0+\int_0^t[\gamma_0(\tau)-2f(\tau)]\mathrm{d}\tau]$, while the width is inversely proportional to $\sqrt{g}\exp\left[\eta_0 + \int_0^t[\gamma_0(\tau) - 2f(\tau)]\mathrm{d}\tau\right]$ so that the total number of BEC atoms $\int_{-\infty}^{+\infty}|\psi(x,t)|^2\mathrm{d}x = \frac{2a(t)}{\ell_0}\frac{1}{\sqrt{g}}\exp\left[\eta_0 + \int_0^t[\gamma_0(\tau) - 2f(\tau)]\mathrm{d}\tau\right]$ is time-dependent.

ii) The centre of the bright soliton is $\xi = \frac{\exp[\int_0^t 2f(\tau)\mathrm{d}\tau]}{\ell_0}(K\int_0^t\exp[-4\int_0^t f(\tau)\mathrm{d}\tau]\mathrm{d}\tau+K_0)$, which satisfies the following equation:

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}t^2} + 2k\xi = 0, \tag{13}$$

meaning that the centre of mass of the macroscopic wave packet behaves like a classical particle, and allows one to manipulate the motion of bright solitons in BEC systems by controlling the external harmonic trapping potential. In what follows, we take some classical examples to demonstrate the dynamics of bright solitons in 1-D BEC systems with different kinds of scattering length, harmonic trapping potential, and feeding/loss parameter.

We start with a constant strength of the magnetic trap $k = -2\lambda^2$ ($\lambda \simeq 0.05$) [42]. Solving the Riccati equation (4a) leads to the particular solutions $f(t) = \pm\lambda$. Choosing $g(t) = \exp[\pm\lambda t]$ [44] and $\gamma_0 = \gamma_{00} = \text{const}$ [17] yields

$$\psi(x,t) = \frac{\ell_0^{-1} a_0 \exp[(\gamma_{00} \mp \lambda)t] \exp[i\theta]}{\cosh\left[\ell_0^{-1} a_0 \exp\left[\left(\gamma_{00} \mp \frac{1}{2}\lambda\right) t\right] \left(x - \ell_0 K_0 \exp[\pm 2\lambda t] \mp \frac{K}{2\lambda\ell_0} \sinh[\pm 2\lambda t]\right)\right]},$$

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \mathrm{d}x = \frac{2a_0}{\ell_0} \exp\left[\left(\gamma_{00} \mp \frac{3}{2}\lambda\right) t\right].$$

Therefore, the total number of BEC atoms increases with time (gain of atoms) if $\gamma_{00} > \pm\frac{3}{2}\lambda$, decreases with time (loss of atoms) if $\gamma_{00} < \pm\frac{3}{2}\lambda$, and remains unchanged if $\gamma_{00} = \pm\frac{3}{2}\lambda$. It is seen from the above expression for $\psi(x,t)$ that the bright soliton always has an increase (decrease) in the peak value and a compression (broadening) in its width for the increasing (decreasing) of $g(t)$ in the feeding (loss) regime. The behavior of the bright soliton propagating in the feeding (loss) regime for the decreasing (increasing) of the absolute value of the s-wave scattering length depends on the choice of parameter γ_{00}. According to eq. (13), the velocity for the bright soliton reads $\frac{\mathrm{d}\xi}{\mathrm{d}t} = \ell_0^{-1} K \cosh[2\lambda t] \pm 2\lambda\ell_0 K_0 \exp[\pm 2\lambda t]$. In particular, when $K_0 = 0$ and $K > 0$, soliton velocity increases as the bright soliton propagates along the longitudinal direction due to the repulsive trapping potential.

As the second example, we consider a temporal periodic modulation of the s-wave scattering length [43] with the strengths of time-dependent two-body interatomic interactions $g(t) = 1 + m\sin[\omega t]$ with $1 < m < 1$. We take the strength of the magnetic trap as $k(t) = -\frac{m\omega^2}{2} \frac{[m+\sin[\omega t]+m\cos^2[\omega t]]}{(1+m\sin[\omega t])^2}$. We then use the above particular solution of the Riccati equation (4a), *i.e.*, $f(t) = -\frac{m\omega}{2} \frac{\cos[\omega t]}{1+m\sin[\omega t]}$. For simplicity, we consider a constant feeding/loss parameter $\gamma_0(t) = \gamma_{00}$ [17]. The corresponding soliton solution (12) then takes the form

$$\psi(x,t) = \frac{\frac{a_0}{\ell_0}\sqrt{1 + m\sin[\omega t]} \exp[\gamma_{00} t]}{\cosh\left[\frac{a_0}{\ell_0}(1 + m\sin[\omega t]) \exp[\gamma_{00} t] \left(x - \frac{\ell_0(KT - K_0)}{1+m\sin[\omega t]}\right)\right]} \exp[i\theta], \tag{14}$$

where $T(t) = \ell_0^{-2}\left(\frac{2+m^2}{2}t + \frac{4m}{\omega}\sin^2\frac{\omega t}{2} - \frac{m^2}{4\omega}\sin[2\omega t]\right)$. For the present example, the total number of BEC atoms is $\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \mathrm{d}x = \frac{2a_0}{\ell_0} \exp[\gamma_{00} t]$, so that the feeding and the loss regimes are associated with $\gamma_{00} > 0$ and $\gamma_{00} < 0$, respectively. The BEC density associated with solution (14) is plotted in fig. 1 showing the dynamics (top plots) and the evolution (bottom plots) of a bright soliton in time-dependent harmonic trapping potential with feeding/loss term is shown in fig. 1. Figures. 1(a) and (c) show the dynamics and the evolution of bright soliton in the feeding regime, while figs. 1(b) and (d) show the dynamics and the evolution of the bright soliton in the loss regime. It is seen from plots of fig. 1 that in the feeding (loss) regime of propagation, the wave has an increase (decrease) in the peak value and a compression (broadening) in its width. It is also seen from the bottom plots that the wave trajectories oscillate due to the temporal periodic modulation of both the s-wave scattering length and trapping potential.

In our third and last example, we consider the case of time-dependent potential corresponding to the strength of the magnetic trap $k(t) = (t+t_0)^{-2}/8$ [16], and use the associated particular solution $f(t) = (t+t_0)^{-1}/4$ of the Riccati equation (4a). Because $f(t) > 0$ for large t, we consider a positive t_0. We then we consider, as in the previous example, the temporal periodic modulation of the s-wave scattering length with the nonlinearity parameter $g(t) = 1 + m\sin[\omega t]$ withe $1 < m < 1$. The gain/loss term is assumed to be constant, *i.e.*, $\gamma_0(t) = \gamma_{00}$. We then obtain

$$\psi(x,t) = \frac{\frac{a_0}{\ell_0}\left(\frac{t_0}{t+t_0}\right)^{\frac{1}{4}} \exp[\gamma_{00} t] \exp[i\theta]}{\cosh\left[\frac{a_0}{\ell_0}\left(\frac{t_0}{t+t_0}\right)^{\frac{1}{4}} \exp[\gamma_{00} t]\sqrt{1 + m\sin[\omega t]}\left(x - \ell_0\left(\frac{t+t_0}{t_0}\right)^{\frac{1}{2}}\left(K\frac{t_0}{\ell_0^2}\ln\frac{t+t_0}{t_0} - K_0\right)\right)\right]}, \tag{15a}$$

$$\int_{-\infty}^{+\infty} |\psi(x,t)|^2 \mathrm{d}x = \frac{2a_0}{\ell_0} \frac{\left(\frac{t_0}{t+t_0}\right)^{\frac{1}{4}} \exp[\gamma_{00} t]}{\sqrt{1 + m\sin[\omega t]}}. \tag{15b}$$

We show in fig. 2 the plot of the BEC density (top panel) and the total number of atoms (bottom panel) associated with soliton solution (15a). The top panel of fig. 2 shows the dynamics of bright soliton propagating in (a) the feeding regime and (b) the loss regime. Plots of the top panel show that the bright soliton in the feeding regime has an increase

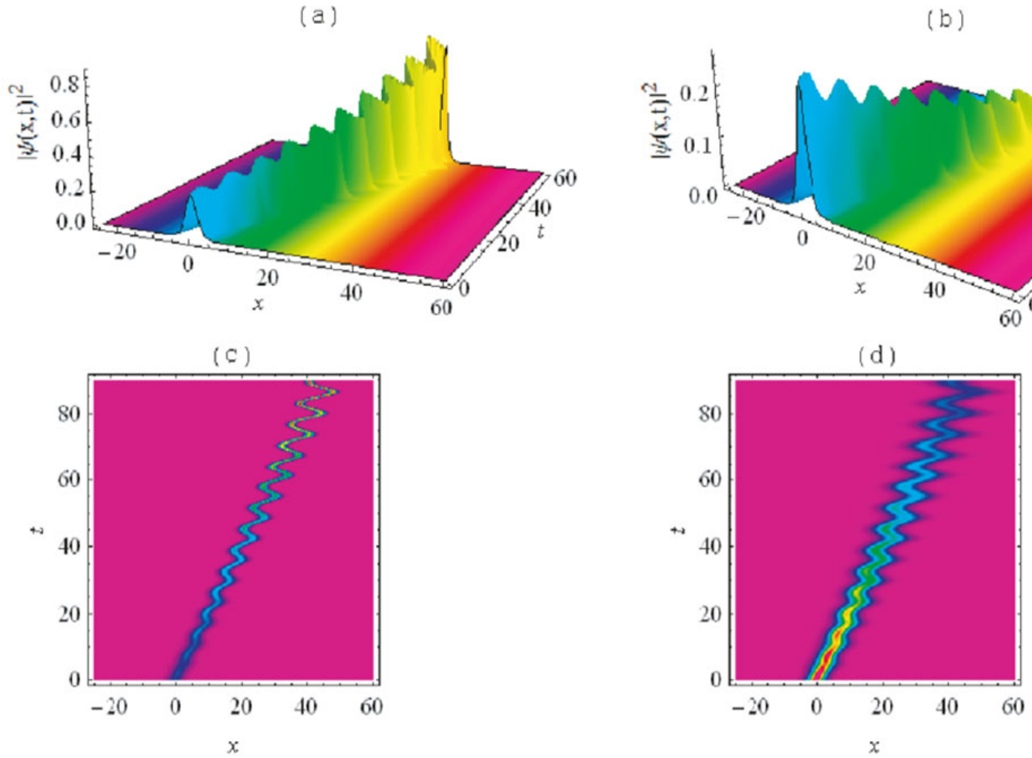

Fig. 1. The dynamics of a bright soliton in a time-dependent harmonic trapping potential with feeding/loss parameter given by eq. (14) with the parameters $a_0 = 0.5$, $K_0 = 0.5$, $\ell_0 = 1$, $K = 0.5$, $m = 0.1$, $\omega = 1$ for different values of γ_{00}. (a), (c): $\gamma_{00} = 0.01$; (b), (d): $\gamma_{00} = -0.01$.

in the peak value, while in the loss regime, the peak value first increases, and then, decreases as the wave propagates. Figure 2(c) shows that in the feeding regime, BEC first losses atoms, and then gradually gains atoms as the time passes. From fig. 2(d), it is seen that in the loss regime, BEC only losses atoms. Figure 2 also shows that the bright soliton peaks and the total number of the BEC atoms oscillate due to the temporal periodic modulation of the s-wave scattering length.

3.1.2 Use of function $u(X)$ given by eq. (9a) (case when the BEC density contains the functional real parameter φ)

Inserting eq. (9a) into transformation (3) under condition $g(t) > 0$ leads to the following exact analytical bright solitary wave-like solution of the GP equation (1) with potential (2):

$$\psi(x,t) = \pm\frac{\sqrt{g^{-1}\left(K^2 - 2\ell^2\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)}}{\ell(t)}\left(1 - \frac{12}{1 + 6\cosh^2\left[\frac{1}{2}\frac{1}{\ell(t)}\sqrt{K^2 - 2\ell^2\frac{\mathrm{d}\varphi}{\mathrm{d}t}}(x - \ell(t)(KT - K_0))\right]}\right)\exp[i\theta], \tag{16}$$

where $\theta = f(t)x^2 + \frac{K}{\ell(t)}x - \varphi(t)$, $f(t)$, $\ell(t)$, $T(t)$ are solution of eqs. (4a), (4b), and (4c), respectively, K_0 is an arbitrary real constant, and constant K and the functional parameter $\varphi(t)$ are to be chosen from condition $K^2 - 2\ell^2\frac{\mathrm{d}\varphi}{\mathrm{d}t} > 0$. In the present example, we aim to show how the functional parameter $\varphi(t)$ can be used to manipulate the soliton motion. It is important to notice the bright solitary wave-like solution (16) does not contain explicitly the feeding/loss parameter $\gamma_0(t)$. Throughout the present analysis, we consider the time-independent harmonic potential which was used in the creation of bright BEC solitons [42] and choose $k(t) = -2\lambda^2$ ($\lambda \simeq 0.05$). For simplicity, we consider the case of an increasing of the absolute value of the s-wave scattering length and take $g(t) = \exp[\lambda t]$. As the solution of the Riccati equation (4a), we use $f(t) = \lambda$. Therefore $\ell(t) = \ell_0\exp[2\lambda t]$ and $T(t) = \frac{1-\exp[-4\lambda t]}{4\ell_0^2\lambda}$, where ℓ_0 is any positive real constant. Figures 3 and 4 show the evolution of the bright solitary wavelike associated with solution (16) for different differential equations leading to the functional parameter $\varphi(t)$. Plots of these two figures show how we may manipulate the soliton motion through the use of the functional parameter $\varphi(t)$. In figs. 3(a) and (b), the solitary

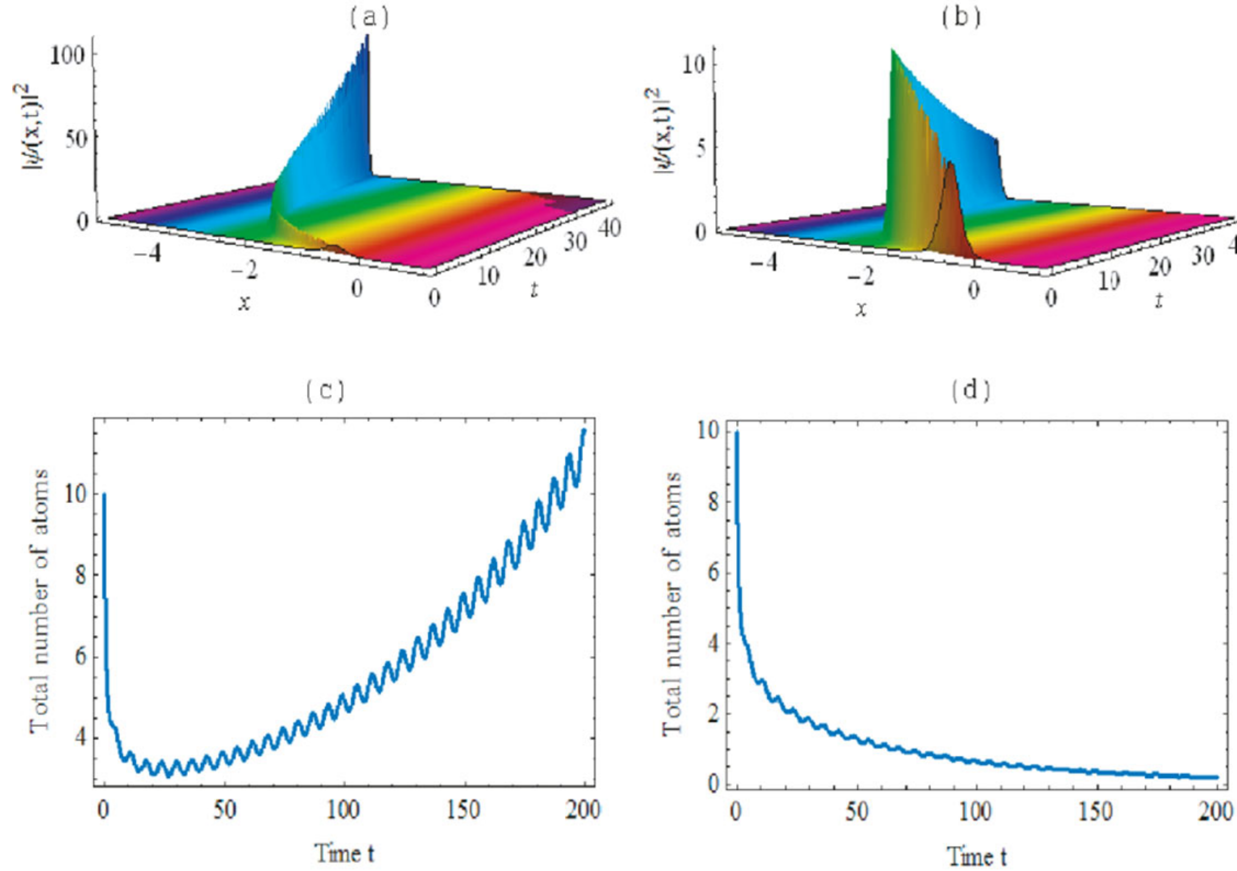

Fig. 2. The dynamics of a bright soliton (top panel) and the total number of atoms (bottom panel) of BEC in time-dependent harmonic trapping potential with feeding/loss term given by eq. (15a) with the parameters $a_0 = 5$, $K_0 = 0.5$, $\ell_0 = 1$, $K = 0.5$, $m = 0.1$, $\omega = 1$, and $t_0 = 0.1$ for different values of γ_{00}. (a): $\gamma_{00} = 0.02$; (b): $\gamma_{00} = -0.02$; (c): $\gamma_{00} = 0.01$; (b): $\gamma_{00} = -0.01$.

wavelike has a constant in the peak value and width. In figs. 3(c) and (d), the solitary wavelike has a decrease (increase) in the peak value and a broadening (compression) in its width during its propagation. For a better understanding, we plotted in fig. 4 the soliton profile at different times for the same equation for $\varphi(t)$ as in figs. 3(c) and (d).

3.2 Dynamics of matter-wave solitons in 1-D Gross-Pitaevskii equation with cubic-quintic nonlinearity

Now, we aim to use exact analytical soliton solutions of the GP equation (1) with external potential (2) for investigating the dynamics of matter-wave solitons in BEC systems with both two- and three-body interatomic interactions. We limit ourselves to the dynamics of bright solitary wavelike in BECs described by eqs. (1) and (2). For simplicity, we consider the exact bright solitary wave-like solution associated with solution (10a) of the EODE (6a). Thus, using eqs. (6a) and (3) and remembering that $\rho(X) = u^2(X)$ yield

$$\psi(x,t) = \sqrt{-\frac{g}{\chi_0}} \frac{1}{\sqrt{1 + 3\cosh^2\left[\frac{g}{\sqrt{-\chi_0}}(x - \ell(t)(KT - K_0))\right]}} \exp[i\theta], \tag{17}$$

where $\theta = f(t)x^2 + \frac{K}{\ell(t)}x - \varphi(t)$, $f(t)$, $\ell(t)$, and $T(t)$ are functions defined by eqs. (4a), (4b), and (4c), respectively, and $\varphi(t)$ is any real solution of the differential equation $2\ell^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} - K^2 = 0$. The bright solitary wave-like solution (17) is defined under conditions $\chi_0 < 0$ and $g(t) > 0$. Because the strength of the three-body interaction is usually very small when compared with strength of the two-body interaction as pointed out by Gammal [14], we consider in this study that $\chi_0(t) \approx -\chi\%$ of $g(t)$ so that $|\chi_0(t)/g(t)| << 1$. Thus, $\chi_0(t) = -\frac{\chi}{100}g(t)$, $0 < \chi < 100$. It is seen from solution (17) that when the strength of repulsive three-body interactions is increased, one observes a decrease in the density of condensates as shown in the top panel of fig. 5. One then understands that the strength of the repulsive three-body interactions can be used to control the total number of BEC atoms. The plots of bottom panel of fig. 5 confirms, as

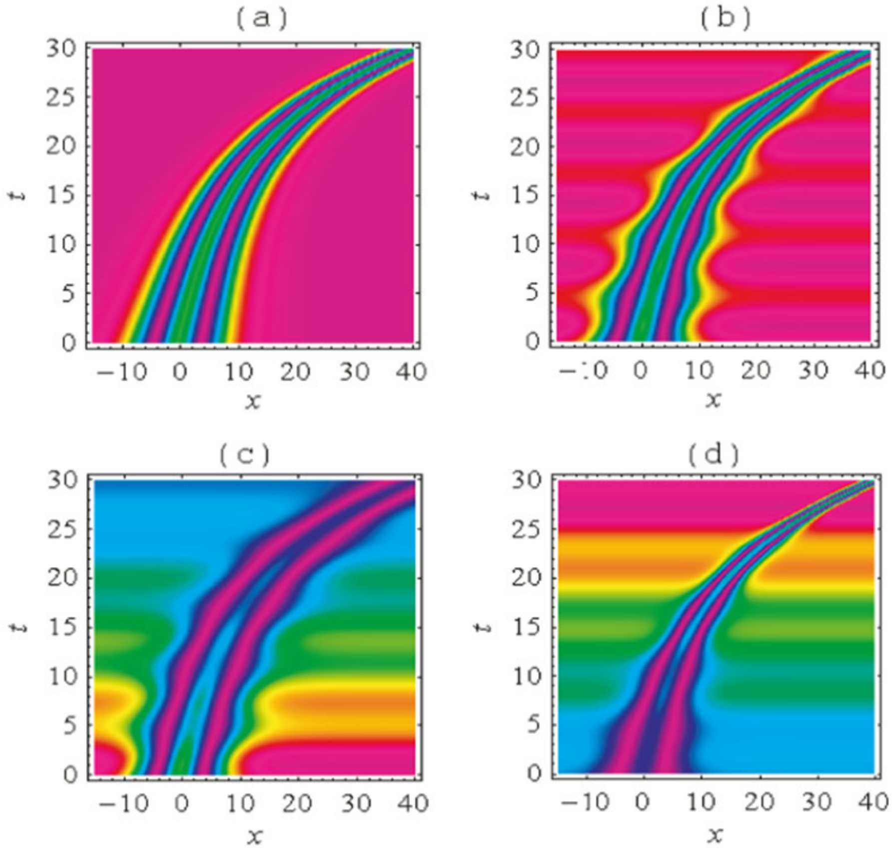

Fig. 3. The evolution plot of a bright solitary wave in a time-dependent harmonic trapping potential given by eq. (16) for different ordinary differential equations defining the functional parameter $\varphi(t)$ with the solution parameters $K = 0.5$, $K_0 = 0.5$, $\ell_0 = 1$. (a): $a_0^2 g\ell^2 + 2\ell^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} - K^2 = 0$; (b): $2\ell^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} + a_0^2 g\ell^2(1 + m\sin[\omega t]) - K^2 = 0$, $0 < m < 1$; (c): $2\ell^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} + a_0^2 g\ell^2(1 + m\sin[\omega t])\exp[-\lambda t] - K^2 = 0$, $0 < m < 1$; (d): $2\ell^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} + a_0^2 g\ell^2(1+m\sin[\omega t])\exp[\lambda t] - K^2 = 0$, $0 < m < 1$. Here, we used $a_0 = 0.4$, $m = 0.1$, $\omega = 1$.

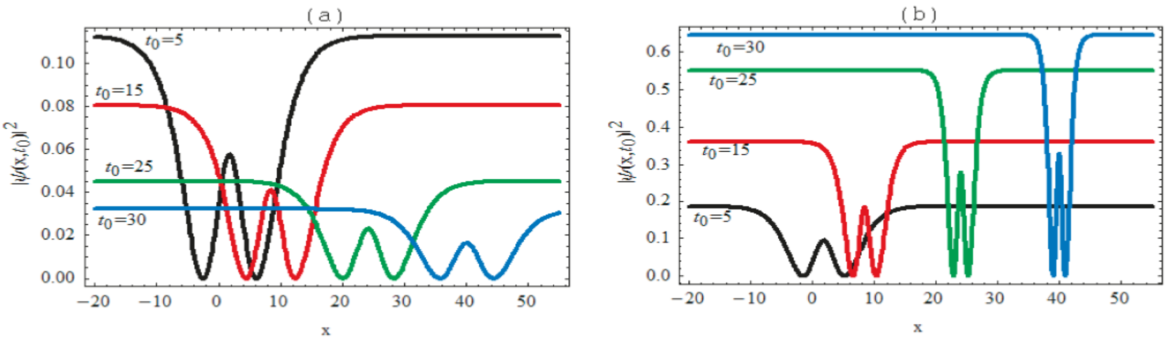

Fig. 4. Plot of the density $|\psi(x,t)|^2$ associated with solution (16) showing the bright soliton profile at different time for two different differential equations defining the functional parameter $\varphi(t)$: (a): $2\ell^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} + a_0^2 g\ell^2(1 + m\sin[\omega t])\exp[-\lambda t] - K^2 = 0$, $0 < m < 1$; (b): $2\ell^2 \frac{\mathrm{d}\varphi}{\mathrm{d}t} + a_0^2 g\ell^2(1 + m\sin[\omega t])\exp[\lambda t] - K^2 = 0$, $0 < m < 1$. We used the same parameters as in fig. 3.

one can see from eq. (17), that the bright soliton will have a broadening in its width when χ increases. Therefore, the strength $\chi_0(t)$ of the repulsive three-body interactions can be used to manage the soliton motion in the BECs described by the GP equation (1) with external potential (2). It is important to notice that during its propagation, the bright soliton obtained from the exact analytical solution (17) has a constant in the peak value, meaning that, despite the presence of the feeding/loss term in potential (2), the total number of BEC atoms remains unchanged. This means that the expulsive three-body interaction can be used to ensure the stability of the condensates over the time.

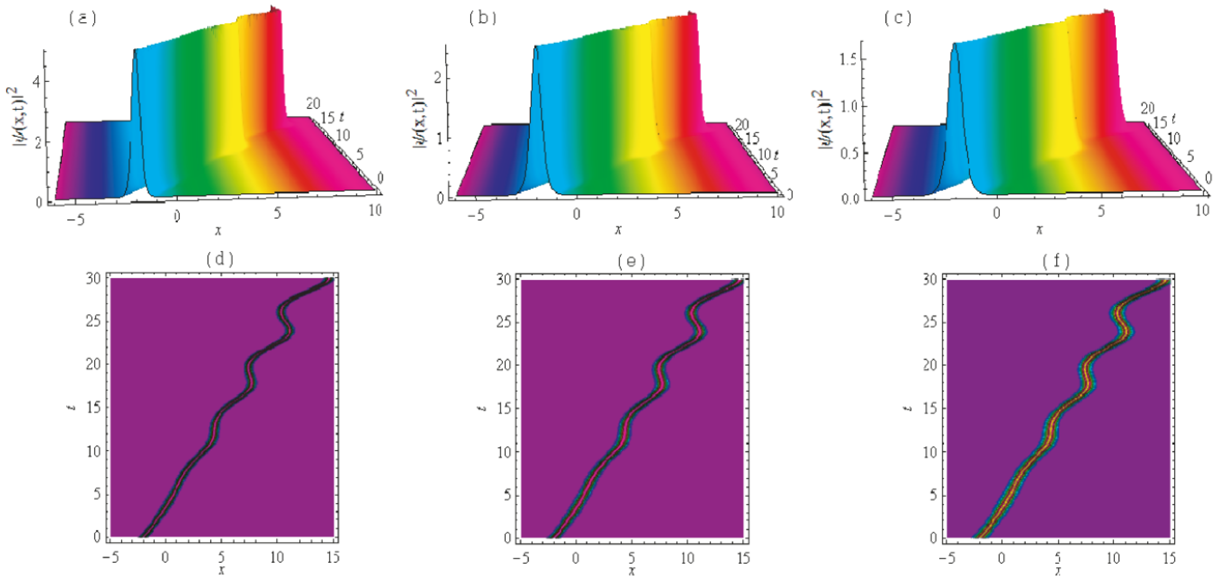

Fig. 5. Plots of the BEC density $|\psi(x,t)|^2$ associated with solution (17) for $g(t)=1+m\sin[\omega t]$, $k(t)=-\frac{m\omega^2}{2}\frac{[m+\sin[\omega t]+m\cos^2[\omega t]]}{(1+m\sin[\omega t])^2}$, and $f(t)=-\frac{m\omega}{2}\frac{\cos[\omega t]}{1+m\sin[\omega t]}$, $(1<m<1)$, leading to $\ell(t)=\frac{\ell_0}{1+m\sin[\omega t]}$ and $T(t)=\ell_0^{-2}(\frac{2+m^2}{2}t+\frac{4m}{\omega}\sin^2\frac{\omega t}{2}-\frac{m^2}{4\omega}\sin[2\omega t])$ with different strengths of the three-body interatomic interactions. (a), (d): $\chi_0(t)=-5\%g(t)$; (b), (e): $\chi_0(t)=-10\%g(t)$; (c), (f): $\chi_0(t)=-15\%g(t)$. We used the parameters $\ell_0=1$, $m=0.1$, and $\omega=1$.

4 Conclusion

In this paper, we have considered a 1-D Gross-Pitaevskii equation which may describe the dynamics of the BEC matter-wave solitons with the time-dependent s-wave scattering length and time-dependent harmonic trapping potential with a feeding/loss term. With the help of the modified lens-type transformation, we reduced the one-dimensional GP equation (1) with external complex potential (2) to an elliptic ordinary differential equation, and derived exact analytical bright, kink, and dark one-solitary waves on a vanishing cw background. The methodology presented in this work is powerful for systematically finding an large number of BEC solitary wave-like solutions by exactly matching the two- and three-body interatomic interactions and external harmonic trapping complex potential. These exact analytical solutions imply that control of the two- and three-body interactions, the external harmonic trapping complex potential, and the solution functional parameter allows us to manipulate the motion of solitons in BEC systems. Our analytical investigations show that the amplitude of solitons depends on the parameters of two- and three-body interactions while their motion depends on the external trapping potential and the functional parameter $\varphi(t)$. Our investigations also showed that decreasing the strength of the three-body interactions generates low density condensates.

Appendix A.

It is known [40,41] that solutions to eq. (6a) are given by

$$u(X)=\rho_0+\frac{\sqrt{R(\rho_0)}\frac{\mathrm{d}\wp(X;g_2,g_3)}{\mathrm{d}X}+\frac{1}{2}R'(\rho_0)\left[\wp(X;g_2,g_3)-\frac{1}{24}R''(\rho_0)\right]+\frac{1}{24}h(\rho_0)R'''(\rho_0)}{2\left[\wp(X;g_2,g_3)-\frac{1}{24}R''(\rho_0)\right]^2-\frac{1}{48}R(\rho_0)R''''(\rho_0)}\,, \tag{A.1}$$

where ρ_0 is any real function of τ (not necessary a root of polynomial $R(y)$) and $\wp(X;g_2,g_3)$ is the Weierstrass' elliptic function with invariants $g_2=-4\beta\delta+3\gamma^2$ and $g_3=2\beta\gamma\delta-\alpha\delta^2-\gamma^3$.

If $\Delta=-\delta^2[27\alpha^2\delta^2+4\beta(16\beta^2-27\alpha\gamma)\delta+18\gamma^2(3\alpha\gamma-2\beta^2)]=0$, $g_2\geq 0$ and $g_3\leq 0$, $u(X)$ is solitary wave-like and given [41]

$$u(X)=\rho_0+\frac{R'(\rho_0)}{4\left[e_1-\frac{R''(\rho_0)}{24}+3e_1\cosh^2[\sqrt{3e_1}X]\right]}\,,\qquad e_1=\frac{1}{2}\sqrt[3]{-g_3}\,, \tag{A.2}$$

if ρ_0 is a simple root of polynomial $R(\rho)$. When polynomial $R(\rho)$ does not possess real simple roots, solitary wave-like solutions for eq. (6a) are obtained from eq. (A.1) with the use of [41]

$$\wp(X;g_2,g_3)=e_1\left(1+\frac{3}{\sinh^2(\sqrt{3e_1}X)}\right).$$

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