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# Partially relativistic self-gravitating Bose-Einstein condensates with a stiff equation of state

Pierre-Henri Chavanis[a]

Laboratoire de Physique Théorique, Université Paul Sabatier, 118 route de Narbonne 31062 Toulouse, France

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**Abstract.** Because of their superfluid properties, some compact astrophysical objects such as neutron stars may contain a significant part of their matter in the form of a Bose-Einstein condensate (BEC). We consider a partially relativistic model of self-gravitating BECs where the relation between the pressure and the rest-mass density is assumed to be quadratic (as in the case of classical BECs) but pressure effects are taken into account in the relation between the energy density and the rest-mass density. At high densities, we get a stiff equation of state in which the speed of sound equals the speed of light. We determine the maximum mass of general relativistic BEC stars described by this equation of state using the formalism of Tooper (1965). This maximum mass is slightly larger than the maximum mass obtained by Chavanis and Harko (2012) using a fully relativistic model. We also consider the possibility that dark matter is made of BECs and apply the partially relativistic model of BECs to cosmology. In this model, we show that the universe experiences a stiff matter era, followed by a dust matter era, and finally by a dark energy era due to the cosmological constant. Interestingly, the Friedmann equations can be solved analytically in that case and provide a simple generalization of the $\Lambda$CDM model including a stiff matter era. We point out, however, the limitations of the partially relativistic model for BECs and show the need for a fully relativistic one.

## 1 Introduction

Bose-Einstein condensates (BECs) play a major role in condensed matter physics [1]. Recently, it has been suggested that they could play an important role also in astrophysics and cosmology. Indeed, dark matter halos could be quantum objects made of BECs. The wave properties of dark matter may stabilize the system against gravitational collapse providing halo cores instead of cuspy profiles that are predicted by the cold dark matter (CDM) model [2] but not observed [3,4]. The resulting coherent configuration may be understood as the ground state of some gigantic bosonic atom where the boson particles are condensed in a single macroscopic quantum state $\psi(\mathbf{r})$. In the BEC model, the formation of dark matter structures at small scales is suppressed by quantum mechanics. This property could alleviate the problems of the CDM model such as the cusp problem [3] and the missing satellite problem [5]. At the scale of galaxies, Newtonian gravity can be used so the evolution of the wave function $\psi(\mathbf{r},t)$ is governed by the Gross-Pitaevskii-Poisson (GPP) system. The Gross-Pitaevskii (GP) equation [6–10] is valid at $T=0$. Using the Madelung [11] transformation, the GP equation turns out to be equivalent to hydrodynamic (Euler) equations involving an isotropic pressure due to short-range interactions (scattering) and an anisotropic quantum pressure arising from the Heisenberg uncertainty principle. At large scales, quantum effects are negligible and one recovers the classical hydrodynamic equations of the CDM model which are remarkably successful in explaining the large-scale structure of the universe [12]. At small scales, the pressure arising from the Heisenberg uncertainty principle or from the repulsive scattering of the bosons may stabilize dark matter halos against gravitational collapse and lead to smooth core densities instead of cuspy density profiles in agreement with the observations [3]. Quantum mechanics may therefore be a way to solve the problems of the CDM model.

The possibility that dark matter could be in the form of BECs has a long history (see recent reviews in [13–15]). In some works [16–39], it is assumed that the bosons have no self-interaction. In that case, gravitational collapse is prevented by the Heisenberg uncertainty principle which is equivalent to a quantum pressure. This leads to a mass-radius relation $MR = 9.95\hbar^2/Gm^2$ [18,36,40]. In order to account for the mass and size of dwarf dark matter halos ($r_h = 33\,\mathrm{pc}$, $M_h = 0.39\times 10^6\,M_\odot$), the mass of the bosons must be extremely small, of the order of $m \sim 2.57\times$

[a] e-mail: chavanis@irsamc.ups-tlse.fr

$10^{-20}\,\mathrm{eV}/c^2$ (see appendix D of [41]). Ultralight scalar fields like axions may have such small masses (multidimensional string theories predict the existence of bosonic particles down to masses of the order of $m \sim 10^{-33}\,\mathrm{eV}/c^2$). This corresponds to "fuzzy cold dark matter" [23]. In other works [17, 35, 36, 42–56], it is assumed that the bosons have a repulsive self-interaction measured by the scattering length $a_s > 0$. In that case, gravitational collapse is prevented by the pressure arising from the scattering. In the Thomas-Fermi (TF) approximation which amounts to neglecting the quantum pressure, the resulting structure is equivalent to a polytrope of index $n = 1$ with an equation of state $P = 2\pi\hbar^2 a_s \rho^2/m^3$ [57]. Its radius is given by $R = \pi(a_s\hbar^2/Gm^3)^{1/2}$ [35, 44, 46, 47], independent of its mass $M$. In order to account for the size of dwarf dark matter halos, the ratio between the mass and the scattering length of the bosons is fixed at $(\mathrm{fm}/a_s)^{1/3}(mc^2/\mathrm{eV}) = 0.654$ (see appendix D of [41]). For $a_s = 10^6\,\mathrm{fm}$, corresponding to the value of the scattering length observed in terrestrial BEC experiments [57], this gives a boson mass $m = 65.4\,\mathrm{eV}/c^2$ much larger than the mass $m \sim 2.57 \times 10^{-20}\,\mathrm{eV}/c^2$ required in the absence of self-interaction[1]. This may be more realistic from a particle physics point of view. The general mass-radius relation of self-gravitating BECs at $T = 0$ with an arbitrary scattering length $a_s$, connecting the non-interacting limit ($a_s = 0$) to the TF limit ($GM^2 m a_s/\hbar^2 \gg 1$), has been determined analytically and numerically in [35, 36]. These papers also provide the general density profile of dark matter halos interpreted as self-gravitating BECs at $T = 0$[2]. The effect of a finite temperature has been considered in [62–69] using different approaches.

Since atoms like $^7$Li have negative scattering lengths in terrestrial BEC experiments [57], it may be relevant to consider the possibility of self-gravitating BECs with an attractive self-interaction ($a_s < 0$). In that case, there exists a maximum mass $M_{\max} = 1.01\hbar/\sqrt{|a_s|Gm} = 5.07 M_P/\sqrt{|\lambda|}$, where $\lambda = 8\pi a_s mc/\hbar$ is the self-interaction constant and $M_P = (\hbar c/G)^{1/2}$ is the Planck mass, above which the BEC collapses [35, 36]. In most applications, this mass is extremely small (when $|\lambda| \sim 1$ it is of the order of the Planck mass $M_P = 2.18 \times 10^{-8}\,\mathrm{kg}$!) so that the collapse of the BEC is very easily realized in the presence of an attractive self-interaction. This may lead to the formation of supermassive black holes at the center of galaxies [15]. On the other hand, when the BEC hypothesis is applied in a cosmological context, an attractive self-interaction can enhance the Jeans instability and accelerate the formation of structures in the universe [70].

Self-gravitating BECs have also been proposed to describe boson stars [40, 61, 71–92]. For these compact objects, we must use general relativity and couple the Klein-Gordon equation to the Einstein field equations. Initially, the study of boson stars was motivated by the axion field, a pseudo-Nambu-Goldstone boson of the Peccei-Quinn phase transition, that was proposed as a possible solution to the strong CP problem in QCD. In the early works of Kaup [71] and Ruffini and Bonazzola [40], it was assumed that the bosons have no self-interaction. This leads to a maximum mass of boson stars equals to $M_{\mathrm{Kaup}} = 0.633 M_P^2/m$ (and a minimum radius $R_{\min} = 9.53\,GM_{\mathrm{Kaup}}/c^2$). Above that mass no equilibrium configuration exists. In that case, the system collapses into a black hole. This maximum mass is much smaller than the maximum mass $M_{\mathrm{OV}} = 0.376 M_P^3/m^2$ of fermion stars (with corresponding radius $R_{\mathrm{OV}} = 9.36\,GM_{\mathrm{OV}}/c^2$) determined by Oppenheimer and Volkoff [93] in general relativity. They differ by a factor $m/M_P \ll 1$. This is because boson stars are stopped from collapsing by Heisenberg's uncertainty principle while, for fermion stars, gravitational collapse is avoided by Pauli's exclusion principle. For $m \sim 1\,\mathrm{GeV}/c^2$, corresponding to the typical mass of the neutrons, the Kaup mass $M_{\mathrm{Kaup}} = 8.48 \times 10^{-20} M_\odot$ (associated with $R_{\min} = 1.20 \times 10^{-18}\,\mathrm{km}$) is very small. This corresponds to mini boson stars like axion black holes. The mass of these mini boson stars may be too small to be astrophysically relevant. They could play a role, however, if they exist in the universe in abundance or if the axion mass is extraordinary small leading to macroscopic objects with a mass $M_{\mathrm{Kaup}}$ comparable to the mass of the sun (or even larger) [89]. It has also been proposed that stable boson stars with a boson mass $m \sim 10^{-17}\,\mathrm{eV}/c^2$ could mimic supermassive black holes ($M \sim 10^6\,M_\odot$, $R \sim 10^7\,\mathrm{km}$) that reside at the center of galaxies [87, 90]. On the other hand, Colpi *et al.* [75] assumed that the bosons have a repulsive self-interaction. In the Thomas-Fermi approximation, this leads to a maximum mass $M_{\max} = 0.0612\sqrt{\lambda} M_P^3/m^2$ (and a minimum radius $R_{\min} = 6.25\,GM_{\max}/c^2$) which,

[1] Actually, using the constraint $4\pi a_s^2/m < 1.25\,\mathrm{cm}^2/\mathrm{g}$ set by the Bullet Cluster [58], implying $(a_s/\mathrm{fm})^2(\mathrm{eV}/mc^2) < 1.77\times10^{-8}$, one finds the upper bounds $a_s = 1.73 \times 10^{-5}\,\mathrm{fm}$ and $m = 1.69 \times 10^{-2}\,\mathrm{eV}/c^2$ (see appendix D of [41]) in agreement with the limit $m < 1.87\,\mathrm{eV}/c^2$ obtained from cosmological considerations [59].

[2] For a value of the boson mass in the range $2.57 \times 10^{-20}\,\mathrm{eV}/c^2 < m < 1.69 \times 10^{-2}\,\mathrm{eV}/c^2$, we have $T \ll T_c$ (where $T_c$ is the condensation temperature) for all the dark matter halos of the universe (with mass $M \sim 10^6$–$10^{11}\,M_\odot$), so they can be considered to be at $T = 0$ [41, 60]. They have a core-halo structure with a solitonic core (BEC), which is a stationary solution of the GP equation, surrounded by a halo of scalar radiation in which the density decreases as $r^{-3}$ similarly to the Navarro-Frank-White (NFW) [2] and Burkert [3] profiles. This core-halo structure results from a process of gravitational cooling [61]. Dwarf dark matter halos are compact objects that have just a solitonic core (BEC) without atmosphere. Therefore, their size is equal to the size of the soliton. By contrast, large dark matter halos are extended objects with a core-halo structure. It is the radiative atmosphere that fixes the size of large dark matter halos. The atmosphere can be much larger than the size of the soliton (core). The presence of the radiative atmosphere solves the apparent paradox that BEC halos at $T = 0$ should all have the same radius (in the self-interacting case) or that their radius should decrease with their mass (in the non-interacting case), in contradiction with the observations. The studies [35, 36] do not take into account the halo of scalar radiation that surrounds the solitonic core (condensate), so they are applicable only to dwarf halos which do not possess such an atmosphere.

for $\lambda \sim 1$, is of the order of the maximum mass of fermion stars $M_{\rm OV} = 0.376 M_P^3/m^2$. The self-interaction has the same effect on the bosons as the exclusion principle on the fermions. It plays the role of an interparticle repulsion (for $\lambda > 0$) that dominates over uncertainty pressure and prevents catastrophic gravitational collapse. Therefore, for $m \sim 1\,{\rm GeV}/c^2$ and $\lambda \sim 1$, we get a maximum mass of the order of the solar mass $M_\odot$ (and a minimum radius $R_{\rm min} \sim 10\,{\rm km}$), similar to the mass of neutron stars, which is much larger than the maximum mass $M_{\rm Kaup} \sim 10^{-19} M_\odot$ obtained in the absence of self-interaction (an interpolation formula giving the maximum mass for any value of the self-interaction constant $\lambda$ is given in appendix B.5 of [35]). On the other hand, for a boson mass of the order of $m \sim 1\,{\rm MeV}/c^2$ and a self-interaction constant $\lambda \sim 1$, we get $M_{\rm max} \sim 10^6\, M_\odot$ and $R_{\rm min} \sim 10^7\,{\rm km}$. These parameters are reminiscent of supermassive black holes in active galactic nuclei, so that stable self-interacting boson stars with $m \sim 1\,{\rm MeV}/c^2$ could be an alternative to black holes at the center of galaxies [85]. Therefore, a self-interaction can significantly change the physical dimensions of boson stars, making them much more astrophysically interesting.

Recently, Chavanis and Harko [91] have proposed that, because of the superfluid properties of the core of neutron stars, the neutrons (fermions) could form Cooper pairs and behave as bosons of mass $2m_n$, where $m_n = 0.940\,{\rm GeV}/c^2$ is the mass of the neutrons. Therefore, neutron stars could actually be BEC stars[3]. Since the maximum mass of BEC stars $M_{\rm max} = 0.0612\sqrt{\lambda} M_P^3/m^2 = 0.307\,\hbar c^2\sqrt{a_s}/(Gm)^{3/2}$ depends on the self-interaction constant $\lambda$ (or scattering length $a_s$), this allows to overcome the (fixed) maximum mass of neutron stars $M_{\rm OV} = 0.376\, M_P^3/m^2 = 0.7\, M_\odot$ determined by Oppenheimer and Volkoff [93] by modeling a neutron star as an ideal gas of fermions of mass $m_n$ (the corresponding radius is $R = 9.36\, GM_{\rm OV}/c^2 = 9.6\,{\rm km}$ and the corresponding density is $\rho = 5 \times 10^{15}\,{\rm g/cm^3}$). By taking a scattering length of the order of 10–20 fm (hence $\lambda/8\pi \sim 95.2$–$190$), we obtain a maximum mass of the order of $2M_\odot$, a central density of the order of $1$–$3 \times 10^{15}\,{\rm g/cm^3}$, and a radius in the range 10–20 km. This could account for the recently observed neutron stars with masses in the range of 2–2.4 $M_\odot$ [94–99] much larger than the Oppenheimer-Volkoff limit [93]. For $M > M_{\rm max}$, nothing prevents the gravitational collapse of the star which becomes a black hole.

Self-gravitating BECs may also find applications in the physics of black holes [15]. It has been proposed recently that microscopic quantum black holes could be BECs of gravitons stuck at a critical point [100,101]. These results can be easily understood in terms of the Kaup mass and Kaup radius [15]. Self-gravitating BECs have also been advocated in cosmology in relation to the dark energy problem. It has been proposed [102] that the wave function of a BEC, via the quantum potential it produces, gives rise to a cosmological constant that may account for the correct dark energy in our universe. Therefore, self-gravitating BECs can have many applications in astrophysics, cosmology and black hole physics with promising perspectives.

In this paper, we show that certain approximations that have been made in the study of self-gravitating BECs are too restrictive, and we improve them.

In their study of general relativistic BEC stars, Chavanis and Harko [91] first presented qualitative arguments giving the fundamental scalings of the maximum mass $M_* \sim \hbar c^2\sqrt{a_s}/(Gm)^{3/2}$, minimum radius $R_* \sim (a_s\hbar^2/Gm^3)^{1/2}$, and maximum density $\rho_* \sim m^3c^2/2\pi a_s\hbar^2$ of BEC stars. Then, they developed two models in order to obtain the numerical values of the prefactors. They first developed a semi-relativistic model in which gravity is treated by general relativity using the Tolman-Oppenheimer-Volkoff (TOV) equation but the relation between the pressure and the energy density is given by the quadratic equation of state $P = 2\pi\hbar^2 a_s\epsilon^2/m^3c^4$ obtained from the classical GP equation after identifying the energy density with the rest-mass density ($\epsilon = \rho c^2$). This is a particular case of a polytropic equation of state of type I studied by Tooper [103] in general relativity. This semi-relativistic model leads to a maximum mass $M_{\rm max} = 0.5001\,\hbar c^2\sqrt{a_s}/(Gm)^{3/2}$. This treatment is approximate first because the energy density is not always dominated by the rest-mass density and also because the relation between the pressure and the rest-mass density is altered by relativistic effects. Chavanis and Harko [91] also developed a fully relativistic model in which the relation between the pressure and the energy density is obtained from the Klein-Gordon-Einstein equations [75]. In the dense core, the equation of state reduces to $P \sim \epsilon/3$ which is similar to the equation of state of the radiation or to the equation of state that prevails in the core of neutron stars modeled as an ideal gas of fermions at $T = 0$. In the envelope, we recover the equation of state $P = 2\pi\hbar^2 a_s\epsilon^2/m^3c^4$ of a classical BEC. This fully relativistic model leads to a maximum mass $M_{\rm max} = 0.307\,\hbar c^2\sqrt{a_s}/(Gm)^{3/2}$. This is the correct value of the maximum mass of BEC stars. In this paper, we shall compare these results with a partially relativistic model of self-gravitating BECs where the relation between the pressure and the rest-mass density is assumed to be given by $P = 2\pi\hbar^2 a_s\rho^2/m^3$ (as for a classical BEC) but pressure effects are taken into account in the relation between the energy density and the rest-mass density ($\epsilon = \rho c^2 + P$). This is a particular case of a polytropic equation of state of type II studied by Tooper [104] in general relativity. In the dense core, the equation of state reduces to $P \sim \epsilon$. This is a stiff equation of state for which the speed of sound $c_s = \sqrt{P'(\epsilon)}c$ is equal to the speed of light ($c_s = c$)[4]. In the envelope, we recover the

[3] We note that the BEC star model may have application, besides neutron stars, to boson stars, dark matter stars, and to the core of dark matter halos [15].

[4] This type of equation of state was originally introduced by Zel'dovich [105] in the context of baryon stars in which the baryons interact through a vector meson field.

equation of state $P = 2\pi\hbar^2 a_s \epsilon^2/m^3c^4$ of a classical BEC. This partially relativistic model leads to a maximum mass $M_{\max} = 0.4104\,\hbar c^2\sqrt{a_s}/(Gm)^{3/2}$ intermediate between the two models considered by Chavanis and Harko [91]. This treatment is, however, approximate because the relation between the pressure and the rest-mass density is altered by relativistic effects.

Self-gravitating BECs have also been considered in cosmology. Harko [106] and Chavanis [70] independently developed cosmological models in which dark matter is made of BECs. They solved the Friedmann equations by using a semi-relativistic model, assuming that the equation of state relating the pressure to the energy density is given by $P = 2\pi\hbar^2 a_s \epsilon^2/m^3c^4$. However, as discussed above, this equation of state is not valid when the BEC is strongly relativistic. Therefore, their approach gives wrong results in the very early universe where relativistic effects are important. Here, we improve their study by developing a partially relativistic model in which the relation between the pressure and the rest-mass density is assumed to be given by $P = 2\pi\hbar^2 a_s \rho^2/m^3$ (as for a classical BEC) but pressure effects are taken into account in the relation between the energy density and the rest-mass density ($\epsilon = \rho c^2 + P$). This leads to a cosmological model where the universe experiences a stiff matter era (due to the self-interaction of the bosons) followed by a dust matter era, and finally by a dark energy era due to the cosmological constant[5]. Interestingly, the Friedmann equations can be solved analytically in that case and provide a simple generalization of the $\Lambda$CDM model including a stiff matter era. We point out, however, the limitations of this partially relativistic model for BECs and the need for a fully relativistic one based on the Klein-Gordon-Einstein equations [108, 109].

The paper is organized as follows. In sect. 2, we recall the basic equations describing Newtonian self-gravitating BECs at $T = 0$. We also recall the qualitative arguments of Chavanis and Harko [91] giving the scaling of the maximum mass, minimum radius, and maximum density of relativistic self-gravitating BECs. In sect. 3, we determine the maximum mass of general relativistic BECs using a partially relativistic model and compare the results with the ones obtained by Chavanis and Harko [91] using a semi-relativistic model and a fully relativistic model. We confront these theoretical results to the observations of neutron stars. We also discuss the analogies and the differences between models that treat neutron stars as fermion stars or as BEC stars. In sect. 4, we develop a cosmological model in which dark matter is made of BECs with a partially relativistic equation of state. We show that this system experiences a stiff matter era in the primordial universe. We provide new analytical solutions of the Friedmann equations exhibiting a stiff matter era, a radiation era, a matter era, and a dark energy era (due to the cosmological constant). They generalize the analytical solutions describing the Einstein-de Sitter (EdS) and $\Lambda$CDM models. We also discuss the effect of the BEC equation of state on the evolution of the universe.

## 2 Self-gravitating Bose-Einstein condensates

### 2.1 The Gross-Pitaevskii-Poisson system

At $T = 0$, in the Newtonian regime, a self-gravitating BEC with short-range interactions is described by the Gross-Pitaevskii-Poisson system

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + m\Phi\psi + \frac{4\pi a_s\hbar^2 N}{m}|\psi|^2\psi, \tag{1}$$

$$\Delta\Phi = 4\pi G N m|\psi|^2, \tag{2}$$

where $\rho(\mathbf{r}, t) = Nm|\psi|^2$ is the mass density ($N$ is the number of bosons and $m$ is their mass), $\psi(\mathbf{r}, t)$ is the wave function, $\Phi(\mathbf{r}, t)$ is the gravitational potential, and $a_s$ is the s-scattering length of the bosons. These equations are valid in a mean field approximation which is known to be exact for systems with long-range interactions (such as self-gravitating systems) when $N \to +\infty$.

Using the Madelung [11] transformation

$$\psi = \sqrt{\frac{\rho}{Nm}}e^{iS/\hbar}, \qquad \mathbf{u} = \frac{1}{m}\nabla S, \tag{3}$$

where $S(\mathbf{r}, t)$ is an action and $\mathbf{u}(\mathbf{r}, t)$ is an irrotational velocity field, we can rewrite the GPP system (1) and (2) in

[5] We note, for historical reasons, that the existence of a primordial stiff matter era was predicted by Zel'dovich [107] who assumed that initially, near the cosmological singularity, the universe is filled with cold baryons. However, this model was discarded by Zel'dovich himself after the success of the hot Big Bang theory. It is interesting to note that a stiff matter era reappears in the context of BEC dark matter.

the form of hydrodynamic equations

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{4}$$

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla P - \nabla\Phi - \frac{1}{m}\nabla Q, \tag{5}$$

$$\Delta\Phi = 4\pi G\rho, \tag{6}$$

where

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{7}$$

is the quantum potential and

$$P = \frac{2\pi a_s\hbar^2}{m^3}\rho^2 \tag{8}$$

is the pressure arising from the short-range interactions. It corresponds to a polytropic equation of state

$$P = K\rho^\gamma, \qquad \gamma = 1 + \frac{1}{n}\,, \tag{9}$$

with a polytropic index $n = 1$ (*i.e.* $\gamma = 2$) and a polytropic constant

$$K = \frac{2\pi\hbar^2 a_s}{m^3}\,. \tag{10}$$

Equations (4)–(7) form the quantum barotropic Euler-Poisson system [35, 36].

The condition of hydrostatic equilibrium ($\partial_t = 0$ and $\mathbf{u} = \mathbf{0}$) writes

$$\nabla P + \rho\nabla\Phi + \frac{\rho}{m}\nabla Q = \mathbf{0}. \tag{11}$$

It expresses the balance between the gravitational attraction and the repulsion due to the scattering pressure and the quantum pressure. Combining this equation with the Poisson equation (6), and using eqs. (7) and (8), we obtain

$$\frac{4\pi a_s\hbar^2}{m^3}\Delta\rho + 4\pi G\rho - \frac{\hbar^2}{2m^2}\Delta\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 0. \tag{12}$$

This differential equation determines the density profile of a self-gravitating BEC. It is equivalent to the stationary solution of the GPP system, sometimes called a soliton [15, 35]. It has been solved analytically (approximately) and numerically (exactly) in refs. [35] and [36] for arbitrary values of the scattering length $a_s$ and boson mass $m$.

### 2.2 The Thomas-Fermi approximation

In the TF approximation valid when $GM^2 m a_s/\hbar^2 \gg 1$, we can neglect the contribution of the quantum potential. In that case, the condition of hydrostatic equilibrium reduces to the usual form

$$\nabla P + \rho\nabla\Phi = \mathbf{0}, \tag{13}$$

and the differential equation (12) becomes

$$\frac{4\pi a_s\hbar^2}{m^3}\Delta\rho + 4\pi G\rho = 0. \tag{14}$$

Writing $\rho = \rho_0\theta$ and $r = (a_s\hbar^2/Gm^3)^{1/2}\xi$, where $\rho_0$ is the central density, and considering a spherically symmetric system, this equation can be put in the form of the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta, \tag{15}$$

$$\theta(0) = 1, \qquad \theta'(0) = 0, \tag{16}$$

for a polytrope of index $n = 1$ [110]. It has the analytical solution

$$\theta(\xi) = \frac{\sin\xi}{\xi}. \tag{17}$$

The radius of the configuration is defined by the condition $\theta(\xi_1) = 0$, giving $\xi_1 = \pi$. Therefore the radius $R$ of the self-gravitating BEC is given by

$$R = \pi\sqrt{\frac{a_s\hbar^2}{Gm^3}}. \tag{18}$$

It is independent of the central density and of the mass of the system, and depends only on the physical characteristics of the condensate (the mass $m$ and the scattering length $a_s$ of the bosons). Actually, it is fixed by the ratio $a_s/m^3$.

The mass of a self-gravitating BEC with a quartic non-linearity is given as a function of the central density and coherent scattering length $a_s$ by

$$M = 4\pi\left(\frac{a_s\hbar^2}{Gm^3}\right)^{3/2}\rho_0\xi_1^2\left|\theta'(\xi_1)\right|, \tag{19}$$

yielding

$$M = 4\pi^2\left(\frac{a_s\hbar^2}{Gm^3}\right)^{3/2}\rho_0, \tag{20}$$

where we have used $|\theta'(\xi_1)| = 1/\pi$. Using eq. (18), it can be expressed in terms of the radius and central density by

$$M = \frac{4}{\pi}\rho_0 R^3, \tag{21}$$

which shows that the mean density of the configuration $\overline{\rho} = 3M/4\pi R^3$ is related to the central density by the relation $\overline{\rho} = 3\rho_0/\pi^2$. Other quantities of interest such as the energy and the moment of inertia of the self-gravitating BEC are derived in [35].

### 2.3 Maximum mass of relativistic BEC stars: qualitative treatment and fundamental scalings

The Newtonian treatment of a self-gravitating BEC is appropriate to describe dark matter halos. However, general relativistic effects are important in the case of BEC stars describing compact objects such as neutron stars, boson stars, dark matter stars, black holes, or the core of dark matter halos [15].

The radius of a Newtonian BEC star is given by eq. (18). In the Newtonian treatment, there is no limit on the mass of the BEC. However, the Newtonian treatment breaks down when the radius of the star approaches the Schwarzschild radius $R_S = 2GM/c^2$. Equating the two radii, namely writing $M = Rc^2/2G$ with $R$ given by eq. (18), and ignoring the prefactors that are necessarily inexact, we obtain the scaling of the maximum mass and minimum radius of a relativistic BEC star [91]

$$M_* = \frac{\hbar c^2\sqrt{a_s}}{(Gm)^{3/2}} = 1.420\,\kappa\,M_\odot, \tag{22}$$

$$R_* = \frac{GM_*}{c^2} = \left(\frac{a_s\hbar^2}{Gm^3}\right)^{1/2} = 2.106\,\kappa\,\mathrm{km}, \tag{23}$$

where we have introduced the dimensionless parameter

$$\kappa = \left(\frac{a_s}{1\,\mathrm{fm}}\right)^{1/2}\left(\frac{m}{2m_n}\right)^{-3/2}. \tag{24}$$

From eqs. (22) and (23), we obtain the scaling of the maximum central density

$$\rho_* = \frac{m^3c^2}{2\pi a_s\hbar^2} = 4.846\times 10^{16}\,\kappa^{-2}\,\mathrm{g/cm^3}, \tag{25}$$

where the factor $2\pi$ has been introduced for future convenience.

We note that the expression of the scaled radius $R_*$ is the same as in the Newtonian regime (see eq. (18)) (it is independent of $c$) while the scaling of the mass and density are determined by relativistic effects.

## 3 General relativistic Bose-Einstein condensate stars

For a correct determination of the maximum mass of BEC stars, we cannot ignore the effects induced by the space-time curvature, and a general relativistic treatment is necessary.

### 3.1 The Tolman-Oppenheimer-Volkoff equation

For a static spherically symmetric star, the interior line element is given by

$$\mathrm{d}s^2 = e^{\nu(r)}c^2\mathrm{d}t^2 - e^{\lambda(r)}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right). \tag{26}$$

The equations describing a general relativistic compact star are the mass continuity equation and the Tolman-Oppenheimer-Volkoff (TOV) equation. They write [111]:

$$\frac{\mathrm{d}M}{\mathrm{d}r} = 4\pi\frac{\epsilon}{c^2}r^2, \tag{27}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G(\epsilon+P)\left[4\pi Pr^3/c^2 + M(r)\right]}{r^2c^2\left[1-2GM(r)/c^2r\right]}, \tag{28}$$

where $\epsilon$ is the energy density and $M(r)$ is the total mass interior to $r$. The mass of the star is $M = M(R)$ where $R$ is its radius. These equations extend the classical condition of hydrostatic equilibrium for a self-gravitating gas to the context of general relativity. The system of eqs. (27)–(28) must be closed by choosing the equation of state $P = P(\epsilon)$ for the thermodynamic pressure. At the center of the star, the mass must satisfy the boundary condition $M(0) = 0$. For the thermodynamic pressure $P$, we assume that it vanishes on the surface: $P(R) = 0$.

The exterior of the star is characterized by the Schwarzschild metric, describing the vacuum ($P = \epsilon = 0$) outside the star, and given by [111]:

$$\left(e^{\nu}\right)^{\mathrm{ext}} = \left(e^{-\lambda}\right)^{\mathrm{ext}} = 1 - \frac{2GM}{c^2r}, \quad r \geq R. \tag{29}$$

The interior solution must match with the exterior solution on the vacuum boundary of the star.

The components of the metric tensor are determined by

$$e^{-\lambda(r)} = 1 - \frac{2GM(r)}{rc^2}, \tag{30}$$

$$\frac{\mathrm{d}P}{\mathrm{d}r} + \frac{P+\epsilon}{2}\frac{\mathrm{d}\nu}{\mathrm{d}r} = 0, \qquad e^{\nu(R)} = 1 - \frac{2GM}{Rc^2}. \tag{31}$$

The boundary condition on $e^{\nu}$ has been chosen so that this component is continuous with the exterior solution at $r = R$.

### 3.2 Maximum mass of relativistic BEC stars with short-range interactions: Partially relativistic treatment

We consider a partially relativistic model (see appendix B.3) in which the BEC star is described in general relativity by the equation of state

$$P = K\rho^2, \qquad \epsilon = \rho c^2 + P, \tag{32}$$

where $K$ is given by eq. (10). Here, $\epsilon$ is the energy density and $\rho$ is the rest-mass density. It is related to the number density $n$ by $\rho = mn$. The pressure can be expressed as a function of the energy density as (see appendix B.3):

$$P = \frac{c^4}{4K}\left(\sqrt{1+\frac{4K\epsilon}{c^4}} - 1\right)^2. \tag{33}$$

In the non-relativistic regime ($\epsilon \to 0$), we recover the classical equation of state of a BEC star $P \sim K\epsilon^2/c^4 \sim K\rho^2$. In the ultra-relativistic regime ($\epsilon \to +\infty$), we obtain a stiff equation of state $P \sim \epsilon$ in which the speed of sound is equal to the speed of light. A stiff equation of state was first introduced by Zel'dovich [105] in the context of baryon stars in which the baryons interact through a vector meson field (see sect. 3.5). We know that a linear equation of state $P = q\epsilon$ leads to a mass-central density relation that presents damped oscillations, and to a mass-radius relation that has a spiral structure [112,113]. Therefore, the series of equilibria of BEC stars described by the equation of state (33) will exhibit this behavior. This is similar to the series of equilibria of neutron stars modeled as a gas of relativistic fermions that have a linear equation of state $P \sim \epsilon/3$ for $\epsilon \to +\infty$ (see sect. 3.4) [93, 114, 115]. This is also similar to the series of equilibria of isothermal spheres described by a linear equation of state $P = \rho k_B T/m$ in Newtonian gravity [116].

The equation of state (32) is a particular case, corresponding to a polytropic index $n = 1$, of the class of equations of state (of type II) studied by Tooper [104] in general relativity. We shall use his formalism and notations. Therefore, we set

$$\rho = \rho_0 \theta, \qquad r = \frac{\xi}{A}, \qquad \sigma = \frac{K\rho_0}{c^2}, \tag{34}$$

$$M(r) = \frac{4\pi\rho_0}{A^3} v(\xi), \quad A = \left(\frac{2\pi G}{K}\right)^{1/2}, \tag{35}$$

where $\rho_0$ is the central rest-mass density and $\sigma$ is the relativity parameter. In terms of these variables, the TOV equation and the mass continuity equation become

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{(1+2\sigma\theta)(v+\sigma\xi^3\theta^2)}{\xi^2(1-4\sigma v/\xi)}, \tag{36}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = \theta\xi^2(1+\sigma\theta). \tag{37}$$

For a given value of the relativity parameter $\sigma$, they have to be solved with the initial condition $\theta(0) = 1$ and $v(0) = 0$. Since $v \sim \xi^3$ as $\xi \to 0$, it is clear that $\theta'(0) = 0$. On the other hand, the density vanishes at the first zero $\xi_1$ of $\theta$: $\theta(\xi_1) = 0$. This determines the boundary of the star. In the non-relativistic (Newtonian) limit $\sigma \to 0$, the system of eqs. (36) and (37) reduces to the Lane-Emden equation (15) with $n = 1$.

From the foregoing relations, we find that the radius, the mass and the central density of the configuration are given by

$$R = \xi_1 R_*, \qquad M = 2\sigma v(\xi_1) M_*, \qquad \rho_0 = \sigma\rho_*, \tag{38}$$

where

$$R_* = \left(\frac{K}{2\pi G}\right)^{1/2}, \qquad M_* = \left(\frac{Kc^4}{2\pi G^3}\right)^{1/2}, \qquad \rho_* = \frac{c^2}{K}. \tag{39}$$

For the value of $K$ given by eq. (10), one can check that the fundamental scaling parameters $R_*$, $M_*$ and $\rho_*$ are given by eqs. (22)–(25). By varying $\sigma$ from 0 to $+\infty$, we obtain the series of equilibria in the form $M(\rho_0)$ and $R(\rho_0)$. We can then plot the mass-radius relation $M(R)$ parameterized by $\rho_0$.

Using the Poincaré theorem [117] (see also [118, 119]), one can show [112, 113] that the series of equilibria becomes unstable after the first mass peak and that a new mode of instability appears at each turning point of mass in the series of equilibria (see [120] for an alternative derivation of these results based on the equation of pulsations). These results of dynamical stability for general relativistic stars are similar to results of dynamical and thermodynamical stability for Newtonian self-gravitating systems [116, 121].

The series of equilibria corresponding to the equation of state (32) is represented in figs. 1–3. These figures, respectively, give the mass-central density relation, the radius-central density relation, and the mass-radius relation. Some density profiles are plotted in fig. 4. The series of equilibria is parameterized by the relativity parameter $\sigma$ going from $\sigma = 0$ (non-relativistic) to $\sigma \to +\infty$ (ultra-relativistic). The configurations are stable for $\sigma \le \sigma_c$ and unstable for $\sigma \ge \sigma_c$ where

$$\sigma_c = 0.318 \tag{40}$$

corresponds to the first turning point of mass: $M'(\sigma_c) = 0$. The values of $\xi_1$ and $v(\xi_1)$ at this point are

$$\xi_1 = 1.914, \qquad v(\xi_1) = 0.6453. \tag{41}$$

The corresponding values of radius, mass and central density are

$$R_{\mathrm{min}} = 1.914 \left(\frac{a_s\hbar^2}{Gm^3}\right)^{1/2} = 4.03\,\kappa\,\mathrm{km}, \tag{42}$$

$$M_{\mathrm{max}} = 0.4104 \frac{\hbar c^2 \sqrt{a_s}}{(Gm)^{3/2}} = 0.583\,\kappa\,M_\odot, \tag{43}$$

$$(\rho_0)_{\mathrm{max}} = 0.318 \frac{m^3 c^2}{2\pi a_s \hbar^2} = 1.54 \times 10^{16}\,\kappa^{-2}\,\mathrm{g/cm^3}, \tag{44}$$

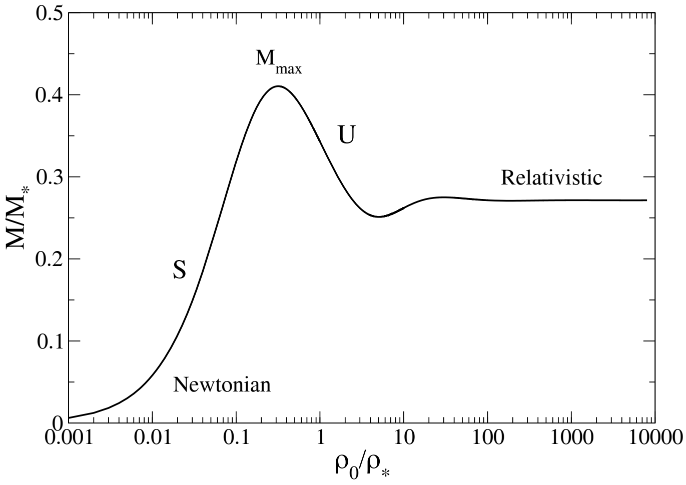


**Fig. 1.** Dimensionless mass-central density relation of a relativistic BEC with short-range interactions modeled by the equation of state (32). There exists a maximum mass $M_{\max}/M_* = 0.4104$ at which the series of equilibria becomes dynamically unstable. The speed of sound is always smaller than the speed of light. We note that the mass-central density relation presents damped oscillations at high densities similarly to neutron stars described by a fermionic equation of state [93, 112–115].

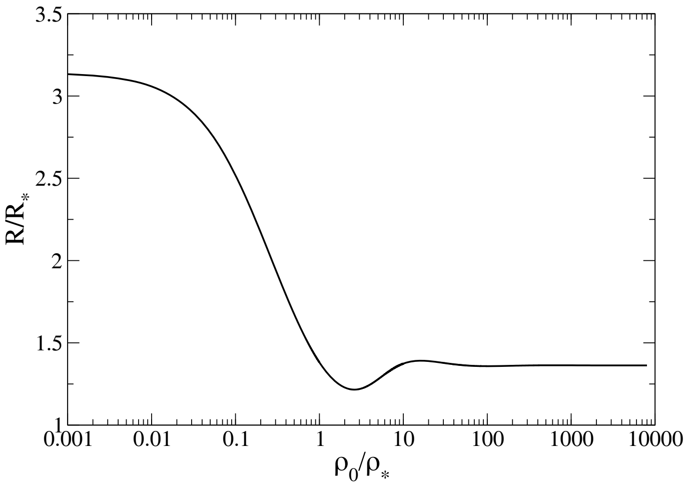


**Fig. 2.** Dimensionless radius-central density relation of a relativistic BEC with short-range interactions modeled by the equation of state (32).

respectively. They define the minimum radius, the maximum mass, and the maximum rest-mass density of the stable configurations.

The energy density is related to the rest-mass density by eq. (32) which can be rewritten as

$$\epsilon/c^2 = \rho\left(1 + \rho\frac{2\pi\hbar^2 a_s}{m^3c^2}\right). \tag{45}$$

Using eqs. (44) and (45), the maximum energy density is

$$(\epsilon_0)_{\max}/c^2 = 0.419\frac{m^3c^2}{2\pi a_s\hbar^2} = 2.03\times 10^{16}\,\kappa^{-2}\,\text{g/cm}^3. \tag{46}$$

We note that the radius of a relativistic BEC star is necessarily smaller than

$$R_{\max} = \pi\sqrt{\frac{\hbar^2 a_s}{Gm^3}} = 6.61\,\kappa\,\text{km}, \tag{47}$$

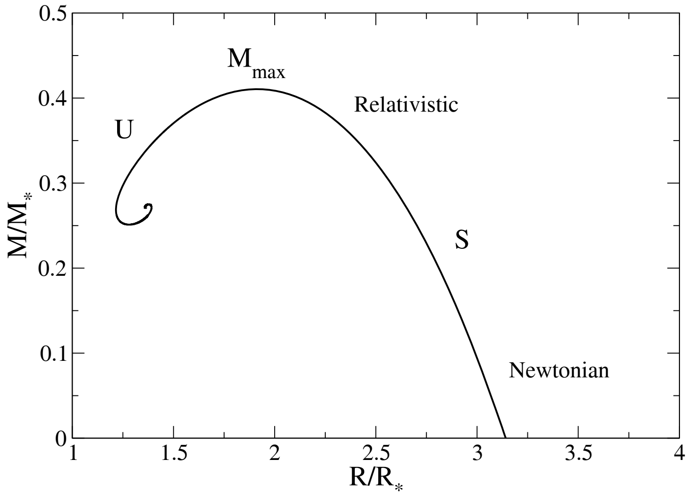


**Fig. 3.** Dimensionless mass-radius relation of a relativistic BEC with short-range interactions modeled by the equation of state (32). The series of equilibria is parameterized by the relativity parameter $\sigma$. The mass-radius relation presents a snail-like structure (spiral) at high densities similarly to neutron stars described by a fermionic equation of state [93, 112–115]. There exists a maximum mass $M_{\rm max}/M_* = 0.4104$ and a minimum radius $R_{\rm min}/R_* = 1.914$ corresponding to a maximum central density $(\rho_0)_{\rm max} = 0.318\rho_*$. There also exists a maximum radius $R_{\rm max}/R_* = \pi$ corresponding to the Newtonian limit $\sigma \to 0$.

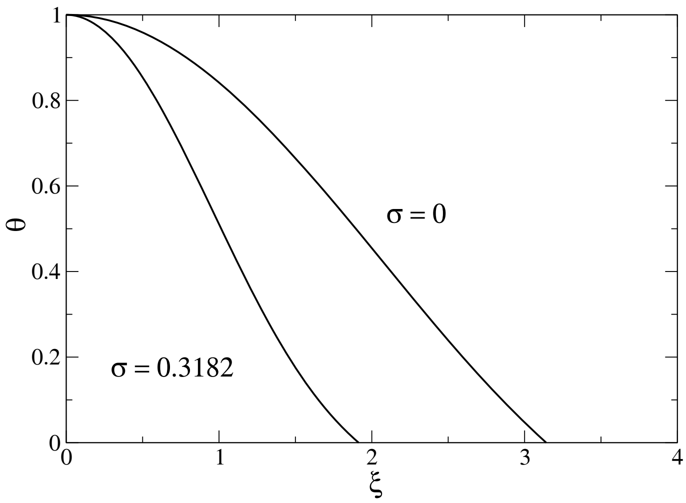


**Fig. 4.** Dimensionless density profiles corresponding to $\sigma = 0$ (Newtonian) and $\sigma = \sigma_c = 0.318$ (maximum mass).

corresponding to the Newtonian limit ($\sigma \to 0$). The Newtonian approximation is valid for small masses $M \ll M_{\rm max}$. The radius decreases as $M$ increases until the maximum mass and the minimum radius are reached. When $M > M_{\rm max}$, there is no equilibrium state and the BEC star is expected to collapse and form a black hole. When $M < M_{\rm max}$, there exist stable equilibrium states with $R_{\rm min} < R < R_{\rm max}$ that correspond to BEC stars for which gravitational collapse is prevented by quantum mechanics (the self-interaction of the bosons). We note that the radius of the BEC star is very much constrained as it lies in the range $4.03\,\kappa \le R(\,{\rm km}) \le 6.61\,\kappa$.

A quantity of physical interest is the mass-radius ratio

$$\frac{2GM}{Rc^2} = \frac{4\sigma v(\xi_1)}{\xi_1}\,. \tag{48}$$

At the critical point, the value of the mass-radius ratio is 0.429. We check that it is smaller than the Buchdahl maximum bound $2GM/Rc^2 = 8/9 = 0.888$ corresponding to constant density stars [122].

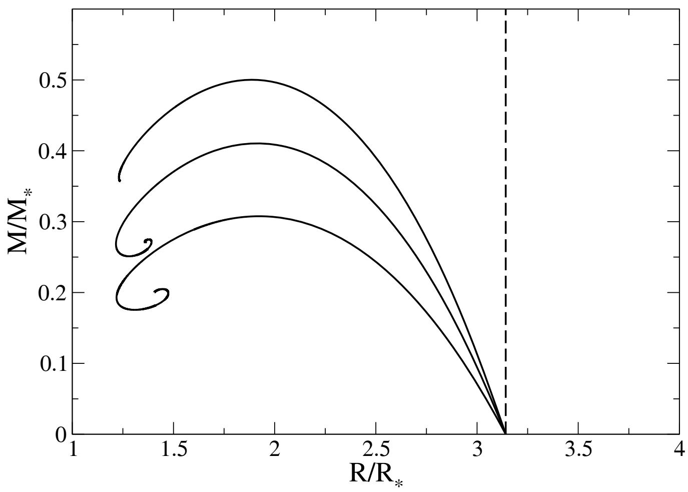


**Fig. 5.** Comparison between the mass-radius relations corresponding to the fully relativistic model (lower curve), partially relativistic model (intermediate curve), and semi-relativistic model (upper curve). The vertical dashed line corresponds to the Newtonian limit.

### 3.3 Comparison between the different models

The values of the maximum mass, minimum radius, and maximum central energy density of general relativistic BEC stars can be written as

$$R_{\min} = A_1 \left(\frac{a_s \hbar^2}{Gm^3}\right)^{1/2} = A_1' \,\kappa\, \mathrm{km}, \tag{49}$$

$$M_{\max} = A_2 \frac{\hbar c^2 \sqrt{a_s}}{(Gm)^{3/2}} = A_2' \,\kappa\, M_\odot, \tag{50}$$

$$(\epsilon_0)_{\max}/c^2 = A_3 \frac{m^3 c^2}{2\pi a_s \hbar^2} = A_3' \times 10^{16}\, \kappa^{-2}\, \mathrm{g/cm}^3, \tag{51}$$

where $\kappa$ is defined by eq. (24). These scalings are fundamental for BEC stars [91]. However, the values of the prefactors depend on the relativistic model.

The best model is the one based on the equation of state (B.16) considered in sect. VI.C. of Chavanis and Harko [91] because this equation of state can be derived from the Klein-Gordon-Einstein equations [75]. Therefore, this model is fully relativistic, both regarding the equation of state and the treatment of gravity. In that model, the prefactors are $A_1 = 1.923$, $A_1' = 4.047$, $A_2 = 0.307$, $A_2' = 0.436$, $A_3 = 0.398$, and $A_3' = 1.929$. They can be considered as being the exact prefactors for relativistic BEC stars.

The model based on the equation of state (B.30) considered in sect. VI.B. of Chavanis and Harko [91] is very approximate because it is based on an equation of state $P = 2\pi\hbar^2 a_s \rho^2/m^3$ derived from the classical GP equation and it furthermore assumes that the energy density is dominated by the rest-mass density so that $\epsilon = \rho c^2$. Therefore, this model is semi-relativistic because the equation of state is classical while gravity is treated in the framework of general relativity. In that model, the prefactors are $A_1 = 1.888$, $A_1' = 3.974$, $A_2 = 0.5001$, $A_2' = 0.710$, $A_3 = 0.42$, and $A_3' = 2.035$.

The model based on the equation of state (B.26) is intermediate between the two previous models. It is based on an equation of state $P = 2\pi\hbar^2 a_s \rho^2/m^3$ derived from the classical GP equation but it takes into account the difference between the energy density and the rest-mass density due to pressure effects: $\epsilon = \rho c^2 + P$ [104]. Therefore, this model is partially relativistic. In that model, the prefactors are $A_1 = 1.914$, $A_1' = 4.03$, $A_2 = 0.4104$, $A_2' = 0.583$, $A_3 = 0.419$, and $A_3' = 2.03$.

The mass-radius relation of general relativistic BEC stars at $T = 0$ corresponding to these different models is plotted in fig. 5. We note that the values of the prefactors do not differ much from one model to the other. The maximum mass varies between $\sim 0.3 M_*$ and $\sim 0.5 M_*$ while the minimum radius and the maximum energy density almost do not change.

As discussed specifically in sect. 3.4, general relativistic BEC stars may describe neutron stars with a superfluid core. This is why we have normalized the mass of the bosons by $2m_n$ (Cooper pair) in eq. (24). However, general relativistic BEC stars may describe other compact objects such as boson stars, dark matter stars, black holes, or the

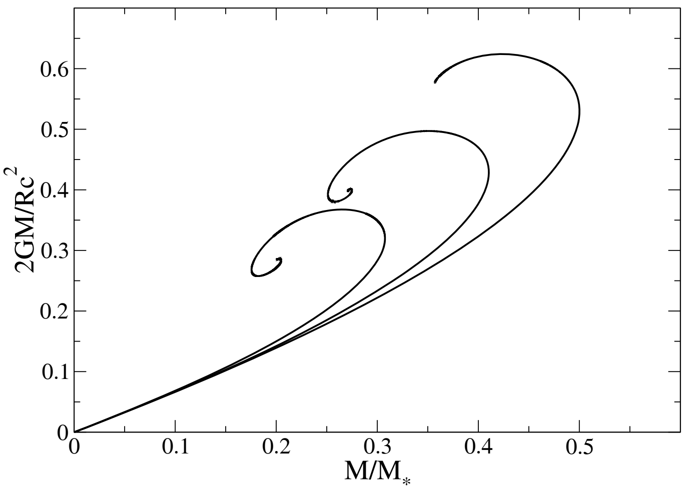


**Fig. 6.** Mass-radius ratio of general relativistic BEC stars corresponding to the fully relativistic model (left curve), partially relativistic model (middle curve), and semi-relativistic model (right curve). The Buchdahl maximum bound $2GM/Rc^2 = 8/9 = 0.888$ is much higher.

**Table 1.** Observational values of the mass and radius of neutron stars.

| Ref. | $M/M_\odot$ | $R$/km | $2GM/Rc^2$ |
|---|---|---|---|
| [124] | 1.3 | 8 | 0.479 |
| [124] | 1.95 | 12 | 0.479 |
| [125] | 2.1 | 12.5 | 0.4956 |
| [126] | 1.58 | 9.11 | 0.512 |
| [127] | 1.9 | 15.2 | 0.369 |

core of dark matter halos [15]. In this respect, it may be convenient to write the maximum mass, minimum radius, and maximum central density as[6]

$$\frac{M_{\rm max}}{M_\odot} = A\left(\frac{a}{\rm fm}\right)^{1/2}\left(\frac{{\rm GeV}/c^2}{m}\right)^{3/2}, \tag{52}$$

$$\frac{R_{\rm min}}{\rm km} = B\,\frac{M_{\rm max}}{M_\odot}, \tag{53}$$

$$\frac{(\epsilon_0)_{\rm max}/c^2}{10^{16}\,{\rm g/cm^3}} = C\left(\frac{M_\odot}{M_{\rm max}}\right)^2. \tag{54}$$

For the fully relativistic model, $A = 1.12$, $B = 9.27$, and $C = 0.364$. For the semi-relativistic model, $A = 1.83$, $B = 5.59$, and $C = 1.02$. For the partially relativistic model, $A = 1.50$, $B = 6.91$, and $C = 0.689$. The maximum radius $R_{\rm max}$ of the star, corresponding to the radius of a Newtonian BEC given by eq. (47), can be written as

$$\frac{R_{\rm max}}{\rm km} = 17.0\left(\frac{a}{\rm fm}\right)^{1/2}\left(\frac{{\rm GeV}/c^2}{m}\right)^{3/2}. \tag{55}$$

Finally, the value of the mass-radius ratio $2GM/Rc^2 = 2.95(M/M_\odot)({\rm km}/R)$ of general relativistic BEC stars at the critical point is 0.319 in the fully relativistic model, 0.529 in the semi-relativistic model, and 0.429 in the partially relativistic model. It varies between $\sim 0.3$ and $\sim 0.5$ depending on the model. The mass-radius ratio is plotted as a function of $M/M_*$ in fig. 6 for the different models. We note that the value of $2GM/Rc^2$ at the critical point provides the maximum value of the mass-radius ratio for the stable part of the series of equilibria.

The observations of neutron stars compiled by Mukherjee *et al.* [123] give a value of the mass-radius ratio $2GM/Rc^2 \sim 0.5$ (see table 1). This is substantially larger than the value 0.319 predicted from the fully relativistic equation of state (B.16). In other words, the predicted radius of the neutron stars is larger than observed. This led

[6] We note that $R_{\rm min}$ and $(\epsilon_0)_{\rm max}$ depend only on $M_{\rm max}$ as a result of relativity. Furthermore, $M_{\rm max}$ depends only on $m$ and $a_s$ through the ratio $\kappa \propto a_s^{1/2}/m^{3/2}$. Therefore, there is only one free parameter $\kappa$ in the theory.

Mukherjee *et al.* [123] to conclude that the BEC model is ruled out. However, their conclusion may be too pessimistic because several effects can alter the equation of state of the BEC. For example, as they note, the interior of neutron stars could be a composition of BECs of kaons or pions. This may change the mass-radius relation of neutron stars. On the other hand, we note that the value 0.5 is relatively close to the values 0.429 and 0.529 obtained from the equations of state (B.26) and (B.30). This agreement is puzzling because these equations of state are less well justified theoretically than the equation of state (B.16). This may be a further motivation to study these equations of state, independently of the BEC model. We finally note that the value 0.369 obtained in [127] is relatively close to the value 0.319 of the fully relativistic BEC model. Additional observations may be necessary to reach definite conclusions.

### 3.4 On the maximum mass of neutron stars

In their seminal paper, Oppenheimer and Volkoff [93] modeled neutron stars as a completely degenerate ideal gas of relativistic fermions without self-interaction. In that case, gravitational collapse is prevented by the Pauli exclusion principle. Since these objects are very compact, one must use general relativity. Therefore, the equilibrium configurations of neutron stars in this model are obtained by solving the TOV equations (27) and (28) with the equation of state $P(\epsilon)$ corresponding to a relativistic fermionic gas at $T=0$. This equation of state is given in parametric form by [110]:

$$P = Af(x), \qquad \rho c^2 = 8Ax^3, \tag{56}$$

$$\epsilon = \rho c^2 + E_{\text{kin}} = 8A\left[x^3 + \frac{1}{8}g(x)\right], \quad A = \frac{\pi m_n^4 c^5}{3h^3}, \tag{57}$$

$$f(x) = x(2x^2-3)(x^2+1)^{1/2} + 3\sinh^{-1} x, \tag{58}$$

$$g(x) = 8x^3\left[(x^2+1)^{1/2} - 1\right] - f(x). \tag{59}$$

In the non-relativistic limit ($\epsilon \to 0$), we get

$$\epsilon \sim \rho c^2, \qquad P \sim \frac{1}{5}\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{m_n^{8/3} c^{10/3}} \epsilon^{5/3}, \qquad P \sim \frac{1}{5}\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{m_n^{8/3}} \rho^{5/3}, \tag{60}$$

corresponding to a polytrope $n = 3/2$. In the ultra-relativistic limit ($\epsilon \to +\infty$), we get

$$\epsilon \sim \frac{3}{4}\left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{m_n^{4/3}} \rho^{4/3}, \qquad P \sim \frac{1}{3}\epsilon, \qquad P \sim \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{m_n^{4/3}} \rho^{4/3}, \tag{61}$$

corresponding to a polytrope $n = 3$. In this limit, the equation of state is linear: $P \sim \epsilon/3$.

The mass-central density relation of fermion stars presents damped oscillations and the mass-radius relation has a snail-like (spiral) structure (see figs. 7 and 8) [93, 112–115]. The maximum mass, minimum radius, and maximum energy density are[7]

$$M_{\text{OV}} = 0.384\left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m_n^2} = 0.709\, M_\odot, \tag{62}$$

$$R_{\text{OV}} = 8.735 \frac{GM_{\text{OV}}}{c^2} = 9.15\,\text{km}, \tag{63}$$

$$(\epsilon_c)_{\text{OV}}/c^2 = 3.44 \times 10^{-3} \frac{c^6}{G^3 M_{\text{OV}}^2} = 4.22 \times 10^{15}\,\text{g/cm}^3. \tag{64}$$

The mass-radius ratio at the critical point is

$$\frac{2GM_{\text{OV}}}{R_{\text{OV}} c^2} = 0.229. \tag{65}$$

In this fermionic model, the maximum mass of a neutron star is determined by fundamental constants and by the mass $m_n$ of the neutrons. As a result, there is no indetermination and the maximum mass predicted by Oppenheimer and Volkoff [93] has a well-specified value $M_{\text{OV}} = 0.7\, M_\odot$.

[7] We note that Tooper [104], in his sect. IX.b, considers a simplified model of neutron stars by using the non-relativistic equation of state for fermions $P = (1/5)(3/8\pi)^{2/3} h^2/m_n^{8/3} \rho^{5/3}$ (corresponding to a polytrope of index $n = 3/2$) with the relativistic relation $\epsilon = \rho c^2 + (3/2)P$ between the energy density and the rest-mass density. This is the fermionic counterpart of the partially relativistic model of BEC stars (see appendix B.3).

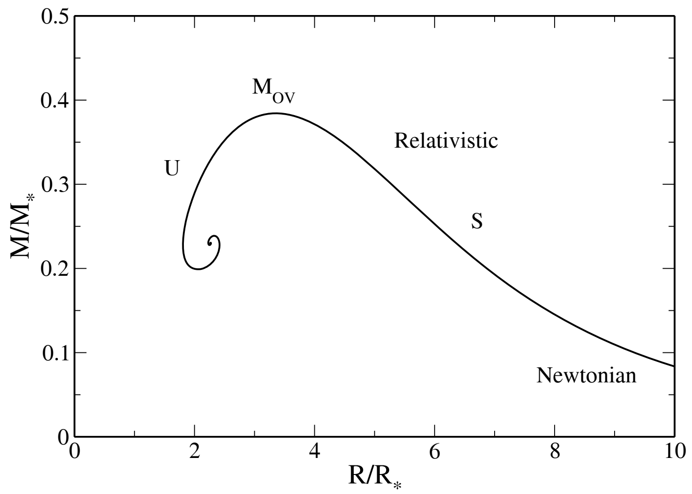


**Fig. 7.** Dimensionless mass-radius relation of neutron stars interpreted as fermion stars (Oppenheimer-Volkoff model) [93]. There exists a maximum mass $M_{\rm OV} = 0.384 M_*$, a minimum radius $R_{\rm OV} = 3.36 R_*$, and a maximum central energy density $\epsilon_{\rm OV}/c^2 = 2.33 \times 10^{-2} \epsilon_*$ with $M_* = (\hbar c/G)^{3/2}/m_n^2$, $R_* = GM_*/c^2 = (\hbar^3/Gc)^{1/2}/m_n^2$ and $\epsilon_*/c^2 = M_*/R_*^3 = m_n^4 c^3/\hbar^3$.

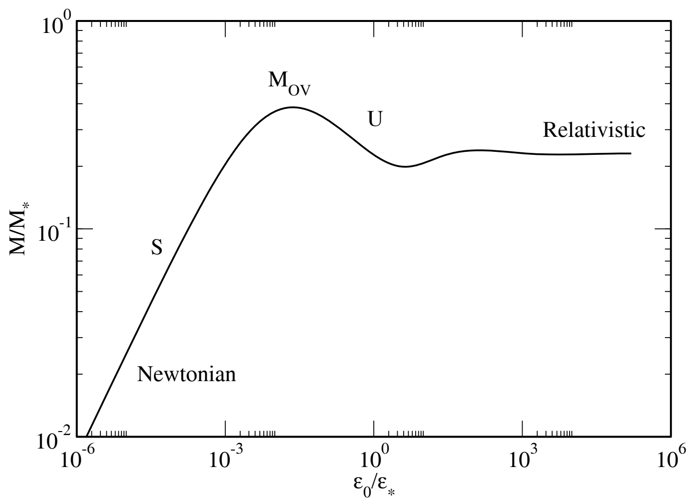


**Fig. 8.** Dimensionless mass-central energy density relation of neutron stars interpreted as fermion stars (Oppenheimer-Volkoff model) [93].

However, neutron stars with a mass in the range 2–2.4 $M_\odot$, well above the Oppenheimer-Volkoff limit, have recently been observed [94–99]. These observations question the validity of the fermionic model. Therefore, alternative models of neutron stars should be constructed[8]. In this respect, Chavanis and Harko [91] have proposed that, because of their superfluid cores, neutron stars could actually be BEC stars[9]. Indeed, the neutrons (fermions) could form Cooper pairs and behave as bosons of mass $m = 2m_n$, where $m_n = 0.940\,\mathrm{GeV}/c^2$ is the mass of the neutrons. They can then make a BEC through the BCS/BEC crossover mechanism. Since the maximum mass of BEC stars $M_{\rm max} = 0.307\,\hbar c^2 \sqrt{a_s}/(Gm)^{3/2}$ [91] depends on the scattering length $a_s$ that is not well known, it can be larger than the Oppenheimer-Volkoff limit $M_{\rm OV} = 0.376 M_P^3/m_n^2 = 0.7\,M_\odot$ [93] obtained by assuming that neutron stars can be modeled as an ideal gas of fermions. By taking a scattering length of the order of 10–20 fm (giving $\kappa \sim 3.16$–$4.47$), we obtain a maximum mass of the order of $2M_\odot$, a central density of the order of $1$–$3 \times 10^{15}\,\mathrm{g/cm}^3$, and a radius in the

[8] Another possibility is to remain within the fermionic model and take non-ideal effects into account [94,128]. Indeed, nuclear matter is a strongly interacting system and, because of self-interaction, neutrons have an effective mass of the order of 70% of their bare mass. This raises the maximum mass of the neutron stars from the Oppenheimer-Volkoff value up to 1.5–2 $M_\odot$ depending on the interaction model. The rotation of neutron stars (pulsars) has also the effect of increasing the value of the maximum mass [129].

[9] See appendix A for the motivations and the limitations of this model.

range 10–20 km [91]. This could account for the recently observed neutron stars with masses in the range 2–2.4 $M_\odot$ larger than the Oppenheimer-Volkoff limit. For $M < M_{\rm max}$, there exist stable equilibrium states of BEC stars with $R_{\rm min} < R < R_{\rm max}$ for which gravitational collapse is prevented by the pressure arising from the scattering length of the bosons. For $M > M_{\rm max}$, nothing prevents the gravitational collapse of the star that becomes a black hole.

It is interesting to come back to the analogies and differences between fermion and boson stars (in the fully relativistic model of [91]). In the ultra-relativistic limit, they are both described by a polytropic equation of state $P \sim K'\rho^{4/3}$ corresponding to an index $n = 3$ but the polytropic constant is different. In the case of fermions $K' = (1/4)(3/8\pi)^{1/3}hc/m^{4/3}$ and in the case of bosons $K' = (\pi/8)^{1/3}\hbar^{2/3}a_s^{1/3}c^{4/3}/m$ (see appendix B.2). In the two cases, the relation between the pressure and the energy density is $P \sim \epsilon/3$. This is the relation that enters in the TOV equation. This linear equation of state is responsible for the damped oscillations of the mass-central density relation and for the snail-like structure (spiral) of the mass-radius relation. In the non-relativistic limit, fermion stars are described by a polytropic equation of state $P \sim K\rho^{5/3}$ of index $n = 3/2$. The polytropic constant is $K = (1/5)(3/8\pi)^{2/3}h^2/m^{8/3}$. This leads to the mass-radius relation $MR^3 = 1.49 \times 10^{-3}\, h^6/(G^3m^8)$. Therefore, there exist configurations of arbitrarily large radius and arbitrarily small mass (see fig. 7). By contrast, in the non-relativistic limit, BEC stars are described by a polytropic equation of state $P = K\rho^2$ of index $n = 1$. The polytropic constant is $K = 2\pi a_s\hbar^2/m^3$. This fixes the radius of the configuration to the value $R = \pi(a_s\hbar^2/Gm^3)^{1/2}$. Therefore, there is no configuration of radius larger than this value (see fig. 5). This is a difference between fermion stars and BEC stars. On the other hand, in the case of ideal fermion stars, the equation of state depends (apart from fundamental constants) only on the mass $m$ of the fermions. For neutron stars, this is the mass of the neutrons $m_n$ whose value is perfectly known. Therefore, the maximum mass of neutron stars modeled as ideal fermion stars has an unambiguous value $0.7\, M_\odot$. By contrast, in the case of BEC stars, the equation of state depends on $m$ and $a_s$ through the combination $\kappa^2 \propto a_s/m^3$. As a result, the maximum mass of neutron stars modeled as BEC stars depends on this parameter $\kappa$ (compare eqs. (50) and (62)). Since the value of this parameter is not well known, it may be possible to overcome the Oppenheimer-Volkoff limit.

### 3.5 An upper limit on the maximum mass of relativistic stars

In sect. 3.2 we have shown that partially relativistic BEC stars have a stiff equation of state at high densities. The stiff equation of state $P = \epsilon$ was introduced by Zel'dovich [105] in the context of cold baryon stars. Although Zel'dovich's model is now outdated, it remains important from a historical perspective. It provides a simple example of models where self-interaction between particles is included. In addition, since it is based on the stiffest equation of state consistent with the requirements of relativity (the speed of sound cannot be larger than the speed of light), it can be used to obtain an upper limit on the maximum mass of relativistic stars. Therefore, we devote a short paragraph to determine the maximum mass of relativistic stars in Zel'dovich's model.

Zel'dovich [105] considered a gas of baryons interacting through a vector meson field and showed that the equation of state of this system is of the form of eq. (32) with a polytropic constant

$$K = \frac{g^2h^2}{2\pi m_m^2 m_b^2 c^2}\,, \tag{66}$$

where $g$ is the baryon charge, $m_m$ is the meson mass, and $m_b$ is the baryon mass. Zel'dovich [105] introduced this equation of state as an example to show how the speed of sound could approach the speed of light at very high pressures and densities (see appendix B.3).

The equation of state (32) has been studied by Tooper [104] in relation to baryon stars (see his sect. IX.c). Our treatment is a little more accurate and provides the following values for the maximum mass, minimum radius, and maximum density of baryon stars:

$$M_{\rm max} = 0.4104\frac{g\hbar c}{m_m m_b G^{3/2}} = 3.80\, M_\odot, \tag{67}$$

$$R_{\rm min} = 1.914\frac{g\hbar}{m_m m_b c G^{1/2}} = 26.2\,{\rm km}, \tag{68}$$

$$(\rho_0)_{\rm max} = 0.318\frac{m_m^2 m_b^2 c^4}{2\pi g^2\hbar^2} = 3.64\times 10^{14}\,{\rm g/cm^3}. \tag{69}$$

The maximum energy density is $(\epsilon_0)_{\rm max}/c^2 = 1.32\,(\rho_0)_{\rm max} = 4.80\times 10^{14}\,{\rm g/cm^3}$. To make the numerical application, we have taken $g^2/hc \sim 1$, $m_b \sim m_n$, and $m_m = m_b/2$ [104]. The maximum mass $M_{\rm max} = 3.80\, M_\odot$ obtained from Zel'dovich's model represents an upper limit on the maximum mass of relativistic stars. It is of the order of the maximum mass $M_{\max} = 3.2\, M_\odot$ obtained by Rhoades and Ruffini [130]. Of course, our estimate is very qualitative so the numerical value of this upper bound could be refined.

Zel'dovich's model is also considered in sect. 4.2 of [113] in the case where the equation of state (32) is approximated by its asymptotic form $P = \epsilon$ valid at high densities, and the system is enclosed within a spherical box of radius $R$ to make its mass finite. In this simplified setting, it is found that the critical mass-radius ratio $2GM/Rc^2$ is equal to 0.544 instead of 0.429 (see sect. 3.2). The agreement is relatively satisfying in view of the crudeness of the box model. Some analogies between stiff stars and black holes are pointed out in [113].

## 4 Cosmology of a BEC fluid with a stiff equation of state

Harko [106] and Chavanis [70] considered the possibility that dark matter is made of self-interacting BECs and independently studied the cosmological implications of this model[10]. If dark matter is made of BECs, it has a non-vanishing pressure even at $T = 0$, unlike the CDM model. This affects the evolution of the scale factor of the universe. In most applications, the pressure arises from the self-interaction (the quantum pressure due to the Heisenberg uncertainty principle is negligible) so we can make the TF approximation. Harko [106] and Chavanis [70] considered a polytropic equation of state of the form of eq. (B.30) and solved the corresponding Friedmann equations. However, this equation of state is not valid in the strongly relativistic regime so the extrapolation of their results to the very early universe is not correct[11]. In this section, we solve the Friedmann equations with the equation of state (B.26). It leads to very different results in the early universe showing that the precise form of the equation of state of the BEC is crucial in cosmology[12]. We stress that the equation of state (B.26) is itself not exact so the results of this section should be considered with caution. However, it is interesting to compare the effect of different equations of state on the evolution of the universe. Furthermore, the equation of state (B.26) is interesting because it leads to a cosmological model involving a stiff matter era. The same stiff matter era occurs in the cosmological model of Zel'dovich [107] where the early universe is assumed to be made of a gas of cold baryons. Even though Zel'dovich's model has been abandoned, it remains important from a historical perspective[13], so we shall briefly discuss it. The comparison between the different equations of state of BEC dark matter is made in sect. 4.4.

### 4.1 The Friedmann equations

We assume that the universe is homogeneous and isotropic, and contains a uniform perfect fluid of energy density $\epsilon(t)$ and isotropic pressure $P(t)$. The radius of curvature of the 3-dimensional space, or scale factor, is noted $a(t)$ and the curvature of space is noted $k$. The universe is closed if $k > 0$, flat if $k = 0$, and open if $k < 0$. We assume that the universe is flat ($k = 0$) in agreement with the observations of the cosmic microwave background (CMB) [136]. In that case, the Einstein equations can be written as [137]

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0, \tag{70}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\epsilon + 3P) + \frac{\Lambda}{3}, \tag{71}$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\epsilon + \frac{\Lambda}{3}, \tag{72}$$

where we have introduced the Hubble parameter $H = \dot{a}/a$ and accounted for a possible non-zero cosmological constant $\Lambda$. The cosmological constant is equivalent to a fluid with a constant energy density (dark energy)

$$\epsilon_\Lambda = \rho_\Lambda c^2 = \frac{\Lambda c^2}{8\pi G} \tag{73}$$

[10] Harko [106] considered repulsive self-interactions while Chavanis [70] considered repulsive and attractive self-interactions.

[11] Note that the equation of state (B.30) is interesting in its own right in cosmology. Generalized polytropic equations of state of the form $P = \alpha\epsilon + k\epsilon^{1+1/n}$ have been studied in full generality in refs. [131–134]. For a negative polytropic pressure ($k < 0$), they lead to interesting cosmological models exhibiting a phase of early inflation and a phase of late accelerating expansion bridged by a phase of decelerating expansion. However, the justification of these equations of state may not be connected with BECs as initially thought.

[12] It is less crucial in the context of BEC stars since the precise form of the equation of state only slightly changes the prefactor of the maximum mass (see sect. 3.3).

[13] In addition, a stiff equation of state has several interesting properties in cosmology [135] that deserve to be better explored.

and an equation of state $P = -\epsilon$. Equations (70)–(72) are the well-known Friedmann equations describing a non-static universe. Among these three equations, only two are independent. The first equation can be viewed as an equation of continuity. For a given barotropic equation of state $P = P(\epsilon)$, it determines the relation between the energy density $\epsilon$ and the scale factor $a$. Then, the evolution of the scale factor $a(t)$ is given by eq. (72).

### 4.2 The equation of state of a partially relativistic BEC fluid

We assume that dark matter is made of a fluid at $T = 0$, or an adiabatic fluid, with an equation of state $P(\rho)$. In that case, the relation between the energy density $\epsilon$ and the rest-mass density $\rho$ is given by the first law of relativistic thermodynamics (see appendix B.1):

$$\mathrm{d}\epsilon = \frac{P+\epsilon}{\rho}\mathrm{d}\rho. \tag{74}$$

Combining this relation with the continuity equation (70), we get

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + 3\frac{\dot{a}}{a}\rho = 0. \tag{75}$$

We note that this equation is exact for a fluid at $T = 0$ (or for an isentropic fluid) and that it does not depend on the explicit form of the equation of state $P(\rho)$. It can be integrated into

$$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3, \tag{76}$$

where $\rho_0$ is the present value of the rest-mass density and $a_0$ is the present value of the scale factor. This equation expresses the conservation of the rest-mass density (or number density $n = \rho/m$).

We now assume that dark matter is made of BECs at $T = 0$ described by the equation of state

$$P = K\rho^2, \tag{77}$$

where $K$ is given by eq. (10). This polytropic equation of state of index $n = 1$ corresponds to the partially relativistic model of appendix B.3. For the equation of state (77), eq. (74) can be integrated easily. The relation between the energy density and the rest-mass density is given by (see appendix B.1):

$$\epsilon = \rho c^2 + K\rho^2. \tag{78}$$

Combining eqs. (76) and (78), we get

$$\epsilon = \rho_0 c^2 \left(\frac{a_0}{a}\right)^3 + K\rho_0^2 \left(\frac{a_0}{a}\right)^6. \tag{79}$$

This relation can also be obtained by solving the continuity equation (70) with the equation of state (B.26) as detailed in appendix E.

In the early universe ($a \to 0$), we have

$$\epsilon \sim K\rho_0^2 \left(\frac{a_0}{a}\right)^6, \qquad \epsilon \sim K\rho^2, \qquad P \sim \epsilon. \tag{80}$$

These equations describe a stiff fluid ($P = \epsilon$) for which the speed of sound is equal to the speed of light.

In the late universe ($a \to +\infty$), we have

$$\epsilon \sim \rho_0 c^2 \left(\frac{a_0}{a}\right)^3, \qquad \epsilon \sim \rho c^2, \qquad P \sim \frac{K}{c^4}\epsilon^2. \tag{81}$$

These equations describe a classical BEC fluid with a polytropic equation of state of index $n = 1$ ($P = K\epsilon^2/c^4$). Actually, for very large values of the scale factor, we recover the results of the CDM model ($P = 0$) since $\epsilon \propto a^{-3}$.

### 4.3 The solution of the Friedmann equation for a partially relativistic BEC fluid

Substituting eq. (79) in the Friedmann equation (72), we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\left[\rho_0 c^2 \left(\frac{a_0}{a}\right)^3 + K\rho_0^2 \left(\frac{a_0}{a}\right)^6 + \rho_\Lambda c^2\right]. \tag{82}$$

This first-order differential equation determines the temporal evolution of the scale factor $a(t)$. Its formal solution is

$$\int_0^{a(t)/a_0} \frac{\mathrm{d}x}{x\sqrt{\frac{\rho_0 c^2}{x^3}+\frac{K\rho_0^2}{x^6}+\rho_\Lambda c^2}} = \left(\frac{8\pi G}{3c^2}\right)^{1/2} t. \tag{83}$$

This equation has the same form as eq. (F.5) obtained under the assumption that the universe is made of three non-interacting fluids corresponding to stiff matter, dust matter, and dark energy. Therefore, we can immediately transpose the results of appendix F to the present context. The universe starts from a primordial singularity and it successively undergoes a stiff matter era ($\epsilon \propto a^{-6}$), a dust matter era ($\epsilon \propto a^{-3}$), and a dark energy era ($\epsilon \simeq \rho_\Lambda c^2$).

The evolution of the scale factor is explicitly given by

$$\frac{a}{a_0} = \left[\left(\frac{\rho_0}{\rho_\Lambda} + 2\sqrt{\frac{K\rho_0^2}{\rho_\Lambda c^2}}\right)\sinh^2\left(\sqrt{6\pi G\rho_\Lambda}\, t\right) + \sqrt{\frac{K\rho_0^2}{\rho_\Lambda c^2}}\left(1 - e^{-2\sqrt{6\pi G\rho_\Lambda}\, t}\right)\right]^{1/3}. \tag{84}$$

The evolution of the energy density is given by

$$\epsilon = \left[\left(\frac{1}{2}\sqrt{\frac{\rho_0^2 c^2}{\rho_\Lambda}} + \sqrt{K\rho_0^2}\right)\sinh\left(2\sqrt{6\pi G\rho_\Lambda}\, t\right) + \sqrt{K\rho_0^2}\, e^{-2\sqrt{6\pi G\rho_\Lambda}\, t}\right]^2 \Big/ \left(\frac{a}{a_0}\right)^6. \tag{85}$$

This is a simple generalization of the $\Lambda$CDM model for a BEC universe assumed to be described by the equation of state (77). For $t \to +\infty$, we recover the de Sitter solution

$$\frac{a}{a_0} \sim \left(\frac{\rho_0}{\rho_\Lambda} + 2\sqrt{\frac{K\rho_0^2}{\rho_\Lambda c^2}}\right)^{1/3} \frac{1}{2^{2/3}} e^{\sqrt{\frac{8\pi G\rho_\Lambda}{3}}\, t}, \tag{86}$$

with a prefactor affected by the BEC.

In the absence of a cosmological constant ($\rho_\Lambda = 0$), the solution of eq. (82) is

$$\frac{a}{a_0} = \left(6\pi G\rho_0 t^2 + 2\sqrt{\frac{6\pi G K\rho_0^2}{c^2}}\, t\right)^{1/3}, \tag{87}$$

$$\epsilon = \frac{c^2}{6\pi G t^2}\left(\frac{1+\sqrt{\frac{K}{6\pi G}}\frac{1}{ct}}{1+2\sqrt{\frac{K}{6\pi G}}\frac{1}{ct}}\right)^2. \tag{88}$$

This is a simple generalization of the Einstein-de Sitter (EdS) model for a BEC universe assumed to be described by the equation of state (77).

Returning to the case $\Lambda \geq 0$ and considering the formal limit $K \to +\infty$, equivalent to the case of appendix F where we can neglect dust matter in front of stiff matter, we get

$$\frac{a}{a_0} = \left(\frac{K\rho_0^2}{\rho_\Lambda c^2}\right)^{1/6} \sinh^{1/3}\left(2\sqrt{6\pi G\rho_\Lambda}\, t\right), \tag{89}$$

$$\epsilon = \frac{\rho_\Lambda c^2}{\sinh^2\left(2\sqrt{6\pi G\rho_\Lambda}\, t\right)}. \tag{90}$$

In the absence of cosmological constant ($\Lambda = 0$), the foregoing equations reduce to

$$\frac{a}{a_0} = \left(2\sqrt{\frac{6\pi G K\rho_0^2}{c^2}}\, t\right)^{1/3}, \qquad \epsilon = \frac{c^2}{24\pi G t^2}. \tag{91}$$

This corresponds to a model of universe dominated by stiff matter.

We can also recover well-known models as particular cases of the foregoing equations. In the absence of BECs ($K=0$), we recover the $\Lambda$CDM model

$$\frac{a}{a_0} = \left(\frac{\rho_0}{\rho_\Lambda}\right)^{1/3} \sinh^{2/3}\left(\sqrt{6\pi G\rho_\Lambda}\, t\right), \tag{92}$$

$$\epsilon = \frac{\rho_\Lambda c^2}{\sinh^2\left(\sqrt{6\pi G\rho_\Lambda}\, t\right)}. \tag{93}$$

In the absence of BECs and cosmological constant ($K=\Lambda=0$), we recover the EdS universe

$$\frac{a}{a_0} = \left(6\pi G\rho_0 t^2\right)^{1/3}, \qquad \epsilon = \frac{c^2}{6\pi G t^2}. \tag{94}$$

The equation of state (77), with a polytropic constant $K$ given by eq. (66), also appears in the cosmological model of Zel'dovich [107] where the early universe is assumed to be made of a cold gas of baryons. This model presents a stiff matter era that follows the cosmological singularity (Big Bang). Actually, the complete equation of state in Zel'dovich's model is of the form

$$P = K\rho^2 + K'\rho^{4/3}, \tag{95}$$

where the first term is the pressure coming from the self-interaction between the particles and the second term is the quantum (Fermi) pressure. For the equation of state (95), we find from eq. (B.14) that the relation between the energy density and the rest-mass density is

$$\epsilon = \rho c^2 + K\rho^2 + 3K'\rho^{4/3}. \tag{96}$$

Substituting eq. (76) in eq. (96), we obtain

$$\epsilon = \rho_0 c^2 \left(\frac{a_0}{a}\right)^3 + K\rho_0^2 \left(\frac{a_0}{a}\right)^6 + 3K'\rho_0^{4/3} \left(\frac{a_0}{a}\right)^4. \tag{97}$$

When combined with the Friedmann equation (72), we obtain a model of universe exhibiting a stiff matter era ($\epsilon \propto a^{-6}$), a radiation era ($\epsilon \propto a^{-4}$), a dust matter era ($\epsilon \propto a^{-3}$), and a dark energy era ($\epsilon \sim \rho_\Lambda c^2$) as discussed in appendix F.

*Remark.*

In this paper, we have considered the case of a repulsive self-interaction ($K \geq 0$). The case of an attractive self-interaction ($K<0$) is treated in [138]. In that case, the primordial universe is non-singular and bouncing. We have also assumed that the cosmological constant is positive. The case of a negative cosmological constant, leading to an oscillatory universe, is also considered in [138].

### 4.4 Comparison between the different models of BEC cosmology

We now compare the different models of BEC cosmology depending on the considered equation of state.

Harko [106] and Chavanis [70] assumed that BEC dark matter is described by an equation of state of the form of eq. (B.30) and solved the corresponding Friedmann equations. This corresponds to the semi-relativistic model of appendix B.4. For a repulsive self-interaction ($a_s > 0$) they found that the universe starts with a new form of singularity in which the energy density is infinite while the scale factor is finite. At sufficiently late times, the universe returns the usual $\Lambda$CDM model in which the universe experiences an EdS era ($a \propto t^{2/3}$, $\epsilon \propto t^{-2}$) followed by a de Sitter era ($a \propto e^{\sqrt{\Lambda/3}t}$, $\epsilon = \rho_\Lambda c^2$). However, this model differs from the $\Lambda$CDM model in the intermediate era because of the contribution of the BEC. In particular, it is found that the scale factor increases more rapidly when dark matter is made of BEC instead of pressureless matter [70, 106].

In this paper, we have assumed that BEC dark matter is described by the equation of state (B.23). This corresponds to the partially relativistic model of appendix B.3. Solving the corresponding Friedmann equation, we have found that the early universe behaves as a stiff fluid ($P \sim \epsilon$) due to the self-interaction of the bosons. The scale factor increases as $a \propto t^{1/3}$ while the energy density decreases as $\epsilon \propto t^{-2}$. The universe starts from a singularity at $t=0$ in which the energy density is infinite while the scale factor vanishes. At later times, the universe behaves as a non-relativistic BEC with an equation of state $P \sim K\epsilon^2/c^4$.

We stress, however, that the previous models are incorrect in the very early universe because they use an approximate relativistic equation of state. A better model of relativistic BEC dark matter should be based on the equation of state (B.16) arising from the Klein-Gordon-Einstein equations [75]. This corresponds to the fully relativistic model of appendix B.2. In that case, the early universe has an equation of state $P \sim \epsilon/3$ similar to the equation of state of radiation. The scale factor increases as $a \propto t^{1/2}$ while the energy density decreases as $\epsilon \propto t^{-2}$. The universe starts from a singularity at $t = 0$ in which the energy density is infinite while the scale factor vanishes. At later times, the universe behaves as a non-relativistic BEC with an equation of state $P \sim K\epsilon^2/c^4$.

Actually, the fully relativistic BEC model has been recently studied by Li *et al.* [108] using a scalar field theory based of the Klein-Gordon-Einstein equations. They found that the universe undergoes successively a primordial stiff matter era (due to the intrinsic properies of the scalar field), followed by a radiation era (due to the self-interaction of the scalar field), and finally a matter era. The stiff matter era (described by an equation of state $P = \epsilon$) occurs when the oscillations of the scalar field are slower than the Hubble expansion while the radiation and matter eras (described by the equation of state (B.16)) occur when the oscillations of the scalar field are faster than the Hubble expansion. The same model has been investigated by Suárez and Chavanis [109] using a hydrodynamical representation of the Klein-Gordon-Einstein equations. It is important to note that the justification of the stiff matter era in the fully relativistic BEC model is different from that given in the partially relativistic model. However, the analytical results obtained in appendix F for a cosmology including a stiff matter era, a radiation era, a matter era, and a dark matter era may be useful in the context of the fully relativistic BEC model [108, 109], even if the equations determining the transitions between these different phases are different in the two models[14].

*Remark.*

Although the evolution of the early universe is very sensitive to the equation of state of the BEC, it should be recalled that BECs do form only when the temperature has sufficiently decreased. Therefore, we should be careful when extrapolating the solutions to the early universe. If we view BEC dark matter as a small correction to pressureless matter ($\Lambda$CDM model), all the equations of state of appendix B reduce to $P \sim K\epsilon^2/c^4$ for sufficiently late times and give equivalent results. In this sense, the cosmological models of Harko [106] and Chavanis [70] are justified for sufficiently late times after the primordial singularity and after the appearance of BECs.

## 5 Conclusion

In this paper, we have compared different models of relativistic self-gravitating BECs.

Concerning general relativistic BEC stars, we have shown that the partially relativistic model of appendix B.3 (leading to a stiff equation of state at high densities) gives a maximum mass that is smaller than the one obtained from the semi-relativistic model of appendix B.4 but larger than the one obtained from the fully relativistic model of appendix B.2. However, the difference is relatively small (the main indetermination on the maximum mass being the value of the scattering length of the particles) so that the three models provide a fair description of general relativistic BEC stars. Of course, the fully relativistic treatment is the most relevant one on a physical point of view. However, the observed mass-radius ratio of neutron stars seems to be closer to the value obtained from the partially relativistic model or to the value obtained from the semi-relativistic model than to the value obtained from the fully relativistic model. Therefore, the equations of state (B.26) and (B.30) may be useful to describe neutron stars, independently of the BEC model. In this respect, we can note that they are special cases of the two polytropic models developed by Tooper [103, 104] for an index $n = 1$. However, more observations may be necessary to determine the precise value of the mass-radius ratio of neutron stars and ascertain these conclusions. We also stress that the application of the BEC model to neutron stars, although interesting in itself [91], has some limitations discussed in appendix A.

Concerning the evolution of a universe made of BEC dark matter, the precise form of the equation of state is crucial in the very early universe. If dark matter is described by the equation of state (B.30), corresponding to the semi-relativistic model, the universe starts from a primordial singularity in which the scale factor is finite and the density is infinite. On the other hand, if dark matter is described by the equation of state (B.26) corresponding to the partially relativistic model, the early universe undergoes a stiff matter era in which the scale factor increases as $a \propto t^{1/3}$ and the energy density decreases as $\epsilon \propto a^{-6}$. Finally, if dark matter is described by the equation of state (B.16) corresponding to the fully relativistic model, the early universe undergoes a radiation era in which the scale factor increases as $a \propto t^{1/2}$ and the energy density decreases as $\epsilon \propto a^{-4}$. Actually, in the fully relativistic model [108, 109], the universe first undergoes an intrinsic stiff matter era, followed by a radiation era, and finally by a matter era. In principle, only the fully relativistic model is relevant in the very early universe. However, at later times, all the models give equivalent results.

[14] For a relativistic scalar field without self-interaction (fuzzy dark matter), there is only a stiff matter era, a matter era, and a dark energy era (no radiation era). This situation is similar to the one leading to the analytical solutions considered at the beginning of appendix F.

## Appendix A. On the relevance of Bose-Einstein condensation in neutron stars

The possibility that the core of neutron stars is made of BECs has been proposed recently by Chavanis and Harko [91], and subsequently considered by several authors [123, 139–142][15]. In this appendix, we regroup several arguments to support the BEC model for neutron stars, but also stress its limitations.

Superconductivity and superfluidity are believed to exist in neutron stars [143, 144]. The phenomenon of superfluidity is usually related to Bose-Einstein condensation. Indeed, a strongly correlated pair of fermions can be treated approximately like a boson, leading to superfluidity. Similarly, superconductivity can be described as the condensation of electrons or holes into Cooper pairs. The possibility of BECs in neutron stars is discussed in the book of Glendenning [111]. The BECs could be made of negatively charged mesons, kaons/anti-kaons [145, 146], pions [147–151], or H-dibaryon [152, 153] made of hyperons [154, 155]. Neutrino superfluidity, as suggested by Kapusta [156], may also lead to Bose-Einstein condensation [157].

The neutrons inside neutron stars are likely to form pairs due to the strong nuclear forces between them, giving rise to a superfluid state. The particles are bound in Cooper pairs and can be treated as composite bosons with an effective mass $m = 2m_n$, which can form a BEC. The treatment of superfluidity in nuclear matter is well established [158]. A microscopically exact way of treating Bose-Einstein condensation in nuclear matter is provided by the theory of BCS-BEC crossover [159, 160], which describes a transition from the quantum state of superfluidity (BCS phase) to a BEC. The theory describes the pairing mechanism between neutrons, allowing for a coexistence of single neutrons and neutron pairs in a mixed state of Fermi and Bose fluids [161]. If the attractive interaction between particles is strong, the fermions may condense into the bosonic zero mode, forming a BEC. If the interaction is weak, the fermions exist in a BCS state. Nishida and Abuki [162] explain the basic concepts of this crossover and show that the BCS and BEC states are smoothly connected without a phase transition.

In their study, Chavanis and Harko [91] assume that the pairing of the neutrons is strong enough to be able to consider them as perfectly bosonic, and do not take into account the presence of single neutrons. By contrast, in the literature, it is usually considered that the coherence length of the superfluid is much larger than the interparticle distance so that the high-density neutron matter (above the nuclear matter density) is deep in the BCS regime of pairing. For a realistic description, it is necessary to set up the exact theory of the BCS-BEC crossover. The cases of a pure BCS phase and a pure BEC phase are obtained as limits of this general crossover theory. Therefore, considering the limit of a BEC fluid is a first step towards a unifying crossover theory. The full crossover would unify the different physical behavior of the BCS and BEC regimes into one theory, and would apply to both fermion and boson stars simultaneously in the respective limits of the theory. Therefore, the simplified approach of Chavanis and Harko [91] can be regarded as a first approximate solution of this general problem.

## Appendix B. The equation of state of a relativistic BEC

### Appendix B.1. General results

The local form of the first law of thermodynamics can be expressed as

$$\mathrm{d}\left(\frac{\epsilon}{\rho}\right) = -P\mathrm{d}\left(\frac{1}{\rho}\right) + T\mathrm{d}\left(\frac{s}{\rho}\right), \tag{B.1}$$

where $\rho = nm$ is the mass density, $n$ is the number density, and $s$ is the entropy density in the rest frame. For a system at $T = 0$, or for an adiabatic evolution such that $\mathrm{d}(s/\rho) = 0$ (which is valid for a perfect fluid [137]), the first law of thermodynamics reduces to

$$\mathrm{d}\epsilon = \frac{P+\epsilon}{\rho}\mathrm{d}\rho. \tag{B.2}$$

For a given equation of state, eq. (B.2) can be integrated to obtain the relation between the energy density $\epsilon$ and the rest-mass density $\rho$.

If the equation of state is prescribed under the form $P = P(\epsilon)$, eq. (B.2) can be immediately integrated into

$$\ln\rho = \int \frac{\mathrm{d}\epsilon}{P(\epsilon)+\epsilon}\,. \tag{B.3}$$

[15] The possibility of the existence of Bose-Einstein condensed matter inside compact astrophysical objects such as neutron stars, or even the existence of stars formed entirely from a BEC, cannot be excluded a priori. In addition, the BEC model is interesting in its own right, and for its possible application to other systems such as boson stars, dark matter stars, the core of dark matter halos, and even black holes [15].

If, as an example, we consider the "gamma law" equation of state [163, 164]:

$$P = (\gamma - 1)\epsilon, \tag{B.4}$$

we get

$$P = K\rho^\gamma, \qquad \epsilon = \frac{K}{\gamma - 1}\rho^\gamma\,, \tag{B.5}$$

where $K$ is a constant of integration.

We now assume that the equation of state is prescribed under the form $P = P(\rho)$. In that case, eq. (B.2) reduces to the first-order linear differential equation

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho} - \frac{1}{\rho}\epsilon = \frac{P(\rho)}{\rho}\,. \tag{B.6}$$

Using the method of the variation of the constant, we obtain

$$\epsilon = A\rho c^2 + \rho\int^\rho \frac{P(\rho')}{\rho'^2}\,\mathrm{d}\rho', \tag{B.7}$$

where $A$ is a constant of integration.

As an example, we consider the polytropic equation of state [110]:

$$P = K\rho^\gamma, \qquad \gamma = 1 + \frac{1}{n}\,. \tag{B.8}$$

For $\gamma = 1$, we get

$$\epsilon = A\rho c^2 + K\rho\ln\rho. \tag{B.9}$$

For $\gamma \neq 1$, we obtain

$$\epsilon = A\rho c^2 + \frac{K}{\gamma - 1}\rho^\gamma = A\rho c^2 + nP. \tag{B.10}$$

Taking $A = 0$, we recover eqs. (B.4) and (B.5). We now assume $n > 0$ (*i.e.* $\gamma > 1$). In that case, we determine the constant $A$ by requiring that $\epsilon \sim \rho c^2$ when $\rho \to 0$. This gives $A = 1$. As a result, eq. (B.10) takes the form

$$\epsilon = \rho c^2 + \frac{K}{\gamma - 1}\rho^\gamma = \rho c^2 + nP. \tag{B.11}$$

For $\rho \to 0$ (non-relativistic limit), we get

$$\epsilon \sim \rho c^2, \qquad P \sim K(\epsilon/c^2)^\gamma. \tag{B.12}$$

For $\rho \to +\infty$ (ultra-relativistic limit), we get

$$\epsilon \sim nK\rho^\gamma, \qquad P \sim \epsilon/n \sim (\gamma - 1)\epsilon. \tag{B.13}$$

These limits are reversed when $n < 0$ (*i.e.* $\gamma < 1$).

For a general equation of state $P(\rho)$ such that $P \sim \rho^\gamma$ with $\gamma > 1$ when $\rho \to 0$, we determine the constant $A$ in eq. (B.7) by requiring that $\epsilon \sim \rho c^2$ when $\rho \to 0$. This gives[16]

$$\epsilon = \rho c^2 + \rho\int_0^\rho \frac{P(\rho')}{\rho'^2}\,\mathrm{d}\rho' = \rho c^2 + u(\rho). \tag{B.14}$$

We note that $u(\rho)$ may be interpreted as an internal energy density [121]. Therefore, the energy density $\epsilon$ is the sum of the rest-mass energy $\rho c^2$ and the internal energy $u(\rho)$. The rest-mass energy is positive while the internal energy can be positive or negative. Of course, the total energy $\epsilon = \rho c^2 + u(\rho)$ is always positive.

*Remark.*

According to eq. (B.2), the speed of sound defined by $c_s^2/c^2 = P'(\epsilon)$ satisfies the identity

$$\frac{c_s^2}{c^2} = \frac{\rho\frac{\mathrm{d}^2\epsilon}{\mathrm{d}\rho^2}}{\frac{\mathrm{d}\epsilon}{\mathrm{d}\rho}}\,. \tag{B.15}$$

[16] More generally, we take $A = 1$ in eq. (B.7) and set the constant of integration equal to zero.

### Appendix B.2. Fully relativistic model

We consider the equation of state

$$P=\frac{c^4}{36K}\left(\sqrt{1+\frac{12K}{c^4}\epsilon}-1\right)^2, \tag{B.16}$$

where $K$ is given by eq. (10). This equation of state can be derived from the Klein-Gordon-Einstein equations of a self-interacting scalar field in the strong coupling limit [75]. It also applies to a relativistic self-interacting BEC at $T=0$ in the TF approximation [91]. It provides a fully relativistic BEC model. For $\epsilon\to 0$ (non-relativistic limit), we recover the polytropic equation of state $P=K(\epsilon/c^2)^2$ of a classical BEC. For $\epsilon\to+\infty$ (ultra-relativistic limit), we obtain a linear equation of state $P=\epsilon/3$ similar to the one describing the core of neutron stars modeled by the ideal Fermi gas (see sect. 3.4) [93, 112–115].

For the equation of state (B.16), eq. (B.3) becomes

$$\frac{1}{3}\ln\rho=\int^{12K\epsilon/c^4}\frac{\mathrm{d}x}{\left(\sqrt{1+x}-1\right)^2+3x}. \tag{B.17}$$

Using the identity

$$\int\frac{\mathrm{d}x}{(\sqrt{1+x}-1)^2+3x}=\frac{1}{3}\ln\left(\sqrt{1+x}-1\right)+\frac{1}{6}\ln\left(1+2\sqrt{1+x}\right), \tag{B.18}$$

and requiring that $\epsilon\sim\rho c^2$ for $\rho\to 0$, we obtain the following relation between the rest-mass density and the energy density

$$\rho=\frac{c^2}{6\sqrt{3}K}\left(\sqrt{1+\frac{12K}{c^4}\epsilon}-1\right)\left[1+2\sqrt{1+\frac{12K}{c^4}\epsilon}\right]^{1/2}. \tag{B.19}$$

For $\rho\to 0$ (non-relativistic limit), we get

$$\epsilon\sim\rho c^2,\qquad P\sim\frac{K}{c^4}\epsilon^2,\qquad P\sim K\rho^2, \tag{B.20}$$

corresponding to a polytrope $n=1$. This returns the equation of state (8) of a classical BEC. For $\rho\to+\infty$ (ultra-relativistic limit), we get

$$\epsilon\sim\frac{3c^{4/3}K^{1/3}}{2^{4/3}}\rho^{4/3},\qquad P\sim\frac{1}{3}\epsilon,\qquad P\sim\frac{c^{4/3}K^{1/3}}{2^{4/3}}\rho^{4/3}, \tag{B.21}$$

corresponding to a polytrope $n=3$. This is similar to the equation of state of an ultra-relativistic Fermi gas at $T=0$ (core of neutron star) but the polytropic constant is different (see sect. 3.4).

For the equation of state (B.16), the speed of sound is given by

$$\frac{c_s^2}{c^2}=P'(\epsilon)=\frac{1}{3}\left(1-\frac{1}{\sqrt{1+12K\epsilon/c^4}}\right). \tag{B.22}$$

We always have $c_s<c$. For $\epsilon\to+\infty$, $c_s\to c/\sqrt{3}$.

### Appendix B.3. Partially relativistic model

We consider the equation of state

$$P=K\rho^2, \tag{B.23}$$

where $K$ is given by eq. (10). This equation of state can be derived from the classical GP equation. It describes a non-relativistic self-interacting BEC at $T=0$ in the TF approximation. We assume that this relation remains valid in the relativistic regime. This is not exact but it provides a partially relativistic BEC model.

Since the equation of state (B.23) corresponds to a polytrope $n=1$, eq. (B.11) reduces to

$$\epsilon=\rho c^2+P=\rho c^2+K\rho^2. \tag{B.24}$$

This equation can be reversed to give

$$\rho=\frac{c^2}{2K}\left(\sqrt{1+\frac{4K\epsilon}{c^4}}-1\right). \tag{B.25}$$

Combining eqs. (B.23) and (B.25), we obtain the relation between the pressure and the energy density

$$P = \frac{c^4}{4K}\left(\sqrt{1+\frac{4K\epsilon}{c^4}}-1\right)^2. \tag{B.26}$$

This equation of state has a form similar to eq. (B.16) but the coefficients are different (see appendix D). We note that eq. (B.23) with eq. (B.24) is a particular case of the class of equations of state (of type II) studied by Tooper [104] in general relativity. For $\epsilon \to 0$ (non-relativistic limit), we recover the polytropic equation of state $P = K(\epsilon/c^2)^2$ of a classical BEC. For $\epsilon \to +\infty$ (ultra-relativistic limit), we obtain a linear equation of state $P = \epsilon$. This is a stiff equation of state in which the speed of sound is equal to the speed of light ($c_s = c$). This type of equations of state has been introduced by Zel'dovich [105] in the context of baryon stars in which the baryons interact through a vector meson field (see sect. 3.5).

For $\rho \to 0$ (non-relativistic limit), we get

$$\epsilon \sim \rho c^2, \qquad P \sim \frac{K}{c^4}\epsilon^2, \qquad P = K\rho^2. \tag{B.27}$$

For $\rho \to +\infty$ (ultra-relativistic limit), we get

$$\epsilon \sim K\rho^2, \qquad P \sim \epsilon, \qquad P = K\rho^2. \tag{B.28}$$

For the equation of state (B.26), the speed of sound is given by

$$\frac{c_s^2}{c^2} = P'(\epsilon) = 1 - \frac{1}{\sqrt{1+4K\epsilon/c^4}}\,. \tag{B.29}$$

We always have $c_s \le c$. For $\epsilon \to +\infty$, $c_s \to c$.

### Appendix B.4. Semi-relativistic model

We consider the equation of state

$$P = \frac{K}{c^4}\epsilon^2, \tag{B.30}$$

where $K$ is given by eq. (10). This equation of state was studied as a preliminary model by Chavanis and Harko [91] before treating the fully relativistic model corresponding to the equation of state (B.16). The equation of state (B.30) is obtained from eq. (8), derived from the classical GP equation, by replacing the rest mass density by the energy density (*i.e.* by making the approximation $\epsilon \sim \rho c^2$), and by assuming that the resulting equation remains valid in the relativistic regime. This is not exact but it provides a semi-relativistic BEC model. We note that eq. (B.30) is a particular case of the class of polytropic equations of state (of type I) studied by Tooper [103] in general relativity.

For the equation of state (B.30), eq. (B.3) becomes

$$\ln\rho = \int \frac{\mathrm{d}\epsilon}{\left(\frac{K\epsilon}{c^4}+1\right)\epsilon}\,. \tag{B.31}$$

Performing the integral and requiring that $\epsilon \sim \rho c^2$ for $\rho \to 0$, we obtain the following relation between the mass density and the energy density:

$$\rho = \frac{\epsilon/c^2}{\frac{K\epsilon}{c^4}+1}\,. \tag{B.32}$$

This equation can be reversed to give

$$\epsilon = \frac{\rho c^2}{1-\frac{K\rho}{c^2}}\,. \tag{B.33}$$

Combining eqs. (B.30) and (B.33) we obtain

$$P = \frac{K\rho^2}{(1-K\rho/c^2)^2}\,. \tag{B.34}$$

We note that the pressure diverges when $\rho = c^2/K$. Therefore, there is a maximum density

$$\rho_{\max} = \frac{c^2}{K} = \frac{m^3 c^2}{2\pi a_s \hbar^2}\,. \tag{B.35}$$

For $\rho \to 0$ (non-relativistic regime), we get

$$\epsilon \sim \rho c^2, \qquad P \sim \frac{K}{c^4}\epsilon^2, \qquad P \sim K\rho^2. \tag{B.36}$$

For $\rho \to c^2/K$, we get

$$\epsilon \sim \frac{c^4/K}{1-\frac{K\rho}{c^2}}\,, \qquad P \sim \frac{K}{c^4}\epsilon^2, \qquad P \sim \frac{c^4/K}{(1-K\rho/c^2)^2}\,. \tag{B.37}$$

For the equation of state (B.30), the speed of sound is given by

$$c_s^2 = \frac{2K}{c^2}\epsilon. \tag{B.38}$$

The speed of sound can be mathematically larger than the speed of light but such configurations are dynamically unstable [91].

*Remark.*

For a general polytropic equation of state of the form $P = K(\epsilon/c^2)^\gamma$, we get

$$\rho = \frac{\epsilon/c^2}{\left[1+\frac{K}{c^{2\gamma}}\epsilon^{\gamma-1}\right]^{\frac{1}{\gamma-1}}}\,, \quad \epsilon = \frac{\rho c^2}{\left(1-\frac{K}{c^2}\rho^{\gamma-1}\right)^{\frac{1}{\gamma-1}}}\,, \tag{B.39}$$

and

$$P = \frac{K\rho^\gamma}{\left(1-\frac{K}{c^2}\rho^{\gamma-1}\right)^{\frac{\gamma}{\gamma-1}}}\,. \tag{B.40}$$

## Appendix C. The Newtonian value of the maximum mass of a relativistic BEC star

It is interesting to compare the value of the maximum mass of a relativistic BEC star obtained by using general relativity (see [91]) with the one obtained by using Newtonian gravity. We consider the fully relativistic equation of state of appendix B.2. In general relativity, we have to substitute in the TOV equations (27) and (28) the equation of state (B.16) relating the pressure $P$ to the energy density $\epsilon$. This leads to the maximum mass of eq. (50) with $A_2 = 0.307$ [91]. In Newtonian gravity, we have to substitute in the equation of hydrostatic equilibrium (13) the equation of state defined by eqs. (B.16) and (B.19) relating the pressure $P$ to the mass density $\rho$. In order to determine the maximum mass of a relativistic BEC star in the Newtonian framework, it is sufficient to consider the limiting form of this equation of state in the ultra-relativistic limit $\rho \to +\infty$[17]. Equation (B.21) can be rewritten as

$$P = \frac{\pi^{1/3} c^{4/3} \hbar^{2/3} a_s^{1/3}}{2m}\rho^{4/3}. \tag{C.1}$$

This is the equation of state of a polytrope of index $n = 3$ [110]. Combining the condition of hydrostatic equilibrium (13) with the Poisson equation (6), we obtain

$$\nabla\cdot\left(\frac{\nabla P}{\rho}\right) = -4\pi G\rho. \tag{C.2}$$

Substituting the equation of state (C.1) in eq. (C.2), we get

$$\frac{c^{4/3}\hbar^{2/3}a_s^{1/3}}{2\pi^{2/3}Gm}\Delta\rho^{1/3} = -\rho. \tag{C.3}$$

Defining

$$\rho = \rho_0\theta^3, \qquad \xi = \left(\frac{2\pi^{2/3}Gm\rho_0^{2/3}}{c^{4/3}\hbar^{2/3}a_s^{1/3}}\right)^{1/2} r, \tag{C.4}$$

[17] We use a treatment similar to the one performed by Chandrasekhar [165] in the case of relativistic white dwarf stars treated with Newtonian gravity.

where $\rho_0$ is the central density, we obtain the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^3, \tag{C.5}$$

$$\theta(0)=1, \qquad \theta'(0)=0, \tag{C.6}$$

for a polytrope of index $n = 3$ [110]. This equation has to be solved numerically. The function $\theta(\xi)$ vanishes at $\xi_1 = 6.89685$. At that point $\omega_3 \equiv -\xi_1^2\theta'(\xi_1) = 2.01824$. The mass of the star $M = \int_0^R \rho 4\pi r^2\,\mathrm{d}r$ is given by

$$M = \sqrt{2}\omega_3\,\frac{\hbar c^2\sqrt{a_s}}{(Gm)^{3/2}}\,, \tag{C.7}$$

with $\sqrt{2}\omega_3 = 2.854$. We note that the radius $R$ of a star described by the equation of state (C.1) can take arbitrary values while its mass $M$ is fixed. The relation between the radius and the central density of the star is

$$R = \frac{\xi_1}{\sqrt{2}\pi^{1/3}}\,\frac{c^{2/3}\hbar^{1/3}a_s^{1/6}}{m^{1/2}G^{1/2}\rho_0^{1/3}}\,, \tag{C.8}$$

with $\xi_1/\sqrt{2}\pi^{1/3} = 3.33$. We have the relation

$$\rho_0 R^3 = \frac{\xi_1^3}{4\pi\omega_3}\,M, \tag{C.9}$$

with $\xi_1^3/4\pi\omega_3 = 12.9$. The mean density of the star $\overline{\rho} = 3M/4\pi R^3$ is related to the central density by the relation $\overline{\rho} = 3\omega_3\rho_0/\xi_1^3 = 1.85\times 10^{-2}\rho_0$.

It is interesting to contrast the calculations of this appendix to those of sect. 2.2. For a polytrope of index $n = 1$, the radius of the star is fixed but its mass is unspecified. Inversely, for a polytrope of index $n = 3$, the mass of the star is fixed but its radius is unspecified. For other values of the polytropic index, the radius of the star is a function of its mass. The general mass-radius relation of a polytropic star with an equation of state $P = K\rho^{1+1/n}$ is [110]:

$$M^{(n-1)/n}R^{(3-n)/n} = \frac{K(n+1)}{G(4\pi)^{1/n}}\omega_n^{(n-1)/n}, \tag{C.10}$$

where $\omega_n = -\xi_1^{(n+1)/(n-1)}\theta'(\xi_1)$. For $n = 1$ and $n = 3$, we recover the results of sect. 2.2 and of the present appendix.

When the full equation of state $P(\rho)$ defined by eqs. (B.16) and (B.19) is considered, we find that the limiting configuration corresponding to the ultra-relativistic limit determined by the equation of state (C.1) has a radius $R = 0$ and an infinite density $\rho_0 \to +\infty$ [60]. This configuration corresponds to a Dirac peak of mass $M$. Therefore, the mass defined by eq. (C.7) corresponds to the maximum mass of a relativistic BEC star in Newtonian gravity. It has the correct scaling of eq. (22) but the prefactor 2.854 is very different from the exact prefactor 0.307 obtained in general relativity. This shows that general relativity is crucial to determine the maximum mass of relativistic BEC stars. The same conclusion is reached for neutron stars considered as fermion stars. The general relativistic approach of Oppenheimer and Volkoff [93] leads to a maximum mass equal to $M_{\max} = 0.376\,(\hbar c/G)^{3/2}m_n^{-2} = 0.7\,M_\odot$ while the Newtonian treatment of Chandrasekhar leads to a maximum mass equal to $M'_{\max} = 3.10\,(\hbar c/G)^{3/2}m_n^{-2} = 5.76\,M_\odot$ [165, 166].

## Appendix D. Generalized equation of state and alternative form of the differential equations (36) and (37)

Let us consider the equation of state

$$P = \frac{q^2c^4}{4K}\left(\sqrt{1+\frac{4K}{qc^4}\epsilon}-1\right)^2. \tag{D.1}$$

When $\epsilon \to 0$ (non-relativistic regime), we recover the quadratic equation of state $P \sim K\epsilon^2/c^4$ of a classical BEC. When $\epsilon \to +\infty$ (ultra-relativistic regime), we obtain a linear equation of state $P \sim q\epsilon$. The equation of state (D.1) generalizes the equations of state (B.16) and (B.26) corresponding to $q = 1/3$ and $q = 1$, respectively.

The relation between the rest-mass density and the energy density is given by eq. (B.3) which can be rewritten as

$$q\ln\rho = \int^{4K\epsilon/qc^4}\frac{\mathrm{d}x}{\left(\sqrt{1+x}-1\right)^2+\frac{x}{q}}\,. \tag{D.2}$$

The integral can be calculated analytically. Setting $y = \sqrt{x+1}$, we obtain

$$\int \frac{\mathrm{d}x}{\left(\sqrt{1+x}-1\right)^2 + \frac{x}{q}} = \frac{A}{B}\,, \tag{D.3}$$

with

$$A = q(y-1)[1+y+q(y-1)]\{(1+q)\ln(y-1) - (q-1)\ln[1+y+q(y-1)]\} \tag{D.4}$$

and

$$B = (1+q)[x + q(2+x-2y)]. \tag{D.5}$$

For $q = 1/3$, we recover eq. (B.18), and for $q = 1$ the expression in the right hand side of eq. (D.2) reduces to $\ln(y-1)$ returning the equations of appendix B.3 obtained the other way round.

In sect. 3.2, we have written the TOV equations associated with the equation of state (32), equivalent to eq. (33), by introducing the variable $\theta$ defined with the rest-mass density $\rho$. Alternatively, we can introduce a variable $\Theta$ defined with the energy density $\epsilon$ as in [91] (be careful to the change of notations). If we set

$$\epsilon = \epsilon_0 \Theta, \qquad r = \frac{\xi}{A}\,, \qquad \sigma = \frac{K\epsilon_0}{c^4}\,, \tag{D.6}$$

$$M(r) = \frac{4\pi\epsilon_0}{A^3 c^2} v(\xi), \quad A = \left(\frac{2\pi G}{K}\right)^{1/2}, \tag{D.7}$$

and substitute the equation of state (D.1) in the TOV equations (27) and (28), using

$$P'(\epsilon) = q\left[1 - \frac{1}{\sqrt{1+4K\epsilon/qc^4}}\right], \tag{D.8}$$

we obtain

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = -\frac{\frac{2}{q}\left[\frac{q^2}{4}(\sqrt{1+4\sigma\Theta/q}-1)^2 + \sigma\Theta\right]\left[v + \frac{q^2\xi^3}{4\sigma}(\sqrt{1+4\sigma\Theta/q}-1)^2\right]}{\xi^2(1-4\sigma v/\xi)(1-1/\sqrt{1+4\sigma\Theta/q})}\,, \tag{D.9}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\xi} = \Theta\xi^2. \tag{D.10}$$

For $q = 1/3$ these equations reduce to eqs. (87) and (88) of [91].

## Appendix E. Alternative derivation of eq. (79)

In this appendix, we check that eq. (79) can be obtained directly from the equation of continuity (70) with the equation of state (B.26).

Substituting eq. (B.26) in eq. (70), and simplifying some terms, we get

$$\frac{2K}{c^4}\frac{\mathrm{d}\epsilon}{\mathrm{d}a} + \frac{3}{a}\left(\frac{4K\epsilon}{c^4} + 1 - \sqrt{1+\frac{4K\epsilon}{c^4}}\right) = 0. \tag{E.1}$$

With the change of variables $x = (1+4K\epsilon/c^4)^{1/2}$, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}a} + \frac{3}{a}(x-1) = 0. \tag{E.2}$$

This equation can be integrated into

$$x = 1 + \frac{A}{a^3}\,, \tag{E.3}$$

where $A$ is a constant. Returning to the original variables, we obtain

$$\epsilon = \frac{c^4}{4K}\left(\frac{2A}{a^3} + \frac{A^2}{a^6}\right), \tag{E.4}$$

which can be written as eq. (79).

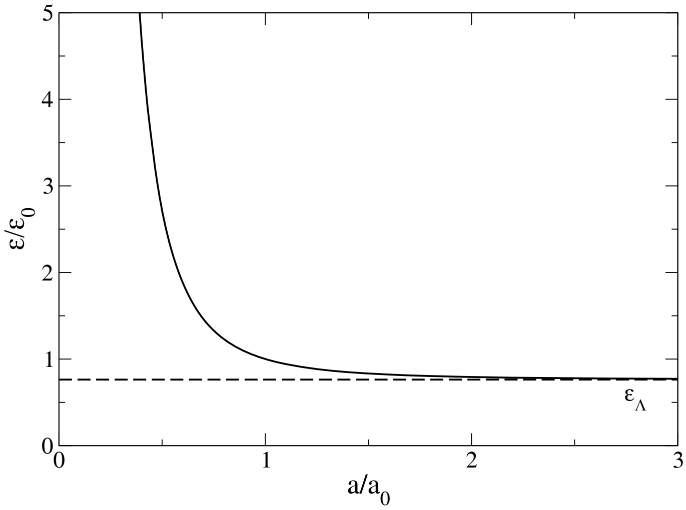


**Fig. 9.** Energy density as a function of the scale factor. We have taken $\Omega_{m,0} = 0.237$, $\Omega_{\Lambda,0} = 0.763$, and $\Omega_{s,0} = 10^{-3}$ (we have chosen a relatively large value of the density of stiff matter $\Omega_{s,0} = 10^{-3}$ for a better illustration of the results).

## Appendix F. A universe with a stiff matter era

In this appendix, we assume that the universe is made of one or several fluids each of them described by a linear equation of state $P = \alpha\epsilon$. The equation of continuity (70) implies that the energy density is related to the scale factor by $\epsilon = \epsilon_0(a_0/a)^{3(1+\alpha)}$, where the subscript 0 denotes present-day values of the quantities. A linear equation of state can describe dust matter ($\alpha = 0$, $\epsilon_m \propto a^{-3}$), radiation ($\alpha = 1/3$, $\epsilon_{rad} \propto a^{-4}$), stiff matter ($\alpha = 1$, $\epsilon_s \propto a^{-6}$), vacuum energy ($\alpha = -1$, $\epsilon = \epsilon_P$), and dark energy ($\alpha = -1$, $\epsilon = \epsilon_\Lambda$).

We consider a universe made of stiff matter, radiation, dust matter and dark energy treated as non-interacting species. Summing the contribution of each species, the total energy density can by written as

$$\epsilon = \frac{\epsilon_{s,0}}{(a/a_0)^6} + \frac{\epsilon_{rad,0}}{(a/a_0)^4} + \frac{\epsilon_{m,0}}{(a/a_0)^3} + \epsilon_\Lambda. \tag{F.1}$$

In this model, the stiff matter dominates in the early universe. This is followed by the radiation era, by the dust matter era and, finally, by the dark energy era. Writing $\epsilon_{\alpha,0} = \Omega_{\alpha,0}\epsilon_0$ for each species, we get

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{s,0}}{(a/a_0)^6} + \frac{\Omega_{rad,0}}{(a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}. \tag{F.2}$$

The energy density starts from $\epsilon = +\infty$ at $a = 0$, decreases, and tends to $\epsilon_\Lambda$ for $a \to +\infty$. The relation between the energy density and the scale factor is shown in fig. 9. The proportions of stiff matter, dust matter and dark energy as a function of the scale factor are shown in fig. 10. Using eq. (F.2), the Friedmann equation (72) takes the form

$$\frac{H}{H_0} = \sqrt{\frac{\Omega_{s,0}}{(a/a_0)^6} + \frac{\Omega_{rad,0}}{(a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}}\,, \tag{F.3}$$

with $\Omega_{s,0} + \Omega_{rad,0} + \Omega_{m,0} + \Omega_{\Lambda,0} = 1$ and $H_0 = (8\pi G\epsilon_0/3c^2)^{1/2}$. Note that we have taken $\Lambda = 0$ in eq. (72) and accounted for the effect of the cosmological constant in the dark energy density $\epsilon_\Lambda$. We also note the relation

$$\frac{\epsilon}{\epsilon_0} = \left(\frac{H}{H_0}\right)^2 \tag{F.4}$$

that will be used later. The evolution of the scale factor is given by

$$\int_0^{a/a_0} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{s,0}}{x^6} + \frac{\Omega_{rad,0}}{x^4} + \frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = H_0 t. \tag{F.5}$$

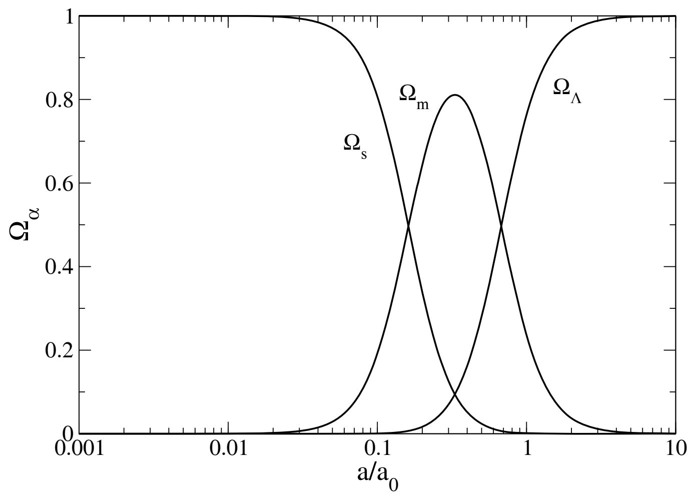


**Fig. 10.** Evolution of the proportion $\Omega_\alpha = \epsilon_\alpha/\epsilon$ of the different components of the universe with the scale factor.

We first ignore radiation ($\Omega_{rad,0} = 0$) and consider a universe made of stiff matter, dust matter, and dark energy. In that case, the Friedmann equation (F.5) reduces to

$$\int_0^{a/a_0} \frac{\mathrm{d}x}{x\sqrt{\frac{\Omega_{s,0}}{x^6} + \frac{\Omega_{m,0}}{x^3} + \Omega_{\Lambda,0}}} = H_0 t. \tag{F.6}$$

Using the identity

$$\int \frac{\mathrm{d}x}{x\sqrt{\frac{a}{x^3} + \frac{b}{x^6} + c}} = \frac{1}{3\sqrt{c}} \ln\left[a + 2cx^3 + 2\sqrt{c}\sqrt{b + ax^3 + cx^6}\right], \tag{F.7}$$

eq. (F.6) can be solved analytically to give

$$\frac{a}{a_0} = \left[\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} + 2\sqrt{\frac{\Omega_{s,0}}{\Omega_{\Lambda,0}}}\right)\sinh^2\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t\right) + \sqrt{\frac{\Omega_{s,0}}{\Omega_{\Lambda,0}}}\left(1 - e^{-3\sqrt{\Omega_{\Lambda,0}}H_0 t}\right)\right]^{1/3}. \tag{F.8}$$

From eq. (F.8), we can compute $H = \dot{a}/a$ leading to

$$\left(\frac{a}{a_0}\right)^3 \frac{H}{H_0} = \left(\frac{\Omega_{m,0}}{2\sqrt{\Omega_{\Lambda,0}}} + \sqrt{\Omega_{s,0}}\right)\sinh\left(3\sqrt{\Omega_{\Lambda,0}}H_0 t\right) + \sqrt{\Omega_{s,0}}e^{-3\sqrt{\Omega_{\Lambda,0}}H_0 t}. \tag{F.9}$$

The energy density $\epsilon/\epsilon_0$ is then given by eq. (F.4) where $H/H_0$ can be obtained from eq. (F.9) with eq. (F.8).

At $t = 0$ the universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$. The scale factor increases with time. For $t \to +\infty$, we obtain

$$\frac{a}{a_0} \sim \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} + 2\sqrt{\frac{\Omega_{s,0}}{\Omega_{\Lambda,0}}}\right)^{1/3} \frac{1}{2^{2/3}} e^{\sqrt{\Omega_{\Lambda,0}}H_0 t}. \tag{F.10}$$

The energy density decreases with time and tends to $\epsilon_\Lambda$ for $t \to +\infty$. The expansion is decelerating during the stiff matter era and the dust matter era while it is accelerating during the dark energy era. The transition takes place at

$$\frac{a_c}{a_0} = \left(\frac{\Omega_{m,0} + \sqrt{\Omega_{m,0}^2 + 32\Omega_{\Lambda,0}\Omega_{s,0}}}{4\Omega_{\Lambda,0}}\right)^{1/3}. \tag{F.11}$$

The temporal evolution of the scale factor and energy density are shown in figs. 11 and 12.

We consider a universe made of stiff matter and dust matter. In the absence of dark energy ($\Omega_{\Lambda,0} = 0$), using the identity

$$\int \frac{\mathrm{d}x}{x\sqrt{\frac{a}{x^3} + \frac{b}{x^6}}} = \frac{2}{3a}\sqrt{b + ax^3}, \tag{F.12}$$

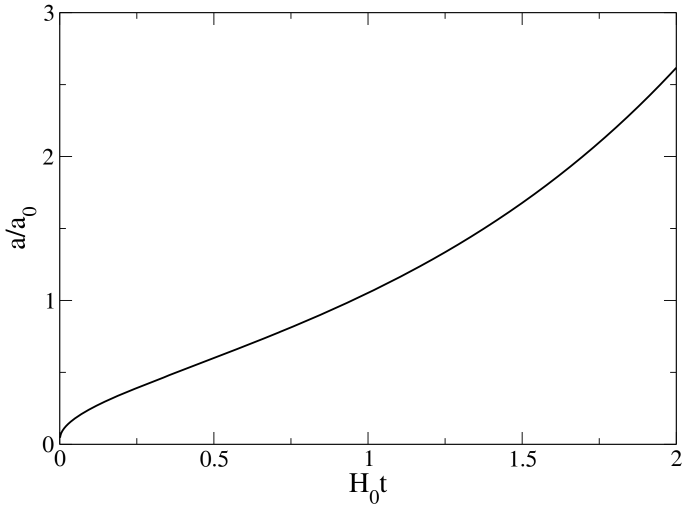


**Fig. 11.** Evolution of the scale factor as a function of time.

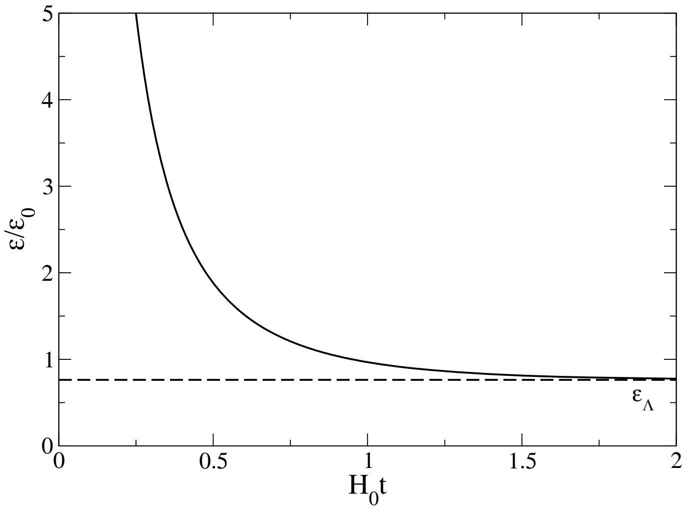


**Fig. 12.** Evolution of the energy density as a function of time.

we obtain

$$\frac{a}{a_0} = \left(\frac{9}{4}\Omega_{m,0}H_0^2t^2 + 3\sqrt{\Omega_{s,0}}H_0t\right)^{1/3}, \tag{F.13}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{4}{9H_0^2t^2}\left(\frac{1+\frac{2\sqrt{\Omega_{s,0}}}{3\Omega_{m,0}H_0t}}{1+\frac{4\sqrt{\Omega_{s,0}}}{3\Omega_{m,0}H_0t}}\right)^2. \tag{F.14}$$

We consider a universe made of stiff matter and dark energy. In the absence of matter ($\Omega_{m,0} = 0$), using the identity

$$\int \frac{\mathrm{d}x}{x\sqrt{\frac{b}{x^6}+c}} = \frac{1}{3\sqrt{c}}\ln\left[2cx^3 + 2\sqrt{c}\sqrt{b+cx^6}\right], \tag{F.15}$$

or setting $X = b/cx^6$ and using the identity

$$\int \frac{\mathrm{d}X}{X\sqrt{X+1}} = \ln\left(\frac{\sqrt{1+X}-1}{\sqrt{1+X}+1}\right), \tag{F.16}$$

we get

$$\frac{a}{a_0} = \left(\frac{\Omega_{s,0}}{\Omega_{\Lambda,0}}\right)^{1/6} \sinh^{1/3}\left(3\sqrt{\Omega_{\Lambda,0}}H_0 t\right), \tag{F.17}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\Lambda,0}}{\tanh^2\left(3\sqrt{\Omega_{\Lambda,0}}H_0 t\right)}. \tag{F.18}$$

We consider a universe made of stiff matter. In the absence of dust matter and dark energy ($\Omega_{m,0} = \Omega_{\Lambda,0} = 0$), we find that

$$\frac{a}{a_0} = \left(3\sqrt{\Omega_{s,0}}H_0 t\right)^{1/3}, \qquad \frac{\epsilon}{\epsilon_0} = \frac{1}{9H_0^2 t^2}. \tag{F.19}$$

We consider a universe made of dust matter and dark energy. In the absence of stiff matter ($\Omega_{s,0} = 0$), using the identity

$$\int \frac{\mathrm{d}x}{x\sqrt{\frac{a}{x^3} + c}} = \frac{1}{3\sqrt{c}} \ln\left[a + 2cx^3 + 2\sqrt{c}\sqrt{ax^3 + cx^6}\right], \tag{F.20}$$

or setting $X = a/cx^3$ and using eq. (F.16), we obtain

$$\frac{a}{a_0} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \sinh^{2/3}\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t\right), \tag{F.21}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\Lambda,0}}{\tanh^2\left(\frac{3}{2}\sqrt{\Omega_{\Lambda,0}}H_0 t\right)}. \tag{F.22}$$

This solution coincides with the $\Lambda$CDM model.

We consider a universe made of dark energy. In the absence of stiff matter and dust matter ($\Omega_{s,0} = \Omega_{m,0} = 0$), we obtain

$$a(t) = a(0)e^{\sqrt{\frac{\Lambda}{3}}t}, \qquad \epsilon = \epsilon_\Lambda. \tag{F.23}$$

This is de Sitter's solution.

We consider a universe made of dust matter. In the absence of stiff matter and dark energy ($\Omega_{s,0} = \Omega_{\Lambda,0} = 0$), we obtain

$$\frac{a}{a_0} = \left(\frac{9}{4}\Omega_{m,0}H_0^2 t^2\right)^{1/3}, \qquad \frac{\epsilon}{\epsilon_0} = \frac{4}{9H_0^2 t^2}. \tag{F.24}$$

This is the Einstein-de Sitter (EdS) solution.

We now come back to the general equation (F.3) including the contribution of radiation.

The transition between the stiff matter era and the radiation era is obtained by taking $\Omega_{m,0} = \Omega_{\Lambda,0} = 0$ in eq. (F.3). In that case, the integral in eq. (F.5) can be performed analytically leading to

$$2\sqrt{\Omega_{rad,0}}\frac{a}{a_0}\sqrt{\Omega_{s,0} + \Omega_{rad,0}\left(\frac{a}{a_0}\right)^2} - 2\Omega_{s,0}\ln\left[\Omega_{rad,0}\frac{a}{a_0} + \sqrt{\Omega_{rad,0}}\sqrt{\Omega_{s,0} + \Omega_{rad,0}\left(\frac{a}{a_0}\right)^2}\right]$$
$$+ \Omega_{s,0}\ln(\Omega_{s,0}\Omega_{rad,0}) = 4(\Omega_{rad,0})^{3/2}H_0 t. \tag{F.25}$$

Using the identity $\sinh^{-1}(x) = \ln(x + \sqrt{x^2+1})$, we can rewrite the foregoing expression as

$$\sqrt{\frac{\Omega_{rad,0}}{\Omega_{s,0}}}\frac{a}{a_0}\sqrt{1 + \frac{\Omega_{rad,0}}{\Omega_{s,0}}\left(\frac{a}{a_0}\right)^2} - \sinh^{-1}\left(\sqrt{\frac{\Omega_{rad,0}}{\Omega_{s,0}}}\frac{a}{a_0}\right) = \frac{2(\Omega_{rad,0})^{3/2}}{\Omega_{s,0}}H_0 t. \tag{F.26}$$

At $t = 0$ the universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$. The scale factor increases with time while the energy density decreases with time. The universe is always decelerating. The temporal evolution of the scale factor and energy density is shown in fig. 13.

The transition between the radiation era and the matter era is obtained by taking $\Omega_{s,0} = \Omega_{\Lambda,0} = 0$ in eq. (F.3). In that case, the integral in eq. (F.5) can be performed analytically leading to[18]

$$H_0 t = -\frac{2}{3}\frac{1}{(\Omega_{m,0})^{1/2}}\left(\frac{2\Omega_{rad,0}}{\Omega_{m,0}} - \frac{a}{a_0}\right)\sqrt{\frac{\Omega_{rad,0}}{\Omega_{m,0}} + \frac{a}{a_0}} + \frac{4}{3}\frac{(\Omega_{rad,0})^{3/2}}{(\Omega_{m,0})^2}. \tag{F.27}$$

[18] We have determined the constant of integration in eq. (F.27) such that $a = 0$ at $t = 0$. This implicitly assumes that there is no stiff matter in the early universe. Otherwise, we need to determine the constant of integration by matching the solution (F.27) with the solution (F.19) of the stiff matter era.

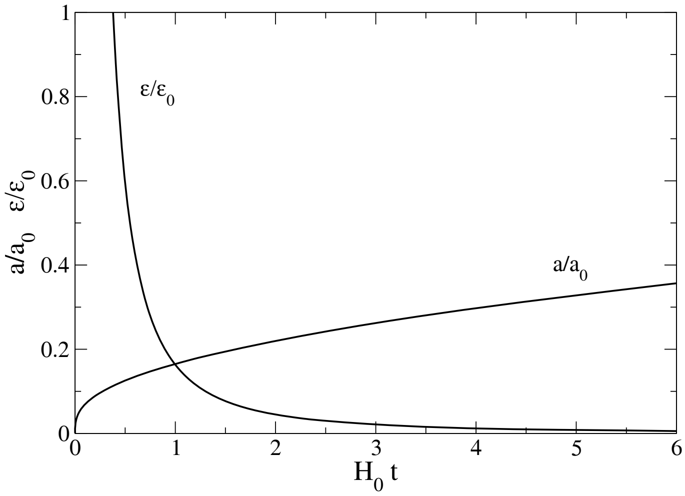


**Fig. 13.** Evolution of the scale factor and energy density as a function of time. We have taken $\Omega_{s,0} = 10^{-6}$ and $\Omega_{rad,0} = 8.48 \times 10^{-5}$.

Equation (F.27) can also be written as

$$\left(\frac{a}{a_0}\right)^3 - 3\frac{\Omega_{rad,0}}{\Omega_{m,0}}\left(\frac{a}{a_0}\right)^2 = \frac{9}{4}\Omega_{m,0}H_0^2t^2 - 6\frac{\Omega_{rad,0}^{3/2}}{\Omega_{m,0}}H_0t. \tag{F.28}$$

This is a cubic equation for $a/a_0$. At $t = 0$ the universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$. The scale factor increases with time while the energy density decreases with time. The universe is always decelerating.

For mathematical completeness, we also give the equations corresponding to a universe containing only radiation and dark energy ($\Omega_{s,0} = \Omega_{m,0} = 0$). They write

$$\frac{a}{a_0} = \left(\frac{\Omega_{rad,0}}{\Omega_{\Lambda,0}}\right)^{1/4} \sinh^{1/2}\left(2\sqrt{\Omega_{\Lambda,0}}H_0t\right), \tag{F.29}$$

$$\frac{\epsilon}{\epsilon_0} = \frac{\Omega_{\Lambda,0}}{\tanh^2\left(2\sqrt{\Omega_{\Lambda,0}}H_0t\right)}. \tag{F.30}$$

At $t = 0$ the universe starts from a singular state at which the scale factor $a = 0$ while the energy density $\epsilon \to +\infty$. The scale factor increases with time while the energy density decreases with time and tends to $\epsilon_\Lambda$ for $t \to +\infty$. The universe is decelerating during the radiation era and accelerating during the dark energy era. The transition takes place at $a_c/a_0 = (\Omega_{rad,0}/\Omega_{\Lambda,0})^{1/4}$.

For a universe containing only radiation ($\Omega_{s,0} = \Omega_{m,0} = \Omega_{\Lambda,0} = 0$) we get

$$\frac{a}{a_0} = \Omega_{rad,0}^{1/4}\sqrt{2H_0t}, \qquad \frac{\epsilon}{\epsilon_0} = \frac{1}{(2H_0t)^2}. \tag{F.31}$$

Finally, we can propose a simple generalization of eq. (F.3) that includes a phase of early inflation. Using the model of inflation developed in [131, 132, 134], we get

$$\frac{H}{H_0} = \sqrt{\frac{\Omega_{s,0}}{(a/a_0)^6 + (a_*/a_0)^6} + \frac{\Omega_{rad,0}}{(a/a_0)^4} + \frac{\Omega_{m,0}}{(a/a_0)^3} + \Omega_{\Lambda,0}}, \tag{F.32}$$

where the constant $a_*$ is determined by the relation $\epsilon_P a_*^6 = \epsilon_{s,0}a_0^6$ where $\epsilon_P = \rho_P c^2$ is the Planck energy density ($\rho_P = c^5/G^2\hbar = 5.16 \times 10^{99}\,\mathrm{g/m^3}$). The transition between the inflation era and the stiff matter era is obtained by taking $\Omega_{rad,0} = \Omega_{m,0} = \Omega_{\Lambda,0} = 0$ in eq. (F.32) leading to

$$\frac{H}{H_0} = \sqrt{\frac{\Omega_{s,0}}{(a/a_0)^6 + (a_*/a_0)^6}}. \tag{F.33}$$

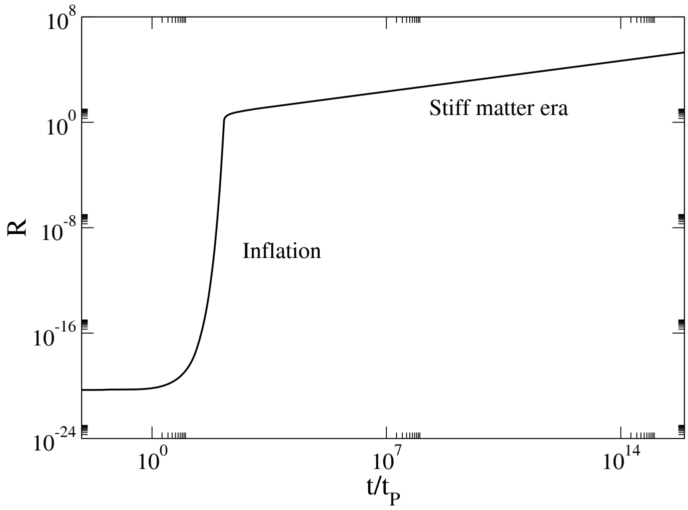


**Fig. 14.** Evolution of the scale factor as a function of time.

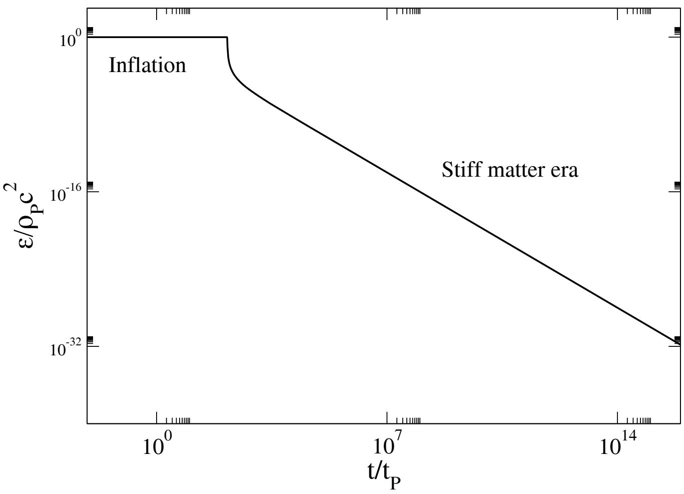


**Fig. 15.** Evolution of the energy density as a function of time.

The equation of state generating this model is [131]

$$P = \epsilon - \frac{2\epsilon^2}{\rho_P c^2}\,. \tag{F.34}$$

The corresponding speed of sound is

$$\frac{c_s^2}{c^2} = 1 - \frac{4\epsilon}{\rho_P c^2}\,. \tag{F.35}$$

The energy density evolves with the scale factor as

$$\epsilon = \frac{\rho_P c^2}{1+R^6}\,, \tag{F.36}$$

where we have defined $R = a/a_*$. Equation (F.33) can be integrated analytically to give [131]

$$\sqrt{R^6+1} - \ln\left(\frac{1+\sqrt{R^6+1}}{R^3}\right) = 3\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P} + C, \tag{F.37}$$

where $t_P = (G\rho_P)^{-1/2} = (\hbar G/c^5)^{1/2} = 5.39\times 10^{-44}\,\mathrm{s}$ is the Planck time and $C$ is a constant of integration that can be determined by requiring that $a = l_P$ at $t = 0$, where $l_P = (G\hbar/c^3)^{1/2} = 1.62\times 10^{-35}\,\mathrm{m}$ is the Planck length. The time evolution of the scale factor and energy density is represented in figs. 14 and 15.

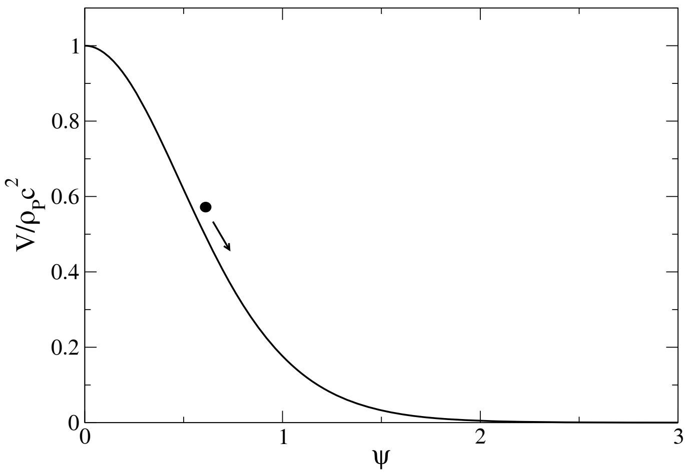


**Fig. 16.** Potential of the scalar field.

For $R \ll 1$,

$$\epsilon \simeq \rho_P c^2, \qquad R \simeq \frac{l_P}{a_*} e^{(\frac{8\pi}{3})^{1/2} t/t_P}, \tag{F.38}$$

so the energy density is constant (equal to the Planck density) and the scale factor increases exponentially rapidly with time (inflation era, $P \simeq -\epsilon$, $c_s^2/c^2 \simeq -3$). For $R \gg 1$,

$$\epsilon \sim \frac{\rho_P c^2}{R^6}, \quad R \sim \left[3\left(\frac{8\pi}{3}\right)^{1/2} \frac{t}{t_P}\right]^{1/3}, \tag{F.39}$$

so the energy density and the scale factor evolve algebraically with time (stiff matter era, $P \sim \epsilon$, $c_s^2/c^2 \simeq 1$). The transition between the inflation era and the stiff matter era takes place at $R = R_* = 1$ (*i.e.* $a = a_*$). The universe is accelerating when $R < R_c$ and decelerating when $R > R_c$, where $R_c = 2^{-1/6}$. The speed of sound is imaginary ($c_s^2 < 0$) when $R < R_e$ and real ($c_s^2 > 0$) when $R > R_e$, where $R_e = 3^{1/6}$. The pressure is negative when $R < R_w$ and positive when $R > R_w$, where $R_w = 1$. It has a maximum value $P_e/\rho_P c^2 = 1/8$ at $R = R_e$ [131].

The phase of inflation in the very early universe is usually described by a scalar field $\phi(t)$ (inflaton) that evolves according to the equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0, \tag{F.40}$$

where $V(\phi)$ is the potential of the scalar field. The scalar field tends to run down the potential towards lower energies. The energy density and the pressure of the scalar field are given by

$$\epsilon = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{F.41}$$

The model of inflation developed in [131,132,134], corresponding to the quadratic equation of state (F.34), is equivalent to a scalar field with a potential (see fig. 16),

$$V(\psi) = \frac{\rho_P c^2}{\cosh^4\psi}, \quad \psi = \left(\frac{12\pi G}{c^2}\right)^{1/2} \phi. \tag{F.42}$$

For $\psi \to 0$,

$$\frac{V(\psi)}{\rho_P c^2} \simeq 1 - 2\psi^2 + \frac{7}{3}\psi^4, \tag{F.43}$$

which is consistent with the symmetry breaking scalar field potential used to describe inflation. For $\psi \to +\infty$,

$$\frac{V(\psi)}{\rho_P c^2} \sim 16 e^{-4\psi}. \tag{F.44}$$

The relation between the scalar field and the scale factor is [132]

$$R^3 = \sinh\psi. \tag{F.45}$$

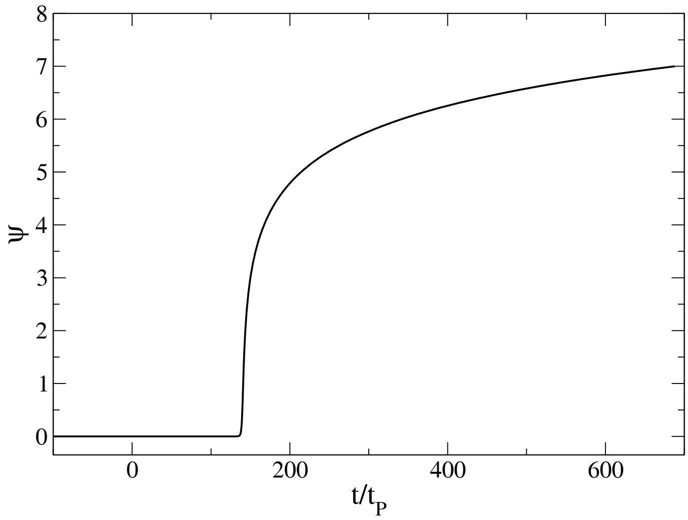


**Fig. 17.** Evolution of the scalar field as a function of time.

The end of the inflation, and the beginning of the stiff matter era, corresponds to $R = R_* = 1$, hence to $\psi = \psi_* = \sinh^{-1}(1) = \ln(1+\sqrt{2}) = 0.881374$. Combining eqs. (F.34), (F.36) and (F.45), we find that the energy density and the pressure of the universe are related to the scalar field by

$$\epsilon = \frac{\rho_P c^2}{\cosh^2\psi}, \quad P = \frac{\rho_P c^2}{\cosh^2\psi}\left(1 - \frac{2}{\cosh^2\psi}\right). \tag{F.46}$$

On the other hand, using eqs. (F.37) and (F.45), we find that the temporal evolution of the scalar field is given by (see fig. 17)

$$\cosh\psi - \ln\left(\frac{1+\cosh\psi}{\sinh\psi}\right) = 3\left(\frac{8\pi}{3}\right)^{1/2}\frac{t}{t_P} + C. \tag{F.47}$$

For $t \to -\infty$ (*i.e.* $R \to 0$), we find that

$$\psi \sim \left(\frac{l_P}{a_*}\right)^3 e^{3\left(\frac{8\pi}{3}\right)^{1/2} t/t_P} \to 0. \tag{F.48}$$

For $t \to +\infty$ (*i.e.* $R \to +\infty$), we find that

$$\psi \simeq \ln(6) + \frac{1}{2}\ln\left(\frac{8\pi}{3}\right) + \ln\left(\frac{t}{t_P}\right) \to +\infty. \tag{F.49}$$

Details and generalizations of this inflationary model can be found in [131, 132, 134].

## References


1. C.J. Pethick, H. Smith, *Bose-Einstein condensation in dilute gases* (Cambridge, Cambridge University Press, 2008).
2. J.F. Navarro, C.S. Frenk, S.D.M. White, Mon. Not. R. Astron. Soc. **462**, 563 (1996).
3. A. Burkert, Astrophys. J. **447**, L25 (1995).
4. P. Salucci, A. Burkert, Astrophys. J. **537**, L9 (2000).
5. G. Kauffmann, S.D.M. White, B. Guiderdoni, Mon. Not. R. Astron. Soc. **264**, 201 (1993).
6. E.P. Gross, Ann. Phys. **4**, 57 (1958).
7. E.P. Gross, Nuovo Cimento **20**, 454 (1961).
8. E.P. Gross, J. Math. Phys. **4**, 195 (1963).
9. L.P. Pitaevskii, Sov. Phys. JETP **9**, 830 (1959).
10. L.P. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961).
11. E. Madelung, Z. Phys. **40**, 322 (1927).
12. P.J.E. Peebles, B. Ratra, Rev. Mod. Phys. **75**, 559 (2003).
13. A. Suárez, V.H. Robles, T. Matos, Astrophys. Space Sci. Proc. **38**, 107 (2014).
14. T. Rindler-Daller, P.R. Shapiro, Astrophys. Space Sci. Proc. **38**, 163 (2014).

15. P.H. Chavanis, *Self-gravitating Bose-Einstein condensates*, in *Quantum Aspects of Black Holes*, edited by X. Calmet (Springer, 2015).
16. M.R. Baldeschi, G.B. Gelmini, R. Ruffini, Phys. Lett. B **122**, 221 (1983).
17. M.Yu. Khlopov, B.A. Malomed, Ya.B. Zel'dovich, Mon. Not. R. Astron. Soc. **215**, 575 (1985).
18. M. Membrado, A.F. Pacheco, J. Sanudo, Phys. Rev. A **39**, 4207 (1989).
19. S.J. Sin, Phys. Rev. D **50**, 3650 (1994).
20. F.E. Schunck, astro-ph/9802258.
21. T. Matos, F.S. Guzmán, Astron. Nachr. **320**, 97 (1999).
22. F.S. Guzmán, T. Matos, Class. Quantum Grav. **17**, L9 (2000).
23. W. Hu, R. Barkana, A. Gruzinov, Phys. Rev. Lett. **85**, 1158 (2000).
24. T. Matos, L.A. Ureña-López, Phys. Rev. D **63**, 063506 (2001).
25. A. Arbey, J. Lesgourgues, P. Salati, Phys. Rev. D **64**, 123528 (2001).
26. M.P. Silverman, R.L. Mallett, Class. Quantum Grav. **18**, L103 (2001).
27. M. Alcubierre, F.S. Guzmán, T. Matos, D. Núñez, L.A. Ureña-López, P. Wiederhold, Class. Quantum Grav. **19**, 5017 (2002).
28. M.P. Silverman, R.L. Mallett, Gen. Relativ. Gravit. **34**, 633 (2002).
29. A. Bernal, T. Matos, D. Núñez, Rev. Mex. Astron. Astrofis. **44**, 149 (2008).
30. P. Sikivie, Q. Yang, Phys. Rev. Lett. **103**, 111301 (2009).
31. T. Matos, A. Vazquez-Gonzalez, J. Magana, Mon. Not. R. Astron. Soc. **393**, 1359 (2009).
32. J.W. Lee, Phys. Lett. B **681**, 118 (2009).
33. T.P. Woo, T. Chiueh, Astrophys. J. **697**, 850 (2009).
34. J.W. Lee, S. Lim, J. Cosmol. Astropart. Phys. **01**, 007 (2010).
35. P.H. Chavanis, Phys. Rev. D **84**, 043531 (2011).
36. P.H. Chavanis, L. Delfini, Phys. Rev. D **84**, 043532 (2011).
37. G. Manfredi, P.A. Hervieux, F. Haas, Class. Quantum Grav. **30**, 075006 (2013).
38. H.Y. Schive, T. Chiueh, T. Broadhurst, Nat. Phys. **10**, 496 (2014).
39. H.Y. Schive *et al.*, Phys. Rev. Lett. **113**, 261302 (2014).
40. R. Ruffini, S. Bonazzola, Phys. Rev. **187**, 1767 (1969).
41. P.H. Chavanis, M. Lemou, F. Méhats, arXiv:1409.7840.
42. J.W. Lee, I. Koh, Phys. Rev. D **53**, 2236 (1996).
43. P.J.E. Peebles, Astrophys. J. **534**, L127 (2000).
44. J. Goodman, New Astron. **5**, 103 (2000).
45. J. Lesgourgues, A. Arbey, P. Salati, New Astron. Rev. **46**, 791 (2002).
46. A. Arbey, J. Lesgourgues, P. Salati, Phys. Rev. D **68**, 023511 (2003).
47. C.G. Böhmer, T. Harko, J. Cosmol. Astropart. Phys. **06**, 025 (2007).
48. F. Briscese, Phys. Lett. B **696**, 315 (2011).
49. T. Harko, J. Cosmol. Astropart. Phys. **05**, 022 (2011).
50. M.O.C. Pires, J.C.C. de Souza, J. Cosmol. Astropart. Phys. **11**, 024 (2012).
51. V.H. Robles, T. Matos, Mon. Not. R. Astron. Soc. **422**, 282 (2012).
52. T. Rindler-Daller, P.R. Shapiro, Mon. Not. R. Astron. Soc. **422**, 135 (2012).
53. V. Lora, J. Magaña, A. Bernal, F.J. Sánchez-Salcedo, E.K. Grebel, J. Cosmol. Astropart. Phys. **02**, 011 (2012).
54. A.X. González-Morales, A. Diez-Tejedor, L.A. Ureña-López, O. Valenzuela, Phys. Rev. D **87**, 021301 (2013).
55. F.S. Guzmán, F.D. Lora-Clavijo, J.J. González-Avilés, F.J. Rivera-Paleo, J. Cosmol. Astropart. Phys. **09**, 034 (2013).
56. D. Bettoni, M. Colombo, S. Liberati, J. Cosmol. Astropart. Phys. **02**, 004 (2014).
57. F. Dalfovo, S. Giorgini, L.P. Pitaevskii, S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
58. S.W. Randall, M. Markevitch, D. Clowe, A.H. Gonzalez, M. Bradac, Astrophys. J. **679**, 1173 (2008).
59. T. Fukuyama, M. Morikawa, T. Tatekawa, J. Cosmol. Astropart. Phys. **06**, 033 (2008).
60. P.H. Chavanis, *Mass-radius relation of self-gravitating Bose-Einstein condensates*, preprint.
61. E. Seidel, W.M. Suen, Phys. Rev. Lett. **72**, 2516 (1994).
62. G. Ingrosso, R. Ruffini, Nuovo Cimento **101**, 369 (1988).
63. G. Ingrosso, M. Merafina, R. Ruffini, Nuovo Cimento **105**, 977 (1990).
64. N. Bilic, H. Nikolic, Nucl. Phys. B **590**, 575 (2000).
65. T. Matos, A. Suárez, EPL **96**, 56005 (2011).
66. T. Harko, E. Madarassy, J. Cosmol. Astropart. Phys. **01**, 020 (2012).
67. Z. Slepian, J. Goodman, Mon. Not. R. Astron. Soc. **427**, 839 (2012).
68. T. Harko, G. Mocanu, Phys. Rev. D **85**, 084012 (2012).
69. V.H. Robles, T. Matos, Astrophys. J. **763**, 19 (2013).
70. P.H. Chavanis, Astron. Astrophys. **537**, A127 (2012).
71. D.J. Kaup, Phys. Rev. **172**, 1331 (1968).
72. W. Thirring, Phys. Lett. B **127**, 27 (1983).
73. J.D. Breit, S. Gupta, A. Zaks, Phys. Lett. B **140**, 329 (1984).
74. E. Takasugi, M. Yoshimura, Z. Phys. C **26**, 241 (1984).
75. M. Colpi, S.L. Shapiro, I. Wasserman, Phys. Rev. Lett. **57**, 2485 (1986).

76. J.J. van der Bij, M. Gleiser, Phys. Lett. B **194**, 482 (1987).
77. M. Gleiser, Phys. Rev. D **38**, 2376 (1988).
78. M. Gleiser, R. Watkins, Nucl. Phys. B **319**, 733 (1989).
79. E. Seidel, W.M. Suen, Phys. Rev. D **42**, 384 (1990).
80. F.V. Kusmartsev, E.W. Mielke, F.E. Schunck, Phys. Lett. A **157**, 465 (1991).
81. F.V. Kusmartsev, E.W. Mielke, F.E. Schunck, Phys. Rev. D **43**, 3895 (1991).
82. T.D. Lee, Y. Pang, Phys. Rep. **221**, 251 (1992).
83. P. Jetzer, Phys. Rep. **220**, 163 (1992).
84. J. Balakrishna, E. Seidel, W.M. Suen, Phys. Rev. D **58**, 104004 (1998).
85. F.E. Schunck, A.R. Liddle, *Black Holes: Theory and Observation*, in *Proceedings of the 179th W. E. Heraeus Seminar*, edited by Friedrich W. Hehl, Claus Kiefer, Ralph J.K. Metzler (Springer, 1998) p. 285.
86. E.W. Mielke, F.E. Schunck, Nucl. Phys. B **564**, 185 (2000).
87. D.F. Torres, S. Capozziello, G. Lambiase, Phys. Rev. D **62**, 104012 (2000).
88. X.Z. Wang, Phys. Rev. D **64**, 124009 (2001).
89. F.E. Schunck, E.W. Mielke, Class. Quantum Grav. **20**, R301 (2003).
90. F.S. Guzmán, Phys. Rev. D **73**, 021501 (2006).
91. P.H. Chavanis, T. Harko, Phys. Rev. D **86**, 064011 (2012).
92. C. Herdeiro, E. Radu, Phys. Rev. Lett. **112**, 221101 (2014).
93. J.R. Oppenheimer, G.M. Volkoff, Phys. Rev. **55**, 374 (1939).
94. J.M. Lattimer, M. Prakash, in *From Nuclei to Stars*, edited by S. Lee (World Scientific, Singapore, 2011) arXiv:1012.3208.
95. P.B. Demorest, T. Pennucci, S.M. Ransom, M.S.E. Roberts, J.W.T. Hessels, Nature **467**, 1081 (2010).
96. O. Barziv, L. Karper, M.H. van Kerkwijk, J.H. Telging, J. van Paradijs, Astron. Astrophys. **377**, 925 (2001).
97. H. Quaintrell, A.J. Norton, T.D.C. Ash, P. Roche, B. Willems, T.R. Bedding, I.K. Baldry, R.P. Fender, Astron. Astrophys. **401**, 303 (2003).
98. M.H. van Kerkwijk, R. Breton, S.R. Kulkarni, Astrophys. J. **728**, 95 (2011).
99. J. Antoniadis *et al.*, Science **340**, 6131 (2013).
100. G. Dvali, C. Gomez, Fortschr. Phys. **61**, 742 (2013).
101. R. Casadio, A. Orlandi, J. High Energy Phys. **8**, 25 (2013).
102. S. Das, R.K. Bhaduri, Class. Quantum Grav. **32**, 105003 (2015).
103. R.F. Tooper, Astrophys. J. **140**, 434 (1964).
104. R.F. Tooper, Astrophys. J. **142**, 1541 (1965).
105. Ya.B. Zel'dovich, Soviet Phys. JETP **14**, 1143 (1962).
106. T. Harko, Mon. Not. R. Astron. Soc. **413**, 3095 (2011).
107. Ya.B. Zel'dovich, Mon. Not. R. Astron. Soc. **160**, 1 (1972).
108. B. Li, T. Rindler-Daller, P.R. Shapiro, Phys. Rev. D **89**, 083536 (2014).
109. A. Suárez, P.H. Chavanis, Phys. Rev. D **92**, 023510 (2015).
110. S. Chandrasekhar, *An Introduction to the Study of Stellar Structure* (Dover, 1958).
111. N.K. Glendenning, *Compact Stars, Nuclear Physics, Particle Physics and General Relativity* (Springer, New York, 2000).
112. P.H. Chavanis, Astron. Astrophys. **381**, 709 (2002).
113. P.H. Chavanis, Astron. Astrophys. **483**, 673 (2008).
114. C.W. Misner, H.S. Zapolsky, Phys. Rev. Lett. **12**, 635 (1964).
115. D.W. Meltzer, K.S. Thorne, Astrophys. J. **145**, 514 (1966).
116. P.H. Chavanis, Astron. Astrophys. **381**, 340 (2002).
117. H. Poincaré, Acta Math. **7**, 259 (1885).
118. J. Katz, Mon. Not. R. Astron. Soc. **183**, 765 (1978).
119. P.H. Chavanis, Int. J. Mod. Phys. B **20**, 3113 (2006).
120. S.L. Shapiro, S.A. Teukolsky, *Black holes, white dwarfs and neutron stars* (Wiley, New York, 1983).
121. P.H. Chavanis, Astron. Astrophys. **451**, 109 (2006).
122. H.A. Buchdahl, Phys. Rev. **116**, 1027 (1959).
123. A. Mukherjee, S. Shah, S. Bose, Phys. Rev. D **91**, 084051 (2015).
124. F. Ozel, G. Baym, T. Guver, Phys. Rev. D **82**, 101301 (2010).
125. F. Ozel, A. Gould, T. Guver, Astrophys. J. **748**, 5 (2012).
126. T. Guver, P. Wroblewski, L. Camarota, F. Ozel, Astrophys. J. **719**, 1807 (2010).
127. S. van Straaten, E.C. Ford, M. van der Klis, M. Mendez, P. Kaaret, astro-ph/0001480.
128. P. Haensel, A.Y. Potekhin, D.G. Yakovlev, *Neutron Stars: Equation of State and Structure* (Springer, 2007).
129. N. Chamel, P. Haensel, J.L. Zdunik, A.F. Fantina, Int. J. Mod. Phys. E **22**, 1330018 (2013).
130. C.E. Rhoades, R. Ruffini, Phys. Rev. Lett. **32**, 324 (1974).
131. P.H. Chavanis, Eur. Phys. J. Plus **129**, 38 (2014).
132. P.H. Chavanis, Eur. Phys. J. Plus **129**, 222 (2014).
133. P.H. Chavanis, arXiv:1208.1185.
134. P.H. Chavanis, arXiv:1309.5784.
135. J.D. Barrow, Nature **272**, 211 (1978).
136. J. Binney, S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, NJ, 1987).

137. S. Weinberg, *Gravitation and Cosmology* (John Wiley, 2002).
138. P.H. Chavanis, arXiv:1412.0743.
139. D. Pugliese, H. Quevedo, J.A. Rueda, R. Ruffini, Phys. Rev. D **88**, 024053 (2013).
140. C. Gruber, A. Pelster, Eur. Phys. J. D **68**, 341 (2014).
141. T. Harko, M.J. Lake, Phys. Rev. D **91**, 045012 (2015).
142. B. Danila, T. Harko, Z. Kovacs, arXiv:1504.06014.
143. D. Page, M. Prakash, J.M. Lattimer, A.W. Steiner, Phys. Rev. Lett. **106**, 081101 (2011).
144. D. Page, M. Prakash, J.M. Lattimer, A.W. Steiner, Phys. Rev. Lett. **106**, 081101 (2011).
145. S. Banik, D. Bandyopadhyay, Phys. Rev. D **67**, 123003 (2003).
146. S. Banik, M. Hanauske, D. Bandyopadhyay, W. Greiner, Phys. Rev. D **70**, 123004 (2004).
147. R.F. Sawyer, Phys. Rev. Lett. **29**, 382 (1972).
148. S. Barshay, G. Vagradov, G.E. Brown, Phys. Lett. B **43**, 359 (1973).
149. G. Baym, Phys. Rev. Lett. **30**, 1340 (1973).
150. N.K. Glendenning, B. Banerjee, M. Gyulassy, Ann. Phys. **149**, 1 (1983).
151. N.K. Glendenning, P. Hecking, V. Ruck, Ann. Phys. **149**, 1 (1983).
152. T. Sakai, K. Yazaki, K. Shimizu, Nucl. Phys. A **594**, 247 (1995).
153. T. Sakai, K. Shimizu, K. Yazaki, Prog. Theor. Phys. Suppl. **137**, 121 (2000).
154. S. Weissenborn, D. Chatterjee, J. Schaffner-Bielich, Nucl. Phys. A **881**, 62 (2012).
155. S. Weissenborn, D. Chatterjee, J. Schaffner-Bielich, Phys. Rev. C **85**, 065802 (2012).
156. J.I. Kapusta, Phys. Rev. Lett. **93**, 251801 (2004).
157. H. Abuki, Nucl. Phys. A **791**, 117 (2007).
158. D.M. Brink, R.A. Broglia, *Nuclear Superfluidity: pairing in finite systems* (Cambridge University Press, 2010).
159. C.A.R. Sá de Melo, M. Randeria, J.R. Engelbrecht, Phys. Rev. Lett. **71**, 3202 (1993).
160. J.R. Engelbrecht, M. Randeria, C.A.R. Sá de Melo, Phys. Rev. B **55**, 15153 (1997).
161. L. Salasnich, J. Phys.: Conf. Ser. **497**, 0122026 (2014).
162. Y. Nishida, H. Abuki, Phys. Rev. D **72**, 096004 (2005).
163. H. Bondi, Proc. R. Soc. (London) A **282**, 303 (1964).
164. B.K. Harrison, K.S. Thorne, M. Wakano, J.A. Wheeler, *Gravitation Theory and Gravitational Collapse* (Chicago, University of Chicago Press, 1965).
165. S. Chandrasekhar, Astrophys. J. **74**, 81 (1931).
166. S. Chandrasekhar, Mon. Not. R. Astron. Soc. **74**, 81 (1931).