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Einstein spaces modeling nonminimal modified gravity

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Abstract. Off-diagonal vacuum and nonvacuum configurations in the Einstein gravity can mimic physical effects of modified gravitational theories of $f(R, T, R_{\mu\nu}T^{\mu\nu})$ type. To prove this statement, exact and approximate solutions are constructed in the paper, which encode certain models of covariant Hořava-type gravity with dynamical Lorentz symmetry breaking. The corresponding FLRW cosmological dynamics with possible nonholonomic deformations and the reconstruction procedure of certain actions closely related with the standard ΛCDM universe are studied. Off-diagonal generalizations of de Sitter universes are constructed which are generated through nonlinear gravitational polarization of fundamental physical constants and which model interactions with nonconstant exotic fluids and effective matter. The problem of possible matter instability for such off-diagonal deformations in (modified) gravity theories is briefly discussed.

1 Introduction

There are several motivations for the study of modified gravity theories. At short scales it seems clear that General Relativity (GR) needs to be modified in order to take into consideration quantum effects, which leads to second- and higher-order terms in the curvature R, and these same modifications might be also very useful to solve problems at large scales, as the acceleration of the universe expansion and other, like the nature itself of dark matter (DM) and dark energy (DE). Not the least, it is an attempt to formulate a self-consistent (in some particular way, possibly) theory of quantum gravity. Moreover, an increasing amount of more and more accurate and constraining observational data will help to discriminate among the different, alternative modifications of the gravity theory in the search for a better description of our universe.

Among the different classes of extensions of general relativity some of the most popular are $f(R)$, $f(R,T)$, and $f(\mathbf{R},T,F)$ —which we will here generically call $f(R,\ldots)$-modified theories, being R the Ricci scalar and T the metric torsion. In these approaches, the standard Lagrangian for GR, namely as $\mathcal{L} = R$, on a pseudo-Riemannian manifold, V —where R is the Ricci scalar curvature for the Levi-Civita connection, ∇— is modified by the addition of a functional, $f(R,\ldots)$, of the Ricci scalar only, in the first case, of R and the torsion tensor, $T^{\alpha}_{\beta\gamma}$, the energy-momentum tensor for matter, $T_{\beta\gamma}$, and/or its trace $T = T^{\alpha}_{\alpha}$ (in the second), and of a generalized Ricci scalar $\mathbf{R}$, and a Finsler generating function, F, in the third case (such values may be defined on the tangent bundle TV), etc. Classes of modified theories of these kinds can be successfully constructed, and also the corresponding reconstruction procedures, able to mimic the ΛCDM model including the dark energy epochs and the transitions between the different main stages of the universe evolution, thus providing a unified description of the entire cosmological history. For reviews of some of the most important results along this line, see refs. [1–14].

Several of these model constructions of modified theories are actually related with different forms of the so-called covariant Hořava gravity associated with a dynamical breaking of Lorentz invariance [15–18], and with further develop-

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ments, as well, including in particular generic off-diagonal solutions, Lagrange-Hamilton-Finsler–like generalizations, deformation quantization, A-brane models, and gauge-like gravity [19–23]. In some simplified approaches, theories of this kind can be constructed in a power-counting renormalizable form or as nonholonomic brane configurations which correspond to power-law versions of actions of type $f(R, T, R_{\mu\nu}T^{\mu\nu})$ [24–26]. In general, the spacetime geometries can be of Finsler type, with commutative and/or noncommutative parameters, and off-diagonal metrics for wapred/trapped solutions [27–31]. This also includes effects as Lorentz violations, nonlinear dispersion relations and locally anisotropic re-scaling, and effective polarizations of constants, which provide a deeper understanding of the possible connections between these $f(R, \ldots)$ modified theories, and Hořava-Lifshitz and Finsler theories (see also refs. [32–39], in this respect).

Field equations for gravitational and matter field interactions in GR and various modified theories of the types described above usually consist of very sophisticated systems of nonlinear partial differential equations (PDEs). No wonder they request advanced numeric, analytic and geometric techniques for constructing exact and approximate solutions. The most important physical solutions, for black hole configurations, observable cosmological scenarios, etc., have been therefore constructed with the simplifying diagonalizable ansatz for the corresponding metric (obtained by appropriate coordinate transformations and frame rotations) with Killing symmetries. After a series of assumptions of “high symmetry” of the relevant interactions (for spherical, cylindrical or torus ansätze, with a possible additional Lie group interior symmetry), the systems of resulting nonlinear PDEs are usually transformed into much more simplified systems of nonlinear ordinary differential equations (ODEs), what is a great advantage, indeed. In such cases, some classes of exact solutions can be obtained in explicit form (see the monographs [40, 41] for reviews of some results in GR). Even then, it actually took more than half a century to understand the fundamental physical implications of these solutions; for instance, of the Schwarzschild and Kerr black hole metrics, and to finally elaborate the Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological scenario. At present, it is already possible to construct more general classes of exact solutions in (modified) gravity theories depending generically on two or three variables in a “less symmetric” form for four-dimensional (4-d), and extra-dimensional models using advanced geometric, analytic and numerical methods. We can provide a plausible physical interpretation when such solutions are defined by certain symmetry transforms or small deformations of some well-known classes of solutions. However, to derive a new class of exact solutions of a gravity theory is by itself not of main interest for physicists, neither to applied mathematicians, unless such construction does result in new and interesting classical or quantum-physical effects, or does provide a convincing fitting framework to cosmological data coming from the newest surveys.

In a series of works [27–31], the so-called anholonomic frame deformation method (AFDM) for the construction of exact solutions in gravity has been developed. It provides a general geometric technique which allows to integrate PDEs for gravitational and matter fields interactions, for generic off-diagonal metrics with generalized connections, or for the torsionless Levi-Civita one. Such solutions may depend on all spacetime coordinates via various classes of generating and integration functions, commutative and noncommutative parameters, and so on. In particular, the corresponding metrics and connections can exhibit anisotropic ellipsoidal or toroidal symmetries, or encode certain generalized solitonic hierarchies, coming from an effective nonholonomic (with nonintegrable constraints) dynamics with nontrivial topological configurations. It is furthermore possible to analyze the physical implications of geometric constructions of this kind, provided they describe certain “small parameter” deformations related to well-defined black hole objects in cosmological models, or particle physics interactions with broken symmetries.

In brief, the AFDM is based on a quite surprising decoupling property of the vacuum and on certain classes of nonvacuum fundamental field equations in GR and modified gravities. The main idea is to work with an “auxiliary” connection when physically important systems of nonlinear PDE decouple in certain classes of nonholonomic frames. This allows to integrate systems of this kind of very general form. Usually, the auxiliary connection which is needed involves nontrivial nonholonomically induced torsion, which in GR is determined by certain generic off-diagonal terms of the metric and corresponding classes of nonholonomic (equivalently, anholonomic, *i.e.* nonintegrable) constraints on gravitational and matter field dynamics. A nonholonomically induced torsion is different from that in the Einstein-Cartan or string gravity theory, where torsion fields are subject to additional algebraic or dynamical field equations. Having constructed certain general integral varieties of solutions, we can almost generically consider different classes of constraints where the auxiliary connection transforms into the Levi-Civita one. Here we note that it is important to impose such zero-torsion constraints after certain classes of generalized solutions are found in general form, but not before applying the AFDM. Generic off-diagonal solutions for nonlinear systems can be restricted to torsionless configurations, provided certain nontrivial solutions have been already found. If nonholonomic constraints are imposed from the very beginning (for instance, spherical symmetries and a simplified diagonal ansatz for the metric), then one excludes from the analysis more general classes of nonlinear interactions.

The crucial importance of generic off-diagonal solutions in GR and modified theories is determined by a series of geometric and analytic properties of the associated systems of nonlinear PDEs which are used for elaborating the cosmological models and for performing the quantization of the associated gravity theories. In this respect we should emphasize three key issues: 1) A number of physical effects and observed cosmological data can already be explained in GR or in the context of modified gravity theories (for instance, within $f(R)$ gravity or with nontrivial massive terms, see [42, 43]) or can alternatively be modelled with the help of off-diagonal interactions in different types of modified theories. 2) Generic off-diagonal solutions encode configurations with a nontrivial parametric vacuum and effective matter field interactions, also with gravitational polarizations of the interaction and a cosmological constant, and for nontrivial generating and integration functions. For certain well-defined conditions, such models describe broken fundamental symmetries —as, for instance, locally anisotropic interactions, terms deploying violation of local Lorentz symmetry, and warped and trapping configurations— which points towards new methods of quantization and results in alternative scenarios of accelerating and anisotropic cosmological theories, with different solutions for the dark energy and dark matter physical problems. 3) Considering off-diagonal configurations one can model classical and quantum $f(R)$ modified gravities and the like, and also Hořava-Lifshitz and Finsler like theories in a suggestive, unified geometric way [19–22, 27–31].

The aim of this paper is to apply the anholonomic frame deformation method for the constructing of exact off-diagonal solutions corresponding to cosmological models of modified gravity of the general form $f(R, T, R_{\mu\nu}T^{\mu\nu})$, and to study the conditions under which such configurations can be alternatively modeled as effective Einstein spaces with nontrivial off-diagonal parametric vacuum and nonvacuum configurations. The FLRW cosmological dynamics and a reconstruction procedure of the ΛCDM universe will be investigated. To be noted is that we will not work with exotic anisotropic fluid configurations as in [15–18, 24–26], but rather with off-diagonal deformations of de Sitter solutions, as in [19–22]. The problem of matter instabilities in modified and deformed GR theories will be analyzed and solutions will be obtained for certain classes of nonholonomic configurations.

2 Off-diagonal interactions in modified gravity and cosmology

In this section, we formulate a geometric approach to $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity and summarize the anholonomic frame deformation method [27–31]. The geometric constructions will be adapted to nonholonomic distributions with associated nonlinear connection (N-connection) structure[1]. The N-connections formalism will be used for constructing certain classes of N-adapted frames with respect to which the gravitational and matter field equations decouple in very general forms (see sect. 3 below). This is possible for metric compatible linear connections with nonholonomically induced torsions completely defined by metric tensors. Imposing additional constraints, we can generalize zero torsion configurations for the Levi-Civita connection, ∇.

2.1 Modeling dark energy with off-diagonal metrics

In order to motivate our approach, we discuss simple FLRW cosmology and dark energy and dark matter models which are extended for generic off-diagonal solutions in GR and modifications. Working in a spatially flat spacetime with diagonal quadratic form

$$\mathrm{d}s^2 = \mathring{g}_\alpha(t)(\mathrm{d}u^\alpha)^2 = \mathring{a}^2(t)\left[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}y^3)^2\right] - \mathrm{d}t^2, \tag{1}$$

for local coordinates $u^\alpha = (x^i, y^3, y^4 = t)$, when $i = 1, 2$, the FLRW equations are

$$\frac{3}{\kappa^2}\mathring{H}^2 = \mathring{\rho} \qquad \text{and} \qquad \mathring{\rho}^\diamond + 3\mathring{H}(\mathring{\rho} + \mathring{p}) = 0,$$

where $\mathring{\rho}$ and $\mathring{p}$ are, respectively, the total energy and pressure of a perfect fluid (pressureless or just radiation), $\mathring{H} := \mathring{a}^\diamond/\mathring{a}$ for $\mathring{a}^\diamond := \partial\mathring{a}/\partial t = \partial_4\mathring{a} = \partial_t\mathring{a}$, and κ^2 is related to the gravitational (Newton) constant[2]. To explain the observational data of an accelerating universe, and dark energy and matter, various models have been studied (see reviews and references in [1–14]), with effective or exotic matter with an equation of state (EoS) of phantom kind, $p = \varpi\rho$, with $\varpi < -1$. The simplest model of phantom DE is given by

$$\frac{3}{\kappa^2}H_{\mathrm{DE}}^2 = \rho_{\mathrm{DE}} \qquad \text{and} \qquad \rho_{\mathrm{DE}}^\diamond + 3H_{\mathrm{DE}}(1+\varpi)\rho_{\mathrm{DE}} = 0,$$

which for $\varpi < -1$ admits an exact solution

$$H_{\mathrm{DE}} = \frac{2}{3(1+\varpi)(t_s - t)}\,. \tag{2}$$

[1] It should be noted that in generalized Finsler like theories, the N-connection structure is given by a set of three fundamental geometric objects which, for certain models, is completely defined by the so-called Lagrange/Finsler generating function. We do not study in this work Finsler like modifications of GR.

[2] We use a system of notations different from that in standard cosmology, as this will be convenient for constructing cosmological models with generic off-diagonal metrics, and also in order to follow the conventions in our previous works.

This solution has a finite-time future singularity (Big Rip) at $t = t_s$.

Some models have been considered where the Hubble function $H(t)$ is determined by a phantom DE coupled with DM, via a coupling constant, Q, which results in the conservation law

$$\rho^{\diamond}_{\mathrm{DE}} + 3H(1+\varpi)\rho_{\mathrm{DE}} = -Q\rho_{\mathrm{DE}}, \qquad \rho^{\diamond}_{\mathrm{DM}} + 3H\rho_{\mathrm{DM}} = Q\rho_{\mathrm{DM}}.$$

The solutions of these equations can be expressed as

$$\rho_{\mathrm{DE}} = {}^{0}\rho_{\mathrm{DE}}\, e^{-3(1+\varpi)}e^{-Qt} \qquad \text{and} \qquad \rho_{\mathrm{DM}}a^3 = Q\ {}^{0}\rho_{\mathrm{DE}} \int^t \mathrm{d}t' e^{-3\varpi}e^{-Qt},$$

respectively, where ${}^{0}\rho_{\mathrm{DE}}$ is an integration constant and the EoS is taken to be $p = \varpi\rho_{\mathrm{DE}}$. These functions can be used for the second FLRW equation,

$$-\frac{1}{\kappa^2}(2H^{\diamond} + 3H^2) = p.$$

We have

$$H = -Q/3(1+\varpi), \tag{3}$$

for the exact solution of this equation, which corresponds to the evolution for de Sitter space, $a(t) = a_0 e^{-Qt/3(1+\varpi)}$, where a_0 is determined from $a_0^{3(1+\varpi)} = -\frac{3\kappa^2}{Q^2}(1+\varpi)^2\varpi\ {}^{0}\rho_{\mathrm{DE}}$. The value of H in (3) is positive for $\varpi < -1$, what does not mean that the Big Rip singularity in (2) can be avoided, but just shows that the coupling of the phantom DE and DM gives a possibility that the universe could evolve as a de Sitter phase. More than that, the first FLRW equation,

$$\frac{3}{\kappa^2}H^2 = \rho_{\mathrm{DE}} + \rho_{\mathrm{DM}}, \tag{4}$$

imposes the relation $\rho_{\mathrm{DM}} = (1+\varpi)\rho_{\mathrm{DE}}$. Considering a de Sitter solution as an attractor, with $\varpi \sim -4/3$, we obtain $-(1+\varpi) \sim 1/3$, which is almost independent from the initial condition, *i.e.*, it solves for free the so-called coincidence problem[3].

Since the DE-DM coupling does not always remove the singularity and there is no such fluid with constant EoS parameter, models were considered which are proportional to a power of the scalar curvature, for instance, $p_{fluid} \propto R^{1+\epsilon}$, for $\epsilon > 0$. In that case the total EoS parameter is greater than -1 and a Big Rip does not occur for large curvature. Two variants of theories have been exploited where this kind of inhomogeneous effective fluid matter is realized, by a conformal anomaly and other quantum effects or by some modified model of gravity, for instance, when the gravitational Lagrange density $R \to f(R) = R + R^{\varkappa}$. In the case $1 < \varkappa < 2$, we have that solutions with

$$Ht \sim -\frac{(\varkappa-1)(2\varkappa-1)}{\varkappa-2} \qquad \text{and} \qquad w_{\mathrm{eff}} \sim -1 - 2H^{\diamond}/3H^2 > -1$$

do not result in a Big Rip or any other kind of future singularity. Similar classical and quantum arguments were considered as motivations to study $f(R)$ modified gravity theories [1–18, 24–26, 42, 43].

In a series of works [19–22, 27–31, 44–47], various classes of off-diagonal solutions were studied which can be constructed by geometric methods in modified gravity theories. We proved that, for instance, certain important effects and cosmological models related to $f(R)$ modified theories and the like can alternatively be explained by nonlinear off-diagonal gravitational and matter field interactions with respect to nonholonomic frames. Let us briefly recall the main ideas supporting such an approach. Off-diagonal ansätze for metrics (see, for instance, (59)),

$$g_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} g_1 + \omega^2({w_1}^2 h_3 + {n_1}^2 h_4) & \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & \omega^2 w_1 h_3 & \omega^2 n_1 h_4 \\ \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & g_2 + \omega^2({w_2}^2 h_3 + {n_2}^2 h_4) & \omega^2 w_2 h_3 & \omega^2 n_2 h_4 \\ \omega^2 w_1 h_3 & \omega^2 w_2 h_3 & \omega^2 h_3 & 0 \\ \omega^2 n_1 h_4 & \omega^2 n_2 h_4 & 0 & \omega^2 h_4 \end{bmatrix}, \tag{5}$$

[3] If DE does not couple with DM, we have $\rho_{\mathrm{DM}} \sim a^{-3}$ and $\rho_{\mathrm{DE}} \sim a^{-3(1+\varpi)}$, which do not satisfy the observed 1/3 ratio of DE and DM and does result in a coincidence problem.

where the coefficients are parameterized by functions of the type $g_1 = g_2 \sim e^{\psi(x^i)}$ and $n_i(x^k)$ (we can fix certain constants for corresponding classes of generating, $\Phi(x^k,t)$, and integration functions), thus $h_a[\Phi(x^k,t)] \sim h_a(t)$, [for $a = 3,4$], $w_i[\Phi(x^k,t)] \sim w_i(t)$ and $\omega(x^k,t) \sim \omega(t)$, were found to generate exact (in general, nonhomogeneous) cosmological solutions in modified gravity theories. Such generic off-diagonal metrics[4] can be represented in the form

$$\mathrm{d}s^2 = a^2(t)[(e^1)^2 + (e^2)^2] + a^2(t)\widehat{h}_3(t)(\widehat{\mathbf{e}}^3)^2 + (\widehat{\mathbf{e}}^4)^2, \tag{6}$$

with respect to so-called N-adapted frames (eqs. (12) and (13) are used, in general)

$$\widehat{\mathbf{e}}^3 = \mathrm{d}y^3 + n_i\,\mathrm{d}x^i, \qquad \widehat{\mathbf{e}}^4 = \mathrm{d}t + w_i(t)\mathrm{d}x^i.$$

For certain well-defined conditions (see sect. 3), we can consider off-diagonal deformations $\mathring{g}_\alpha(t) \to g_{\underline{\alpha\beta}}(x^k,t) \sim g_{\underline{\alpha\beta}}(t)$ defining new classes of cosmological models which mimic contributions from $f(R)$ modified gravity encoded into the data for $\omega(t)$, $w_i(t)$, etc. The corresponding formulas are nonlinear functionals relating generating functions to the (effective) matter sources. Such off-diagonal configurations are equivalently modeled as solutions of some effective field equations $\check{\mathbf{R}}^\alpha{}_\beta = \check{\Lambda}\delta^\alpha{}_\beta$. In this way, various classes of cosmological solutions of modified gravities can be alternatively modeled by metrics of the type (6), when the scaling factor $a(t)$ is nonlinearly determined by the coefficients $w_i(t)$ and $h_a(t)$ via a generating function $\Phi(t)$ and an effective source $\Upsilon(t)$. We can model ΛCDM cosmology and analogously DE and DM effects with $\rho_{\mathrm{DE}} + \rho_{\mathrm{DM}}$ encoded into $\Phi(t)$ and $\Upsilon(t)$, but with respect to the adapted frames $\widehat{\mathbf{e}}^a(t)$. Solutions with off-diagonal metrics may be interpreted in accordance with observational data if $a(t)$ is chosen to determine, for instance, an effective $H(t)$ (3) with cosmological evolution from a spacetime background encoding $f(R)$-modifications. To prove such results in a rigorous mathematical way we need to apply advanced methods from the geometry of nonholonomic manifolds. For our purposes, such manifolds can be considered as usual pseudo-Riemannian spacetimes, endowed with additional nonintegrable distributions and frame structures.

Both classes of metrics (1) and (6) can be characterized, respectively, by scaling factors $\mathring{a}(t)$ and $a(t)$. Let us suppose that we have found a cosmological solution of type (6) in a given theory of modified gravity and analyze how this metric can be formally diagonalized for deformations of a small real parameter ε (when $0 \le \varepsilon \ll 1$). We can consider "homogeneous" approximations of type $\widehat{h}_3(t) \approx 1 + \varepsilon\widehat{\chi}_3(t)$, $w_i(t) \sim \varepsilon\check{w}_i(t)$ and $n_i \sim \varepsilon\check{n}_i$[5]. In explicit form, such a metric, with small off-diagonal deformations on ε and rescaling $\mathring{a}(t) \to a(t)$, can be written as

$$\mathrm{d}s^2 = a^2(t)\left[(e^1)^2 + (e^2)^2\right] + a^2(t)\left[1 + \varepsilon\widehat{\chi}_3(t)\right]\left(\mathrm{d}y^3 + \varepsilon\check{n}_i\mathrm{d}x^i\right)^2 + \left(\mathrm{d}t + \varepsilon\check{w}_i(t)\mathrm{d}x^i\right)^2. \tag{7}$$

See below how it is possible to construct subclasses of off-diagonal configurations in a $\widehat{\mathbf{f}}(\widehat{\mathbf{R}},\ldots)$ gravity where $\widehat{\boldsymbol{\Upsilon}}$ (32) goes into $\check{\Lambda}$ (33), and $\check{\Phi}^2 = \check{\Lambda}^{-1}[\widehat{\Phi}^2|\widehat{\boldsymbol{\Upsilon}}| + \int \mathrm{d}\zeta\,\widehat{\Phi}^2\partial_\zeta|\widehat{\boldsymbol{\Upsilon}}|]$ (61) results in $\widehat{\mathbf{f}} \to \check{\mathbf{f}} = \check{\mathbf{R}}$, an effective $\check{\mathbf{R}}^\alpha{}_\beta = \check{\Lambda}\delta^\alpha{}_\beta$ which admits LC solutions with zero torsion. We will be able to reproduce the ΛCDM model provided the metric (7) defines certain classes of solutions constructed for a corresponding effective action in GR, namely

$$S = \frac{1}{\kappa^2}\int \delta^4 u\sqrt{|{}^\varepsilon\mathbf{g}_{\alpha\beta}|}\,({}^\varepsilon\check{\mathbf{R}} - 2\check{\Lambda} + {}_m\mathcal{L}({}^\varepsilon\mathbf{g}_{\alpha\beta},\ {}_m\Psi)). \tag{8}$$

In this action, the Ricci scalar ${}^\varepsilon\check{\mathbf{R}} = \check{\mathbf{R}}(a,\varepsilon)$ is constructed for ${}^\varepsilon\mathbf{g}_{\alpha\beta}$ with coefficients of (7), $\check{\Lambda}$ is an effective cosmological constant used for nonholonomic deformations, and ${}_m\mathcal{L}$ is considered for certain effective matter fields with certain pressure ${}_mp$ and energy density ${}_m\rho$. The EoS are chosen, for simplicity, to correspond to an effective de Sitter configuration determined by $\check{\Lambda}$, where $\check{\varpi} := \check{p}_\Lambda/\check{\rho}_\Lambda = -1$, with pressure $\check{p}_\Lambda$ and energy density $\check{\rho}_\Lambda$.

We can describe the theories determined by (8) and (7) with respect to nonholonomic (nonintegrable) dual frames $\widehat{\mathbf{e}}^\alpha = (e^i, \widehat{\mathbf{e}}^a)$, which is convenient for constructing off-diagonal solutions, or to redefine the constructions with respect to local coordinate coframes $\mathrm{d}u^\alpha = (\mathrm{d}x^i, \mathrm{d}y^a)$, where certain analog of the FLRW metric and ΛCDM like theories can be analyzed. For $\varepsilon \to 0$, the metric (7) transforms into

$$\mathrm{d}s^2 = a^2(t)\left[(e^1)^2 + (e^2)^2 + (\mathrm{d}y^3)\right]^2 + \mathrm{d}t^2, \tag{9}$$

which is just (1) but with a rescaled factor because of the nonholonomic transformations $\widehat{\boldsymbol{\Upsilon}} \to \check{\Lambda}$ and $\widehat{\Phi} \to \check{\Phi}$.

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[4] Which cannot be diagonalized by coordinate transformations.

[5] For information on inhomogeneity effects in cosmology, see [48]. In a more general context, it is possible to consider also "small" local anisotropic deformations depending on space-like coordinates when $\widehat{\chi}_3(x^k,t)$, $w_i(t) \sim \varepsilon\check{w}_i(x^k,t)$ and $n_i \sim \varepsilon\check{n}_i(x^k)$. Some amount of anisotropy is compatible with observational data in various gravity and cosmological theories. See [49,50], for reviews of various approaches related to GR and generalizations of Bianchi, Kasner and Gödel type configurations, refs. [44–47], for off-diagonal configurations and ref. [51] for $f(R)$-modified gravity theories. We note also that the approximation $\widehat{h}_3(t) \approx 1 + \varepsilon\widehat{\chi}_3(t)$ can be very restrictive —one can consider more general classes of solutions with arbitrary $\widehat{h}_3(t)$.

The corresponding Einstein equations with respect to the nonholonomic frames are

$$
\begin{aligned}
3H^2 &= \kappa^2\ {}_m\rho + \check{\Lambda}, \\
2H^{\diamond} &= -\kappa^2({}_m\rho + {}_mP + \check{\Lambda}),
\end{aligned} \tag{10}
$$

where $H^{\diamond} := a^{\diamond}/a$. We can express ${}^{\varepsilon}\check{\mathbf{R}} + {}_m\mathcal{L} = {}^{a}\check{\mathbf{R}} + {}^{0}_{m}\mathcal{L} + \varepsilon\ {}^{1}_{m}\mathcal{L}$, where ${}^{a}\check{\mathbf{R}}$ and ${}^{0}_{m}\mathcal{L}$ are computed for the metric (9) and ${}^{1}_{m}\mathcal{L}$ include all ε-deformations in (8). Such ${}^{1}_{m}\mathcal{L}$ results in the effective splitting ${}_m\rho = {}^{0}_{m}\rho + \varepsilon\ {}^{1}_{m}\rho$ and ${}_mp = {}^{0}_{m}p + \varepsilon\ {}^{1}_{m}p$. In this way, we can encode the off-diagonal components as certain additional terms into the matter source, or either consider them as a polarization of the effective cosmological constant $\Lambda := \check{\Lambda} + \varepsilon\ {}^{1}\check{\Lambda}$[6]. We also note that possible small inhomogeneous and locally anisotropic contributions, and concordance with observational data, can be estimated similarly to those presented, *e.g.*, in [51]. This could be a ground for further investigations of such "slightly" $f(R)$-modified off-diagonal cosmological models. In this subsection we provide only a few qualitative estimations, in order to demonstrate that "realistic" ΛCDM like cosmological theories can be equivalently modeled both in terms of nonholononomic frames and of coordinate frames, if small off-diagonal deformations are considered, only, and then the limit $\varepsilon \to 0$ is taken.

In coordinate frames, eqs. (10) are written as

$$
\begin{aligned}
3H^2 &= \kappa^2\ {}^{0}_{m}\rho + \Lambda, \\
2H^{\diamond} &= -\kappa^2({}^{0}_{m}\rho + {}^{0}_{m}P + \Lambda).
\end{aligned}
$$

For $\varepsilon \to 0$, the diagonalized solutions are determined by a (and not by $\mathring{a}$ in (1)) and can be parameterized to define and effective ΛCDM like model where $a = a_c e^{H_c t}$, for a positive constant a_c. Thus, modified gravities with equivalent off-diagonal encodings of $f(R)$-modified gravity seem to result in realistic cosmological models, at least for small parametric ε-deformations.

The main goal of this work is to study possible nonlinear gravitational and matter field interactions which result in the encoding of modified gravities into generic off-diagonal metrics defining effective Einstein spaces, without certain special assumptions on the linearization of some associated systems of PDEs and their solutions. Surprisingly enough, the AFDM allows us to find such "nonperturbative" solutions in explicit form, by using geometrical methods. The values $a(t)$, $\widehat{h}_3(t)$, $w_i(t)$ and n_i are nonlinearly determined by generating functions and sources of type $\Phi(t)$ and $\Upsilon(t)$. In general, such nonlinear modifications of a "prime" $\mathring{a}(t)$ are not small. Even if we can introduce an effective scaling factor $a(t)$, this value describes a nonlinear and inhomogeneous evolution with respect to nonholonomic (nonintegrable) dual frames $\widehat{\mathbf{e}}^{\alpha} = (e^i, \widehat{\mathbf{e}}^a)$. All generic off-diagonal cosmological models can be also redefined with respect to local coordinate coframes $\mathrm{d}u^{\alpha} = (\mathrm{d}x^i, \mathrm{d}y^a)$. In local coordinate form, we are not able to analyze common and different properties of diagonalizable and nondiagonalizable models only by comparing the evolutions of $a(t)$ and $\mathring{a}(t)$. From the physical point of view, we can consider the Universe as an aether with a complex vacuum and nonvacuum nonlinear structure determined by possible $f(R)$-modifications. An observer acquires experimental/observational data with respect to a local comoving frame $\widehat{\mathbf{e}}^{\alpha} = (e^i, \widehat{\mathbf{e}}^a)$ where generic off-diagonal gravitational and matter field interactions are taken into consideration. For certain parametric resonant dependencies, even the smallest nonlinearities can result in substantial polarizations of the gravitational vacuum aether, with possible Lie group or solitonic symmetries, or without any anisotropic symmetry prescribed in advance. Such cosmological models are described by more sophisticate geometries than the FLRW cosmology (for references, see [48, 50]).

The key idea of our work is that, within certain assumptions, various possible $f(R)$-nonlinear modifications can be encoded into off-diagonal terms and some effective $a(t)$, $\widehat{h}_3(t)$, $w_i(t)$ via nonlinear interactions. This can be done for more general classes of cosmological solutions with nonlinear gravitational interactions restructuring the spacetime aether before considering certain small ε-parameters. Such nonlinear cosmological evolution is determined by three functions of a time like variable, t, characterizing a more complex model then the FLRW one. We get, indeed: 1) a scaling factor $a(t)$; 2) a diagonal inhomogeneity function $\widehat{h}_3(t)$; and 3) off-diagonal deformations via $w_i(t)$. Having constructed a class of off-diagonal solutions then, with certain additional assumptions for the effective linearization in terms of ε, one can study possible observable ε-small inhomogeneous or locally anisotropic contributions. The physical effects of small ε-deformations can be compared, for instance, with those for a scaling factor $\mathring{a}(t)$, although this will not be the aim of this paper. The main results and conclusions of it (see sects. 4 and 5) will have to do with certain special properties of off-diagonal nonlinear systems with nonintegrable constraints and with the exact solutions. Even for the physical interpretations of observable cosmological data at a fixed time $t = t_0$, we can take the limit $\varepsilon \to 0$, where $\widehat{h}_3(t) \to 1$, and w_i and n_i may vanish or result in a nonholonomic frame structure; a generalized nonlinear cosmological evolution by such generalized solutions may result in a modified scaling factor $a(t)$ encoding both $f(R)$-modifications and off-diagonal nonlinear interactions for $t < t_0$.

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[6] We do not provide here explicit formulas for the corrections proportional to ε because, in the end, we shall take smooth limits $\varepsilon \to 0$. The main constructions for nonholonomic off-diagonal transforms are based on rescaling $\mathring{a}(t) \to a^2(t)$ generated by the solutions with $\widehat{\boldsymbol{\Upsilon}} \to \check{\Lambda}$ and $\widehat{\Phi} \to \check{\Phi}$.

2.2 Conventions and geometric preliminaries

2.2.1 Nonlinear connections and N-adapted frames

Let us consider a pseudo-Riemannian manifold V, $\dim V = n+m$, $(n,m \geq 2)$. A Whitney sum $\mathbf{N}$ is defined for its tangent space TV,

$$\mathbf{N} : TV = hTV \oplus vTV. \tag{11}$$

Conventionally, this states a nonholonomic (equivalently, nonintegrable, or anholonomic) horizontal (h) and vertical (v) splitting, or a nonlinear connection (*N-connection*) structure. In local form, it is determined by its coefficients $\mathbf{N} = \{N_i^a(u)\}$, when $\mathbf{N} = N_i^a(x,y)\mathrm{d}x^i \otimes \partial/\partial y^a$ for certain local coordinates $u = (x,y)$, or $u^\alpha = (x^i, y^a)$, and h-indices $i,j,\ldots = 1,2,\ldots n$ and v-indices $a,b,\ldots = n+1, n+2, \ldots, n+m$[7]. Such a h-v decomposition can be naturally associated with some N-adapted frame or, respectively, dual frame structures, $\mathbf{e}_\nu = (\mathbf{e}_i, e_a)$ and $\mathbf{e}^\mu = (e^i, \mathbf{e}^a)$,

$$\mathbf{e}_i = \partial/\partial x^i - N_i^a(u)\partial/\partial y^a, \qquad e_a = \partial_a = \partial/\partial y^a, \tag{12}$$

and

$$e^i = \mathrm{d}x^i, \qquad \mathbf{e}^a = \mathrm{d}y^a + N_i^a(u)\mathrm{d}x^i. \tag{13}$$

The nonholonomy relations hold

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = W_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \tag{14}$$

with nontrivial anholonomy coefficients $W_{ia}^b = \partial_a N_i^b$, $W_{ji}^a = \Omega_{ij}^a = \mathbf{e}_j(N_i^a) - \mathbf{e}_i(N_j^a)$. The coefficients Ω_{ij}^a define the N-connection curvature.

2.2.2 Distinguished metric structures

Any metric structure $\mathbf{g}$ on $\mathbf{V}$ (for physical applications, we consider pseudo-Euclidean signatures of type $(+,+,+,-)$) can be written in two equivalent ways: 1) with respect to a dual local coordinate basis,

$$\mathbf{g} = \underline{g}_{\alpha\beta}\mathrm{d}u^\alpha \otimes \mathrm{d}u^\beta, \tag{15}$$

where

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e g_{be} & g_{ab} \end{bmatrix}, \tag{16}$$

or 2) as a distinguished metric (in brief, *d-metric*), *i.e.* in N-adapted form,

$$\mathbf{g} = g_\alpha(u)\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_i(x^k)\mathrm{d}x^i \otimes \mathrm{d}x^i + g_a(x^k, y^b)\mathbf{e}^a \otimes \mathbf{e}^a. \tag{17}$$

To prove the decoupling of fundamental gravitational equations in modified gravity is possible for d-metrics and working with respect to N-adapted frames.

2.2.3 Distinguished connections

A linear connection is called distinguished, *d-connection*, $\mathbf{D} = (hD, vD)$, if it preserves under parallelism a prescribed N-connection splitting (11). Any $\mathbf{D}$ defines an operator of covariant derivation, $\mathbf{D}_\mathbf{X}\mathbf{Y}$, for a d-vector field $\mathbf{Y}$ in the direction of a d-vector $\mathbf{X}$. We note that any vector $Y(u) \in T\mathbf{V}$ can be parameterized as a d-vector, $\mathbf{Y} = \mathbf{Y}^\alpha \mathbf{e}_\alpha = \mathbf{Y}^i \mathbf{e}_i + \mathbf{Y}^a e_a$, or $\mathbf{Y} = (hY, vY)$, with $hY = \{\mathbf{Y}^i\}$ and $vY = \{\mathbf{Y}^a\}$, where the N-adapted base vectors and duals, or covectors, are chosen in N-adapted form (12) and (13). The local coefficients of $\mathbf{D}_\mathbf{X}\mathbf{Y}$ can be computed for $\mathbf{D} = \{\boldsymbol{\Gamma}^\gamma_{\ \alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})\}$ and h-v components of $\mathbf{D}_{\mathbf{e}_\alpha}\mathbf{e}_\beta := \mathbf{D}_\alpha \mathbf{e}_\beta$ using $\mathbf{X} = \mathbf{e}_\alpha$ and $\mathbf{Y} = \mathbf{e}_\beta$[8]. We can characterize a d-connection by three fundamental geometric objects: the d-torsion, $\mathcal{T}$, the nonmetricity, $\mathcal{Q}$, and the d-curvature, $\mathcal{R}$, respectively, defined by

$$\begin{aligned} &\mathcal{T}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_\mathbf{X}\mathbf{Y} - \mathbf{D}_\mathbf{Y}\mathbf{X} - [\mathbf{X},\mathbf{Y}], \qquad \mathcal{Q}(\mathbf{X}) := \mathbf{D}_\mathbf{X}\mathbf{g}, \\ &\mathcal{R}(\mathbf{X},\mathbf{Y}) := \mathbf{D}_\mathbf{X}\mathbf{D}_\mathbf{Y} - \mathbf{D}_\mathbf{Y}\mathbf{D}_\mathbf{X} - \mathbf{D}_{[\mathbf{X},\mathbf{Y}]}. \end{aligned} \tag{18}$$

[7] The Einstein rule on index summation will be applied if the contrary is not stated. For convenience, "primed" and "underlined" indices will be used, and boldface letters to emphasize that an N-connection spitting is considered on a spacetime manifold $\mathbf{V} = (V, \mathbf{N})$.

[8] We shall use the terms d-vector, d-tensor, etc., for any vector, tensor valued with coefficients defined in a N-adapted form with respect to the necessary types of tensor products of bases, (12) and (13), and necessary h-v decompositions.

The N-adapted coefficients,

$$\mathcal{T} = \{\mathbf{T}^{\gamma}{}_{\alpha\beta} = \left(T^{i}{}_{jk}, T^{i}{}_{ja}, T^{a}{}_{ji}, T^{a}{}_{bi}, T^{a}{}_{bc}\right)\}, \qquad \mathcal{Q} = \{\mathbf{Q}^{\gamma}{}_{\alpha\beta}\},$$
$$\mathcal{R} = \{\mathbf{R}^{\alpha}{}_{\beta\gamma\delta} = \left(R^{i}{}_{hjk}, R^{a}{}_{bjk}, R^{i}{}_{hja}, R^{c}{}_{bja}, R^{i}{}_{hba}, R^{c}{}_{bea}\right)\},$$

of such fundamental geometric objects are computed by introducing $\mathbf{X} = \mathbf{e}_{\alpha}$ and $\mathbf{Y} = \mathbf{e}_{\beta}$, and $\mathbf{D} = \{\boldsymbol{\Gamma}^{\gamma}{}_{\alpha\beta}\}$ into the formulas above (see [27–31] for details).

2.2.4 Preferred d-metric and d-connection structures

A d-connection $\mathbf{D}$ is compatible with a d-metric $\mathbf{g}$ if and only if $\mathcal{Q} = \mathbf{D}\mathbf{g} = 0$. Any metric structure $\mathbf{g}$ on $\mathbf{V}$ is characterized by a unique metric compatible and torsionless linear connection called the Levi-Civita (LC) connection, ∇. It should be noted that ∇ is not a d-connection because it does not preserve under parallelism the N-connection splitting (11). Nevertheless, such a h-v decomposition allows us to define N-adapted distortions of any d-connection $\mathbf{D}$,

$$\mathbf{D} = \nabla + \mathbf{Z}, \tag{19}$$

with respective conventional "nonboldface" and "boldface" symbols for the coefficients: $\nabla = \{\Gamma^{\alpha}{}_{\beta\gamma}\}$ and, for the distortion d-tensor, $\mathbf{Z} = \{\mathbf{Z}^{\alpha}{}_{\beta\gamma}\}$.

This stands for any prescribed $\mathbf{N}$ and $\mathbf{g} = h\mathbf{g} + v\mathbf{g}$, but alternatively to ∇, on $\mathbf{V}$, we can work with the so-called canonical d-connection, $\widehat{\mathbf{D}}$, when

$$(\mathbf{g}, \mathbf{N}) \ \rightarrow \ \begin{array}{llll} \boldsymbol{\nabla} : \boldsymbol{\nabla}\,\mathbf{g} = 0; & {}^{\nabla}\mathcal{T} = 0; & & \\ \widehat{\mathbf{D}} : \widehat{\mathbf{D}}\mathbf{g} = 0; & h\widehat{\mathcal{T}} = 0, & v\widehat{\mathcal{T}} = 0, & hv\widehat{\mathcal{T}} \neq 0; \end{array}$$

are completely defined *by the same* metric structure. The canonical distortion d-tensor $\widehat{\mathbf{Z}}$ in the distortion relation of type (11), $\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$, is an algebraic combination of the coefficients of the corresponding torsion d-tensor $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}\}$. The respective coefficients of the torsions, $\widehat{\mathcal{T}}$ and ${}^{\nabla}\mathcal{T} = 0$, and curvatures, $\widehat{\mathcal{R}} = \{\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\delta}\}$ and ${}^{\nabla}\mathcal{R} = \{R^{\alpha}{}_{\beta\gamma\delta}\}$, of $\widehat{\mathbf{D}}$ and ∇ can be defined and computed using formulas similar to (18). We note that the coefficients $\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$ are not trivial but nonholonomically induced by anholonomy coefficients $W^{\gamma}_{\alpha\beta}$ (14) and certain off-diagonal coefficients of the metric (16)[9].

The Ricci tensors of $\widehat{\mathbf{D}}$ and ∇ are computed in the standard form,

$$\widehat{\mathcal{R}}ic = \left\{\widehat{\mathbf{R}}\ {}_{\beta\gamma} := \widehat{\mathbf{R}}^{\gamma}{}_{\alpha\beta\gamma}\right\} \qquad \text{and} \qquad Ric = \left\{R\ {}_{\beta\gamma} := R^{\gamma}{}_{\alpha\beta\gamma}\right\}.$$

With respect to N-adapted coframes (13), the Ricci d-tensor $\widehat{\mathcal{R}}ic$ is characterized by four h-v N-adapted coefficients

$$\widehat{\mathbf{R}}_{\alpha\beta} = \left\{\widehat{R}_{ij} := \widehat{R}^{k}{}_{ijk},\ \widehat{R}_{ia} := -\widehat{R}^{k}{}_{ika},\ \widehat{R}_{ai} := \widehat{R}^{b}{}_{aib},\ \widehat{R}_{ab} := \widehat{R}^{c}{}_{abc}\right\}, \tag{20}$$

and (an alternative to the LC-scalar curvature, $R := \mathbf{g}^{\alpha\beta}R_{\alpha\beta}$) scalar curvature,

$$\widehat{\mathbf{R}} := \mathbf{g}^{\alpha\beta}\widehat{\mathbf{R}}_{\alpha\beta} = g^{ij}\widehat{R}_{ij} + g^{ab}\widehat{R}_{ab}. \tag{21}$$

We emphasize that any (pseudo) Riemannian geometry can be equivalently described by both geometric data $(\mathbf{g}, \boldsymbol{\nabla})$ and $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$. For instance, there are canonical distortion relations

$$\widehat{\mathcal{R}} = {}^{\nabla}\mathcal{R} + {}^{\nabla}\mathcal{Z} \qquad \text{and} \qquad \widehat{\mathcal{R}}ic = Ric + \widehat{\mathcal{Z}}ic,$$

where the respective distortion d-tensors ${}^{\nabla}\mathcal{Z}$ and $\widehat{\mathcal{Z}}ic$ are computed by introducing $\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$ into the corresponding formulas (18) and (20). The canonical data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ provide an example of nonholonomic (pseudo-) Riemannian manifold which is a standard one but enabled with a nonholonomic distribution determined by $(\mathbf{g}, \mathbf{N})$. If the coefficients $\Omega^{a}_{ij} = 0$, such a distribution is holonomic, *i.e.* integrable.

[9] In the Riemann-Cartan geometry, such a torsion is for a general metric compatible linear connection, D, which is not necessarily completely defined by the data $(\mathbf{g}, \mathbf{N})$.

Nevertheless, physical theories formulated in terms of data as $(\mathbf{g}, \mathbf{\nabla})$, or $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$, are not equivalent if certain additional conditions are not imposed. Let us consider an explicit example. We can introduce the Einstein d-tensor of $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{E}}_{\alpha\beta} := \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\ \widehat{\mathbf{R}}, \tag{22}$$

and construct a N-adapted energy momentum tensor for a Lagrange density ${}^m\mathcal{L}$ of the matter fields, $\widehat{\mathbf{T}}_{\alpha\beta} := -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\ {}^m\widehat{\mathcal{L}})}{\delta \mathbf{g}^{\alpha\beta}}$, performing a N-adapted variational calculus with respect to N-elongated (co) frames (12) and (13), and consider that $\widehat{\mathbf{D}}$ is used as covariant derivative instead of ∇. In this way a nonholonomic deformation of Einstein's gravity is constructed, being $\nabla \to \widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$, with gravitational field equations

$$\widehat{\mathbf{R}}_{\alpha\beta} = \kappa^2\left(\widehat{\mathbf{T}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\widehat{\mathbf{T}}\right), \tag{23}$$

for a conventional gravitational constant κ^2 and $\widehat{\mathbf{T}} := \mathbf{g}^{\mu\nu}\widehat{\mathbf{T}}_{\mu\nu}$. Such equations are different from the standard Einstein equations in GR because, in general, $\widehat{\mathbf{R}}_{\alpha\beta} \neq R_{\alpha\beta}$ and $\widehat{\mathbf{T}}_{\alpha\beta} \neq T_{\alpha\beta}$, where $T_{\alpha\beta} := -\frac{2}{\sqrt{|\mathbf{g}_{\mu\nu}|}}\frac{\delta(\sqrt{|\mathbf{g}_{\mu\nu}|}\ {}^m\mathcal{L})}{\delta \mathbf{g}^{\alpha\beta}}$ for ${}^m\mathcal{L}[\mathbf{g}_{\alpha\beta}, \nabla] \neq {}^m\widehat{\mathcal{L}}[\mathbf{g}_{\alpha\beta}, \widehat{\mathbf{D}}]$.

LC-configurations can be extracted from certain classes of solutions of eqs. (23) if additional conditions are imposed, resulting in zero values for the canonical d-torsion, $\widehat{\mathcal{T}} = 0$. In N-adapted coefficient form, such condition is equivalent to

$$\widehat{T}^i{}_{jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj}, \qquad \widehat{T}^i{}_{ja} = \widehat{C}^i_{jb}, \qquad \widehat{T}^a{}_{ji} = -\Omega^a{}_{ji}, \qquad \widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(N^c_j), \qquad \widehat{T}^a{}_{bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb}. \tag{24}$$

It should be emphasized that we are able to find generic off-diagonal solutions of the Einstein equations in GR depending on three and more coordinates for $\widehat{\mathbf{D}} \to \nabla$, when $\widehat{\mathbf{R}}_{\alpha\beta} \to R_{\alpha\beta}$ and $\widehat{\mathbf{T}}_{\alpha\beta} \to T_{\alpha\beta}$, if the nonholonomic constraints (24) are imposed after certain classes of solutions were found for $\widehat{\mathbf{D}} \neq \nabla$. But we are not able to decouple such systems of nonlinear PDEs if the zero torsion condition for ∇ is imposed from the very beginning.

2.3 Nonholonomic structures in $\mathbf{f(R, \ldots)}$-modified gravity theories

In general, different models of modified gravity are formulated for independent metric and linear connection fields with a corresponding Palatini-type variational formulation (see [1–14]). The gravitational and matter field equations in modified gravities consist in very sophisticate systems of nonlinear PDEs for which finding exact solutions is a very difficult technical task, even for the simplest diagonal ansätze with the coefficients of the metrics and connections depending on just one (time or space) variable. Nevertheless, the anholonomic frame deformation method [27–31] seems to work efficiently and allows to construct off-diagonal solutions in modified gravity theories [44–47].

2.3.1 Equivalent modeling of modified gravity

Consider three classes of equivalent theories of modified gravity defined for the same metric field $\mathbf{g} = \{g_{\mu\nu}\}$ but with different actions (and related functionals) for gravity, ${}^gS$, and matter, ${}^mS$, fields,

$$\begin{aligned}\mathcal{S} = {}^g\mathcal{S} + {}^m\mathcal{S} &= \frac{1}{2\kappa^2}\int f(R, T, R_{\alpha\beta}T^{\alpha\beta})\sqrt{|g|}\mathrm{d}^4u + \int {}^m\mathcal{L}\sqrt{|g|}\mathrm{d}^4u \\ = {}^g\widehat{\mathbf{S}} + {}^m\widehat{\mathbf{S}} &= \frac{1}{2\kappa^2}\int \widehat{\mathbf{f}}(\widehat{\mathbf{R}}, \widehat{\mathbf{T}}, \widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta})\sqrt{|\widehat{\mathbf{g}}|}\mathbf{d}^4u + \int {}^m\widehat{\mathbf{L}}\sqrt{|\widehat{\mathbf{g}}|}\mathbf{d}^4u \\ = {}^g\check{\mathbf{S}} + {}^m\check{\mathbf{S}} &= \frac{1}{2\kappa^2}\int \check{\mathbf{R}}\sqrt{|\check{\mathbf{g}}|}\mathbf{d}^4u + \check{\Lambda}\int \sqrt{|\check{\mathbf{g}}|}\mathbf{d}^4u.\end{aligned} \tag{25}$$

Here, we use boldface $\mathbf{d}^4u$ in order to emphasize that the integration volume is for N-elongated partial derivatives (13), κ^2 is the gravitational coupling constant, the values with "ˆ" are computed for a canonical d-connection $\widehat{\mathbf{D}}$ and the values with "∨" for re-defined geometric data $(\check{\mathbf{g}}, \check{\mathbf{N}}, \check{\mathbf{D}})$ for certain nonholonomic frame transforms and nonholonomic deformations $g_{\alpha\beta} \sim \widehat{\mathbf{g}}_{\alpha\beta} \sim \check{\mathbf{g}}_{\alpha\beta}$[10]. For simplicity, we consider matter actions which only depend on the coefficients of a metric field and not on their derivatives,

$$\widehat{\mathbf{T}}^{\alpha\beta} = {}^m\widehat{\mathbf{L}}\ \widehat{\mathbf{g}}^{\alpha\beta} + 2\delta({}^m\widehat{\mathbf{L}})/\delta\widehat{\mathbf{g}}_{\alpha\beta}.$$

[10] We shall give details in sects. 3.1.3 and 3.2.

Such variations can be performed with respect to coordinate frames for $g_{\mu\nu}$ or in various N-adapted forms for $\widehat{\mathbf{g}}_{\mu\nu}$ and $\check{\mathbf{g}}_{\mu\nu}$.

2.3.2 Off-diagonal deformations of FLRW metrics

We assume that the matter content of the universe can be approximated by a perfect (pressureless) fluid, where

$$\widehat{\mathbf{T}}_{\alpha\beta} = p\widehat{\mathbf{g}}_{\alpha\beta} + (\rho + p)\widehat{\mathbf{v}}_\alpha \widehat{\mathbf{v}}_\beta \tag{26}$$

is defined for certain (effective) energy and pressure densities, respectively, $\widehat{\mathbf{v}}_\alpha$ being the four-velocity of the fluid for which $\widehat{\mathbf{v}}_\alpha \widehat{\mathbf{v}}^\alpha = -1$ and $\widehat{\mathbf{v}}^\alpha = (0,0,0,1)$ in N-adapted comoving frames/coordinates. Here frame deformations/transforms of metrics of type $\widehat{\mathbf{g}}_{\alpha\beta} = \mathbf{e}^{\alpha'}{}_\alpha \mathbf{e}^{\beta'}{}_\beta \mathring{g}_{\alpha'\beta'}$, will be studied, being the FLRW diagonalized element

$$\begin{aligned} \mathrm{d}\mathring{s}^2 &= \mathring{g}_{\alpha'\beta'} \mathrm{d}u^{\alpha'} \mathrm{d}u^{\beta'} = \mathring{a}^2(t)\left[\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \mathrm{d}\varphi^2\right] - \mathrm{d}t^2, \\ &= \mathring{a}^2(t)\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] - \mathrm{d}t^2, \end{aligned} \tag{27}$$

where the scale factor $\mathring{a}(t)$ (we use also the value $\mathring{H} := \mathring{a}/\mathring{a}$, for $\mathring{a}^\diamond := \mathrm{d}\mathring{a}/\mathrm{d}t$) with signature $(+,+,+,-)$, and a parametrization of coordinates in the form $u^{\alpha'} = (x^{1'} = r, x^{2'} = \theta, y^{3'} = \varphi, y^{4'} = t)$, or as Cartesian coordinates $(x^{1'} = x, x^{2'} = y, y^{3'} = z, y^{4'} = t)$. For such cosmological metrics, the main issues of the Einstein and modified Universes are encoded into energy-momentum tensor $\mathring{T}_{\alpha\beta} = \mathring{p}\mathring{g}_{\alpha\beta} + (\mathring{\rho} + \mathring{p})\mathring{v}_\alpha \mathring{v}_\beta$ (we omit primes or other distinctions in the coordinate indices if there is no ambiguity) arising from a matter Lagrangian ${}^m\mathring{\mathcal{L}}$ through

$$\mathring{T}(t) = \mathring{T}^\alpha{}_\alpha = -\mathring{\rho}, \qquad \mathring{P}(t) = \mathring{R}_{\alpha\beta}\mathring{T}^{\alpha\beta} = \mathring{R}_{44}\mathring{T}^{44} = -3\mathring{\rho}(\mathring{H}^2 + \mathring{H}^\diamond), \tag{28}$$

for $\mathring{T}^\alpha{}_\beta = \mathrm{diag}[0,0,0,-\mathring{\rho}]$.

We will consider nonhomogeneous and locally anisotropic cosmological solutions of type (16) and/or (17) generated by off-diagonal deformations of (27)

$$\begin{aligned} g_i &= g_i(x^k) = \eta_i(x^k, y^4)\mathring{g}_i(x^k, y^4) = e^{\psi(x^k)}, \\ g_a &= \omega^2(x^k, y^4) h_a(x^k, y^4) = \omega^2(x^k, y^4)\eta_a(x^k, y^4)\mathring{g}_a(x^k, y^4), \\ N_i^3 &= n_i(x^k), N_i^4 = w_i(x^k, y^4). \end{aligned} \tag{29}$$

In eqs. (29) there is no summation on repeated indices, $\eta_\alpha = (\eta_i, \eta_a)$ are polarization functions, the N-connection coefficients are determined by n_i and w_i, the vertical conformal factor ω may depend on all spacetime coordinates and $\mathring{g}_\alpha = (\mathring{g}_i, \mathring{g}_a)$ define the "prime" diagonal metric if $\eta_\alpha = 1$ and $N_i^a = 0$. The "target" off-diagonal metrics are with Killing symmetry on $\partial/\partial y^3$ when the coefficients (29) do not depend on y^3 [11],

$$\begin{aligned} \mathrm{d}s^2 = {} & a^2(x^k,t)[\eta_1(x^k,t)(\mathrm{d}x^1)^2 + \eta_2(x^k,t)(\mathrm{d}x^2)^2] + a^2(x^k,t)\widehat{h}_3(x^k,t)(\widehat{\mathbf{e}}^3)^2 \\ & + \omega^2(x^k,t) h_4(x^k,t)(\widehat{\mathbf{e}}^4)^2, \end{aligned} \tag{30}$$

when $a^2(x^k,t)\eta_i(x^k,t) = e^{\psi(x^k)}$, for $i = 1,2$; $a^2\ \widehat{h}_3 = \omega^2(x^k,t)h_3(x^k,t)$, and

$$\widehat{\mathbf{e}}^3 = \mathrm{d}y^3 + n_i(x^k)\mathrm{d}x^i, \qquad \widehat{\mathbf{e}}^4 = \mathrm{d}y^4 + w_i(x^k,t)\mathrm{d}x^i.$$

Functions η_i, η_a, a, ψ, ω, n_i, w_i will be found such that, via nonholonomic transforms (29), when $\mathring{g}_{\alpha'\beta'}(t)$ (27) $\to$ $\widehat{\mathbf{g}}_{\alpha\beta}(x^k,t)$ (30), off-diagonal nonhomogeneous cosmological solutions are generated in a model of modified gravity (25). We can consider subclasses of off-diagonal cosmological solutions but with deformed symmetries when certain nontrivial limits $\widehat{\mathbf{g}}_{\alpha\beta}(x^k,t) \to \widehat{\mathbf{g}}_{\alpha\beta}(t)$ can be found and define viable cosmological models.

[11] We can consider nonholonomic deformations with non-Killing symmetries when, for instance, $\omega(x^k,y^4) \to \omega(x^k,y^3,y^4)$, which results in a more cumbersome calculus and geometric techniques. For simplicity, we do not study such generalizations in this work (see examples in [27–30]).

2.3.3 Field equations for nonholonomic modified gravities and FLRW cosmology

Applying an N-adapted variational procedure with respect to a nonholonomic basis (12) and (13) for the action $\mathcal{S} = {}^g\widehat{\mathbf{S}} + {}^m\widehat{\mathbf{S}}$, which is similar to that in [24–26] but for $\nabla \to \widehat{\mathbf{D}}$ and matter source $\widehat{\mathbf{T}}_{\alpha\beta}$ (26), we obtain the field equations for the corresponding modified gravity theory

$$\begin{aligned} &\widehat{\mathbf{R}}_{\alpha\beta}\ {}^1\widehat{\mathbf{f}} - \frac{1}{2}\ \widehat{\mathbf{g}}_{\alpha\beta}\widehat{\mathbf{f}} + (\widehat{\mathbf{g}}_{\alpha\beta}\widehat{\mathbf{D}}^{\mu}\widehat{\mathbf{D}}_{\mu} - \widehat{\mathbf{D}}_{\alpha}\widehat{\mathbf{D}}_{\beta})\ {}^1\widehat{\mathbf{f}} + (\widehat{\mathbf{T}}_{\alpha\beta} + \boldsymbol{\Theta}_{\alpha\beta})\ {}^2\widehat{\mathbf{f}} \\ &+ \boldsymbol{\Xi}_{\alpha\beta}\ {}^3\widehat{\mathbf{f}} + \frac{1}{2}(\widehat{\mathbf{D}}^{\mu}\widehat{\mathbf{D}}_{\mu}\widehat{\mathbf{T}}_{\alpha\beta}\ {}^3\widehat{\mathbf{f}} + \widehat{\mathbf{g}}_{\alpha\beta}\widehat{\mathbf{D}}_{\mu}\widehat{\mathbf{D}}_{\nu}\widehat{\mathbf{T}}^{\mu\nu}\ {}^3\widehat{\mathbf{f}}) - \widehat{\mathbf{D}}_{\nu}\widehat{\mathbf{D}}_{(\alpha}\widehat{\mathbf{T}}_{\beta)}^{\ \nu}\ {}^3\widehat{\mathbf{f}} = \kappa^2\ \widehat{\mathbf{T}}_{\alpha\beta}, \end{aligned} \tag{31}$$

for

$$\boldsymbol{\Theta}_{\alpha\beta} = p\ \widehat{\mathbf{g}}_{\alpha\beta} - 2\widehat{\mathbf{T}}_{\alpha\beta}, \qquad \boldsymbol{\Xi}_{\alpha\beta} = 2\ \widehat{\mathbf{E}}^{\nu}{}_{(\alpha}\widehat{\mathbf{T}}_{\beta)\nu} - p\ \widehat{\mathbf{E}}_{\alpha\beta} - \frac{1}{2}\widehat{\mathbf{R}}\widehat{\mathbf{T}}_{\alpha\beta},$$

with respective d-tensors defined by eqs. (20), (21) and (22), where ${}^1\widehat{\mathbf{f}} := \partial\widehat{\mathbf{f}}/\partial\widehat{\mathbf{R}}$, ${}^2\widehat{\mathbf{f}} := \partial\widehat{\mathbf{f}}/\partial\widehat{\mathbf{T}}$ and ${}^3\widehat{\mathbf{f}} := \partial\widehat{\mathbf{f}}/\partial\widehat{\mathbf{P}}$, when $\widehat{\mathbf{P}} = \widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta}$ and $(\alpha\beta)$ denotes symmetrization of the indices.

In general, the divergence with $\widehat{\mathbf{D}}$ and/or ∇ of eqs. (31) is not zero. Also eqs. (23) have a similar property. In the last case, we can obtain the continuity equations as in GR and then deform them by using the distortions (19), which for the canonical d-connections are completely determined by the metric structure. There are certain types of conservation laws for matter fields with additional nonholonomic constraints. We can consider the field equations (31) and the equations derived by taking the divergence with $\widehat{\mathbf{D}}$ as nonholonomic distortions of similar systems of nonlinear functional PDEs considered in [24–26] for ∇. Remarkably, such sophisticate nonholonomic and nonlinear systems can be solved in very general off-diagonal forms, by applying the anholonomic frame deformation method. In order to compare these results and to find possible applications in modern cosmology, we will consider a particular equation of state (EoS) $p = \varpi\rho$ with $\varpi = \mathrm{const}$, and study the cosmology of off-diagonal distortions of certain FLRW models considered in the framework of GR and its modifications. In both cases, by exploring some particular classes of solutions, the dynamics of the matter sector of generalized $f(R, T, R_{\mu\nu}T^{\mu\nu})$ gravity (with respect to N-adapted frames) may lead to similar cosmological scenarios as GR, but with nonholonomic constraints and deformations.

3 The anholonomic frame deformation method and exact solutions in modfied gravities

A surprising property of eqs. (23) and (31) is that they can be integrated in very general form with generic off-diagonal metrics when their coefficients depend on all spacetime coordinates via various classes of generating and integration functions and constants. In particular, we can consider such generating and integration functions when $\widehat{\mathbf{g}}_{\alpha\beta}(x^k, t)$ (30) result in off-diagonal metrics of type $\widehat{\mathbf{g}}_{\alpha\beta}(t)$ depending on the parameters and possible (non)commutative Lie algebra or algebroid symmetries.

3.1 Off-diagonal FLRW-like cosmological models

We shall study cosmological models with sources of type (26) when the four-velocity $\widehat{\mathbf{v}}_{\alpha}$ is reparameterized in a way that for some frame transforms as

$$\begin{aligned} \widehat{\mathcal{Y}}_{\alpha\beta} &:= \kappa^2\left(\widehat{\mathbf{T}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\widehat{\mathbf{T}}\right) \\ &\to \mathrm{diag}\left[\Upsilon_1 = \Upsilon_2, \Upsilon_2 = {}^h\Upsilon(x^i), \Upsilon_3 = \Upsilon_4, \Upsilon_4 = {}^v\Upsilon(x^i, t)\right] && (32) \\ &\to \widehat{\Lambda}\mathbf{g}_{\alpha\beta}\ \text{(redefining the generating functions and sources)}, && (33) \end{aligned}$$

for effective h- and v-polarized sources, respectively, ${}^h\Upsilon(x^i)$ and $\Upsilon_4 = {}^v\Upsilon(x^i, t)$, or an effective cosmological constant $\widehat{\Lambda}$. For simplicity, we can consider effective matter sources and "prime" metrics with Killing symmetry on ∂/∂_3, *i.e.* when the effective matter sources and d-metrics do not depend on the coordinate y^3 [12]. In brief, the partial derivatives $\partial_{\alpha} = \partial/\partial u^{\alpha}$ on a 4-d manifold will be written as $s^{\bullet} = \partial s/\partial x^1$, $s' = \partial s/\partial x^2$, $s^* = \partial s/\partial y^3$, $s^{\diamond} = \partial s/\partial y^4$.

[12] The method can be extended to account for y^3 dependence and non-Killing configurations (see [27–30]). In this paper the local coordinates and ansätze for d-metrics are parameterized in different forms than in previous works, what is more convenient for the study of cosmological models.

The nontrivial components of the Ricci d-tensor (20) and nonholonomic Einstein equations (22), with source (32) parameterized with respect to N-adapted bases (12) and (13), for a d-metric ansätze (17) with coefficients (29), are

$$-\widehat{R}^1_1 = -\widehat{R}^2_2 = \frac{1}{2g_1g_2}\left[g_2^{\bullet\bullet} - \frac{g_1^{\bullet}g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1'g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}\right] = {}^h\Upsilon, \tag{34}$$

$$-\widehat{R}^3_3 = -\widehat{R}^4_4 = \frac{1}{2h_3h_4}\left[h_3^{\diamond\diamond} - \frac{(h_3^{\diamond})^2}{2h_3} - \frac{h_3^{\diamond}h_4^{\diamond}}{2h_4}\right] = {}^v\Upsilon, \tag{35}$$

$$\widehat{R}_{3k} = \frac{h_3}{2h_4}n_k^{\diamond\diamond} + \left(\frac{h_3}{h_4}h_4^{\diamond} - \frac{3}{2}h_3^{\diamond}\right)\frac{n_k^{\diamond}}{2h_4} = 0, \tag{36}$$

$$\widehat{R}_{4k} = \frac{w_k}{2h_3}\left[h_3^{\diamond\diamond} - \frac{(h_3^{\diamond})^2}{2h_3} - \frac{h_3^{\diamond}h_4^{\diamond}}{2h_4}\right] + \frac{h_3^{\diamond}}{4h_3}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k h_3^{\diamond}}{2h_3} = 0, \tag{37}$$

where $\widehat{\mathbf{R}}^{\alpha}{}_{\beta}$ are computed for $\omega = 1$ and then the formulas are generalized for $\omega \neq 1$ via v-conformal transforms (see refs. [24–26] for details),

$$\mathbf{e}_i\omega = \partial_i\omega - n_i\ \omega^* - w_i\omega^{\diamond} = 0. \tag{38}$$

The d-torsion (24) vanishes if the (Levi-Civita, LC) conditions $\widehat{L}^c_{aj} = e_a(N^c_j)$, $\widehat{C}^i_{jb} = 0$, $\Omega^a{}_{ji} = 0$, are satisfied for

$$\begin{aligned} w_i^{\diamond} &= (\partial_i - w_i\partial_4)\ln\sqrt{|h_4|}, \qquad (\partial_i - w_i\partial_4)\ln\sqrt{|h_3|} = 0,\\ \partial_k w_i &= \partial_i w_k, \qquad n_i^{\diamond} = 0, \qquad \partial_i n_k = \partial_k n_i. \end{aligned} \tag{39}$$

The above system of equations can be integrated in very general situations, for instance, for d-metrics with Killing symmetry on ∂_3.

3.1.1 Decoupling of PDEs for inhomogeneous cosmological metrics

The system of equations (34)–(38) has an important decoupling property. To show this explicitly, we rewrite it as nonlinear PDE which posses an important decoupling property, allowing integration step by step of such equations. For $h_a^{\diamond} \neq 0$, ${}^h\Upsilon$, ${}^v\Upsilon \neq 0$, Killing symmetry on ∂_3 and parameterizations (29), these equations can be written as

$$\psi^{\bullet\bullet} + \psi'' = 2\ {}^h\Upsilon \tag{40}$$

$$\phi^{\diamond}h_3^{\diamond} = 2h_3h_4\ {}^v\Upsilon \tag{41}$$

$$n_i^{\diamond\diamond} + \gamma n_i^{\diamond} = 0, \tag{42}$$

$$\beta w_i - \alpha_i = 0, \tag{43}$$

$$\partial_i\omega - (\partial_i\phi/\phi^{\diamond})\omega^{\diamond} = 0, \tag{44}$$

for

$$\alpha_i = h_3^{\diamond}\partial_i\phi, \qquad \beta = h_3^{\diamond}\,\phi^{\diamond}, \qquad \gamma = \left(\ln|h_3|^{3/2}/|h_4|\right)^{\diamond}, \tag{45}$$

where

$$\phi = \ln|h_3^{\diamond}/\sqrt{|h_3h_4|}|, \qquad \text{and/or} \qquad \Phi := e^{\phi}, \tag{46}$$

is considered as a generating function. Equation (44) is just eq. (38) for a nontrivial solution of (43) with coefficients (45), when

$$w_i = \partial_i\phi/\phi^{\diamond}. \tag{47}$$

The decoupling property of the above system of equations follows from the facts that: 1) integrating the 2-d Laplace equation (40) one finds solutions for the h-coefficients of the d-metric, and 2) the solutions for the coefficients of the d-metric can be found from (41) and (46). 3) Then the N-connection coefficients w_i and n_i can be found from (42) and (43), respectively.

3.1.2 Cosmological solutions with nonholonomically induced torsion

Equations (40) and (43) can be solved, respectively, for any source ${}^h\Upsilon(x^k)$ and generating function $\phi(x^k,t)$. The system (41) and (46) can be written under the form

$$h_3h_4 = \phi^{\diamond}h_3^{\diamond}/2\ {}^v\Upsilon \qquad \text{and} \qquad |h_3h_4| = (h_3^{\diamond})^2e^{-2\phi},$$

for any nontrivial source ${}^v\Upsilon(x^i,t)$ in (41). Introducing the first equation into the second, one finds $|h_3^\diamond| = \frac{(e^{2\phi})^\diamond}{4|\ {}^v\Upsilon|} = \frac{\Phi^\diamond\Phi}{2|\ {}^v\Upsilon|}$, *i.e.* $h_3 = {}^0h_3(x^k) + \frac{\epsilon_3\epsilon_4}{4}\int \mathrm{d}t \frac{(\Phi^2)^\diamond}{{}^v\Upsilon}$, where ${}^0h_3(x^k)$ and $\epsilon_3, \epsilon_4 = \pm 1$. Using again the first equation, we obtain

$$h_4 = \frac{\phi^\diamond(\ln\sqrt{|h_3|})^\diamond}{2\ {}^v\Upsilon} = \frac{1}{2\ {}^v\Upsilon}\frac{\Phi^\diamond}{\Phi}\frac{h_3^\diamond}{h_3}\,. \tag{48}$$

We can simplify such formulas for h_3 and h_4 if we redefine the generating function, $\Phi \to \widehat{\Phi}$, where $(\Phi^2)^\diamond/|\ {}^v\Upsilon| = (\widehat{\Phi}^2)^\diamond/\Lambda$, *i.e.*

$$\Phi^2 = \Lambda^{-1}\left[\widehat{\Phi}^2|\ {}^v\Upsilon| + \int \mathrm{d}t\ \widehat{\Phi}^2|\ {}^v\Upsilon|^\diamond\right], \tag{49}$$

for an effective cosmological constant Λ which may take positive or negative values. We can integrate on t, include the integration function ${}^0h_3(x^k)$ in $\widehat{\Phi}$ and write

$$h_3[\widehat{\Phi}] = \widehat{\Phi}^2/4\Lambda. \tag{50}$$

Introducing this formula and (49) into (48), we compute

$$h_4[\widehat{\Phi}] = \frac{(\ln|\Phi|)^\diamond}{4|\ {}^v\Upsilon|} = \frac{(\widehat{\Phi}^2)^\diamond}{8}\left[\widehat{\Phi}^2|\ {}^v\Upsilon| + \int \mathrm{d}t\ \widehat{\Phi}^2|\ {}^v\Upsilon|^\diamond\right]^{-1}. \tag{51}$$

As next step, we need solve eq. (42) by integrating on t twice. We obtain

$$n_k = {}_1n_k + {}_2n_k\int \mathrm{d}t\ h_4/(\sqrt{|h_3|})^3, \tag{52}$$

where ${}_1n_k(x^i)$, ${}_2n_k(x^i)$ are integration functions and $h_a[\widehat{\Phi}]$ are given by formulas (50) and (51). If we fix ${}_2n_k = 0$, we shall be able to find $n_k = {}_1n_k(x^i)$ which have zero torsion limits (see examples in subsect. 3.1.3).

The solutions of (43) are given by (47), which for different types of generating functions are parameterized as

$$w_i = \frac{\partial_i\Phi}{\Phi^\diamond} = \frac{\partial_i(\Phi^2)}{(\Phi^2)^\diamond}\,, \tag{53}$$

where the integral functional $\Phi[\widehat{\Phi},\ {}^v\Upsilon]$ is given by (49).

We can introduce certain polarization functions η_α in order to write the d-metric of such solutions in the form (30). Let us fix $\omega^2 = |h_4|^{-1}$ to satisfy the condition (44), which for a generating function $\Phi[\phi]$ is equivalent to

$$\Phi^\diamond\partial_i h_4 - \partial_i\Phi\, h_4^\diamond = 0. \tag{54}$$

These first order PDE equations impose certain conditions on the class of generating function Φ and source ${}^v\Upsilon$. For instance, we can choose such a system of coordinates where ${}^v\Upsilon = \frac{1}{4}(e^{-\phi})^\diamond$ which transforms eq. (51) into $h_4 = \Phi$, *i.e.* this coefficient of the d-metric is considered as a generating function, and eqs. (54) are solved. In general, the integral varieties of such equations cannot be expressed in explicit holonomic form.

A modification of the scale factor $\mathring{a}(t) \to a(x^k,t)$, for the FLRW metric (27) (with for $\mathring{g}_1 = \mathring{g}_2 = \mathring{g}_3 = \mathring{a}^2$, $\mathring{g}_4 = -1$), has to be chosen in order to explain observational cosmological data. For any prescribed functions $a(x^k,t)$ and $\omega^2 = |h_4|^{-1}$ and solutions $e^{\psi(x^k)}$, (see (40)) and $h_a[\widehat{\Phi}]$, $n_k(x^i)$, $w_i[\widehat{\Phi}]$ (given, respectively, by formulas (50)–(53)), we can compute the polarization functions $\eta_i = a^{-2}e^\psi$, $\eta_3 = \mathring{a}^{-2}h_3$, $\eta_4 = 1$ and function $\widehat{h}_3 = h_3/a^2|h_4|$. Such coefficients (see the data (29)), define off-diagonal metrics of type (30),

$$\mathrm{d}s^2 = a^2(x^k,t)\left[\eta_1(x^k,t)(\mathrm{d}x^1)^2 + \eta_2(x^k,t)(\mathrm{d}x^2)^2\right] + a^2(x^k,t)\widehat{h}_3(x^k,t)\left[\mathrm{d}y^3 + n_i(x^k)\mathrm{d}x^i\right]^2 - \left[\mathrm{d}t + \frac{\partial_i\Phi[\widehat{\Phi},\ {}^v\Upsilon]}{\Phi^\diamond[\widehat{\Phi},\ {}^v\Upsilon]}\mathrm{d}x^i\right]^2. \tag{55}$$

Choosing any generating functions $a^2(x^k,t)$, $\psi(x^i)$ and $\Phi[\widehat{\Phi},\ {}^v\Upsilon]$ and integration functions $n_i(x^k)$, we generate a nonhomogeneous cosmological model with nonholonomically induced torsion (24). More general torsions can be induced if $n_i(x^k,t)$ is taken with two types of integration functions ${}_1n_i(x^k)$ and ${}_2n_i(x^k)$ (see eqs. (52)). Having constructed this solution, we can now consider certain subclasses of generating and integration functions where $a(x^k,t) \to a(t) \neq \mathring{a}(t)$, $w_i \to w_i(t)$, $n_i \to$ const, etc. In this way generic off-diagonal cosmological metrics are generated (because there are nontrivial anholonomy coefficients W_{ia}^b in (14)).

3.1.3 Levi-Civita off-diagonal cosmological configurations

The LC-conditions (39) are given by a set of nonholonomic constraints which cannot be solved in explicit form for arbitrary data (Φ, Υ) and integration functions ${}_1n_k$ and ${}_2n_k$. However, some subclasses of off-diagonal solutions can still be constructed where via frame and coordinate transforms we can chose ${}_2n_k = 0$ and ${}_1n_k = \partial_k n$ with a function $n = n(x^k)$. It should be noted that $(\partial_i - w_i\partial_4)\Phi \equiv 0$ for any $\Phi(x^k, y^4)$ if w_i is defined by (53). Introducing a new functional $B(\Phi)$, we find that $(\partial_i - w_i\partial_4)B = \frac{\partial B}{\partial \Phi}(\partial_i - w_i\partial_4)\Phi = 0$. Using eq. (50) for functionals of type $h_3 = B(|\tilde{\Phi}(\Phi)|)$, we solve eqs. $(\partial_i - w_i\partial_4)h_3 = 0$, what is equivalent to the second system of equations in (39), because $(\partial_i - w_i\partial_4)\ln\sqrt{|h_3|} \sim (\partial_i - w_i\partial_4)h_3$.

We can use a subclass of generating functions $\Phi = \check{\Phi}$ for which

$$(\partial_i\check{\Phi})^\diamond = \partial_i\check{\Phi}^\diamond \tag{56}$$

and get for the left part of the second equation in (39), $(\partial_i - w_i\partial_4)\ln\sqrt{|h_3|} = 0$. The first system of equations in (39) can be solved in explicit form if w_i are determined by formulas (53), and $h_3[\tilde{\Phi}]$ and $h_4[\tilde{\Phi}, \tilde{\Phi}^\diamond]$ are chosen respectively for any $\Upsilon \to \Lambda$. We can consider $\tilde{\Phi} = \tilde{\Phi}(\ln\sqrt{|h_4|})$ for a functional dependence $h_4[\tilde{\Phi}[\check{\Phi}]]$. This allows us to obtain $w_i = \partial_i|\tilde{\Phi}|/|\tilde{\Phi}|^\diamond = \partial_i|\ln\sqrt{|h_4|}|/|\ln\sqrt{|h_4|}|^\diamond$. Taking the derivative ∂_4 on both sides of these equations, we get

$$w_i^\diamond = \frac{(\partial_i|\ln\sqrt{|h_4|}|)^\diamond}{|\ln\sqrt{|h_4|}|^\diamond} - w_i\frac{|\ln\sqrt{|h_4|}|^{\diamond}}{|\ln\sqrt{|h_4|}|^\diamond}.$$

If the mentioned conditions are satisfied, we can construct in explicit form generic off-diagonal configurations with $w_i^\diamond = (\partial_i - w_i\partial_4)\ln\sqrt{|h_4|}$, which is necessary for the zero torsion conditions.

We need als solve for the conditions $\partial_k w_i = \partial_i w_k$ from the second line in (39). We find in explicit form the solutions for such coefficients if

$$\check{w}_i = \partial_i\check{\Phi}/\check{\Phi}^\diamond = \partial_i\widetilde{A}, \tag{57}$$

with a nontrivial function $\widetilde{A}(x^k, y^4)$ depending functionally on the generating function $\check{\Phi}$.

Finally, we conclude that we generate LC-configurations for a class of off-diagonal cosmological metric type (17) for $\Upsilon = \check{\Upsilon} = \Lambda$, $\Phi = \check{\Phi} = \tilde{\Phi}$ and ${}_2n_k = 0$ in (52) which are parameterized by quadratic elements

$$\mathrm{d}s^2 = e^{\psi(x^k)}\left[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2\right] + \frac{\check{\Phi}^2}{4|\Lambda|}\left[\mathrm{d}y^3 + (\partial_k n(x^i))\mathrm{d}x^k\right]^2 - \frac{(\check{\Phi}^\diamond)^2}{|\Lambda|\check{\Phi}^2}\left[\mathrm{d}t + (\partial_i\widetilde{A}[\check{\Phi}])\mathrm{d}x^i\right]^2. \tag{58}$$

We can re-write such solutions in the form (55), which provides us a general procedure of off-diagonal deformations with $\mathring{a}(t) \to a(x^k, t)$ (see the FLRW metric (27)), resulting in nonhomogeneous cosmological metrics in GR. Prescribing a function $a(x^k, t)$, a generating function $\check{\Phi}(x^k, t)$ satisfying the condition (56) and a solution $e^{\psi(x^k)}$ (see (40)), we, respectively, compute the v-conformal factor and the polarization functions for

$$\widehat{h}_3 = h_3/a^2|h_4| = \check{\Phi}^4/4a^2(\check{\Phi}^\diamond)^2, \qquad \omega^2 = |h_4|^{-1} = |\Lambda|\check{\Phi}^2/(\check{\Phi}^\diamond)^2,$$
$$\eta_i = a^{-2}e^\psi, \qquad \eta_3 = \mathring{a}^{-2}h_3 = \check{\Phi}^2/4|\Lambda|\mathring{a}^2, \qquad \eta_4 = 1.$$

Such coefficients (see data (29)) transform the off-diagonal cosmological solutions (58) into metrics of type (30),

$$\mathrm{d}s^2 = a^2(x^k,t)\{[\eta_1(x^k,t)(\mathrm{d}x^1)^2 + \eta_2(x^k,t)(\mathrm{d}x^2)^2] + \widehat{h}_3(x^k,t)[\mathrm{d}y^3 + (\partial_k n(x^i))\mathrm{d}x^k]^2\} - [\mathrm{d}t + (\partial_i\widetilde{A}[\check{\Phi}])\mathrm{d}x^i]^2. \tag{59}$$

The dependence on the source Λ is contained in explicit form in the polarization η_3, for instance. This class of effective Einstein off-diagonal metrics $\mathbf{g}_{\alpha\beta}(x^k,t)$ define new nonhomogeneous cosmological solutions in GR as off-diagonal deformations of the FLRW cosmology. For certain well-defined conditions, one can find limits $\mathbf{g}_{\alpha\beta} \to \mathbf{g}_{\alpha\beta}(t, a(t), \widehat{h}_3(t), \check{\Phi}(t), \eta_i(t))$. This provides explicit geometric models of nonlinear off-diagonal anisotropic cosmological evolution which, with respect to N-adapted frames, describe $a(t)$ with modified rescaling factors.

3.2 Effective FLRW cosmology for f-modified gravity

The anholonomic frame deformation method outlined in previous subsections can be applied for the generation of off-diagonal cosmological solutions of field equations of modified gravities, see (31). Redefining the generating functions via the transforms (49) and (56), $\Phi \to \check{\Phi} \to \tilde{\Phi}$, we can generate off-diagonal cosmological configurations with $\widehat{\mathbf{R}} = 4\Lambda$, see (32) and (33). Such parameterizations of geometric data and sources are possible for certain general conditions via transforms of N-adapted frames when the action functional functionally depends on Λ and on the effective sources, $\widehat{\mathbf{f}}[\widehat{\mathbf{R}}(\Lambda), \widehat{\mathbf{T}}(\Lambda), \widehat{\mathbf{P}}]$, with $\widehat{\mathbf{P}}(t) = \widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta} = -3\mathring{\rho}(H^2 + H^\diamond)$ and $H = a^\diamond/a$ with scaling factor $a(t)$ taken for some limits of a solution (55), or (59). It should be noted that in the variables corresponding to the Levi-Civita connection ∇ the functional $\widehat{\mathbf{f}} \to f(R, T, R_{\mu\nu}T^{\mu\nu})$ describe very general modifications of GR which in our approach are encoded into a very sophisticated off-diagonal effective vacuum structure with nontrivial vacuum constants.

We assume that the density of matter $\rho = \mathring{\rho}$ in $\widehat{\mathbf{T}}_{\alpha\beta}$ (26) is the same as for a standard FLRW metric (27) and does not change under off-diagonal deformations with respect to N-adapted frames. For such configurations, the functions $\boldsymbol{\Theta}_{\alpha\beta}$ and $\boldsymbol{\Xi}_{\alpha\beta}$ are parameterized, respectively, as

$$\boldsymbol{\Theta}^{\alpha}{}_{\beta} = (p - 2\Lambda)\delta^{\alpha}{}_{\beta}, \qquad \boldsymbol{\Xi}^{\alpha}{}_{\beta} = \left(2\Lambda^2 - p\ \Lambda - \frac{1}{2}4\Lambda^2\right)\delta^{\alpha}{}_{\beta} = -p\Lambda\delta^{\alpha}{}_{\beta},$$

where terms with Λ^2 compensate each other in 4-d. We can write $\widehat{\mathbf{D}}_\mu \widehat{\mathbf{T}}_{\alpha\beta} = 0$, $\widehat{\mathbf{D}}_\mu\ {}^1\widehat{\mathbf{f}} \sim \partial^2\widehat{\mathbf{f}}/\partial\widehat{\mathbf{R}}^2_{\dots}\ \mathbf{e}_\mu\Lambda \sim 0$, and (similarly) $\widehat{\mathbf{D}}_\mu\ {}^2\widehat{\mathbf{f}} \sim 0$, $\widehat{\mathbf{D}}_\mu\ {}^3\widehat{\mathbf{f}} \sim 0$, for $\widehat{\mathbf{R}}_{\alpha\beta} \sim \widehat{\mathbf{T}}_{\alpha\beta} \sim \Lambda\delta_{\alpha\beta}$, $\Lambda = \mathrm{const}$, with respect to corresponding classes of N-adapted frames. Equations (31) transform into a system of nonholonomic nonlinear PDEs of type (34)–(37), $\widehat{\mathbf{R}}^{\alpha}{}_{\beta} = \widehat{\boldsymbol{\Upsilon}}\delta^{\alpha}{}_{\beta}$, with effective diagonalized source

$$\widehat{\boldsymbol{\Upsilon}} = \frac{\Lambda}{{}^1\widehat{\mathbf{f}}} + \frac{\widehat{\mathbf{f}}}{2\ {}^1\widehat{\mathbf{f}}} + (2\Lambda - \kappa^{-2}\Lambda - p)\frac{{}^2\widehat{\mathbf{f}}}{{}^1\widehat{\mathbf{f}}} + p\Lambda\frac{{}^3\widehat{\mathbf{f}}}{{}^1\widehat{\mathbf{f}}}\,, \tag{60}$$

which can be parameterized with dependencies on (x^i, t), or on t. These equations can be solved for very general off-diagonal forms, depending on generating and integration functions, following the procedure outlined in previous subsections. Redefining the generation function as in (49), when an effective cosmological constant $\check{\Lambda}$ is generated from $\widehat{\boldsymbol{\Upsilon}}(x^i, t)$, one has

$$\check{\Phi}^2 = \check{\Lambda}^{-1}\left[\widehat{\Phi}^2|\ \widehat{\boldsymbol{\Upsilon}}| + \int \mathrm{d}t\ \widehat{\Phi}^2|\ \widehat{\boldsymbol{\Upsilon}}|^\diamond\right]. \tag{61}$$

Such a generating function defines off-diagonal cosmological solutions of type (55), or (58), as solutions of field equations for an effective (nonholonomic) Einstein space $\check{\mathbf{R}}^{\alpha}{}_{\beta} = \check{\Lambda}\delta^{\alpha}{}_{\beta}$. In this way, a geometric method is provided when the (effective or modified) matter sources transform as $\widehat{\boldsymbol{\Upsilon}}$ (32) $\to \check{\Lambda}$ (33) and the gravitational field equations in modified gravity can be effectively expressed as nonholonomic Einstein spaces when the d-metric coefficients encode the contributions of $\widehat{\mathbf{f}}$, ${}^1\widehat{\mathbf{f}}$, ${}^2\widehat{\mathbf{f}}$ and ${}^3\widehat{\mathbf{f}}$ and of the matter sources.

We can consider inverse transforms with $\check{\Lambda} \to \widehat{\boldsymbol{\Upsilon}}$ for (61) and state that for certain well-defined conditions of type (56) and (57) we can mimic both f-functional contributions and/or massive gravitational theories [44–47]. Here we emphasize that off-diagonal configurations (of vacuum and nonvacuum types) are possible even if the effective sources from modified gravity are constrained to be zero.

4 Off-diagonal modeling of cosmological modified gravity theories

This section has three goals. The first is to provide a reconstruction procedure for off-diagonal effective Einstein and modified gravity cosmological scenarios. The second is to apply these methods in practice and provide explicit examples related to $f(R)$ gravity and cosmology. The third goal is to analyze how matter stability problems for $f(R)$-theories can be solved by nonholonomic frame transforms and deformations and imposing nonintegrable constraints.

4.1 Reconstructing nonholonomic general f(R)-models

Let us construct an effective Einstein space which models a quite general modified gravity theory with $f(R, T, R_{\alpha\beta}T^{\alpha\beta}) = R + F(R_{\alpha\beta}T^{\alpha\beta}) + G(T)$. This theory admits a reconstruction procedure which does not affect the observational constraints when a realistic evolution is studied [24–26]. Following the anholonomic frame deformation method with an auxiliary canonical d-connection $\widehat{\mathbf{D}}$, the modified gravity (31) is formulated for

$$\widehat{\mathbf{f}}(\widehat{\mathbf{R}}, \widehat{\mathbf{T}}, \widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta}) = \widehat{\mathbf{R}} + \widehat{\mathbf{F}}(\widehat{\mathbf{P}}) + \widehat{\mathbf{G}}(\widehat{\mathbf{T}}). \tag{62}$$

We can self-consistently embed this model into a nonholonomic background determined by N-adapted frames (12) and (13) for a generic off-diagonal solution (59) with limits $\widehat{\mathbf{D}} \to \nabla$ and $\mathbf{g}_{\alpha\beta} \to \mathbf{g}_{\alpha\beta}(t, a(t), \widehat{h}_3(t), \check{\Phi}(t), \eta_i(t))$. With respect to such frames, the nonholonomic FLRW equations are similar to those found in sect. III B of [24–26] (see the second paper for details on methods of constructing solutions and speculations on the problem of matter instability)[13].

4.1.1 f-modified off-diagonal FLRW equations

The effective function $a(t)$ defines in our case off-diagonal cosmological evolution scenarios which are different from those where $\mathring{a}(t)$ stands for a standard diagonal FLRW cosmology. For $H := a^{\diamond}/a$, ${}^{1}\widehat{\mathbf{G}} := \mathrm{d}\widehat{\mathbf{G}}/\mathrm{d}\widehat{\mathbf{T}}$ and ${}^{1}\widehat{\mathbf{F}} := \mathrm{d}\widehat{\mathbf{F}}/\mathrm{d}\widehat{\mathbf{P}}$, we have

$$\begin{aligned} &3H^2 + \frac{1}{2}\left[\widehat{\mathbf{f}} + \widehat{\mathbf{G}} - 3(3H^2 - H^{\diamond})\ \rho\ {}^{1}\widehat{\mathbf{F}}\right] - \rho(\kappa^2 - {}^{1}\widehat{\mathbf{G}}) = 0,\\ &-3H^2 - 2H^{\diamond} - \frac{1}{2}\left[\widehat{\mathbf{f}} + \widehat{\mathbf{G}} - \left(\rho\ {}^{1}\widehat{\mathbf{F}}\right)^{\diamond\diamond} - 4H\left(\rho\ {}^{1}\widehat{\mathbf{F}}\right)^{\diamond} - \left(3H^2 + H^{\diamond}\right)\ \rho\ {}^{1}\widehat{\mathbf{F}}\right] = 0. \end{aligned} \tag{63}$$

An observer is here in a nonholonomic basis determined by $N_i^a = \{n_i, w_i(t)\}$ for a nontrivial off-diagonal vacuum with effective polarizations $\eta_\alpha(t)$, and can test cosmological scenarios in terms of the redshift $1 + z = a^{-1}(t)$ for $P = P(z)$ and $T = T(z)$, with a new "shift" derivative when (for instance, for a function $s(t)$) $s^{\diamond} = -(1+z)H\partial_z$.

The system of two equations (63) simplifies by extending it to a set of three equations for four unknown functions $\{\widehat{\mathbf{f}}(z), \widehat{\mathbf{G}}(z), \rho(z), \varsigma(z)\}$ with a new variable $\varsigma(z) := \rho\ {}^{1}\widehat{\mathbf{F}}$,

$$\begin{aligned} &3H^2 + \frac{1}{2}[\widehat{\mathbf{f}}(z) + \widehat{\mathbf{G}}(z)] - \frac{3}{2}[3H^2 - (1+z)H(\partial_z H)]\ \varsigma(z)\frac{3}{2}H^2(1+z)\partial_z\varsigma(z) - \kappa^2\rho(z) = 0,\\ &-3H^2 + (1+z)H(\partial_z H) - \frac{1}{2}\Big\{\widehat{\mathbf{f}}(z) + \widehat{\mathbf{G}}(z) - [3H^2 - (1+z)H(\partial_z H)]\varsigma(z)\\ &+ [3(1+z)H^2 - (1+z)H(\partial_z H)]\partial_z\varsigma(z) + (1+z)^2\partial^2_{zz}\varsigma(z)\Big\} = 0,\\ &(\partial_z\ {}^{1}\widehat{\mathbf{F}})\ \varsigma(z) - \rho(z)\ (\partial_z\ \widehat{\mathbf{f}}) = 0. \end{aligned} \tag{64}$$

Here, by rescaling the generating function, we have fixed the condition $\partial_z\ {}^{1}\widehat{\mathbf{G}}(z) = 0$. Such a nontrivial term must be considered if one wants to transform $\widehat{\mathbf{f}}$ into a standard theory $f(R, T, R_{\alpha\beta}T^{\alpha\beta})$. The functional $\widehat{\mathbf{G}}(\widehat{\mathbf{T}})$, in both holonomic and nonholonomic forms, encodes a new degree of freedom for the evolution of the energy density of type

$$\rho = \rho_0 a^{-3(1+\varpi)} = \rho_0(1+z)a^{3(1+\varpi)}, \tag{65}$$

which is taken for the dust matter approximation ϖ when the evolution reduces to $\rho \sim (1+z)^3$. For the assumption that such an evolution can be considered with respect to N-adapted frames, the solutions of (64) are determined by data $\{\widehat{\mathbf{f}}(z), \widehat{\mathbf{G}}(z), \varsigma(z)\}$ by replacing the second and third equations into the first one and obtaining a single fourth-order equation for $\widehat{\mathbf{f}}(z)$.

4.1.2 Reconstructing $\widehat{\mathbf{f}}$-models and effective Einstein spaces

The reconstruction procedure is restricted to fluids without pressure when such approximation is considered locally with N-adapted frames and the expressions (28) for $(\mathring{a}, \mathring{H}, \mathring{\rho})$ are re-defined in terms of (a, H, ρ); data are written with a script "0" if $z = z_0$, with $\xi = \kappa^2\rho_0/3H_0^2$. One should not confused, *e.g.*, $\mathring{H}$ and H_0, because these values are computed for *different* FLRW solutions, with $\mathring{a}(z)$ determined for a diagonal configuration and $a(z)$ for an off-diagonal one, respectively. We can express

$$\widehat{\mathbf{T}} = \widehat{\mathbf{T}}^{\alpha}{}_{\alpha} = -\xi\frac{3H_0^2}{\kappa^2}(1+z)^3 \qquad \text{and} \qquad \widehat{\mathbf{P}} = \widehat{\mathbf{R}}_{\alpha\beta}\ \widehat{\mathbf{T}}^{\alpha\beta} = -3\xi\frac{3H_0^2}{\kappa^2}(1+z)^3[H^2 - (1+z)H(\partial_z H)].$$

Following the approach outlined in sect. III B of [24–26], we introduce the parameterizations

$$\widehat{\mathbf{F}}(\widehat{\mathbf{P}}) = H_0^2\check{\mathbf{F}}(\check{\mathbf{P}}) \qquad \text{and} \qquad \widehat{\mathbf{G}}(\widehat{\mathbf{T}}) = H_0^2\check{\mathbf{G}}(\check{\mathbf{T}}), \tag{66}$$

[13] In sect. III A of that work, a model with $G(T) = 0$ was investigated in detail. The conclusion was that in order to elaborate a realistic evolution it is necessary to consider nontrivial values for $G(T)$. In nonholonomic variables, such term $\widehat{\mathbf{G}}(\widehat{\mathbf{T}})$ allows to encode $f(R)$ modified theories and related into certain off-diagonal configurations in GR, which simplifies the solution of the problem of matter instability (see subsect. 4.3).

where $\check{\mathbf{P}} = \widehat{\mathbf{P}}/P_0$ and $\check{\mathbf{T}} = \widehat{\mathbf{T}}/T_0$, for $P_0 = -9H_0^4\xi/\kappa^2$ and $T_0 = -3H_0^2\xi/\kappa^2$. In correspondingly N-adapted variables, the off-diagonal cosmological solutions can be associated with a class of de Sitter (dS) solutions with effective cosmological constant $\check{\Lambda}$ (see (61)), where $H(z) = \check{H}_0$ results in $\check{\mathbf{P}} = \check{\mathbf{T}} = (1+z)^3$ for the energy-density (65). In these variables, the solutions of (64) can be written as

$$\begin{aligned}\check{\mathbf{F}} &= c_1\check{\mathbf{P}}^{b_1} + \check{\mathbf{P}}^{b_2/3}\left[c_2\cos\left(\frac{b_3}{3}\ln\check{\mathbf{P}}\right) + c_3\sin\left(\frac{b_3}{3}\ln\check{\mathbf{P}}\right)\right] + c_4 + 3\xi\check{\mathbf{P}},\\ \check{\mathbf{G}} &= \tilde{c}_1\check{\mathbf{T}}^{b_1} + \check{\mathbf{T}}^{b_2/3}\left[\tilde{c}_2\cos\left(\frac{b_3}{3}\ln\check{\mathbf{T}}\right) + \tilde{c}_3\sin\left(\frac{b_3}{3}\ln\check{\mathbf{P}}\right)\right] + \tilde{c}_4 - 3\xi\check{\mathbf{T}},\end{aligned} \tag{67}$$

being the constants $b_1 = -1.327$, $b_2 = 3.414$ and $b_3 = 1.38$. The values c_1, c_2, c_3 and c_4 are integration constants, and the second set of constants $\tilde{c}_1$, $\tilde{c}_2$, $\tilde{c}_3$ and $\tilde{c}_4$ can be expressed via such integration constants, and b_1, b_2 and b_3. We omit explicit formulas because for general solutions they can be included in certain generating or integration functions for the modified gravity equations and ultimately related to real observation data for the associated cosmological models.

For off-diagonal configurations, the $\widehat{\mathbf{f}}(\widehat{\mathbf{R}}, \widehat{\mathbf{T}}, \widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta})$ gravity positively allows for dS solutions in presence of non-constant fluids, not only due to the term $\widehat{\mathbf{P}} = \widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta}$ in (25), and respective gravitational field and cosmological equations. This is possible also because of the off-diagonal nonlinear gravitational interactions in the effective gravitational models. It should be emphasized that the reconstruction procedure elaborated in [24–26], see also references therein, can be extended to more general classes of modified gravity theories, to Finsler-like theories and the ensuing cosmological models [44–46]. Introducing (67) and (66) into (62), we reconstruct a function $\widehat{\mathbf{f}} = \widehat{\mathbf{R}} + \widehat{\mathbf{F}}(\widehat{\mathbf{P}}) + \widehat{\mathbf{G}}(\widehat{\mathbf{T}})$. As a result, we can associate an effective matter source $\widehat{\boldsymbol{\Upsilon}}$ (60), which allows the definition of a corresponding generating function $\check{\Phi}$ (61) (see also Φ and (49)). Finally, we can reconstruct an off-diagonal cosmological solution with nonholonomically induced torsion of type (55) or to model a similar cosmological metric for LC configurations (58) (equivalently, (59)).

4.2 How specific f(R) gravities and the FLRW cosmology are encoded in nonholonomic deformations?

It is well known that any FLRW cosmology can be realized in a specific $f(R)$ gravity (see refs. [52,53] and, for further generalizations, [24–26])[14]. In this subsection we analyze two examples of reconstruction of $f(R)$-gravities where the "e-folding" variable $\zeta := \ln a/a_0 = -\ln(1+z)$ is used instead of the cosmological time t and in related nonholonomic off-diagonal deformations. For such models, we consider $\widehat{\mathbf{f}} = \widehat{\mathbf{f}}(\widehat{\mathbf{R}})$ in (25), use $\widehat{\boldsymbol{\Upsilon}}(x^i,\zeta) = \Lambda/{}^1\widehat{\mathbf{f}} + \widehat{\mathbf{f}}/2\ {}^1\widehat{\mathbf{f}}$ instead of (60) and introduce these values in eq. (61), which can be parameterized with dependencies on (x^i,ζ) (in particular, only on ζ), $\check{\Phi}^2 = \check{\Lambda}^{-1}[\widehat{\Phi}^2|\ \widehat{\boldsymbol{\Upsilon}}| + \int d\zeta\ \widehat{\Phi}^2\partial_\zeta|\ \widehat{\boldsymbol{\Upsilon}}|]$, when $\partial_\zeta = \partial/\partial\zeta$ with $s^\diamond = H\partial_\zeta s$ for any function s. The matter energy density ρ is taken as in (64).

We restrict ourselves to N-adapted frames, (12) and (13), determined by an off-diagonal cosmological solution of the (modified) gravitational field equations, and can repeat all computations leading to eqs. (2)–(7) in [52,53] and prove that a modified gravity with $\widehat{\mathbf{f}}(\widehat{\mathbf{R}})$ realizes the FLRW cosmological model. Such solutions depend on the above source type $\widehat{\boldsymbol{\Upsilon}}(x^i,\zeta)$ and generating function $\check{\Phi}(x^i,\zeta)$; also the nonholonomic background can be modeled to be nonhomogeneous (via w_i and n_i depending, respectively, on x^i and ζ, or only on ζ). The off-diagonal analog of the field equation corresponding to the first FLRW equation is

$$\widehat{\mathbf{f}}(\widehat{\mathbf{R}}) = (H^2 + H\ \partial_\zeta H)\partial_\zeta[\widehat{\mathbf{f}}(\widehat{\mathbf{R}})] - 36H^2\left[4H + (\partial_\zeta H)^2 + H\partial^2_{\zeta\zeta}H\right]\partial^2_{\zeta\zeta}[\widehat{\mathbf{f}}(\widehat{\mathbf{R}})] + \kappa^2\rho.$$

In terms of an effective quadratic Hubble rate, $q(\zeta) := H^2(\zeta)$, and considering that $\zeta = \zeta(\widehat{\mathbf{R}})$ for certain parameterizations, this equation yields

$$\begin{aligned}\widehat{\mathbf{f}}(\widehat{\mathbf{R}}) = &-18q(\zeta(\widehat{\mathbf{R}}))\left[\partial^2_{\zeta\zeta}q(\zeta(\widehat{\mathbf{R}})) + 4\partial_\zeta q(\zeta(\widehat{\mathbf{R}}))\right]\frac{d^2\widehat{\mathbf{f}}(\widehat{\mathbf{R}})}{d\widehat{\mathbf{R}}^2}\\ &+ 6\left[q(\zeta(\widehat{\mathbf{R}})) + \frac{1}{2}\partial_\zeta q(\zeta(\widehat{\mathbf{R}}))\right]\frac{d\widehat{\mathbf{f}}(\widehat{\mathbf{R}})}{d\widehat{\mathbf{R}}} + 2\rho_0 a_0^{-3(1+\varpi)}a^{-3(1+\varpi)\zeta(\widehat{\mathbf{R}})}.\end{aligned} \tag{68}$$

We can construct an off-diagonal cosmological model with metrics of type (55) and nonholonomically induced torsion (when $t \to \zeta$) if a solution $\widehat{\mathbf{f}}(\widehat{\mathbf{R}})$ is used for computing $\widehat{\boldsymbol{\Upsilon}}$ and $\check{\Phi}$. Modeling such nonlinear systems we can consider

[14] We use a system of notations different from that article; here, *e.g.*, N in used for the N-connection and we work with nonholonomic geometric objects.

solutions of the field equations for an effective (nonholonomic) Einstein space $\check{\mathbf{R}}^{\alpha}{}_{\beta} = \check{\Lambda}\delta^{\alpha}{}_{\beta}$, when certain terms of type $\mathrm{d}\widehat{\mathbf{f}}(\check{\mathbf{R}})/\mathrm{d}\check{\mathbf{R}}$ and higher derivatives vanish for a functional dependence $\widehat{\mathbf{f}}(\check{\Lambda})$ with $\partial_\zeta \check{\Lambda} = 0$. The nonholonomic cosmological evolution is determined by off-diagonal coefficients of the metrics and by certain nonexplicit relations for the functionals variables, like $q(\zeta(\widehat{\mathbf{R}}(\check{\Lambda})))$ and (effective/modified) matter sources transform as $\widehat{\boldsymbol{\Upsilon}}$ (32) $\rightarrow \check{\Lambda}$ (33).

LC-configurations can be modeled by off-diagonal cosmological metrics of type (58) when the zero torsion conditions (39) are satisfied. We obtain a standard expression (see [52,53]) for the curvature of ∇,

$$R = 3\partial_\zeta q(\zeta) + 12q(\zeta), \tag{69}$$

if the polarization or generating functions for (58) and the solutions of (68) are taken for diagonal configurations.

4.2.1 Off-diagonal encoding of $f(R)$ gravity and reproduction of the ΛCDM era

We here provide an example of reconstruction of models of $f(R)$ gravity and nonholonomically deformed GR when both reproduce the ΛCDM era. For simplicity, we do not consider a real matter source (if such a source exists, it can be easily encoded into a nontrivial vacuum structure with generic off-diagonal contributions).

With respect to correspondingly N-adapted frames and for $a(\zeta)$ and $H(\zeta)$ determined by an off-diagonal solution (55), with nonholonomically induced torsion, or (59), for LC-configurations, the FLRW equation for ΛCDM cosmology is given by

$$3\kappa^{-2}H^2 = 3\kappa^{-2}H_0^2 + \rho_0 a^{-3} = 3\kappa^{-2}H_0^2 + \rho_0 a_0^{-3}e^{-3\zeta}. \tag{70}$$

This equation looks similar to the one for Einstein gravity for diagonal configurations but contains values determined, in general, for other classes of models with off-diagonal interactions. Thus, H_0 and ρ_0 are fixed to be certain constant values, after the coefficients of off-diagonal solutions are found, and for an approximation were the dependencies on (x^i, ζ) are changed into dependencies on ζ (via a corresponding redefinition of the generating functions and the effective sources). We can relate the first term on the rhs to an effective cosmological constant Λ (33), which in our approach appears via a redefinition (49). The second term in the formula describes, in general, an inhomogeneous distribution of cold dark mater (CDM) with respect to N-adapted frames. In order to keep the similarity with the diagonalizable cosmological models in GR we can choose these integration constants for $\Lambda = 12H_0^2$ to survive in the limit $w_i, n_i \rightarrow 0$. It should be noted that such limit must be computed for "nonlinear" nonholonomic constraints via generating functions and effective sources.

Using (70), the effective quadratic Hubble rate and the modified scalar curvature, $\widehat{\mathbf{R}}$, are computed to be, respectively,

$$q(\zeta) := H_0^2 + \kappa^2\rho_0 a_0^{-3}e^{-3\zeta} \qquad \text{and} \qquad \widehat{\mathbf{R}} = 3\partial_\zeta q(\zeta) + 12q(\zeta) = 12H_0^2 + \kappa^2\rho_0 a_0^{-3}e^{-3\zeta}.$$

These functional formulas can be used for the dependencies on $\widehat{\mathbf{R}}$ if a necessary re-definition of the generation functions, or an approximation $(x^i, \zeta) \rightarrow \zeta$ is performed. Expressing

$$3\zeta = -\ln\left[\kappa^{-2}\rho_0^{-1}a_0^3(\widehat{\mathbf{R}} - 12H_0^2)\right] \qquad \text{and} \qquad X := -3 + \widehat{\mathbf{R}}/3H_0^2,$$

we obtain from eq. (68)

$$X(1-X)\frac{\mathrm{d}^2\widehat{\mathbf{f}}}{\mathrm{d}X^2} + [\chi_3 - (\chi_1 + \chi_2 + 1)X]\frac{\mathrm{d}\widehat{\mathbf{f}}}{\mathrm{d}X} - \chi_1\chi_2\widehat{\mathbf{f}} = 0, \tag{71}$$

for certain constants, for which $\chi_1 + \chi_2 = \chi_1\chi_2 = -1/6$ and $\chi_3 = -1/2$. The solutions of this equation with constant coefficients and for R (69) were found in [52,53] as Gauss hypergeometric function, denoted there by $\widehat{\mathbf{f}} = F(X) := F(\chi_1, \chi_2, \chi_3; X)$, as (for some constants A and B)

$$F(X) = AF(\chi_1, \chi_2, \chi_3; X) + BX^{1-\chi_3}F(\chi_1 - \chi_3 + 1, \chi_2 - \chi_3 + 1, 2 - \chi_3; X).$$

This provides a proof of the statement that $f(R)$ gravity can indeed describe ΛCDM scenarios without the need of an effective cosmological constant. Working with auxiliary connections of the type $\widehat{\mathbf{D}}$, we can generalize the constructions to off-diagonal configurations and various classes of modified gravity theories, where A, B and χ_1, χ_2, χ_3 are appropriate functions of the h coordinates. For instance, reconstruction procedures for Finsler like theories and cosmology models on tangent and Lorentz bundles, and bi-metric/massive gravity models are given in [42–47].

Having chosen $\widehat{\mathbf{f}} = F(X)$ for a modified gravity, we can go further and mimic an off-diagonal configuration when $\widehat{\mathbf{f}} = \widehat{\mathbf{f}}(\widehat{\mathbf{R}})$ is introduced in (25) and the source $\widehat{\boldsymbol{\Upsilon}}(x^i, \zeta) = \Lambda/{}^1\widehat{\mathbf{f}} + \widehat{\mathbf{f}}/2\ {}^1\widehat{\mathbf{f}}$ is considered instead of (60) and (61) for $\check{\Phi}^2 = \check{\Lambda}^{-1}[\widehat{\Phi}^2|\ \widehat{\boldsymbol{\Upsilon}}| + \int \mathrm{d}\zeta\ \widehat{\Phi}^2\partial_\zeta|\ \widehat{\boldsymbol{\Upsilon}}|]$. Nevertheless, recovering nonhomogeneous modified cosmological models cannot be completed for general re-parameterized dependencies on (x^i, ζ) (in particular, on ζ only). This distinguishes explicitly the modified gravity theories of type $f(R)$ from those generated by nonholonomic deformations. For certain homogeneity conditions, we can state an equivalence of some classes of gravities and cosmological models, or analyze their alternative physical implications. But a complete recovering is only possible if all generating and integration functions and the effective sources are correlated with certain observable cosmological effects and further approximations and redefinitions in terms of constant parameters and functionals depending on a time-like coordinate can be effectively performed.

In general, a modified gravity theory is not transformed completely into a nonholonomic off-diagonal Einstein manifold; an overlap between certain classes of solutions and cosmological and quantum gravity models may exist (see the constructions and discussion in [19–22], related to [15–18]). A rigorous theoretical analysis of various types of classical and quantum corrected solutions is necessary and new experimental data are compulsory in order to conclude that an orthodox paradigm with nonholonomic off-diagonal sophisticate (non)vacuum configurations in GR may be enough for elaborating a final, viable cosmological model and to perform a variant of the effective covariant anisotropic quantization, what is indeed missing, up to now. In a more radical case, we will have to modify substantially the GR theory and a number of additional issues may arise on the status of off-diagonal solutions, on methods of quantization and the recovering formalism, on stability issues and some other, in the search for a matching solution.

4.2.2 Nonholonomic configurations mimicking phantom and nonphantom matter in $f(R)$ gravity

The anholonomic frame deformation method allows to reconstruct off-diagonal configurations modeling $f(R)$ gravity and cosmology with nonphantom or phantom matter in GR. With respect to N-adapted frames in an off-diagonal (modified, or not) gravitational background, the FLRW equations can be written as

$$3\kappa^{-2}H^2 = {}_s\rho(x^k)a^{-c(x^k)} + {}_p\rho(x^k)a^{c(x^k)}, \tag{72}$$

where $a(x^k,\zeta)$ and $H(x^k,\zeta)$ are determined by a solution (55), or (59). For reparameterizations or approximations with $(x^i,\zeta)\to\zeta$, we assume that the positive functions ${}_s\rho(x^k)$, ${}_p\rho(x^k)$ and $c(x^k)$ can be considered. The first term on the rhs dominates for small a in the early universe, as in GR with nonphantom matter described by an EoS parameter $w=-1+c/3>-1$. Introducing $q(x^k,\zeta) := H^2(x^k,\zeta)$ and the respective functionals ${}_sq := \frac{\kappa^2}{3}\,{}_s\rho a_0^{-c}$ and ${}_pq := \frac{\kappa^2}{3}\,{}_p\rho a_0^{c}$, for $q = {}_sqe^{-c\zeta} + {}_pqe^{c\zeta}$, in $\widehat{\mathbf{R}} = 3\partial_\zeta q(\zeta) + 12q(\zeta)$, we find

$$e^{c\zeta} = \begin{cases} \left[\widehat{\mathbf{R}} \pm \sqrt{\widehat{\mathbf{R}}^2 - 4(144-9c^2)}\right] \Big/ 6(4+c), & \text{for } c\neq 4;\\ \widehat{\mathbf{R}}/24, & \text{for } c=4. \end{cases} \tag{73}$$

The nonphantom matter may correspond to the case $c=4$ in (73), including radiation with $w=1/3$. eq. (72) transform into a functional equation on Y determined by changing the functional variable $\widehat{\mathbf{R}}^2 = -576\ {}_sq\ {}_pq\ Y$, $4Y(1-Y)\frac{\mathrm{d}^2\widehat{\mathbf{f}}}{\mathrm{d}Y^2} + (3+Y)\frac{\mathrm{d}\widehat{\mathbf{f}}}{\mathrm{d}Y} - 2\widehat{\mathbf{f}} = 0$. This is again a functional variant (if we consider dependencies on x^k) of the generating Gauss' hypergeometric function, similarly to (71), which can be solved in explicit form.

For the case $c\neq 4$ in (73), we come to models with phantom-like dominant components. A similar procedure as for deriving eqs. (22) and (23) in [52,53], results in a functional generalization of the Euler equation, namely

$$\widehat{\mathbf{R}}^2\frac{\mathrm{d}^2\widehat{\mathbf{f}}(\widehat{\mathbf{R}})}{\mathrm{d}\widehat{\mathbf{R}}^2} + A\widehat{\mathbf{R}}\frac{\mathrm{d}\widehat{\mathbf{f}}(\widehat{\mathbf{R}})}{\mathrm{d}\widehat{\mathbf{R}}} + B\widehat{\mathbf{f}}(\widehat{\mathbf{R}}) = 0,$$

for some coefficients $A=-H_0(1+H_0)$ and $B=(1+2H_0)/2$, for $H_0 = 1/3(1+{}_{ph}w)$. Here we consider, for simplicity, homogenous limits and approximations $H^2(t) = \frac{\kappa^2}{3}\,{}_{ph}\rho$ for the phantom EoS fluid-like states, ${}_{ph}p = {}_{ph}w\ {}_{ph}\rho$, with ${}_{ph}w < -1$. In both cases, with a trivial or a nontrivial nonholonomically induced torsion, there are solutions of the nonholonomic Euler equations above which can be expressed in the form $\widehat{\mathbf{f}}(\widehat{\mathbf{R}}) = C_+\widehat{\mathbf{R}}^{m_+} + C_-\widehat{\mathbf{R}}^{m_-}$, for some integration constants $C_\pm$ and $2m_\pm = 1-A\pm\sqrt{(1-A)^2-4B}$. This reproduces with respect to N-adapted frames the phantom dark energy cosmology with a behavior of the type $a(t) = a_0(t_s-t)^{-H_0}$, where t_s is the so-called Rip time. If the generating functions for the off-diagonal cosmological solutions are chosen in a way such that the N-connection coefficients w_i and n_i transform to zero, the solutions describe universes which end at a Big Rip singularity during t_s. Additionally to the former result that in the $f(R)$ theory no phantom fluid is needed, we conclude that for off-diagonal configurations we can effectively model such locally anisotropic cosmological configurations.

4.2.3 On nonholonomic constraints and nonconservation of the effective energy-momentum tensor

One can encode and effectively model various types of cosmological solutions for modified gravity theories with $f(R)$ and/or $f(R,T,R_{\alpha\beta}T^{\alpha\beta})$ functionals and their nonholonomic deformations. The cosmological reconstruction procedures can be elaborated for various types of viable modified gravity which may pass, or not, local gravitational tests and explain observational data for accelerating cosmology, dark energy and dark matter interactions [24–26, 42–47, 52, 53]. Nevertheless, these theories exhibit certain specific problems as nonconservation of the energy-momentum tensors for the effective or physical matter fields.

Let us discuss the "nonconservation" issue which is related to the nonholonomic deformations of GR. Even in the case $\nabla \to \widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$, we have the condition $\widehat{\mathbf{D}}_\gamma \widehat{\mathbf{T}}_{\alpha\beta} \neq 0$, which is a typical one for nonholonomic (subjected to nonintegrable constraints) mechanical or classical field theories. In Lagrange mechanics, for instance, the issue of nonholonomic restrictions is solved by introducing additional integration constants. In such cases, the conservation laws should be re-considered by taking into account various classes of nondynamical functions. Because the distortion tensor $\widehat{\mathbf{Z}}$ is completely defined by the data $(\widehat{\mathbf{g}}, \mathbf{N})$, we can compute in unique form the value $\widehat{\mathbf{D}}_\gamma \widehat{\mathbf{T}}_{\alpha\beta}$ and relate this to the fact that, in general, $\widehat{\mathbf{R}}_{\alpha\beta}$ is not symmetric. This is a consequence of a nonholonomically induced torsion. It is convenient to work with such nonholonomic variables $(\widehat{\mathbf{g}}, \mathbf{N}, \widehat{\mathbf{D}})$ in order to apply the anholonomic frame deformation method and decouple certain modified gravitational equations and generate off-diagonal solutions of (23). After this general, integral variety of solutions has been found, one can redefine the generating functions and sources in order to generate LC-configurations, as was shown in sect. 3.1.3. This way, the problem of "nonconservation" of effective and physical $\widehat{\mathbf{T}}_{\alpha\beta}$ can be solved by encoding into generic off-diagonal configurations with effective conservation laws which are similar to GR.

In explicit form, we explain how the "nonconservation" problem can be solved for off-diagonal solutions with one Killing symmetry in the framework of $f(R,T)$ theories generalizing certain constructions from [54]. Following a similar procedure as in sect. II of that work, but using the operator $\widehat{\mathbf{D}}$ instead of ∇, for $\widehat{\mathbf{f}} = \widehat{\mathbf{f}}(\widehat{\mathbf{R}}, \widehat{\mathbf{T}})$, and considering an N-adapted parametrization of the effective source $\widehat{\boldsymbol{\Upsilon}} = \text{const}$, we prove that

$$\left(1 + \frac{\kappa^2}{{}^2\widehat{\mathbf{f}}}\right) \widehat{\mathbf{D}}^\alpha \widehat{\mathbf{T}}_{\alpha\beta} = \frac{1}{2}\mathbf{g}_{\alpha\beta}\widehat{\mathbf{D}}^\alpha \widehat{\mathbf{T}} - (\widehat{\mathbf{T}}_{\alpha\beta} + \widehat{\Theta}_{\alpha\beta})\widehat{\mathbf{D}}^\alpha \ln({}^2\widehat{\mathbf{f}}) - \widehat{\mathbf{D}}^\alpha \widehat{\Theta}_{\alpha\beta}. \tag{74}$$

In these equations the values ${}^2\widehat{\mathbf{f}} := \partial \widehat{\mathbf{f}}/\partial \widehat{\mathbf{T}}$ and $\widehat{\Theta}_{\alpha\beta} = -2\widehat{\mathbf{T}}_{\alpha\beta} - p\mathbf{g}_{\alpha\beta}$ are used, with an energy-momentum tensor (26) for nonholonomic flows of a perfect fluid. In general, $\widehat{\mathbf{D}}^\alpha \widehat{\mathbf{T}}_{\alpha\beta} \neq 0$ even for nonholonomic deformations of GR. Nevertheless, we can consider a subclass of off-diagonal configurations in $\widehat{\mathbf{f}}(\widehat{\mathbf{R}}, \widehat{\mathbf{T}})$ gravity when $\widehat{\boldsymbol{\Upsilon}}$ (32) $\to \check{\Lambda}$ (33) and $\check{\Phi}^2 = \check{\Lambda}^{-1}[\widehat{\Phi}^2 |\ \widehat{\boldsymbol{\Upsilon}}| + \int d\zeta\ \widehat{\Phi}^2 \partial_\zeta |\ \widehat{\boldsymbol{\Upsilon}}|]$ (61) result in $\widehat{\mathbf{f}} \to \check{\mathbf{f}} = \check{\mathbf{R}}$ and effective $\check{\mathbf{R}}^\alpha_{\ \beta} = \check{\Lambda}\delta^\alpha_{\ \beta}$ which admit LC-solutions with zero torsion. For such nonholonomic distributions with $\widehat{\mathbf{D}} \to \nabla$, $\widehat{\mathbf{D}}^\alpha \widehat{\mathbf{T}}_{\alpha\beta} \to \check{\nabla}\check{\Lambda} = 0$ and all terms on the lhs of (74) vanish, because they are nonholonomically equivalent to functionals of the effective cosmological constant $\check{\Lambda}$. Such conditions are satisfied in correspondingly N-adapted frames and for canonical d-connections. Equations (74) generalize to nonholonomic forms the similar ones derived for the Levi-Civita connection ∇ (see eq. (10) in ref. [54]).

The already mentioned problem of "nonconservation" becomes worse for general $f(R,T,R_{\alpha\beta}T^{\alpha\beta})$ theories. Even if there are certain special cases where it can be solved [24–26, 52, 53], it is the case that no solution can be found in general form. Surprisingly, the anholonomic frame deformation method also suggests a procedure for selecting off-diagonal configurations which can mimic modified gravity theories in a self-consistent way, admitting effective torsions completely determined by observational data $(\widehat{\mathbf{f}},\ \widehat{\boldsymbol{\Upsilon}}, \widehat{\mathbf{g}}, \mathbf{N}, \widehat{\mathbf{D}})$, and/or by constraints to LC-configurations. By redefining the generating functions in the form $\widehat{\boldsymbol{\Upsilon}}$ (32) $\to \check{\Lambda}$ (33), we get the possibility to consider effective sources and off-diagonal Einstein spaces for which the divergence and nonconservation problem become similar to those in GR or (for more geometrically complex configurations) to those in viable $f(R)$ models. Finally, we emphasize the fact that in this way we do not find a general solution for all $f(R,\ldots)$-modified theories but only for those models which admit an encoding in an effective GR system and a general decoupling via the nonholonomic deformations.

4.3 Nonholonomic constraints and matter instability

There is another serious problem in modified gravities which is related to possible matter instabilities related to modifications of the gravitational actions. Even tiny modifications of GR may make the new model to posses unstable interior solutions (see, *e.g.*, [55,56]). It was demonstrated however that there are viable $f(R)$ theories (with appropriated choices of the functional) where such instabilities may not occur [1–14, 57]. In this section, we apply the anholonomic frame deformation method to more general $f(R,T,R_{\alpha\beta}T^{\alpha\beta})$ theories. The corresponding field equations are very difficult to solve even in a linear approximation [24–26], if we work in coordinate frames and general functionals. In the nonholonomic variable formalism, the gravitational field equations in modified gravity theories posses the decoupling property exhibited above, which allows to encode $f(R,\ldots)$-modifications into off-diagonal nonholonomic configurations for the effective Einstein manifolds.

For a stability analysis, the trace equations where (31) are multiplied by $\mathbf{g}^{\mu\nu}$ are to be considered, namely

$$-2\widehat{\mathbf{f}}+(\widehat{\mathbf{R}}+3\widehat{\mathbf{D}}^{\mu}\widehat{\mathbf{D}}_{\mu})\ {}^{1}\widehat{\mathbf{f}}+(\widehat{\mathbf{T}}+\boldsymbol{\Theta})\ {}^{2}\widehat{\mathbf{f}}+\left(\frac{1}{2}\widehat{\mathbf{D}}^{\mu}\widehat{\mathbf{D}}_{\mu}\widehat{\mathbf{T}}+\widehat{\mathbf{D}}_{\mu}\widehat{\mathbf{D}}_{\nu}\widehat{\mathbf{T}}^{\mu\nu}+\boldsymbol{\Xi}\right)\ {}^{3}\widehat{\mathbf{f}}=\kappa^{2}\ \widehat{\mathbf{T}}, \tag{75}$$

where ${}^{1}\widehat{\mathbf{f}}:=\partial\widehat{\mathbf{f}}/\partial\widehat{\mathbf{R}}$, ${}^{2}\widehat{\mathbf{f}}:=\partial\widehat{\mathbf{f}}/\partial\widehat{\mathbf{T}}$ and ${}^{3}\widehat{\mathbf{f}}:=\partial\widehat{\mathbf{f}}/\partial\widehat{\mathbf{P}}$, when $\widehat{\mathbf{P}}=\widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta}$. Let us envisage a trace configuration in the interior of a celestial body, when $\widehat{\mathbf{T}}=\widehat{\mathbf{T}}_0$ and $-2\widehat{\mathbf{f}}+\widehat{\mathbf{R}}_0\ ({}^{1}\widehat{\mathbf{f}})=\kappa^2\ \widehat{\mathbf{T}}_0$. Imposing nonholonomic constraints, we parameterize a LC-configuration in GR and model an interior solution in the presence of some gravitational objects (for instance, the Sun or the Earth). The f-modifications (in general with strong coupling for the curvature and the energy-momentum tensor) may result in a worsening of the stability problems and may prevent $\widehat{\mathbf{T}}_0$ to be a solution of any suitable background equation. It is difficult to find solutions of (75) even for very much simplified cases in the nonlinear situation if we work in coordinate frames for the connection $\widehat{\mathbf{D}}=\nabla$.

A rigorous study of the problem of matter instability for $f(R)$ and more generally $f(R,T,R_{\alpha\beta}T^{\alpha\beta})$ gravities, for certain illustrative cases when ${}^{1}\widehat{\mathbf{f}}=R$, and for restrictive conditions where there is a qualitative description via additional functionals on T and P shows that the appearance of a large instability can actually be avoided. Using the anholonomic frame deformation method, we can consider modified gravity theories with f-modifications which are effectively described by $\check{\mathbf{R}}^{\alpha}{}_{\beta}=\check{\Lambda}\delta^{\alpha}{}_{\beta}$ when the modifications are encoded into polarization functions and N-coefficients. For models generated by

$$\widehat{\mathbf{f}}(\widehat{\mathbf{R}},\widehat{\mathbf{T}},\widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta})=\widehat{\mathbf{f}}_1(\widehat{\mathbf{R}})+\widehat{\mathbf{F}}(\widehat{\mathbf{P}})+\widehat{\mathbf{G}}(\widehat{\mathbf{T}}),$$

we take a constant interior solution with $\widehat{\mathbf{T}}_0=\text{const}$ and $\widehat{\mathbf{P}}_0=\text{const}$, and denote by $\widehat{\mathbf{f}}_1^{(s)}:=\partial^s\widehat{\mathbf{f}}_1/\partial\widehat{\mathbf{R}}^s$ and $\widehat{\mathbf{F}}^{(s)}:=\partial^s\widehat{\mathbf{F}}/\partial\widehat{\mathbf{P}}^s$ for $s=1,2,\ldots$ We can repeat, with respect to the N-frames (12) and (13), the computations presented in detail for eqs. (45)–(48) in [24–26] (see also references therein), and prove that eqs. (75) for linear pertubations can be written in the form

$$\left[\check{\mathbf{D}}^{\mu}\check{\mathbf{D}}_{\mu}+2\frac{\check{\mathbf{T}}_0^{\mu\nu}}{\check{\mathbf{T}}_0}\check{\mathbf{D}}_{\mu}\check{\mathbf{D}}_{\nu}+2\frac{\boldsymbol{\Xi}_0}{\check{\mathbf{T}}_0}+4\frac{\check{\mathbf{P}}_0}{\check{\mathbf{T}}_0}\frac{\widehat{\mathbf{f}}_1^{(1)}}{\widehat{\mathbf{F}}^{(2)}}\right]\delta\check{\mathbf{P}}=\left[\frac{2}{\check{\mathbf{T}}_0}\frac{\widehat{\mathbf{f}}_1^{(1)}}{\widehat{\mathbf{F}}^{(2)}}-\frac{\check{\mathbf{P}}_0}{\check{\mathbf{T}}_0}\frac{\widehat{\mathbf{F}}^{(1)}}{\widehat{\mathbf{F}}^{(2)}}\left(2\ {}^{m}\widehat{\mathcal{L}}-\check{\mathbf{T}}_0\right)\right]\delta\check{\mathbf{R}}.$$

The values of type $\delta\widehat{\mathbf{P}}$ and $\delta\widehat{\mathbf{R}}$ are considered for a small perturbations in the curvature where $\widehat{\mathbf{R}}=\widehat{\mathbf{R}}_0+\delta\widehat{\mathbf{R}}$ and $\widehat{\mathbf{P}}=\widehat{\mathbf{P}}_0+\delta\widehat{\mathbf{P}}$. No instability appears if $\delta\check{\mathbf{P}}=\delta\check{\mathbf{R}}=0$ which is a particular solution of the above equation. We can in fact model a damped oscillator with additional nonholonomic constraints if $\boldsymbol{\Xi}_0+2\check{\mathbf{P}}_0\widehat{\mathbf{f}}_1^{(1)}/\widehat{\mathbf{F}}^{(2)}\geq\check{\mathbf{T}}_0$, which allows to avoid large instabilities in the interior of a spherical body. For some specific functionals $f(R)$ and appropriate $G(T)$, the same behavior as in GR results (with mass stability in the cosmological context), although there are possible large perturbations δR and δP remaining. The ideas how to circumvent the mass instability problem for holonomic configurations has been studied in [58–64]. Redefining the generating functions and sources in a f-modified model into an effective Einsteinian theory, with $\mathcal{S}[\check{\mathbf{R}},\check{\Lambda}]$, one can consider a nonholonomically deformed Hilbert-Einstein action with $\widehat{\mathbf{f}}\to\check{\mathbf{f}}=\check{\mathbf{R}}$. In such cases, $\delta\widehat{\mathbf{R}}=\delta\check{\mathbf{R}}=\mathbf{0}$ and instabilities are not produced, indeed, if we impose the zero torsion conditions (see (24)), we get back to the GR theory. Even if eq. (75) involves not only perturbations of the Ricci scalar $\widehat{\mathbf{R}}$ but also of the Ricci d-tensor $\widehat{\mathbf{R}}_{\alpha\beta}$ (through $\delta\widehat{\mathbf{P}}$), via nonholonomic transforms to effective $\check{\mathbf{R}}^{\alpha}{}_{\beta}=\check{\Lambda}\delta^{\alpha}{}_{\beta}$, the stability of the system is obtained via off-diagonal interactions and the nonholonomic constraints used for an effective modeling of a subclass of $\widehat{\mathbf{f}}$-theories to certain nonholonomic deformations of the Einstein equations with effective cosmological constant $\check{\Lambda}$. This is indeed a remarkable result.

5 Effective field theory for off-diagonal cosmological configurations

Both in particle and condensed matter physics, effective field theory (EFT) methods have proven so far to provide a quick and economic way in order to connect experimental data and phenomenological results with certain fundamental theories (see the reviews [65, 66]). Recently methods of that sort have been applied in cosmology, in particular to inflation [67–69], late-time acceleration [70–72] and dark energy physics [73, 74] (details and references can be found in [75, 76]). The goal of this section is to construct an EFT describing perturbations both over diagonal and off-diagonal cosmological background solutions in modified gravities, in cases where an effective Einsteinian manifold can be associated, and when the matter sector obeys the weak equivalence principle and all modifications of gravity and the matter fields can be encoded into an effective cosmological constant.

5.1 Off-diagonal background evolution and ΛCDM

We shall consider here configurations with redefined generating functions and sources and where the third effective action in (25) is taken to be of the form

$$\mathcal{S} = {}^g\mathbf{S} + {}^m\mathbf{S} = \int \left[\frac{1}{2\overline{\kappa}^2}\Omega(t)\mathbf{R} + \widehat{\Lambda}(t) - c(t)\delta\mathbf{g}^{44}\right]\sqrt{|\mathbf{g}|}\mathbf{d}^4u + {}^m\mathbf{S}[\mathbf{g}_{\alpha\beta}]. \tag{76}$$

This is related to a background FLRW solution constrained, on its turn, by observation to be in close agreement with ΛCDM[15]. In this formula, the value $\delta\mathbf{g}^{44}$ is the perturbation of the time-time like component of the d-metric, and the Ricci scalar $\mathbf{R}$ is defined by a canonical d-connection $\mathbf{D}$ which can be obtained by a finite chain of redefinitions, resulting in $[\widehat{\mathbf{g}}, \widehat{\mathbf{N}}, \widehat{\mathbf{D}}, \widehat{\Upsilon}] \rightarrow [\mathbf{g}, \mathbf{N}, \mathbf{D}, \mathbf{T}_{\alpha\beta}, \widehat{\Lambda}(t)]$ where $\widehat{\Lambda}(t)$ takes a constant value $\Lambda/2\overline{\kappa}^2$ for the h-equations while it is a function of t for the v-equations. The energy-momentum tensor $\mathbf{T}_{\alpha\beta}$ determined by ${}^m\mathbf{S}[\mathbf{g}_{\alpha\beta}]$ does not satisfy, in general, the condition $\mathbf{D}_\alpha\mathbf{T}^{\alpha\beta} \neq 0$, but $\nabla_\alpha\mathbf{T}^{\alpha\beta} \rightarrow 0$ for $\mathbf{D}_\alpha \rightarrow \nabla_\alpha$. For simplicity, effective matter is treated as a perfect fluid, which in the N-adapted model is described by a time dependent average energy density $\overline{\rho}(t)$ and pressure $\overline{p}(t)$. For LC-configurations one has the usual continuity equation

$$\overline{\rho}^\diamond + 3H(\overline{\rho} + \overline{p}) = 0,$$

where H is determined by a scaling factor $a(t)$ for a generic off-diagonal solution of type (55), or (59). We note that in this section $\overline{\rho}^\diamond$ is the derivative with respect to the physical time (the formalism works also in conformal time). The function $a(t)$ can be considered as the limiting result for $\varepsilon \rightarrow 0$ of an integration in a metric of type (7) (where nonlinear interactions are encoded by f-modifications), as we explained before for that formula.

5.1.1 Off-diagonal background evolution

The background evolution is in general off-diagonal (values $\varepsilon \approx 10^{-20}$ do not contradict present experimental data [51]). For simplicity, we can consider a diagonal background with a FLRW metric with zero spatial curvature. We can chose $a(t)$, $\overline{\rho}(t)$ and $\overline{p}(t)$ to be close to ΛCDM evolution if $\varepsilon \rightarrow 0$. As in standard (diagonal) cosmology we can use the Friedmann equations to eliminate the functions $\widehat{\Lambda}(t)$ and $c(t)$ (which can again be considered as a redefinition of the generating functions in our approach) but keep $\Omega(t)$ as a free function, similarly to [75, 76] and [77]. For diagonal configurations, such theory can be formulated in the Jordan frame and dealt with a nonminimal coupling between an effective scalar field and metric. In the Einstein frame, we get a coupling of matter to the effective scalar field. Nevertheless, the scaling factor $a(t)$ has some "memory" of the genuine nonholonomic redefinition of the integration functions and corresponding contributions of the modified gravity theory.

Varying (76) with respect to N-adapted frames, we get an effective Friedman equation, which allows to express

$$-2c(t) = \overline{\rho} + \overline{p} + (2\Omega H^\diamond + \Omega^{\diamond\diamond}/2 - H\Omega^\diamond)/\overline{\kappa}^2$$
$$-\widehat{\Lambda}(t) = \overline{p} + (3\Omega H^2 + 2\Omega H + \Omega^{\diamond\diamond} + H\Omega^\diamond)/\overline{\kappa}^2$$

for any given data $(\Omega, a, \overline{\rho}, \overline{p})$[16]. These equations describe how the background energy and pressure of the DE component evolve over cosmological history, corresponding to the evolution of the N-coefficients $w_i(t)$.

5.1.2 N-adapted perturbations

Assuming that for $\mathbf{D} \rightarrow \nabla$ the weak equivalence principle is true, one can always introduce a conformal (Jordan) frame, when the matter fields couples only to the d-metric $\mathbf{g}_{\alpha\beta}$ and not to the scalar field.

[15] In effective field theories the mass scale $\overline{m}^2 = \overline{\kappa}^{-2}$ can be different from the Plank mass of GR. It is used to render Ω dimensionless.

[16] This can be rewritten in a more conventional form in terms of the dark energy density, ρ_{DE}, and pressure, p_{DE} (cf. (4)), when

$$3\Omega H^2 = \overline{\kappa}^2(\rho_{\mathrm{DE}} + \overline{\rho}) \quad \text{and} \quad 2\Omega H^\diamond = -\overline{\kappa}^2(\rho_{\mathrm{DE}} + p_{\mathrm{DE}} + \overline{\rho} + \overline{p}),$$

for $2c(t) = \rho_{\mathrm{DE}} + p_{\mathrm{DE}} + (H\Omega^\diamond - \Omega^{\diamond\diamond})/\overline{\kappa}^2$ and $\widehat{\Lambda}(t) = p_{\mathrm{DE}} - (H\Omega^\diamond + \Omega^{\diamond\diamond})/\overline{\kappa}^2$.

Starting from an N-adapted form for (76), a procedure similar to that for the construction of effective field theories (see eq. (2.1) of [75, 76]) leads to the general effectively unitary gauge action

$$
\begin{aligned}
\mathcal{S} = \int \sqrt{|\mathbf{g}|}\mathbf{d}^4 u \Big[& \frac{1}{2\overline{\kappa}^2}\Omega(t)\mathbf{R} + \widehat{\Lambda}(t) - c(t)\delta\mathbf{g}^{44} + \frac{M_2^4(t)}{2}(\delta\mathbf{g}^{44})^2 + \frac{M_3^4(t)}{2}(\delta\mathbf{g}^{44})^3 + \dots \\
& - \frac{\overline{M}_1^3(t)}{2}\delta\mathbf{g}^{44}\ \delta\mathbf{K}^{\alpha}_{\ \alpha} - \frac{\overline{M}_2^2(t)}{2}(\delta\mathbf{K}^{\alpha}_{\ \alpha})^2 - \frac{\overline{M}_3^2(t)}{2}\delta\mathbf{K}^{\alpha}_{\ \beta}\delta\mathbf{K}^{\beta}_{\ \alpha} + \dots + {}^1\lambda(t)(\delta\mathbf{R})^2 + {}^2\lambda(t)\delta\mathbf{R}_{\alpha\beta}\ \delta\mathbf{R}^{\alpha\beta} \\
& + {}^1\gamma(t)\mathbf{C}_{\mu\nu\sigma\lambda}\mathbf{C}^{\mu\nu\sigma\lambda} + {}^2\gamma(t)\epsilon^{\mu\nu\sigma\lambda}\mathbf{C}_{\mu\nu}^{\ \ \alpha\beta}\mathbf{C}_{\sigma\lambda\alpha\beta} + \dots \\
& + {}^1m^2(t)\mathbf{n}^{\mu}\mathbf{n}^{\nu}(\mathbf{e}_{\mu}\mathbf{g}^{44})(\mathbf{e}_{\nu}\mathbf{g}^{44}) + {}^2m^2(t)(\mathbf{g}^{\mu\nu} + \mathbf{n}^{\mu}\mathbf{n}^{\nu})(\mathbf{e}_{\mu}\mathbf{g}^{44})(\mathbf{e}_{\nu}\mathbf{g}^{44}) + \dots \Big] + {}^m\mathbf{S}[\mathbf{g}_{\alpha\beta}]. \qquad (77)
\end{aligned}
$$

The value $\mathbf{C}_{\mu\nu\sigma\lambda}$ is the Weyl tensor determined by $\mathbf{D}$, while $\mathbf{n}^{\mu}$ is the vector normal to the surfaces of constant time. Each term in (77) can have a time-dependent coefficient because the background solutions is, in general, off-diagonal and breaks time translation symmetry. The matter action ${}^m\mathbf{S}[\mathbf{g}_{\alpha\beta}]$ can be arbitrary, with sources to be redefined by integration functions. We will fix below the quantities $\delta\mathbf{g}^{44}$, $\delta\mathbf{R}$, $\delta\mathbf{R}_{\alpha\beta}$ and $\delta\mathbf{K}_{\alpha\beta}$ as perturbations, the dots corresponding to various terms which we do not specify of quadratic and higher-order perturbations. N-elongated partial derivatives $\mathbf{e}_{\mu}$ (12) are used instead of ∂_{μ}.

We involve and additional $3+1$ splitting (the variant $2+2$ is convenient for constructing off-diagonal, exact solutions), where the Ricci d-tensor of $\mathbf{D}$ decomposes as

$$\mathbf{R}_{\alpha\beta}\mathbf{n}^{\alpha}\mathbf{n}^{\beta} = \mathbf{K}_{\alpha\beta}\mathbf{K}^{\alpha\beta} - \mathbf{K}_{\alpha}^{\ \beta}\mathbf{K}_{\beta}^{\ \alpha} + \mathbf{D}_{\alpha}\left(\mathbf{n}^{\gamma}\mathbf{D}_{\gamma}\,\mathbf{n}^{\alpha}\right) - \mathbf{D}_{\alpha}\left(\mathbf{n}^{\alpha}\mathbf{D}_{\gamma}\,\mathbf{n}^{\gamma}\right).$$

As a result of the N-adapted construction, the value $\mathbf{n}^{\alpha}\delta\mathbf{K}_{\alpha\beta}$, where $\delta\mathbf{K}_{\alpha\beta} = \mathbf{K}_{\alpha\beta} - {}^0\mathbf{K}_{\alpha\beta} := \mathbf{K}_{\alpha\beta} + 3H(\mathbf{g}^{\mu\nu} + \mathbf{n}^{\mu}\mathbf{n}^{\nu})$ vanishes.

5.1.3 Effective field theory in terms of Stückelberg d-fields

In effective field theories and cosmological models, the Stükelberg technique [67–69, 75, 76] is used when explicit functions of time are modified according to

$$\varphi(t) \to \varphi(t + \pi(u^{\alpha})) \qquad (78)$$

and then Taylor-expanded in $\pi(u^{\alpha})$. Such procedure is applied to the action (77) when the ansatz for off-diagonal solutions (55), or (59), is reparameterized as a $3+1$ form,

$$
\begin{aligned}
\mathrm{d}s^2 = \underline{g}_{\alpha\beta}\,\mathrm{d}u^{\alpha}\,\mathrm{d}u^{\beta} &= a^2(\tilde{g}_{\underline{i}\,\underline{j}} + \varsigma_{\underline{i}\,\underline{j}})\mathrm{d}\widetilde{x}^{\,\underline{i}}\,\mathrm{d}\widetilde{x}^{\,\underline{j}} - \mathrm{d}\widetilde{t}^{\,2}, \text{ synchronous gauge,} \\
&= a^2(\tilde{\mathbf{g}}_{\underline{i}\,\underline{j}} + \tilde{\varsigma}_{\underline{i}\,\underline{j}})\mathrm{d}\widetilde{\mathbf{e}}^{\,\underline{i}}\,\mathrm{d}\widetilde{\mathbf{e}}^{\,\underline{j}} - (\delta\widetilde{t})^2, \text{ N-adapted synchronous gauge,} \qquad (79) \\
\mathrm{d}s^2 &= a^2(1 - 2\widetilde{\phi})\tilde{g}_{\underline{i}\,\underline{j}}\,\mathrm{d}\widetilde{x}^{\,\underline{i}}\,\mathrm{d}\widetilde{x}^{\,\underline{j}} - (1 + 2\psi)\mathrm{d}\widetilde{t}^{\,2}, \text{ Newtonian gauge,} \\
&= a^2(1 - 2\widetilde{\phi})\tilde{\mathbf{g}}_{\underline{i}\,\underline{j}}\,\mathrm{d}\widetilde{\mathbf{e}}^{\,\underline{i}}\,\mathrm{d}\widetilde{\mathbf{e}}^{\,\underline{j}} - (1 + 2\psi)(\delta\widetilde{t}\,)^2, \text{ N-adapted Newtonian gauge,} \qquad (80)
\end{aligned}
$$

where $\mathrm{d}\widetilde{\mathbf{e}}^{\,\underline{i}} = (\mathrm{d}\widetilde{x}^{\,i}, \delta\tilde{e}^3 = \mathrm{d}y^3 + \tilde{n}_i\,\mathrm{d}\widetilde{x}^i)$ and $\delta\widetilde{t} = \mathrm{d}\widetilde{t} + \tilde{w}_i\,\mathrm{d}\widetilde{x}^i$, see (13). The basic premise for this is that we can perform coordinate transformations $\widetilde{t} = t + \pi(u^{\alpha}), \widetilde{x}^{\underline{i}} = x^{\underline{i}}$, were the convention for indices is $\underline{i}, \underline{j}, \ldots = 1, 2, 3$ and the coordinates $u^{\alpha} = (x^i, y^3, t) \to \widetilde{u}^{\alpha} = (\widetilde{x}^{\,\underline{i}}, \widetilde{t})$. We consider $\tilde{g}_{\underline{i}\,\underline{j}}$ to be a time-independent background metric but $a(\widetilde{t})$ and $\varsigma_{\underline{i}\,\underline{j}}[N_i^a, g_i, h_a]$ are certain nonlinear functionals determined by a cosmological off-diagonal solution in the modified theory. The metric (79) is written in a form which allows to describe perturbations in a synchronous gauge. This parametrization can be obtained for any d-metric (17) which with respect to coordinate frames is rewritten in the form (15) with coefficients (16). In the synchronous gauge, the coordinate transforms $u^{\alpha} \to \widetilde{u}^{\alpha}$ are chosen in a for to be satisfied the conditions $\delta g^{44} = g^{4\underline{i}} = 0$ (we note that here we do not use boldface symbols because, in general, such conditions can be imposed for not N-adapted frames). It will be convenient to discuss some phenomena in the Newtonian gauge (80). For small ε-deformations (see (7)), off-diagonal extensions of cosmological metrics can be treated as effective fluctuations which may be nonholonomically constrained, or not, to be parameterized in N-adapted form. In both cases, we can use the formulas derived in [75, 76] but keeping in mind that we work with a d-connection $\mathbf{D}$ which only at the end will be additionally constrained, when $\mathbf{D} \to \nabla$[17].

[17] For simplicity, we do not dub, in an N-adapted form, the proofs from those works but use directly the synchronous gauge representation which is more convenient for studying both perturbations and ε-deformations all included in a term $\varsigma_{\underline{i}\,\underline{j}}(\varepsilon, x^{\underline{i}}, t)$; we omit the tilde on spacetime coordinates when this does not result in ambiguities, and use in brief the term perturbations both for ε-deformations and for fluctuations; in a more general context, perturbations with respect to a fixed N-adapted background can be considered.

We can decompose

$$\varsigma_{\underline{i}\,\underline{j}} = \frac{\varsigma}{3}\tilde{g}_{\underline{i}\,\underline{j}} + \left(\widetilde{\mathbf{D}}_{\underline{i}}\widetilde{\mathbf{D}}_{\underline{j}} - \frac{\tilde{g}_{\underline{i}\,\underline{j}}}{3}\widetilde{\mathbf{D}}^2\right)\eta \qquad \text{and} \qquad \tilde{\varsigma}_{\underline{i}\,\underline{j}} = \frac{\tilde{\varsigma}}{3}\tilde{\mathbf{g}}_{\underline{i}\,\underline{j}} + \left(\widetilde{\mathbf{D}}_{\underline{i}}\widetilde{\mathbf{D}}_{\underline{j}} - \frac{\tilde{\mathbf{g}}_{\underline{i}\,\underline{j}}}{3}\widetilde{\mathbf{D}}^2\right)\tilde{\eta}, \tag{81}$$

where tildes refer to quantities associated with the spatial metric, and express the metric determinant $\sqrt{|\underline{g}|} = a^3\sqrt{|\tilde{g}|}(1+\varsigma/2)$ and $\sqrt{|\mathbf{g}|} = a^3\sqrt{|\tilde{\mathbf{g}}|}(1+\tilde{\varsigma}/2)$. It is convenient to use a definition in momentum space $k^\alpha = (\overrightarrow{k}, \tilde{k})$, where

$$\eta\left(\overrightarrow{k}, t\right) = -k^{-2}\left[\varsigma\left(\overrightarrow{k}, t\right) + 6\mathring{\eta}\left(\overrightarrow{k}, t\right)\right] \tag{82}$$

(see details on similar conventions in [75,76]), where the effective field constructions are also performed in the massive case.

5.2 Linking off-diagonal perturbations to observations

The goal is here to identify and study the full set of perturbations which result in off-diagonal deformations of the FRW background. Linearized equations of motion for certain effective scalar perturbations will be considered, which will allow to determine: 1) the speed of sound; 2) Poisson's equation; 3) the anisotropic stress; 4) the effective Newton constant; and 5) Caldwell's parameter [78]. Reparametrizations of the metric in the respective forms (79) and (80) allow for this association with the standard phenomenological functions and parameters appearing in literature, connecting in this way observational features with the off-diagonal cosmological solutions, for different modified gravities and effective Einstein spaces. In what follows, and as a hint to explicitly show these possibilities, we will consider in brief 1), 4) and 5) above, as obtained for certain classes of cosmological models.

5.2.1 Effective speed of sound in off-diagonal media

The data π, ς and $\mathring{\eta}$ defined by eqs. (78), (81) and (82) are used for writing the so-called π-equation of motion along with the time-time and space-space trace components of the Einstein equation (in our case, modified in terms of $\nabla \to \mathbf{D}$) via the kinetic matrix $\gamma_{X,Y}$ (namely, the coefficient of Y in the X equation of motion). In its turn, this allows to compute the speed of sound of scalar perturbations in the sub-horizon limit using the synchronous gauge for this calculation. The differential operators are transformed into matrix components by changing $z^\diamond \to -i\omega$. This allows to compute the coefficients of (77), following the same procedure as in table 1 and appendix D (see also eqs. (4.1)–(4.5)) of [75,76], where the determinant of the kinetic matrix is set to be zero. Such equations are parameterized in the form

$$\begin{pmatrix} \gamma_{\pi,\pi} & \gamma_{\pi,\varsigma} & \gamma_{\pi,\mathring{\eta}} \\ \gamma_{\widetilde{tt},\pi} & \gamma_{\widetilde{tt},\varsigma} & \gamma_{\widetilde{tt},\mathring{\eta}} \\ \gamma_{ss,\pi} & \gamma_{ss,\varsigma} & \gamma_{ss,\mathring{\eta}} \end{pmatrix} \begin{pmatrix} \pi \\ \varsigma \\ \mathring{\eta} \end{pmatrix} = 0.$$

The resulting expression yields, in general, a number of nonlinear dispersion relations. For instance, the presence of the operator $M_2^4(t)$ results in the dispersion formula for the speed of sound, c,

$$c_s^{-2} = 1 + 2M_2^4\left[c + \frac{3(\Omega^\diamond)^2}{4\overline{\kappa}^2\Omega}\right]^{-1}.$$

The value c is linked to various stability issues, as analyzed in [73,74], where it was concluded that ghost-free conditions are satisfied if

$$2\left[c + \frac{3(\Omega^\diamond)^2}{4\overline{\kappa}^2\Omega}\right] = \rho_{\mathrm{DE}} + p_{\mathrm{DE}} + \overline{\kappa}^{-2}\left[H\Omega^\diamond + \frac{3}{2}\frac{(\Omega^\diamond)^2}{\Omega} - \Omega^{\diamond\diamond}\right] > 0.$$

For certain off-diagonal configurations, the effective EoS of DE can become of phantom type, $w_{\mathrm{DE}} < -1$, even if the nonholonomic deformations/perturbations do not seem to host one (it may depend on the choice of generating function $\Omega(t)$).

5.2.2 The effective gravitational constant and Caldwell's parameter

In the Newtonian limit, we look at off-diagonal deformations and modified theories of gravity that share the horizon $k^2/a^2 \gg H^2$. Such a horizon is indeed shared by different types of solutions in different models, and may show up, for instance, in ε-deformed nonholonomic backgrounds. Similar considerations as for subsect.4.4 in [75,76] result in an expression for the effective Newton constant $4\pi\ {}^{\mathrm{eff}}G := {}^{\mathrm{eff}}\kappa/2$ (note here π is the mathematical constant, and not the function $\pi(u^\alpha)$ from (78))

$$ {}^{\mathrm{eff}}\kappa^2 = \overline{\kappa}^2 \varOmega^{-1}\left[1 + (\varOmega^\diamond)^2/4\varOmega\left(c\overline{\kappa}^2 + \frac{3}{4}\frac{(\varOmega^\diamond)^2}{\varOmega} + a^2\frac{\overline{\kappa}^2}{k^2}M^2\right)\right]. $$

This expression becomes more complicated if more terms, in addition to M^2, are included. There are modifications of ${}^{\mathrm{eff}}G$ even if $\varOmega^\diamond = 0$. We note here the explicit dependence on a^2, which can be considered as the generating function for off-diagonal solutions.

Let us now consider the parameter ${}^c\varpi$ considered by Caldwell *et al.* [78] as the cosmological analog of Eddington's PPN parameter γ, when ${}^c\varpi \approx 1-\gamma$ (we use here a notation which differs from those in other works on the subject, in order not to interfere with ϖ from eq. (3), etc.)[18]. Using background terms, one obtains

$$ {}^c\varpi = \frac{(\varOmega^\diamond)^2}{2\varOmega}\left[c\overline{\kappa}^2 + \frac{(\varOmega^\diamond)^2}{2\varOmega} + a^2\frac{\overline{\kappa}^2}{k^2}M^2\right]^{-1}. $$

Such formula becomes more complicated if further terms are included. In both cases of this and the previous formulas, only the terms $\overline{M}_3$ and $\widehat{\mathbf{M}}^2 = \delta\mathbf{g}^{44}\delta\mathbf{R}^{(3)}$ (for perturbations of the spacial scalar curvature $\mathbf{R}^{(3)}$ of $\mathbf{D}$) contribute, if $\varOmega^\diamond = 0$. Both ${}^{\mathrm{eff}}\kappa^2(a)$ and ${}^c\varpi(a)$ are functionals of the generating functions. In GR, ${}^{\mathrm{eff}}\kappa^2 = 1/m_P$ and ${}^c\varpi = 0$. Nevertheless, these values are nonzero if the configurations are off-diagonal with an effective cosmological constant. A kind of nonlinear classical polarization of the gravitational constant is there obtained at certain scales.

5.2.3 Confronting an off-diagonal cosmology which encodes modified gravity with actual observational results

In sect. 4.6 of [75,76] (see, in particular, tables 4 and 5 there), the contributions from various operators in a generalized model with an effective field theory action were calculated. We will here combine this knowledge on the dark energy models (in our case with off-diagonal interactions modeling the $\varLambda$CDM, $f(\mathbf{R})$, and Hořava-Lifshitz [79] theories) for the nontrivial N-adapted coefficients in (77):

Operator	$\varOmega$	$\varLambda$	c	$\overline{\mathbf{M}}_2^2$	${}^2m^2$
MGTs	$\mathbf{R}$		$\delta\mathbf{g}^{44}$	$(\delta\mathbf{K}^\alpha_{\ \alpha})^2$	$a^{-2}\mathbf{g}^{\underline{i}\underline{j}}(\mathbf{e}_{\underline{i}}\mathbf{g}^{44})(\mathbf{e}_{\underline{j}}\mathbf{g}^{44})$
$\varLambda$CDM	1	$\curlyvee$	0	–	–
$f(\mathbf{R})$	$\curlyvee$	$\curlyvee$	0	–	–
Hořava-Lifshitz	1	$\curlyvee$	0	$\curlyvee$	$\curlyvee$

In the above matrix, the symbols for the respective operators mean that $\curlyvee$ is necessary, the – is not included, and 1, 0 is for unity or exactly vanishing, respectively.

We conclude that off-diagonal modeling of modified gravities can be performed as small perturbations of $\varLambda$CDM, when dark energy exists in the form of a cosmological constant. The off-diagonal redefinitions of generating functions result in a modification of $a(t)$ which is different from that in diagonal cosmological models. Nevertheless, even very small values of ε-off-diagonal deformations may introduce a certain speed of sound and clumping of DE (this is different from diagonal configurations in when such effects are zero and the effective Newtonian constant is just that for GR). Nevertheless, the models can be characterized by different physical values if the off-diagonal modeling is performed for different modified gravities, indeed:

Operator	$\varOmega$	$\overline{\mathbf{M}}_2^2$	${}^2m^2$
MGTs	$\mathbf{R}$	$(\delta\mathbf{K}^\alpha_{\ \alpha})^2$	$a^{-2}\mathbf{g}^{\underline{i}\underline{j}}(\mathbf{e}_{\underline{i}}\mathbf{g}^{44})(\mathbf{e}_{\underline{j}}\mathbf{g}^{44})$
Speed of sound	1	$\curlyvee + k^4$	$*$
Effective Newtonian constant	$\curlyvee$	$\curlyvee + k^2$	$\curlyvee$
Caldwell's parameter ${}^c\varpi$	$\curlyvee$	$\curlyvee$	–

[18] There are bounds of the type ${}^c\varpi = 0.02 \pm 0.07(2\sigma)$, for scales of about 10 Kpc, and $= 0.03 \pm 0.10(2\sigma)$, for hundreds of Kpc, but as of now there are not known limits at the Mpc scale yet.

Here additional labels have been used, for instance, $\curlyvee + k^4$ is for a new scale dependence, and $*$ is set for an operator behaving unusually.

Finally, we note that the effective field theory can be generalized in a form which allows to describe dark energy and modified gravities in the late universe when off-diagonal nonlinear parametric gravitational interactions encode contributions of such modified theories. The action for perturbations (77) of the effective models (76) includes positive small ε-deformations of the ΛCDM models with nonholonomically constrained fluctuations which contribute to the background cosmological evolution. A systematic investigation of the physical effects of N-adapted operators was undertaken which emphasizes the importance of off-diagonal effects when modeling modified gravities with the aim to constrain the possible nature of DE.

6 Short summary of new geometric ideas and methods for the construction of off-diagonal solutions

In the modern era of precision cosmology, a series of evidences have been found pointing towards deviations from the standard big bang cosmology. The universe may be slightly nonhomogeneous and anisotropic at very large scales and the accelerating expansion phases are determined by dark energy and dark matter effects. A number of alternative modifications of gravity have been proposed already, with the aim to elaborate realistic cosmological models. Some of the most intensively exploited are the so-called $f(R,\ldots)$-modified gravity theories [1–14]. The gravitational field equations in GR and the modified gravities consist of very involved systems of nonlinear PDE. A rigorous study of the cosmological evolution in the frames of the different theories requires new analytic, geometric and numerical methods for constructing exact and approximate solutions, reconstructing procedures and effective field models. In scenarios closely related to the standard ΛCDM universe, the ansatz for the alternative metric is usually taken of FLRW diagonal type and the interactions are modeled by effective matter, exotic fluids and modifications of GR. The advantage of the anholonomic frame deformation method, AFDM, [24–26] is that it provides a geometric formalism for constructing exact generic off-diagonal solutions encoding nonlinear parametric effects which mimic scenarios of anisotropic cosmology via generating and integration functions. Such nonlinear gravitational interactions, and the possible cosmological implications thereof, are not considered if we fix from the very beginning the diagonal ansatz, which results in a system of ordinary differential equations. For generic off-diagonal configurations, we can always derive an effective field theory and confront cosmological theories with existing observational data as we showed in sect. 5.

For the equivalent modeling of exact solutions and cosmological scenarios in different classes of modified and GR theories, we consider three different parametrization of the action for gravitational and matter fields (25), with respective Lagrange functionals

$$\begin{aligned}
\mathcal{L} &= {}^{g}\mathcal{L}[f(R,T,R_{\alpha\beta}T^{\alpha\beta})] + {}^{m}\mathcal{L}[g_{\alpha\beta},\nabla,\ {}^{m}\varphi] && (83)\\
&= {}^{g}\widehat{\mathbf{L}}[\widehat{\mathbf{f}}(\widehat{\mathbf{R}},\widehat{\mathbf{T}},\widehat{\mathbf{R}}_{\alpha\beta}\widehat{\mathbf{T}}^{\alpha\beta})] + {}^{m}\widehat{\mathbf{L}}[\widehat{\mathbf{g}}_{\alpha\beta},\widehat{\mathbf{D}},\ {}^{m}\widehat{\varphi}] && (84)\\
&= {}^{g}\check{\mathbf{L}} + {}^{m}\check{\mathbf{L}} = \check{\mathbf{R}} + \check{\Lambda}. && (85)
\end{aligned}$$

In these formulas, ${}^{m}\varphi$ are some (effective) matter fields, which can be approximated by the components of perfect (pressureless) fluids, for instance, with an energy-momentum tensor of the type (26), and the linear connections ∇, $\widehat{\mathbf{D}}$ and $\check{\mathbf{D}}$ are related via distortion relations of the type (19), which are completely determined by the metric field $g_{\alpha\beta} \simeq \widehat{\mathbf{g}}_{\alpha\beta} \simeq \check{\mathbf{g}}_{\alpha\beta}$ up to frame transformations. The functionals ${}^{g}\mathcal{L}[f(\ldots)]$ and ${}^{m}\mathcal{L}[\ldots]$ determine the corresponding model of f-modified theory of gravity. The Lagrange densities are written in different geometric variables because of this allows us to find exact solutions and model physical effects by means the same solutions but in different theories of gravity.

6.1 Decoupling of the generalized Einstein equations and modified cosmological solutions

We can decouple the gravitational and matter field equations (31) in a modified theory of gravity, derived from a Lagrangian density (83), and construct generic off-diagonal solutions depending (in general) on all spacetime variables, if we work with geometric data $(\widehat{\mathbf{g}},\widehat{\mathbf{D}},\widehat{\mathbf{N}})$. The generalized Einstein equations are written in the N-adapted form $\widehat{\mathbf{R}}_{\alpha\beta} = \widehat{\boldsymbol{\Upsilon}}_{\alpha\beta}$ with an effective source $\widehat{\boldsymbol{\Upsilon}}_{\alpha\beta} = \mathrm{diag}[{}^{h}\Upsilon,\ {}^{v}\Upsilon]$ determined by the matter fields and the nonholonomic and f-deformations. For a very general off-diagonal ansatz for metrics with one Killing, or non-Killing, symmetries, the gravitational field equations transform into a system of nonlinear PDEs (40)–(44), which can be solved in their general form for nonhomogeneous and locally anisotropic cosmological configurations (55). In the coordinate frame

$u^{\underline{\alpha}} = (x^k, y^3, y^4 = t)$ (where t is the time-like coordinate and $k = 1, 2$), such metrics $\mathrm{d}s^2 = g_{\underline{\alpha\beta}} \mathrm{d}u^{\underline{\alpha}} \mathrm{d}u^{\underline{\beta}}$ are of the type (5), or (29), and can be written in a form involving a generalized scaling factor $a^2(x^k, t)$,

$$g_{\underline{\alpha\beta}} = \begin{bmatrix} a^2\eta_1 + \omega^2[(n_1)^2 a^2 \widehat{h}_3 - (w_1)^2] & \omega^2[n_1 n_2 a^2 \widehat{h}_3 - w_1 w_2] & \omega^2 n_1 a^2 \widehat{h}_3 & \omega^2 w_1 \\ \omega^2[n_1 n_2 a^2 \widehat{h}_3 - w_1 w_2] & a^2\eta_2 + (n_2)^2 a^2 \widehat{h}_3 - (w_2)^2 & \omega^2 n_2 a^2 \widehat{h}_3 & \omega^2 w_2 \\ \omega^2 n_1 a^2 \widehat{h}_3 & \omega^2 n_2 a^2 \widehat{h}_3 & \omega^2 a^2 \widehat{h}_3 & 0 \\ \omega^2 w_1 & \omega^2 w_2 & 0 & -\omega^2 \end{bmatrix}. \tag{86}$$

By fixing the data for the generating functions: $\widehat{\Phi}(x^k, t)$, $\widehat{\Phi}^\diamond = \partial_t \widehat{\Phi} \neq 0$; $\omega^2(x^k, y^3, t)$ as a solution of $\partial_i \omega - (\partial_i \Phi / \Phi^\diamond)\omega^\diamond = 0$; $e^{\psi(x^i)} = a^2(x^k, t)\eta_1(x^k, t) = a^2(x^k, t)\eta_2(x^k, t)$ as a solution of $\psi^{\bullet\bullet} + \psi'' = 2\ {}^h\Upsilon$, for the nontrivial sources ${}^h\Upsilon(x^k)$, ${}^v\Upsilon(x^k, t)$ and for the effective cosmological constant $\Lambda \neq 0$, we can then express the coefficients of this metric in a certain general form. One further obtains (see the related formulas (50), (51), (52) and (53)) $\widehat{h}_3 = h_3/a^2|h_4|$, with

$$h_3(x^k, t) = \widehat{\Phi}^2/4\Lambda, \qquad h_4 = a^{-2}(x^k, t) = \frac{(\widehat{\Phi}^2)^\diamond}{8}\left[\widehat{\Phi}^2|\ {}^v\Upsilon| + \int \mathrm{d}t\ \widehat{\Phi}^2|\ {}^v\Upsilon|^\diamond\right]^{-1},$$
$$w_i(x^k, t) = \partial_i \Phi[\widehat{\Phi},\ {}^v\Upsilon]/\Phi^\diamond[\widehat{\Phi},\ {}^v\Upsilon], \qquad n_k = {}_1n_k + {}_2n_k \int \mathrm{d}t\ h_4/(\sqrt{|h_3|})^3,$$

where ${}_1n_k(x^i)$, ${}_2n_k(x^i)$ are integration functions. The solutions (86) include, in general, a nonholonomically induced torsion (24), which is important for constructing exact solutions in modified theories with nontrivial torsion fields.

We can always put additional zero-torsion constraints (39) and construct Levi-Civita configurations. Such solutions are extracted by choosing ${}_2n_k = 0$ and ${}_1n_k = \partial_k n$ with a function $n = n(x^k)$ and for a subclass of generating functions $\Phi = \check{\Phi}(x^k, t)$ for which $(\partial_i \check{\Phi})^\diamond = \partial_i \check{\Phi}^\diamond$, see the details for eq. (56). One finds $\check{w}_i = \partial_i \check{\Phi}/\check{\Phi}^\diamond = \partial_i \widetilde{A}$ for a nontrivial function $\widetilde{A}(x^k, y^4)$ taken to be a solution of a first order PDE with effective a sources depending functionally on $\check{\Phi}$. The v-components of geometric and physical object are generated by couples of data $(\Phi,\ {}^v\Upsilon)$ and $(\widehat{\Phi}, \Lambda)$, or related by formulas of the type (49), or (for zero torsion), of the type (61), for a fixed value of the effective cosmological constant $\Lambda = \check{\Lambda}$, with

$$\check{\Lambda}\check{\Phi}^2 = \widehat{\Phi}^2|\ {}^v\Upsilon| + \int \mathrm{d}t\ \widehat{\Phi}^2|\ {}^v\Upsilon|^\diamond. \tag{87}$$

For the data $(\check{\Phi}[\widehat{\Phi},\ {}^v\Upsilon], \check{\Lambda})$, a metric (86) is equivalent to the d-metric (59)[19].

The fact that the AFDM setting allows us to integrate in this general, off-diagonal (with Killing or non-Killing symmetries), both the gravitational field equations of GR and of modified theories, is certainly a very important result in mathematical relativity. This result has also fundamental physical implications for modern standard and modified gravity theories, particle physics and cosmology. The first one is that off-diagonal solutions of (generalized) Einstein equations, depending generically on three or four spacetime coordinates, can be generated by general classes of generating and integration functions. This reflects a specific property of nonlinear and nonholonomic off-diagonal gravitational interactions, where re-definitions of generating functions (for instance, of type (87)) of modified gravity theories with sources $\widehat{\mathcal{Y}}_{\alpha\beta}$ (see (32) and (33)), allow us to describe a large class of these modified gravity interactions as effective Einstein spaces. The effective equations $\check{\mathbf{R}}^\alpha{}_\beta = \check{\Lambda}\delta^\alpha{}_\beta$ are derived for the gravitational Lagrangian (85) with a correspondingly nonholonomically deformed linear connection $\check{\mathbf{D}}$. We have found that we can mimic certain classes of solutions of modified theories as off-diagonal configurations in ordinary GR, and inversely, but using different nonholonomic variables and deformed geometric/physical objects. Such constructions cannot be realized at all if we chose from the very beginning the ubiquitous diagonal ansatz for the metric and consider holonomic configurations. It is not clear yet how to prove the stability of the solution for f-modified theories, with the exception of some vary special cases. Nevertheless, for a very general class of such nonlinear nonholonomic systems redefined as $\widehat{\mathbf{f}}$-theories in terms of $\check{\mathbf{R}}^\alpha{}_\beta$ and $\check{\mathbf{R}}$, the stability can be proven indeed, as for the Einstein spacetime manifolds (see sect. 4.3).

Another important physical implication is that, for correspondingly fixed data, for generating and integration functions solutions of the type $\mathbf{g}_{\alpha\beta}(x^k, t)$ (86) determine new classes of nonhomogeneous cosmological metrics in GR as off-diagonal deformations of the FLRW cosmology, in the limit $\mathbf{g}_{\alpha\beta} \to \mathbf{g}_{\alpha\beta}(t, a(t), \widehat{h}_3(t), \check{\Phi}(t), \eta_i(t))$. We can model accelerating cosmology and dark energy and dark matter effects via nonlinear off-diagonal interactions and nonholonomic constraints induced by corresponding transforms of type (87) and modified re-scaling factor $a(t)$ as in

[19] We emphasize that such locally anisotropic cosmological solutions are generically off-diagonal (because, in general, the anholonomy coefficients W^b_{ia}, see (14), are nonzero). Some of the six independent coefficients of the metric depend on all spacetime coordinates. If we fix $\omega^2 = 1$, we generate solutions with Killing symmetry on ∂_3

d-metrics (55), or (59). The possibility to mimic modified gravities as analogous models in GR via nonlinear transformations of generating functions and sources (87) reflects a fundamental property of the class of off-diagonal solutions of the gravitational field equations. This property holds true for solutions with one Killing symmetry and/or non-Killing symmetries when metrics are generically off-diagonal and depend (in 4 dimensions) on three/four spacetime coordinates. It reflects a specific nonlinear dynamics of the gravitational and matter fields when off-diagonal interactions are taken into consideration for maximally possible six degrees of freedom of the metric and with certain classes of nonholonomic constraints imposed. Working only with diagonalizable, two Killing symmetries and stationary configurations, such nonlinear gravitational physics and cosmologycal effects cannot be encountered. A radically "orthodox" interpretation of this class of nonlinear and nonholonomic configurations and cosmological evolution scenarios is that they may explain the bulk of accelerating cosmology data and related dark energy and dark matter effects. It might be the case that, in order to understand the observed Universe it is not enough to modify GR in a simplistic way but, rather, to bring into consideration, within the standard GR theory, of a richer class of off-diagonal solutions and then take convenient limits leading to effective theories, in the end. This issue still need rigorous theoretical and observational consideration.

6.2 Small off-diagonal f-deformations and effective FLRW-like cosmologies

We can consider the subclass of generating functions, effective sources and cosmological constant $(\check{\Phi}[\widehat{\Phi},\ {}^{v}\Upsilon], \check{\Lambda})$ as in (87), where the cosmological evolution of spacetime regions is approximated by off-diagonal deformations with polarization functions $\eta_\alpha \simeq 1 + \varepsilon\chi_\alpha(x^k,t)$ and the N-coefficients $n_i(x^k)$ and $w_i(x^k,t)$ are proportional to a small parameter ε when $0 \leq \varepsilon \ll 1$. This is motivated by the fact that although possible anisotropic cosmological effects are very small, the modifications of the scale factor $\mathring{a}(t) \to a(x^k,t)$ (for a FLRW metric (27) with for $\mathring{g}_1 = \mathring{g}_2 = \mathring{g}_3 = \mathring{a}^2$, $\mathring{g}_4 = -1$) can be substantial for some intervals of time when the generating functions are of type (87). Prescribing any $a(x^k,t)$ and the solution $e^{\psi(x^k)}$ compatible with the observations, and fixing, for simplicity, $\omega^2 = 1$, as for the d-metric (59), when the polarization functions can be approximated as $1+\varepsilon\chi_i = a^{-2}e^{\psi}$, arbitrary $\chi_3(x^k,t)$ but $\chi_4 = 1$, and the function $\widehat{h}_3 = \eta_3/a^2$, we obtain, up to ε^2,

$$g_{\underline{\alpha}\underline{\beta}} = \begin{bmatrix} a^2(1+\varepsilon\chi_1) + \varepsilon^2[(n_1)^2a^2 - (w_1)^2] & \varepsilon^2[n_1n_2a^2 - w_1w_2] & \varepsilon n_1a^2 & \varepsilon w_1 \\ \varepsilon^2[n_1n_2a^2 - w_1w_2] & a^2(1+\varepsilon\chi_2) + \varepsilon^2[(n_2)^2a^2\widehat{h}_3 - (w_2)^2] & \varepsilon n_2a^2 & \varepsilon w_2 \\ \varepsilon n_1a^2 & \varepsilon n_2a^2 & a^2 & 0 \\ \varepsilon w_1 & \varepsilon w_2 & 0 & -1 \end{bmatrix}. \tag{88}$$

This class of locally anisotropic metrics is of type (86), with Killing symmetry on ∂_3 and small off-diagonal deformations on an anisotropy parameter ε which has to be fixed by experimental data. We can consider the limit $\varepsilon \to 0$ in (88) but even in such cases we have an anisotropic scaling factor $a(x^k,t)$, or $a(t)$, which is different from the standard $\mathring{a}(t)$ FLRW one. This is a consequence of the nonlinear off-diagonal and nonholonomic gravitational interactions, with generating functions and possible modified gravity sources related by transforms (87).

Having constructed a class of LC configurations (88), we can extract certain subclasses of cosmological evolution scenarios with generating and integration functions when $a(x^k,t) \to a(t) \neq \mathring{a}(t)$, $w_i \to w_i(t)$, $n_i \to$ const, etc. In this way we reproduce an effective FLRW like cosmology when the scaling factor $a(t)$ is defined not by exotic dark matter and dark energy interactions but by certain off-diagonal gravitational interactions which mimic contributions of the modified gravity type. In a more general context, metrics of type (88) may encode certain nonholonomic torsion configurations if the LC-constraints (39) are not imposed. Such solutions also contain a small parameter ε but a scaling factor $a(x^k,t) \to a(t) \neq \mathring{a}(t)$ is generated by data $(\Phi[\widehat{\Phi},\ {}^{v}\Upsilon], \Lambda)$ related by formula (49) instead of (87).

7 Concluding remarks and discussion

We have proven in this paper that a wide class of $f(R,\ldots)$ modified gravity theories can be encoded into effective off-diagonal Einstein spaces if nonholonomic deformations and constraints are considered for the nonlinear dynamics of gravity and matter fields. Special attention has been paid to a new version of modified gravity which includes strong coupling of the fields [24–26]. Such modified gravity theories have physical motivations from the covariant Hořava-Lifshitz like gravity models, with dynamical breaking of the Lorentz invariance [15–18], which provides also an example of a covariant, power-counting renormalizable theory and is represented by a simplest power-law $f(R,T,R_{\alpha\beta}T^{\alpha\beta})$ gravity.

We have demonstrated that the gravitational field equations in such modified gravity theories admit a decoupling property with respect to certain classes of nonholonomic frames, which allows us to generate exact solutions for very general off-diagonal forms. The corresponding integral varieties of solutions are parameterized by generating and integration functions and various classes of commutative and noncommutative symmetry parameters. For certain nonholonomic configurations, it is possible to redefine the generating functions and effective sources of matter fields in such a way that the $f(R)$-terms are equivalently encoded into effective Einstein spaces with complex parametric nonlinear structure for the gravitational vacuum. We argue that certain nonholonomic configurations model also covariant gravity theories with nice ultraviolet behaviors and seem to be (super-)renormalizable in the sense of Hořava-Lifshitz gravity [15–22, 44–47].

Notwithstanding the fact that the various $f(R)$ modified theories and general relativity are actually very different theories, the off-diagonal configurations and nonlinear parametric interactions considered in GR may encode various classes of such modified gravity effects and explain alternatively observational data for accelerating cosmology and certain effects in dark energy and dark matter physics. In both cases, it is possible to find cosmological solutions and reconstruct the corresponding action. In the already mentioned classes of modified gravity theories with f-modifications [1–18, 24–26], the dynamics of the matter sector is modeled by a perfect fluid. This is necessary to satisfy the continuity equation and guarantee an evolution which is similar to that in GR. For the alternative models with nonholonomic configurations [19–22, 27–31, 44–47], the behavior of modified gravity theories is determined by the off-diagonal terms and nonintegrable constraints. In general, we cannot distinguish the effects of f-modifications from the off-diagonal ones because nonholonomic frame transforms mix different classes of nonlinear interactions and parametric constraints. Nevertheless, for certain well-defined parameterizations, we can work with effective $a(\zeta)$ and $H(\zeta)$ which, with respect to N-adapted frames and for appropriate types of nonholonomic constraints, mimic ΛCDM cosmology when the gravitational background is generically off-diagonal and with a nontrivial gravitational vacuum structure (which may be a nonholonomically induced torsion) with an effective cosmological constant.

Off-diagonal cosmological solutions can be described by a realistic Hubble parameter but with an anomalous behavior for the barionic dark matter as it is shown in the first subsection of sect. 4. This kind of nonlinear parametric evolution allows to reproduce de Sitter like universes modeled on nonholonomic backgrounds of certain forms, encoding $f(R, T, R_{\alpha\beta}T^{\alpha\beta})$ gravity. For a corresponding class of generating functions, we can model nonholonomic deformations of the ΛCDM universe with a standard evolution for dust matter. It is possible to distinguish corrections with $f(T)$ and/or $f(R_{\alpha\beta}T^{\alpha\beta})$ terms. The priority of the anholonomic frame deformation method (AFDM) is that in such way we can generate analytical and exact formulas for the field and cosmological evolution equations, to formulate equivalent modeling criteria, etc.

Another priority of the AFDM is that we can study the issue of matter instability with various classes of modified gravity theories using geometric methods. The equations for the perturbations are complicated fourth-order differential equations involving linear perturbations of the Ricci scalar and tensor for different classes of linear connections. Nevertheless, we were able to consider specific nonholonomic transforms and constraints which allowed us to avoid matter instabilities. In this approach, some viable effective off-diagonal Einstein models and $f(R)$ gravities could be elaborated to encode the $f(R_{\alpha\beta}T^{\alpha\beta})$ contributions.

An effective field theory approach to off-diagonal cosmological configurations can be elaborated in terms of Stückelberg fields adapted to nonlinear connection structures, see sect. 5. The constructions for generic off-diagonal perturbations of MGTs (in terms of effective speed of sound, gravitational constant, Caldwell's parameter, etc.) are linked and confronted with actual observational data.

We note, finally, that modified gravity theories in general contain ghosts, due to the higher-derivative terms in the action. However, we can select certain ghost-free configurations determined by corresponding classes of nonholonomic deformations or constraints. Such models of bi-metric and massive graviton gravities were recently studied in [42–47]. Together with the results in [15–22], the conclusion is reached that some $f(R, T, R_{\alpha\beta}T^{\alpha\beta})$ models, and their off-diagonal nonholonomic equivalents, may possess nice ultraviolet properties and that interesting connections can be established with viable theories of quantum gravity.

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