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# A general analytical solution for the variance-to-mean Feynman-alpha formulas for a two-group two-point, a two-group one-point and a one-group two-point cases

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**Abstract.** This paper presents a full derivation of the variance-to-mean or Feynman-alpha formula in a two-energy-group and two-spatial-region treatment. The derivation is based on the Chapman-Kolmogorov equation with the inclusion of all possible neutron reactions and passage intensities between the two regions. In addition, the two-group one-region and the two-region one-group Feynman-alpha formulas, treated earlier in the literature for special cases, are extended for further types and positions of detectors. We focus on the possibility of using these theories for accelerator-driven systems and applications in the safeguards domain, such as the differential self-interrogation method and the differential die-away method. This is due to the fact that the predictions from the models which are currently used do not fully describe all the effects in the heavily reflected fast or thermal systems. Therefore, in conclusion, a comparative study of the two-group two-region, the two-group one-region, the one-group two-region and the one-group one-region Feynman-alpha models is discussed.

## 1 Introduction

In detection statistics, the relation between the average number $\langle N\rangle$ of counts during a detection time $t$, and the fluctuations around this value, expressed by the variance $\langle N^2\rangle - \langle N\rangle^2$, *i.e.* the variance-to-mean ratio,

$$Q^2 \sim \frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle},$$

is often used to characterise the statistics of the particle field detected. In the case of neutrons emitted from a radioactive source obeying a simple Poisson statistics, this ratio is obviously equal to unity. However, for a neutron chain in a multiplying medium[1], such as a subcritical reactor with a source or a fissile sample with an inherent neutron source due to spontaneous fission, the branching character represented by the fission process has the consequence that the individual detections will not be independent, rather positive correlations exist between them. Hence the variance-to-mean ratio is larger than unity, and the deviation from unity carries information on the medium in which the branching process (neutron multiplication) took place.

This fact was used by Feynman and de Hoffmann in 1944–1956 [1–3] for the derivation of a formula for a branching process where the variance to mean was above unity, $Q^2 = 1 + Y(t)$. The $Y(t)$-function became called the Feynman $Y$-function, characterising the deviation of the relative variance from unity. Both its time dependence ($e^{-\alpha t}$), expressed by the prompt neutron[2] decay constant $\alpha$, as well as its asymptotic value, carry information on the sought parameters of the system. The original application of these studies was related to the theoretical description of statistical fluctuations of the number of neutrons in multiplying medium or in other words, to the determination of the level of subcriticality. Therefore, the above-mentioned research remained classified for several years.

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[1] A medium, where particles (*e.g.* neutrons) induce reactions which lead to the emission of several particles of the same type.
[2] Neutrons which are emitted in less than $10^{-14}$ s following a nuclear fission event.

The fundamental principles of the Feynman-alpha theory have been extensively described in a number of publications, *e.g.* [4]. About a decade ago the interest in this subject was revived in connection to the on-line measurement of subcritical reactivity (a quantity which is normally used to characterize the multiplication properties of a reactor core) of Accelerator-Driven Systems (ADS)[3]. Whereas the original Feynman-alpha formulas referred to a homogeneous system in a monoenergetic ("one-group", *i.e.* one energy group approximation) description [3], dealing only with one exponent or decay constant, the experiments, *e.g.* the Yalina [5–7], MUSE [8] and FREYA [9,10], showed the appearance of more than one decay constant and, therefore, the possible need of extension of the one-group one-region (also referred to as "one-point") Feynman-alpha formulas to more energy groups and/or spatial regions. Several attempts were made towards the explanation of multiple exponential modes by the spatial effects [11–14]. By that time it was decided that the future ADS-systems will be driven by pulsed neutron and spallation sources[4] which led to the extension of the theory of variance-to-mean formulas for a continuous source with Poisson statistics to the cases of pulsed and spallation neutron sources with different definition of the pulse shapes and pulsing manner [15–28]. Latter analysis [29] showed the close link between the application of Feynman-alpha formulas to subcriticality measurements and Safeguards (nuclear material control and accounting).

In line with the above, the suggestion of the new Safeguards technique for mixed oxide (MOX) fuel/spent fuel assay [30], the Differential Die-away Self-Interrogation (DDSI) technique displayed the interest towards the energy-dependent aspects of neutron counting. In connection with this, the two-group Feynman-alpha theory was elaborated in [31], where delayed neutrons[5] were neglected, and in [32] with inclusion of delayed neutron precursors. However, fast fission and thermal detections were neglected in both papers. The results of further considerations of the importance of the energy-aspect in evaluation of the real systems shows that "a measured variance-to-mean ratio in fast systems may be contaminated by the energy-higher order mode effect except when the system is near-critical [33]".

In the light of recent advances in detector technologies in Safeguards towards the development of fast neutron detection systems with scintillators, the knowledge of the energy-dependent behavior of neutron counting became a very important issue to be taken into account in Feynman-alpha theory. The authors of [34] showed that the short- and long-time behavior of the $Y$-function can be used to assay the amount of $^{240}$Pu and the absolute amount of $^{239}$Pu + $^{241}$Pu in the reprocessed fuel. Therefore, one part of this paper is devoted to the derivation of the general case of one-point two-group Feynman-alpha formulas, when fast fission and thermal detections and delayed neutrons are included. However in some cases, for example, when the fission chambers are used as detectors, the energy importance makes way for the region-dependent aspect. This issue has not well been studied previously, although some expressions for the one-group two-region Feynman-alpha formulas can be found in [35]. However, even these investigations are limited to the case of delayed neutron precursors having been neglected and detections accounted for only in one region. Thus, the second part of this paper is devoted to the derivation of the general case of the two-point one-group Feynman-alpha formulas, when detections and delayed neutrons are accounted for in both regions.

It has to be noted that the present paper does not carry out fully an analysis of the diagnostic value of the obtained formulas the same way as it was made in the traditional works based on a one-group treatment in a single (infinite) homogeneous medium. In the traditional case the time dependence of the Feynman $Y(t)$-function is characterised essentially with one decay constant which can clearly be related to the subcriticality of the system. In the case of using two energy groups and two spatial regions, the number of decay constants increases and each of them becomes a much more involved function of the increased number of material properties (reaction intensities) that the treatment of different regions and energy intervals incurs. The sought system parameters become very involved functions of these decay constants, and no attempt is made in this paper on the investigation of how these parameters can be extracted from the measurements. This is deferred to later work. The objective of the present work is to give a clear and transparent derivation of the various variance-to-mean formulas as functions of the reaction and transition intensities, and to compare the solutions for the different cases.

## 2 The main concept and assumptions

In this paper, the two-point two-group, the two-group one-point (with delayed neutrons) and the one-group two-point (with delayed neutrons) Feynman-alpha formulas were derived by using the Kolmogorov approach. The Kolmogorov equations, originally developed by the Russian mathematician Andrei Kolmogorov [36], also often called "master equations", describe the evolution of the probability distribution of Markovian processes. These can be given in two different forms, called the forward and the backward approaches. In the present work we use the forward Kolmogorov approach [4], mostly because the symbolic computational codes (in our case the Mathematica package [37]) are better suited to solve the coupled system of differential equations (arising from the forward approach) than to calculate the multiple nested integrals which arise from the backward approach.

[3] A subcritical reactor driven by an external source.

[4] Usually meant as a thick target made of a high-$Z$ material which is bombarded by accelerated charged particles, *e.g.* high-energy protons.

[5] Neutrons which are emitted with an exponential decay law after the fission, with mean delay times between 0.1 and 10 s.

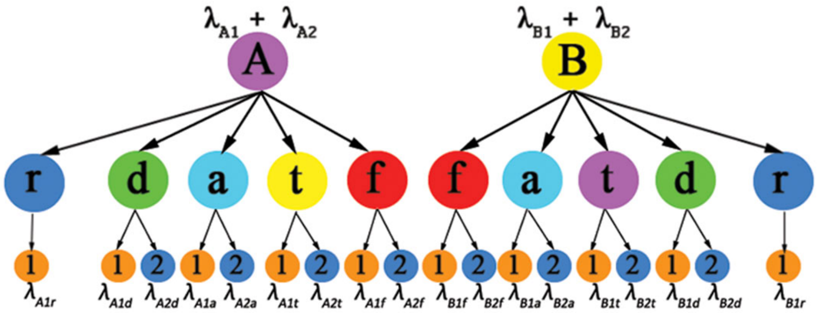


**Fig. 1.** A two-point two-group model of various processes which particles can undergo.

In the general model used for derivations we assume that the neutron population consists of two groups of neutrons: fast (denoted as 1) and thermal (denoted as 2). Fast and thermal neutrons can undergo different reactions ($i$) listed below:

- absorption ($i = a$);
- fission ($i = f$);
- detection ($i = d$);
- removal from the fast group to the thermal ($i = r$).

Unlike in the terminology, used in the traditional one-group treatments, absorbtion here stands only for capture. The decay constant of the delayed neutron precursors is given as $\lambda$. In addition, both the fast and thermal neutrons can transit from one region to the others, in both directions. In all models the source is considered as releasing $n$ particles with probability $p_q(n)$ at an emission event. In this paper a term "*two-point*" has the same meaning as "*two-region*".

## 2.1 The two-group two-point model

For the two-group two-point model it was assumed that two adjacent infinite and homogeneous half-space regions (denoted as A and B) with different independent reaction intensities for absorption of fast and thermal neutrons ($\lambda_{A1a}$, $\lambda_{A2a}$, $\lambda_{B1a}$, $\lambda_{B2a}$), fission induced by fast and thermal neutrons ($\lambda_{A1f}$, $\lambda_{A2f}$, $\lambda_{B1f}$, $\lambda_{B2f}$) and detection of fast and thermal neutrons ($\lambda_{A1d}$, $\lambda_{B1d}$, $\lambda_{A2d}$, $\lambda_{B2d}$). The two regions are coupled by two passage intensities ($\lambda_{A1t}$, $\lambda_{A2t}$, $\lambda_{B1t}$, $\lambda_{B2t}$) in two different directions[6]. Thus, each of the reactions for the different groups of neutrons can be described by transition intensities, as shown in fig. 1. Total intensities including both the reactions and transitions between the regions for the fast and the thermal neutrons are denoted as $\lambda_{A1}$ and $\lambda_{A2}$, $\lambda_{B1}$ and $\lambda_{B2}$ for regions A and B, respectively,

$$\begin{aligned}
\lambda_{A1} &= \lambda_{A1a} + \lambda_{A1f} + \lambda_{A1t} + \lambda_{A1r} + \lambda_{A1d}\\
\lambda_{A2} &= \lambda_{A2a} + \lambda_{A2f} + \lambda_{A2t} + \lambda_{A2d}\\
\lambda_{B1} &= \lambda_{B1a} + \lambda_{B1f} + \lambda_{B1t} + \lambda_{B1r} + \lambda_{B1d}\\
\lambda_{B2} &= \lambda_{B2a} + \lambda_{B2f} + \lambda_{B2t} + \lambda_{B2d}.
\end{aligned}$$

The slowing down process, *i.e.* the removal of neutrons from the fast group to the thermal group is described by the removal reaction intensity $\lambda_{i=r=R}$. In the two-point two-group model we also include two extraneous compound Poisson sources of fast neutrons placed in different regions, A and/or B, with intensities $S_A$ and $S_B$. In the following, two special cases of the above general form will be described briefly. Because in the lower dimensionality of the special cases, inclusion of delayed neutrons is possible.

## 2.2 The two-group one-point model (with delayed neutrons)

In the two-group one-point Feynman-alpha model (fig. 2), we assume that the medium is infinite and homogeneous. The neutron population consists of two groups of neutrons, fast and thermal. A compound Poisson source of fast neutrons with emission intensity $S_1$ is included in the model. Thus, the total transition intensities for the fast and thermal neutrons, denoted as $\lambda_1$ and $\lambda_2$, are given as

$$\begin{aligned}
\lambda_1 &= \lambda_{1a} + \lambda_{1f} + \lambda_R + \lambda_{1d}\\
\lambda_2 &= \lambda_{2a} + \lambda_{2f} + \lambda_{2d}.
\end{aligned}$$

[6] $\lambda_{Ait}$ describes the intensity of particles (group $i$) leaving region A for region B and $\lambda_{Bit}$ is the intensity of particles (group $i$) transferring to region A from region B.

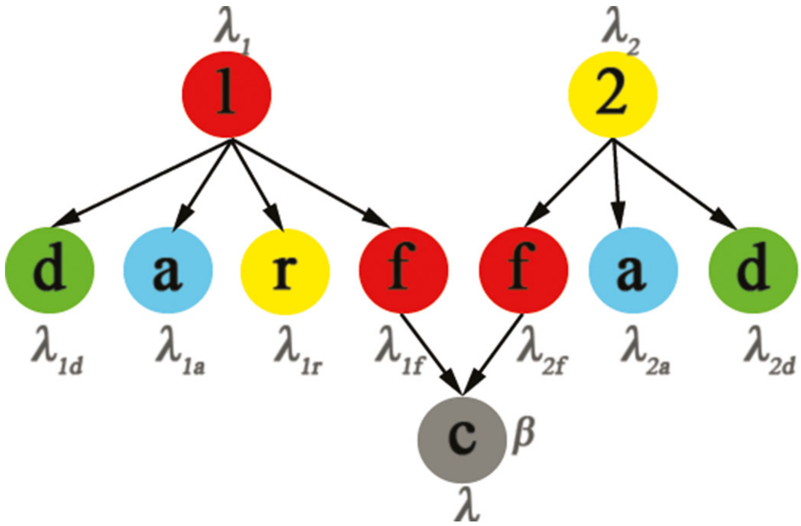


**Fig. 2.** A two-group one-point model of various processes which particles can undergo.

### 2.3 The one-group two-point model (with delayed neutrons)

The assumption behind the one-group two-point model is that the two adjacent homogeneous half-space regions (denoted as A and B) with independent reaction intensities for detection ($\lambda_{Ad}$, $\lambda_{Bd}$), absorption $\lambda_{Aa}$ and $\lambda_{Ba}$, and fission $\lambda_{Af}$ and $\lambda_{Bf}$ are coupled by two passage intensities $\lambda_{At}$ and $\lambda_{Bt}$ in two different directions. The decay constants of delayed neutron precursors are given as $\lambda_{Ac}$ and $\lambda_{Bc}$ for regions A and B, as shown in fig. 3. Thus, total transition intensities for region A and region B are denoted as $\lambda_A$ and $\lambda_B$:

$$\lambda_A = \lambda_{Aa} + \lambda_{Af} + \lambda_{At} + \lambda_{Ad}$$
$$\lambda_B = \lambda_{Ba} + \lambda_{Bf} + \lambda_{Bt} + \lambda_{Bd}.$$

In the model we include two compound Poisson sources of fast neutrons in regions A and B with emission intensities $S_A$ and $S_B$, respectively. The sources are considered as releasing $n$ particles in one emission with the probability distributions of $p_A(n)$ and $p_B(n)$, respectively. For the induced fission reaction, we consider that $k$ neutrons and $l$ delayed neutron precursors are emitted with the probability distributions $f_A(k,l)$ and $f_B(k,l)$ for the fission reaction in region A and region B, respectively.

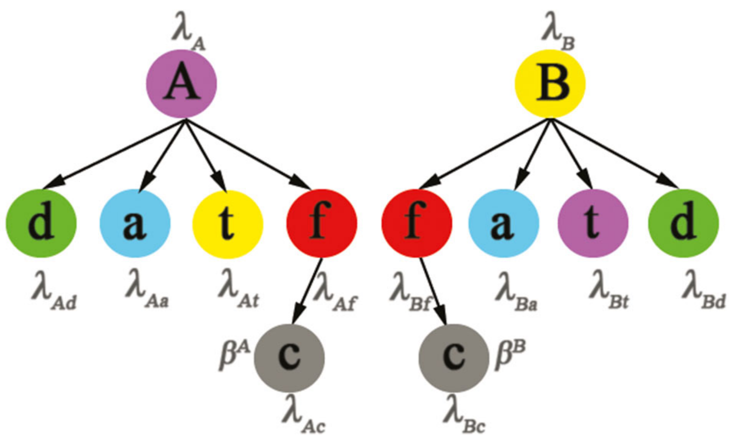


**Fig. 3.** A two-point one-group model of various processes which particles can undergo.

## 3 Theoretical formulas

Based on the main concept and assumptions used the two-group two-region, the two-group one-region and the two-region one-group Feynman-alpha formulas are elaborated as below.

### 3.1 Two-point two-group Feynman-alpha theory

In order to derive the two-point two-group Feynman-alpha theory let us assume that the source $S_A/S_B$ is switched on in the region A/B at the time $t_0 \leq t$, while the detection process is started at the fixed time instant $t_d$, where $t_d \leq t$ and $t_d \geq t_0$. Let the random processes $N_{A1}(t)$, $N_{B1}(t)$, $N_{A2}(t)$ and $N_{B2}(t)$ represent the number of fast neutrons in region A, fast neutrons in region B, thermal neutrons in region A and thermal neutrons in region B at the time $t \geq 0$ and $Z_{A1}(t,t_d)$, $Z_{A2}(t,t_d)$, $Z_{B1}(t,t_d)$, $Z_{B2}(t,t_d)$ —the number of fast and thermal particle detections in the regions A and B in the time interval $[t_d,t]$, respectively. For convenience, we consider $t_d = 0$. Thus, the joint probability of having $N_{A1}$ fast neutrons in region A, $N_{B1}$ fast neutrons in region B, $N_{A2}$ thermal neutrons in region A, $N_{B2}$ thermal neutrons in region B at time $t$, $Z_{A1}$ fast neutrons have been detected in region A, and $Z_{B1}$ fast neutrons have been detected in region B, $Z_{A2}$ thermal neutrons have been detected in region A, and $Z_{B2}$ thermal neutrons have been detected in region B during the period of time $t - t_d \geq 0$ can be defined as

$$P(N_{A1}, N_{A2}, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t|t_0).$$

By summing up the probabilities of all mutually exclusive events of the particle not having or having a specific reaction within the infinitesimally small time interval d$t$, one can write:

$$\begin{aligned}
&\frac{\partial P(N_{A1}, N_{A2}, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t)}{\partial t} = \\
&-(\lambda_{A1}N_{A1} + \lambda_{A2}N_{A2} + \lambda_{B1}N_{B1} + \lambda_{B2}N_{B2} + S_A + S_B)P(N_{A1}, N_{A2}, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{A1a}(N_{A1}+1)P(N_{A1}+1, N_{A2}, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{A2a}(N_{A2}+1)P(N_{A1}, N_{A2}+1, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{B1a}(N_{B1}+1)P(N_{A1}, N_{A2}, N_{B1}+1, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{B2a}(N_{B2}+1)P(N_{A1}, N_{A2}, N_{B1}, N_{B2}+1, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{A1f}\sum_{k}^{N_{A1}+1}(N_{A1}+1-k)f_{A1}(k)P(N_{A1}+1-k, N_{A2}, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{B1f}\sum_{k}^{N_{B1}+1}(N_{B1}+1-k)f_{B1}(k)P(N_{A1}, N_{A2}, N_{B1}+1-k, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{A2f}\sum_{k}^{N_{A1}}(N_{A2}+1)f_{A2}(k)P(N_{A1}-k, N_{A2}+1, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{B2f}\sum_{k}^{N_{B1}}(N_{B2}+1)f_{B2}(k)P(N_{A1}, N_{A2}, N_{B1}-k, N_{B2}+1, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{A1t}(N_{A1}+1)P(N_{A1}+1, N_{A2}, N_{B1}-1, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{B1t}(N_{B1}+1)P(N_{A1}-1, N_{A2}, N_{B1}+1, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{A2t}(N_{A2}+1)P(N_{A1}, N_{A2}+1, N_{B1}, N_{B2}-1, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{B2t}(N_{B2}+1)P(N_{A1}, N_{A2}-1, N_{B1}, N_{B2}+1, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{Ar}(N_{A1}+1)P(N_{A1}+1, N_{A2}-1, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{Br}(N_{B1}+1)P(N_{A1}, N_{A2}, N_{B1}+1, N_{B2}-1, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{A1d}(N_{A1}+1)P(N_{A1}+1, N_{A2}, N_{B1}, N_{B2}, Z_{A1}-1, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{B1d}(N_{B1}+1)P(N_{A1}, N_{A2}, N_{B1}+1, N_{B2}, Z_{A1}, Z_{B1}-1, Z_{A2}, Z_{B2}, t) \\
&+\lambda_{A2d}(N_{A2}+1)P(N_{A1}, N_{A2}+1, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}-1, Z_{B2}, t) \\
&+\lambda_{B2d}(N_{B2}+1)P(N_{A1}, N_{A2}, N_{B1}, N_{B2}+1, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}-1, t) \\
&+S_A\sum_{n}^{N_{A1}}p_A(n)P(N_{A1}-n, N_{A2}, N_{B1}, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t) \\
&+S_B\sum_{n}^{N_{B1}}p_B(n)P(N_{A1}, N_{A2}, N_{B1}-n, N_{B2}, Z_{A1}, Z_{B1}, Z_{A2}, Z_{B2}, t),
\end{aligned}$$

with initial conditions

$$P(N_{A1},N_{A2},N_{B1},N_{B2},Z_{A1},Z_{B1},Z_{A2},Z_{B2},t=t_0 \mid t_0)=\delta_{N_{A1},0}\delta_{N_{A2},0}\delta_{N_{B1},0}\delta_{N_{B2},0}\delta_{Z_{A1},0}\delta_{Z_{A2},0}\delta_{Z_{B1},0}\delta_{Z_{B2},0}$$

and

$$\sum_{N_{A1}}\sum_{N_{A2}}\sum_{N_{B1}}\sum_{N_{B2}}P(N_{A1},N_{A2},N_{B1},N_{B2},Z_{A1},Z_{B1},Z_{A2},Z_{B2},t=t_d \mid t_0)=\delta_{Z_{A1},0}\delta_{Z_{A2},0}\delta_{Z_{B1},0}\delta_{Z_{B2},0}$$

and $f_i(k)$ is the number distribution of neutrons in a fission of type $i$.

This equation can be solved by using the generating function technique in the way similar to as described in [4]. By defining the following generating function for the probability distribution $P(N_{A1},N_{A2},N_{B1},N_{B2},Z_{A1},Z_{B1},Z_{A2},Z_{B2},t)$:

$$\begin{aligned}G(X_A,Y_A,X_B,Y_B,M,N,O,P,t)=&\sum_{N_{A1}}\sum_{N_{A2}}\sum_{N_{B1}}\sum_{N_{B2}}\sum_{Z_{A1}}\sum_{Z_{A2}}\sum_{Z_{B1}}\sum_{Z_{B2}}X_A^{N_{A1}}Y_A^{N_{A2}}X_B^{N_{B1}}Y_B^{N_{B2}}M^{Z_{A1}}N^{Z_{A2}}O^{Z_{B1}}P^{Z_{B2}}\\&*P(N_{A1},N_{A2},N_{B1},N_{B2},Z_{A1},Z_{B1},Z_{A2},Z_{B2},t),\end{aligned}$$

with initial condition for $t_0 \le 0$

$$G(X_A,Y_A,X_B,Y_B,M,N,O,P,t=t_0 \mid t_0)=1$$

and

$$G(1,1,1,1,M,N,O,P,t=t_d \mid t_0)=1.$$

The following partial differential equation is obtained:

$$\begin{aligned}\frac{\partial G}{\partial t}=&\left[\lambda_{A1a}+\lambda_{Ar}Y_A+\lambda_{A1t}X_B+\lambda_{A1d}M-\lambda_{A1}X_A+q_{A1}(X_A)\lambda_{A1f}\right]\frac{\partial G}{\partial X_A}\\&+\left[\lambda_{A2a}+\lambda_{A2t}Y_B+\lambda_{A2d}N-\lambda_{A2}Y_A+q_{A2}(X_A)\lambda_{A2f}\right]\frac{\partial G}{\partial Y_A}\\&+\left[\lambda_{B1a}+\lambda_{Br}Y_B+\lambda_{B1t}X_A+\lambda_{B1d}O-\lambda_{B1}X_B+q_{B1}(X_B)\lambda_{B1f}\right]\frac{\partial G}{\partial X_B}\\&+\left[\lambda_{B2a}+\lambda_{B2t}Y_A+\lambda_{B2d}P-\lambda_{B2}Y_B+q_{B2}(X_B)\lambda_{B2f}\right]\frac{\partial G}{\partial Y_B}\\&+S_A[r_A(X_A)-1]G+S_B[r_B(X_B)-1]G,\end{aligned}$$

where

$$\begin{aligned}q_i(X)&=\sum_k X^k f_{if}(k)\\r(X)&=\sum_n p_q(n)X^n.\end{aligned}$$

For the sake of simplicity, some identities are used in the solution as below ($i=1,2$):

$$\begin{aligned}\left.\frac{\partial}{\partial X}r(X)\right|_{X=1}&=\sum_n np_q(n)\\&=r'\\\left.\frac{\partial^2}{\partial X^2}r(X)\right|_{X=1}&=\sum_n n(n-1)p_q(n)\\&=r''.\end{aligned}$$

Thus, $\nu_{Ai}{}'$ $(q_{Ai}{}')$, $\nu_{Bi}{}'$ $(q_{Bi}{}')$, $\nu_{Ai}{}''$ $(q_{Ai}{}'')$, $\nu_{Bi}{}''$ $(q_{Bi}{}'')$ and $r_A{}'$, $r_B{}'$, $r_A{}''$, $r_B{}''$ stand for the first and second factorial moments[7] of the number of neutrons emitted in a fission process and in a source event, respectively. The index $i=1,2$ denotes fission induced by fast or thermal neutrons, respectively. In a steady subcritical medium with a steady source,

[7] If $f$ is a random variable or a random process, then the expectation $\langle f^n\rangle$ is called the $n$-th ordinary moment of the process, whereas $\langle f[f-1]\dots[f-n+1]\rangle$ is called its $n$-th factorial moment.

when $t_0 \to -\infty$, the following stationary solutions for the neutron populations $\bar{N}_{A1}$, $\bar{N}_{A2}$, $\bar{N}_{B1}$, $\bar{N}_{B2}$, $\bar{Z}_{A1}$, $\bar{Z}_{B1}$, $\bar{Z}_{A2}$, $\bar{Z}_{B2}$ are obtained as below:

$$\begin{aligned}
\bar{N}_{A1} &= \frac{1}{\omega_1\omega_2\omega_3\omega_4}\Big(S_B r'_B\left(\lambda_{A2}\lambda_{B1t}\lambda_{B2} + \lambda_{B2t}\left(-\lambda_{A2t}\lambda_{B1t} + \lambda_{A2f}\lambda_{Br}\nu'_{A2}\right)\right)\\
&\quad + S_A r'_A\Big(\lambda_{A2t}\lambda_{B2t}\left(-\lambda_{B1} + \lambda_{B1f}\nu_{B1}{}'\right) + \lambda_{A2}\left(\lambda_{B1}\lambda_{B2} - \lambda_{B1f}\lambda_{B2}\nu_{B1}{}' - \lambda_{B2f}\lambda_{Br}\nu'_{B2}\right)\Big)\Big)\\
\bar{N}_{B1} &= \frac{1}{\omega_1\omega_2\omega_3\omega_4}\Big(S_A r'_A\left(\lambda_{A1t}\left(\lambda_{A2}\lambda_{B2} - \lambda_{A2t}\lambda_{B2t}\right) + \lambda_{A2t}\lambda_{Ar}\lambda_{B2f}\nu'_{B2}\right)\\
&\quad + S_B r'_B\Big(\lambda_{A1}\left(\lambda_{A2}\lambda_{B2} - \lambda_{A2t}\lambda_{B2t}\right) + \lambda_{A1f}\left(-\lambda_{A2}\lambda_{B2} + \lambda_{A2t}\lambda_{B2t}\right)\nu_{A1}{}' - \lambda_{A2f}\lambda_{Ar}\lambda_{B2}\nu'_{A2}\Big)\Big)\\
\bar{N}_{A2} &= \frac{1}{\omega_1\omega_2\omega_3\omega_4}\Big(S_A r'_A\left(\lambda_{A1t}\lambda_{B2t}\lambda_{Br} + \lambda_{Ar}\left(\lambda_{B1}\lambda_{B2} - \lambda_{B1f}\lambda_{B2}\nu_{B1}{}' - \lambda_{B2f}\lambda_{Br}\nu'_{B2}\right)\right)\\
&\quad + S_B r'_B\Big(\lambda_{Ar}\lambda_{B1t}\lambda_{B2} + \lambda_{B2t}\lambda_{Br}\left(\lambda_{A1} - \lambda_{A1f}\nu_{A1}{}'\right)\Big)\Big)\\
\bar{N}_{B2} &= \frac{1}{\omega_1\omega_2\omega_3\omega_4}\Big(S_A r'_A\left(\lambda_{A1t}\lambda_{A2}\lambda_{Br} + \lambda_{A2t}\lambda_{Ar}\left(\lambda_{B1} - \lambda_{B1f}\nu_{B1}{}'\right)\right)\\
&\quad + S_B r'_B\Big(\lambda_{A2t}\lambda_{Ar}\lambda_{B1t} + \lambda_{Br}\left(\lambda_{A1}\lambda_{A2} - \lambda_{A1f}\lambda_{A2}\nu_{A1}{}' - \lambda_{A2f}\lambda_{Ar}\nu'_{A2}\right)\Big)\Big)\\
\bar{Z}_{A1} &= \lambda_{A1d}\bar{N}_{A1}t\\
\bar{Z}_{B1} &= \lambda_{B1d}\bar{N}_{B1}t\\
\bar{Z}_{A2} &= \lambda_{A2d}\bar{N}_{A2}t\\
\bar{Z}_{B2} &= \lambda_{B2d}\bar{N}_{B2}t.
\end{aligned}$$

By introducing the modified second factorial moment of the random variables $a$ and $b$ as follows:

$$\begin{aligned}
\mu_{aa} &\equiv \langle a(a-1)\rangle - \langle a\rangle^2 = \sigma_a^2 - \langle a\rangle\\
\mu_{ab} &\equiv \langle ab\rangle - \langle a\rangle\langle b\rangle,
\end{aligned}$$

and then taking cross- and auto-derivatives, the following system of differential equations of modified second factorial moments ($\mu_{X_AX_A}$, $\mu_{X_BX_B}$, $\mu_{X_AY_A}$, $\mu_{X_AY_B}$, $\mu_{X_AX_B}$, $\mu_{X_BY_A}$, $\mu_{X_BY_B}$) for the neutron population are obtained as below:

$$\begin{aligned}
\frac{\partial}{\partial t}\mu_{X_AX_A} &= 2\lambda_{B1t}\mu_{X_AX_B} + 2\lambda_{A2f}\nu_{A2}{}'\mu_{X_AY_A} + 2(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1})\mu_{X_AX_A} + S_A r''_A + \lambda_{A2f}\nu_{A2}{}''\bar{N}_{A2} + \lambda_{A1f}\nu_{A1}{}''\bar{N}_{A1}\\
\frac{\partial}{\partial t}\mu_{X_AY_A} &= \lambda_{B1t}\mu_{Y_AX_B} + \lambda_{A2f}\nu_{A2}{}'\mu_{Y_AY_A} + \lambda_{B2t}\mu_{X_AY_B} + \left(\lambda_{A1f}\nu'_{A1} - \lambda_{A1} - \lambda_{A2}\right)\mu_{X_AY_A} + \lambda_{Ar}\mu_{X_AX_A}\\
\frac{\partial}{\partial t}\mu_{X_AX_B} &= \lambda_{B1t}\mu_{X_BX_B} + \lambda_{A2f}\nu_{A2}{}'\mu_{Y_AX_B} + \lambda_{B2f}\nu_{B2}{}'\mu_{X_AY_B} + \left(\lambda_{A1f}\nu'_{A1} - \lambda_{A1}\right)\mu_{X_AX_B}\\
&\quad + \left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right)\mu_{X_AX_B} + \lambda_{A1t}\mu_{X_AX_A}\\
\frac{\partial}{\partial t}\mu_{X_AY_B} &= \lambda_{B1t}\mu_{X_BY_B} + \lambda_{A2f}\nu_{A2}{}'\mu_{Y_AY_B} + \left(\lambda_{A1f}\nu'_{A1} - \lambda_{A1} - \lambda_{B2}\right)\mu_{X_AY_B} + \lambda_{Br}\mu_{X_AX_B} + \lambda_{A2t}\mu_{X_AY_A}\\
\frac{\partial}{\partial t}\mu_{X_BX_B} &= 2\lambda_{B2f}\nu_{B2}{}'\mu_{X_BY_B} + 2(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1})\mu_{X_BX_B} + 2\lambda_{A1t}\mu_{X_AX_B} + S_B r''_B + \lambda_{B2f}\nu_{B2}{}''\bar{N}_{B2} + \lambda_{B1f}\nu_{B1}{}''\bar{N}_{B1}\\
\frac{\partial}{\partial t}\mu_{Y_AX_B} &= \lambda_{B2t}\mu_{X_BY_B} + \lambda_{B2f}\nu_{B2}{}'\mu_{Y_AY_B} + \left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1} - \lambda_{A2}\right)\mu_{Y_AX_B} + \lambda_{Ar}\mu_{X_AX_B} + \lambda_{A1t}\mu_{X_AY_A}\\
\frac{\partial}{\partial t}\mu_{X_BY_B} &= \lambda_{B2f}\nu_{B2}{}'\mu_{Y_BY_B} + \left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1} - \lambda_{B2}\right)\mu_{X_BY_B} + \lambda_{Br}\mu_{X_BX_B} + \lambda_{A2t}\mu_{Y_AX_B} + \lambda_{A1t}\mu_{X_AY_B}\\
\frac{\partial}{\partial t}\mu_{Y_AY_A} &= 2\lambda_{B2t}\mu_{Y_AY_B} - 2\lambda_{A2}\mu_{Y_AY_A} + 2\lambda_{Ar}\mu_{X_AY_A}\\
\frac{\partial}{\partial t}\mu_{Y_AY_B} &= \lambda_{B2t}\mu_{Y_BY_B} - \lambda_{A2}\mu_{Y_AY_B} - \lambda_{B2}\mu_{Y_AY_B} + \lambda_{Br}\mu_{Y_AX_B} + \lambda_{A2t}\mu_{Y_AY_A} + \lambda_{Ar}\mu_{X_AY_B}\\
\frac{\partial}{\partial t}\mu_{Y_BY_B} &= -2\lambda_{B2}\mu_{Y_BY_B} + 2\lambda_{Br}\mu_{X_BY_B} + 2\lambda_{A2t}\mu_{Y_AY_B}.
\end{aligned}$$

This system can be solved in the stationary state (when the left-hand sides are equal to 0). The final expression of two-point two-group Feynman-alpha formulas for fast detections is given as below:

$$\frac{\sigma_{ZZ}^2(t)}{\bar{Z}_{A1/A2/B1/B2}} = 1 + Y(t) = 1 + \sum_{i=1}^{4} Y_i \left(1 - \frac{1 - e^{-\omega_i t}}{\omega_i t}\right).$$

The four roots, namely $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$ can be obtained by solving the forth order characteristic equation in $\omega$ with known coefficients $a$, $b$, $c$, $d$, obtained from the temporal Laplace transform of the time-dependent equations for $\mu_{Z_A Z_B}$, etc.,

$$\omega^4 + a \cdot \omega^3 + b \cdot \omega^2 + c \cdot \omega + d = 0.$$

The coefficients $a$, $b$, $c$ and $d$ are given in appendix A.

If detection of fast neutrons is performed in region A, then the functions $Y_1$, $Y_2$, $Y_3$ and $Y_4$ should be used in the form:

$$-Y_1 = \frac{2\lambda_{A1d}\left(K_0 - \omega_1\left(\omega_1\left(K_3\omega_1 - K_2\right) + K_1\right)\right)}{\bar{N}_{A1}\omega_1\left(\omega_1 - \omega_2\right)\left(\omega_1 - \omega_3\right)\left(\omega_1 - \omega_4\right)}$$

$$-Y_2 = \frac{2\lambda_{A1d}\left(K_0 - \omega_2\left(\omega_2\left(K_3\omega_2 - K_2\right) + K_1\right)\right)}{\bar{N}_{A1}\omega_2\left(\omega_2 - \omega_1\right)\left(\omega_2 - \omega_3\right)\left(\omega_2 - \omega_4\right)}$$

$$-Y_3 = \frac{2\lambda_{A1d}\left(K_0 - \omega_3\left(\omega_3\left(K_3\omega_3 - K_2\right) + K_1\right)\right)}{\bar{N}_{A1}\omega_3\left(\omega_3 - \omega_1\right)\left(\omega_3 - \omega_2\right)\left(\omega_3 - \omega_4\right)}$$

$$-Y_4 = \frac{2\lambda_{A1d}\left(K_0 - \omega_4\left(\omega_4\left(K_3\omega_4 - K_2\right) + K_1\right)\right)}{\bar{N}_{A1}\omega_4\left(\omega_4 - \omega_1\right)\left(\omega_4 - \omega_2\right)\left(\omega_4 - \omega_3\right)}.$$

Analytical expressions for the coefficients $K_0$, $K_1$, $K_2$, $K_3$ and $K_4$ are given in appendix A. It can be shown that

$$\begin{aligned} Y_0 &= Y_1 + Y_2 + Y_3 + Y_4 \\ &= \frac{2K_0\lambda_{A1d}}{\omega_1\omega_2\omega_3\omega_4\bar{N}_{A1}}. \end{aligned}$$

If a thermal neutron detector is placed in region A, then the following $Y_1$, $Y_2$, $Y_3$ and $Y_4$ functions are to be used:

$$-Y_1 = \frac{2\lambda_{A2d}\left(L_0 - \omega_1\left(\omega_1\left(L_3\omega_1 - L_2\right) + L_1\right)\right)}{\omega_1\left(\omega_1 - \omega_2\right)\left(\omega_1 - \omega_3\right)\left(\omega_1 - \omega_4\right)\bar{N}_{A2}}$$

$$-Y_2 = \frac{2\lambda_{A2d}\left(L_0 - \omega_2\left(\omega_2\left(L_3\omega_2 - L_2\right) + L_1\right)\right)}{\omega_2\left(\omega_2 - \omega_1\right)\left(\omega_2 - \omega_3\right)\left(\omega_2 - \omega_4\right)\bar{N}_{A2}}$$

$$-Y_3 = \frac{2\lambda_{A2d}\left(L_0 - \omega_3\left(\omega_3\left(L_3\omega_3 - L_2\right) + L_1\right)\right)}{\omega_3\left(\omega_3 - \omega_1\right)\left(\omega_3 - \omega_2\right)\left(\omega_3 - \omega_4\right)\bar{N}_{A2}}$$

$$-Y_4 = \frac{2\lambda_{A2d}\left(L_0 - \omega_4\left(\omega_4\left(L_3\omega_4 - L_2\right) + L_1\right)\right)}{\omega_4\left(\omega_4 - \omega_1\right)\left(\omega_4 - \omega_2\right)\left(\omega_4 - \omega_3\right)\bar{N}_{A2}},$$

where, analytical expressions for coefficients $L_0$, $L_1$, $L_2$, $L_3$ and $L_4$ are given in appendix A. It can be shown that

$$\begin{aligned} Y_0 &= Y_1 + Y_2 + Y_3 + Y_4 \\ &= \frac{2L_0\lambda_{A2d}}{\omega_1\omega_2\omega_3\omega_4\bar{N}_{A2}}. \end{aligned}$$

For the case when a fast neutron detector is placed in region B, the following $Y_1$, $Y_2$, $Y_3$ and $Y_4$ functions should be used:

$$-Y_1 = \frac{2\lambda_{B1d}\left(M_0 - \omega_1\left(\omega_1\left(M_3\omega_1 - M_2\right) + M_1\right)\right)}{\bar{N}_{B1}\omega_1\left(\omega_1 - \omega_2\right)\left(\omega_1 - \omega_3\right)\left(\omega_1 - \omega_4\right)}$$

$$-Y_2 = \frac{2\lambda_{B1d}\left(M_0 - \omega_2\left(\omega_2\left(M_3\omega_2 - M_2\right) + M_1\right)\right)}{\bar{N}_{B1}\omega_2\left(\omega_2 - \omega_1\right)\left(\omega_2 - \omega_3\right)\left(\omega_2 - \omega_4\right)}$$

$$-Y_3 = \frac{2\lambda_{B1d}\left(M_0 - \omega_3\left(\omega_3\left(M_3\omega_3 - M_2\right) + M_1\right)\right)}{\bar{N}_{B1}\omega_3\left(\omega_3 - \omega_1\right)\left(\omega_3 - \omega_2\right)\left(\omega_3 - \omega_4\right)}$$

$$-Y_4 = \frac{2\lambda_{B1d}\left(M_0 - \omega_4\left(\omega_4\left(M_3\omega_4 - M_2\right) + M_1\right)\right)}{\bar{N}_{B1}\omega_4\left(\omega_4 - \omega_1\right)\left(\omega_4 - \omega_2\right)\left(\omega_4 - \omega_3\right)},$$

where, analytical expressions for coefficients $M_0$, $M_1$, $M_2$, $M_3$ and $M_4$ are given in appendix A. It can be shown that

$$\begin{aligned} Y_0 &= Y_1 + Y_2 + Y_3 + Y_4 \\ &= \frac{2M_0\lambda_{B1d}}{\bar{N}_{B1}\omega_1\omega_2\omega_3\omega_4}. \end{aligned}$$

If a thermal neutron detector is placed in region B, the following $Y_1$, $Y_2$, $Y_3$ and $Y_4$ functions should be used:

$$\begin{aligned} -Y_1 &= \frac{2\lambda_{B2d}\left(P_0 - \omega_1\left(\omega_1\left(P_3\omega_1 - P_2\right) + P_1\right)\right)}{\omega_1\left(\omega_1 - \omega_2\right)\left(\omega_1 - \omega_3\right)\left(\omega_1 - \omega_4\right)\bar{N}_{B2}} \\ -Y_2 &= \frac{2\lambda_{B2d}\left(P_0 - \omega_2\left(\omega_2\left(P_3\omega_2 - P_2\right) + P_1\right)\right)}{\omega_2\left(\omega_2 - \omega_1\right)\left(\omega_2 - \omega_3\right)\left(\omega_2 - \omega_4\right)\bar{N}_{B2}} \\ -Y_3 &= \frac{2\lambda_{B2d}\left(P_0 - \omega_3\left(\omega_3\left(P_3\omega_3 - P_2\right) + P_1\right)\right)}{\omega_3\left(\omega_3 - \omega_1\right)\left(\omega_3 - \omega_2\right)\left(\omega_3 - \omega_4\right)\bar{N}_{B2}} \\ -Y_4 &= \frac{2\lambda_{B2d}\left(P_0 - \omega_4\left(\omega_4\left(P_3\omega_4 - P_2\right) + P_1\right)\right)}{\omega_4\left(\omega_4 - \omega_1\right)\left(\omega_4 - \omega_2\right)\left(\omega_4 - \omega_3\right)\bar{N}_{B2}}, \end{aligned}$$

where, analytical expressions for coefficients $P_0$, $P_1$, $P_2$, $P_3$ and $P_4$ are given in appendix A. It can be shown that

$$\begin{aligned} Y_0 &= Y_1 + Y_2 + Y_3 + Y_4 \\ &= \frac{2P_0\lambda_{B2d}}{\bar{N}_{B2}\omega_1\omega_2\omega_3\omega_4}. \end{aligned}$$

Quantitative examples of the Feynman $Y(t)$-function will be given shortly.

### 3.2 Two-group one-point Feynman-alpha theory (with delayed neutrons)

In order to derive the two-group one-point Feynman-alpha theory let us assume that the source S is switched on at the time $t_0 \le t$, while the detection process is started at the fixed time instant $t_d$, where $t_d \le t$ and $t_d \ge t_0$. For convenience, we consider $t_d = 0$. Let the random processes $N_1(t)$, $N_2(t)$ and $C(t)$ represent the number of fast neutrons, thermal neutrons and delayed neutron precursors at the time $t \ge 0$, and $Z_1(t, t_d)$, $Z_2(t, t_d)$ —the number of fast and thermal particle detections in the time interval $[t_d, t]$, respectively. Thus, the joint probability of having $N_1$ fast neutrons, $N_2$ thermal neutrons and $C$ delayed neutron precursors present in system at time $t$, and that $Z_1$ fast neutrons and $Z_2$ thermal neutrons have been detected during the period $t - t_d \ge 0$ can be defined as $P(N_1, N_2, C, Z_1, Z_2, t|t_0)$. By summing up the probabilities of the mutually exclusive events of the particle not having or having a specific reaction or that there is a source emission within the infinitesimally small time interval d$t$, one can write:

$$\begin{aligned} \frac{\partial P(N_1, N_2, C, Z_1, Z_2, t)}{\partial t} &= -(\lambda_1 N_1 + \lambda_2 N_2 + \lambda C + S_1)P(N_1, N_2, C, Z_1, Z_2, t) \\ &+\lambda_{1a}(N_1+1)P(N_1+1, N_2, C, Z_1, Z_2, t) + \lambda_{2a}(N_2+1)P(N_1, N_2+1, C, Z_1, Z_2, t) \\ &+\lambda_{1f}\sum_{k}^{N_1+1}\sum_{l}^{C}(N_1+1-k)f_{1f}(k,l)P(N_1+1-k, N_2, C-l, Z_1, Z_2, t) \\ &+\lambda_{2f}\sum_{k}^{N_1}\sum_{l}^{C}(N_2+1)f_{2f}(k,l)P(N_1-k, N_2+1, C-l, Z_1, Z_2, t) \\ &+\lambda_R(N_1+1)P(N_1+1, N_2-1, C, Z_1, Z_2, t) \\ &+\lambda_{1d}(N_1+1)P(N_1+1, N_2, C, Z_1-1, Z_2, t) \\ &+\lambda_{2d}(N_2+1)P(N_1, N_2+1, C, Z_1, Z_2-1, t) \\ &+\lambda(C+1)P(N_1-1, N_2, C+1, Z_1, Z_2, t) \\ &+S_1\sum_{n}^{N_1}p_q(n)P(N_1-n, N_2, C, Z_1, Z_2, t), \end{aligned}$$

with initial condition

$$P(N_1, N_2, C, Z_1, Z_2, t = t_0 \mid t_0) = \delta_{N_1,0}\delta_{N_2,0}\delta_{C,0}\delta_{Z_1,0}\delta_{Z_2,0}$$

and

$$\sum_{N_1}\sum_{N_2}\sum_{C} P(N_1, N_2, C, Z_1, Z_2, t = t_d \mid t_0) = \delta_{Z_1,0}\delta_{Z_2,0}.$$

By defining the following generating function for the probability distribution $P(N_1, N_2, C, Z_1, Z_2, t)$:

$$G(X, Y, V, M, N, t) = \sum_{N_1}\sum_{N_2}\sum_{C}\sum_{Z_1}\sum_{Z_2} X^{N_1} Y^{N_2} V^{C} M^{Z_1} N^{Z_2} P(N_1, N_2, C, Z_1, Z_2, t)$$

with initial condition for $t_0 \leq 0$,

$$G(X, Y, V, M, N, t = t_0 \mid t_0) = 1$$

and

$$G(1, 1, 1, M, N, t = t_d \mid t_0) = 1,$$

the following partial differential equation is obtained:

$$\begin{aligned}
\frac{\partial G}{\partial t} &= [\lambda_{1a} + \lambda_R Y + q_1(X, V)\lambda_{1f} + \lambda_{1d} M - \lambda_1 X]\,\frac{\partial G}{\partial X} \\
&\quad + [\lambda_{2a} + q_2(X, V)\lambda_{2f} + \lambda_{2d} N - \lambda_2 Y]\,\frac{\partial G}{\partial Y} \\
&\quad + \lambda(X - V)\frac{\partial G}{\partial V} + S_1[r(X) - 1]G,
\end{aligned}$$

where

$$\begin{aligned}
q_1(X, V) &= \sum_k\sum_l X^k V^l f_{1f}(k, l) \\
q_2(X, V) &= \sum_k\sum_l X^k V^l f_{2f}(k, l) \\
r(X) &= \sum_n p_q(n) X^n.
\end{aligned}$$

Here, $f_{1f}(k, l)$ is the probability of having $k$ prompt neutrons and $l$ delayed neutron precursors produced in a fission event induced by a fast neutron, $f_{2f}(k, l)$ is the probability of having $k$ prompt neutrons and $l$ delayed neutron precursors produced in a fission event induced by a thermal neutron. The effective delayed neutron fraction is $\beta$, $\nu_1'$ and $\nu_2'$ are the average total number of neutrons per fast and thermal induced fission, respectively. For the sake of simplicity, some identities are used in the solution as below ($i = 1, 2$):

$$\begin{aligned}
\left.\frac{\partial}{\partial X} q_i(X, V)\right|_{X=1, V=1} &= \sum_k\sum_l k f_{if}(k, l) \\
&= (1 - \beta)\nu_i' \\
\left.\frac{\partial}{\partial V} q_i(X, V)\right|_{X=1, V=1} &= \sum_k\sum_l l f_{if}(k, l) \\
&= \beta\nu_i'
\end{aligned}$$

and

$$\begin{aligned}
\left.\frac{\partial}{\partial X} r(X)\right|_{X=1} &= \sum_n n p_q(n) \\
&= r' \\
\left.\frac{\partial^2}{\partial X^2} r(X)\right|_{X=1} &= \sum_n n(n-1) p_q(n) \\
&= r''.
\end{aligned}$$

In a steady subcritical medium with a steady source, when $t_0 \to -\infty$, the following stationary solutions for the neutron populations $N_1$, $N_2$ and $C$, and detection counts $Z_1$ and $Z_2$ are obtained as below:

$$
\begin{aligned}
\bar{N}_1 &= \frac{\lambda_2 S_1 r'}{\lambda_1\lambda_2 - \lambda_2\nu_1'\lambda_{1f} - \lambda_R\nu_2'\lambda_{2f}}\\
\bar{N}_2 &= \frac{\lambda_R S_1 r'}{\lambda_1\lambda_2 - \lambda_2\nu_1'\lambda_{1f} - \lambda_R\nu_2'\lambda_{2f}}\\
\bar{C} &= \frac{(\lambda_2\beta\nu_1'\lambda_{1f} + \lambda_R\beta\nu_2'\lambda_{2f})S_1 r'}{\lambda(\lambda_1\lambda_2 - \lambda_2\nu_1'\lambda_{1f} - \lambda_R\nu_2'\lambda_{2f})}\\
&= \frac{\bar{N}_1\beta\nu_1'\lambda_{1f}}{\lambda} + \frac{\bar{N}_2\beta\nu_2'\lambda_{2f}}{\lambda}\\
\bar{Z}_1 &= \lambda_{1d}\bar{N}_1 t\\
\bar{Z}_2 &= \lambda_{2d}\bar{N}_2 t.
\end{aligned}
$$

By introducing the modified second factorial moment of the random variables $a$ and $b$ and then taking cross- and auto-derivatives, the following system of differential equations of modified second factorial moments for the neutron population are obtained as below:

$$
\begin{aligned}
\frac{\partial}{\partial t}\mu_{XX} &= S_1 r'' + \lambda_{2f}\nu_{2pp}\bar{N}_2 + \lambda_{1f}\nu_{1pp}\bar{N}_1 + 2\lambda\mu_{XV} + 2\left[-\lambda_1 + (1-\beta)\lambda_{1f}\nu_1'\right]\mu_{XX} + 2(1-\beta)\lambda_{2f}\nu_2'\mu_{XY}\\
\frac{\partial}{\partial t}\mu_{XY} &= \lambda\mu_{YV} + (1-\beta)\lambda_{2f}\nu_2'\mu_{YY} + \left[(1-\beta)\lambda_{1f}\nu_1' - \lambda_1 - \lambda_2\right]\mu_{XY} + \lambda_R\mu_{XX}\\
\frac{\partial}{\partial t}\mu_{YY} &= -2\lambda_2\mu_{YY} + 2\lambda_R\mu_{XY}\\
\frac{\partial}{\partial t}\mu_{XV} &= \lambda\mu_{VV} + \lambda_{2f}\nu_{2pd}\bar{N}_2 + (1-\beta)\lambda_{2f}\nu_2'\mu_{YV} + \lambda_{1f}\nu_{1pd}\bar{N}_1 + \left[-\lambda_1 + (1-\beta)\lambda_{1f}\nu_1' - \lambda\right]\mu_{XV}\\
&\quad + \beta\lambda_{2f}\nu_2'\mu_{XY} + \beta\lambda_{1f}\nu_1'\mu_{XX}\\
\frac{\partial}{\partial t}\mu_{YV} &= (-\lambda - \lambda_2)\mu_{YV} + \beta\lambda_{2f}\nu_2'\mu_{YY} + \lambda_R\mu_{XV} + \beta\lambda_{1f}\nu_1'\mu_{XY}\\
\frac{\partial}{\partial t}\mu_{VV} &= -2\lambda\mu_{VV} + \lambda_{2f}\nu_{2dd}\bar{N}_2 + 2\beta\lambda_{2f}\nu_2'\mu_{YV} + \lambda_{1f}\nu_{1dd}\bar{N}_1 + 2\beta\lambda_{1f}\nu_1'\mu_{XV}.
\end{aligned}
$$

The three coefficients $\omega_1$, $\omega_2$ and $\omega_3$ can be obtained by solving the third order equation in $\omega$ with known constant coefficients $a$, $b$, $c$:

$$
\omega^3 + a\cdot\omega^2 + b\cdot\omega + c = 0,
$$

where

$$
\begin{aligned}
a &= \beta\nu_1'\lambda_{1f} - \nu_1'\lambda_{1f} + \lambda + \lambda_1 + \lambda_2\\
&= -(\omega_1 + \omega_2 + \omega_3)\\
b &= \beta\lambda_2\nu_1'\lambda_{1f} - \lambda\nu_1'\lambda_{1f} - \lambda_2\nu_1'\lambda_{1f} + \beta\nu_2'\lambda_{2f}\lambda_R - \nu_2'\lambda_{2f}\lambda_R + \lambda\lambda_1 + \lambda_2\lambda_1 + \lambda\lambda_2\\
c &= -\lambda\lambda_2\nu_1'\lambda_{1f} - \lambda\nu_2'\lambda_{2f}\lambda_R + \lambda\lambda_1\lambda_2\\
&= -\omega_1\omega_2\omega_3.
\end{aligned}
$$

The stationary modified variance of the fast particle detections can be obtained from the coupled equation system by using the Laplace transform technique:

$$
\begin{aligned}
\frac{\partial}{\partial t}\mu_{XM} &= \lambda\mu_{VM} + (1-\beta)\lambda_{2f}\nu_2'\mu_{YM} + \left(-\lambda_1 + (1-\beta)\lambda_{1f}\nu_1'\right)\mu_{XM} + \lambda_{1d}\mu_{XX}\\
\frac{\partial}{\partial t}\mu_{YM} &= -\lambda_2\mu_{YM} + \lambda_R\mu_{XM} + \lambda_{1d}\mu_{XY}\\
\frac{\partial}{\partial t}\mu_{VM} &= -\lambda\mu_{VM} + \beta\lambda_{2f}\nu_2'\mu_{YM} + \lambda_{1d}\mu_{XV} + \beta\lambda_{1f}\nu_1'\mu_{XM}\\
\frac{\partial}{\partial t}\mu_{MM} &= 2\lambda_{1d}\mu_{XM}.
\end{aligned}
$$

The same can be done to define the stationary modified variance of the thermal particle detections via solving the following coupled equation system:

$$\begin{aligned}
\frac{\partial}{\partial t}\mu_{XN} &= \lambda\mu_{VN} + (1-\beta)\lambda_{2f}\nu_2'\mu_{YN} + \left(-\lambda_1 + (1-\beta)\lambda_{1f}\nu_1'\right)\mu_{XN} + \lambda_{2d}\mu_{XY} \\
\frac{\partial}{\partial t}\mu_{YN} &= -\lambda_2\mu_{YN} + \lambda_R\mu_{XN} + \lambda_{2d}\mu_{YY} \\
\frac{\partial}{\partial t}\mu_{VN} &= -\lambda\mu_{VN} + \beta\lambda_{2f}\nu_2'\mu_{YN} + \lambda_{2d}\mu_{YV} + \beta\lambda_{1f}\nu_1'\mu_{XN} \\
\frac{\partial}{\partial t}\mu_{NN} &= 2\lambda_{2d}\mu_{YN}.
\end{aligned}$$

Some second moment notations were introduced as follows:

$$\begin{aligned}
\left.\frac{\partial^2}{\partial X^2}q_i(X,V)\right|_{X=1,V=1} &= \sum_k\sum_l k(k-1)f_{if}(k,l) \\
&= \nu_{ipp} \\
\left.\frac{\partial^2}{\partial V^2}q_i(X,V)\right|_{X=1,V=1} &= \sum_k\sum_l l(l-1)f_{if}(k,l) \\
&= \nu_{idd} \\
\left.\frac{\partial^2}{\partial V\partial X}q_i(X,V)\right|_{X=1,V=1} &= \sum_k\sum_l kl f_{if}(k,l) \\
&= \nu_{ipd},
\end{aligned}$$

in which $i = 1, 2$. Thus, the solution for the two-group one-point Feynman-alpha formula for fast and thermal detection particles can be written as below:

$$\begin{aligned}
\frac{\sigma_{ZZ}^2(t)}{\bar{Z}_1\,\bar{Z}_2} &= 1 + Y(t) \\
&= 1 + \sum_{i=1}^{3} Y_i\left(1 - \frac{1-e^{-\omega_i t}}{\omega_i t}\right).
\end{aligned}$$

For fast particle detections the following expressions should be used:

$$\begin{aligned}
-Y_1 &= -\frac{2\lambda_{1d}\left(\omega_1\left(K_2\omega_1 - K_1\right) + K_0\right)}{\bar{N}_1\omega_1\left(\omega_1-\omega_2\right)\left(\omega_1-\omega_3\right)} \\
-Y_2 &= \frac{2\lambda_{1d}\left(\omega_2\left(K_2\omega_2 - K_1\right) + K_0\right)}{\bar{N}_1\left(\omega_1-\omega_2\right)\omega_2\left(\omega_2-\omega_3\right)} \\
-Y_3 &= \frac{2\lambda_{1d}\left(\omega_3\left(K_2\omega_3 - K_1\right) + K_0\right)}{\bar{N}_1\left(\omega_1-\omega_3\right)\omega_3\left(\omega_3-\omega_2\right)},
\end{aligned}$$

with

$$\begin{aligned}
K_2 &= \mu_{XX} \\
K_1 &= -\beta\lambda_{2f}\nu_2'\mu_{XY} + \lambda_{2f}\nu_2'\mu_{XY} + \lambda\mu_{XV} + \lambda\mu_{XX} + \lambda_2\mu_{XX} \\
K_0 &= \lambda\lambda_{2f}\nu_2'\mu_{XY} + \lambda\lambda_2\mu_{XV} + \lambda\lambda_2\mu_{XX}.
\end{aligned}$$

It can be shown that

$$\begin{aligned}
Y_0 &= Y_1 + Y_2 + Y_3 \\
&= \frac{2K_0\lambda_{1d}}{\bar{N}_1\omega_1\omega_2\omega_3}.
\end{aligned}$$

If a thermal neutron detector is used, then the following expressions should be considered:

$$-Y_1 = -\frac{2\lambda_{2d}\left(\omega_1\left(L_2\omega_1 - L_1\right) + L_0\right)}{\omega_1\left(\omega_1 - \omega_2\right)\left(\omega_1 - \omega_3\right)\bar{N}_2}$$

$$-Y_2 = \frac{2\lambda_{2d}\left(\omega_2\left(L_2\omega_2 - L_1\right) + L_0\right)}{\left(\omega_1 - \omega_2\right)\omega_2\left(\omega_2 - \omega_3\right)\bar{N}_2}$$

$$-Y_3 = \frac{2\lambda_{2d}\left(\omega_3\left(L_2\omega_3 - L_1\right) + L_0\right)}{\left(\omega_1 - \omega_3\right)\omega_3\left(\omega_3 - \omega_2\right)\bar{N}_2},$$

with

$$\begin{aligned}
L_2 &= \mu_{YY} \\
L_1 &= \beta\lambda_{1f}\nu_1'\mu_{YY} - \lambda_{1f}\nu_1'\mu_{YY} + \lambda_R\mu_{XY} + \lambda\mu_{YY} + \lambda_1\mu_{YY} \\
L_0 &= -\lambda\lambda_{1f}\nu_1'\mu_{YY} + \lambda\lambda_R\mu_{XY} + \lambda\lambda_R\mu_{YV} + \lambda\lambda_1\mu_{YY}.
\end{aligned}$$

It can be shown that

$$\begin{aligned}
Y_0 &= Y_1 + Y_2 + Y_3 \\
&= \frac{2L_0\lambda_{2d}}{\omega_1\omega_2\omega_3\bar{N}_2}.
\end{aligned}$$

### 3.3 One-group two-point Feynman-alpha theory (with delayed neutrons)

Similarly as in the derivation of two-group one-point version of the Feynman-alpha formula, in the one-group two-point Feynman-alpha theory the joint probability of having $N_A$ neutrons in region A, $N_B$ neutrons in region B, $C_A$ delayed neutron precursors presented in region A, $C_B$ delayed neutron precursors presented in region B at time $t$, $Z_A$ neutrons have been detected in region A, and $Z_B$ neutrons have been detected in region B in the system during the period of time $t - t_d \geq 0$ can be defined as $P(N_A, N_B, C_A, C_B, Z_A, Z_B, t|t_0)$. By summing up all mutually exclusive events of the particle not having or having a specific reaction within the infinitesimally small time interval d$t$, one can write

$$\begin{aligned}
\frac{\partial P(N_A, N_B, C_A, C_B, Z_A, Z_B, t)}{\partial t} &= -(\lambda_A N_A + \lambda_B N_B + \lambda_{Ac} C_A + \lambda_{Bc} C_B + S_A + S_B) P(N_A, N_B, C_A, C_B, Z_A, Z_B, t) \\
&+\lambda_{Aa}(N_A + 1) P(N_A + 1, N_B, C_A, C_B, Z_A, Z_B, t) \\
&+\lambda_{Ba}(N_B + 1) P(N_A, N_B + 1, C_A, C_B, Z_A, Z_B, t) \\
&+\lambda_{Af} \sum_{k}^{N_A+1} \sum_{l}^{C_A} (N_A + 1 - k) f_A(k, l) P(N_A + 1 - k, N_B, C_A - l, C_B, Z_A, Z_B, t) \\
&+\lambda_{Bf} \sum_{k}^{N_B+1} \sum_{l}^{C_B} (N_B + 1 - k) f_B(k, l) P(N_A, N_B + 1 - k, C_A, C_B - l, Z_A, Z_B, t) \\
&+\lambda_{At}(N_A + 1) P(N_A + 1, N_B - 1, C_A, C_B, Z_A, Z_B, t) \\
&+\lambda_{Bt}(N_B + 1) P(N_A - 1, N_B + 1, C_A, C_B, Z_A, Z_B, t) \\
&+\lambda_{Ac}(C_A + 1) P(N_A - 1, N_B, C_A + 1, C_B, Z_A, Z_B, t) \\
&+\lambda_{Bc}(C_B + 1) P(N_A, N_B - 1, C_A, C_B + 1, Z_A, Z_B, t) \\
&+\lambda_{Ad}(N_A + 1) P(N_A + 1, N_B, C_A, C_B, Z_A - 1, Z_B, t) \\
&+\lambda_{Bd}(N_B + 1) P(N_A, N_B + 1, C_A, C_B, Z_A, Z_B - 1, t) \\
&+S_A \sum_{n}^{N_A} p_A(n) P(N_A - n, N_B, C_A, C_B, Z_A, Z_B, t) \\
&+S_B \sum_{n}^{N_B} p_B(n) P(N_A, N_B - n, C_A, C_B, Z_A, Z_B, t),
\end{aligned}$$

with initial condition

$$P(N_A, N_B, C_A, C_B, Z_A, Z_B, t = t_0 \mid t_0) = \delta_{N_A,0}\delta_{N_B,0}\delta_{C_A,0}\delta_{C_B,0}\delta_{Z_A,0}\delta_{Z_B,0}$$

and

$$\sum_{N_A}\sum_{N_B}\sum_{C_A}\sum_{C_B} P(N_A, N_B, C_A, C_B, Z_A, Z_B, t = t_d \mid t_0) = \delta_{Z_A,0}\delta_{Z_B,0}.$$

By defining the following generating function for the probability distribution $P(N_A, N_B, C_A, C_B, Z_A, Z_B, t)$:

$$G(X, Y, V, W, M, N, t) = \sum_{N_A}\sum_{N_B}\sum_{C_A}\sum_{C_B}\sum_{Z_A}\sum_{Z_B} X^{N_A} Y^{N_B} V^{C_A} W^{C_B} M^{Z_A} N^{Z_B} P(N_A, N_B, C_A, C_B, Z_A, Z_B, t),$$

with initial condition for $t_0 \le 0$,

$$G(X, Y, V, W, M, N, t = t_0 \mid t_0) = 1$$

and

$$G(1, 1, 1, 1, M, N, t = t_d \mid t_0) = 1,$$

a partial differential equation in the variables $(X, Y, V, W, M, N)$ in terms of generating function can be obtained:

$$\begin{aligned}\frac{\partial G}{\partial t} &= [\lambda_{Aa} + \lambda_{At}Y + q_A(X, V)\lambda_{Af} + \lambda_{Ad}M - \lambda_A X]\frac{\partial G}{\partial X}\\ &\quad + [\lambda_{Ba} + \lambda_{Bt}X + q_B(Y, W)\lambda_{Bf} + \lambda_{Bd}N - \lambda_B Y]\frac{\partial G}{\partial Y} + \lambda_{Ac}(X - V)\frac{\partial G}{\partial V}\\ &\quad + \lambda_{Bc}(Y - W)\frac{\partial G}{\partial W} + S_A[r_A(X) - 1]G + S_B[r_B(Y) - 1]G,\end{aligned}$$

where

$$\begin{aligned}q_A(X, V) &= \sum_k\sum_l X^k V^l f_A(k, l)\\ q_B(Y, W) &= \sum_k\sum_l Y^k W^l f_B(k, l)\\ r_A(X) &= \sum_n p_A(n) X^n\\ r_B(Y) &= \sum_n p_B(n) Y^n.\end{aligned}$$

Here, $\beta_A$ and $\beta_B$ are the effective delayed neutron fractions in region A and region B, respectively. For the sake of simplicity, some identities are used in the solution as below $(i = A, B)$:

$$\begin{aligned}\left.\frac{\partial}{\partial X} q_A(X, V)\right|_{X=1,V=1} &= \sum_k\sum_l k f_{Af}(k, l)\\ &= (1 - \beta_A)\nu'_A\\ \left.\frac{\partial}{\partial V} q_A(X, V)\right|_{X=1,V=1} &= \sum_k\sum_l l f_{Af}(k, l)\\ &= \beta_A\nu'_A\\ \left.\frac{\partial}{\partial Y} q_B(Y, W)\right|_{Y=1,W=1} &= \sum_k\sum_l k f_{Bf}(k, l)\\ &= (1 - \beta_B)\nu'_B\\ \left.\frac{\partial}{\partial W} q_B(Y, W)\right|_{Y=1,W=1} &= \sum_k\sum_l l f_{Bf}(k, l)\\ &= \beta_B\nu'_B\end{aligned}$$

and

$$
\begin{aligned}
\left.\frac{\partial}{\partial X} r_A(X)\right|_{X=1} &= \sum_n n p_A(n) \\
&= r'_A \\
\left.\frac{\partial^2}{\partial X^2} r_A(X)\right|_{X=1} &= \sum_n n(n-1) p_A(n) \\
&= r''_A \\
\left.\frac{\partial}{\partial Y} r_B(Y)\right|_{Y=1} &= \sum_n n p_B(n) \\
&= r'_B \\
\left.\frac{\partial^2}{\partial Y^2} r_B(Y)\right|_{Y=1} &= \sum_n n(n-1) p_B(n) \\
&= r''_B .
\end{aligned}
$$

In a steady subcritical medium with a steady source, when $t_0 \to -\infty$, the following stationary solutions for the neutron population and detection counts are obtained as follows:

$$
\begin{aligned}
\bar{N}_A &= \frac{S_A\left(\lambda_B - \lambda_{Bf}\nu'_B\right) r'_A + S_B \lambda_{Bt} r'_B}{\left(\lambda_A - \lambda_{Af}\nu'_A\right)\left(\lambda_B - \lambda_{Bf}\nu'_B\right) - \lambda_{At}\lambda_{Bt}} \\
\bar{N}_B &= \frac{S_A \lambda_{At} r'_A + S_B\left(\lambda_A - \lambda_{Af}\nu'_A\right) r'_B}{\left(\lambda_A - \lambda_{Af}\nu'_A\right)\left(\lambda_B - \lambda_{Bf}\nu'_B\right) - \lambda_{At}\lambda_{Bt}} \\
\bar{C}_A &= \frac{\beta_A \lambda_{Af}\nu'_A}{\lambda_{Ac}} \bar{N}_A \\
\bar{C}_B &= \frac{\beta_B \lambda_{Bf}\nu'_B}{\lambda_{Bc}} \bar{N}_B \\
\bar{Z}_A &= \lambda_{1d}\bar{N}_1 t \\
\bar{Z}_B &= \lambda_{2d}\bar{N}_2 t .
\end{aligned}
$$

By introducing the modified second factorial moments and then taking cross- and auto-derivatives, the following system of differential equations of modified second factorial moments for the neutron population are obtained as below:

$$
\begin{aligned}
\frac{\partial}{\partial t}\mu_{XX} &= 2\left[(1-\beta_A)\lambda_{Af}\nu'_A - \lambda_A\right]\mu_{XX} + 2\lambda_{Bt}\mu_{XY} + 2\lambda_{Ac}\mu_{XV} + S_A r''_A + \lambda_{Af}\nu_{App}\bar{N}_A \\
\frac{\partial}{\partial t}\mu_{XY} &= \lambda_{Ac}\mu_{YV} + \lambda_{Bc}\mu_{XW} + \lambda_{Bt}\mu_{YY} + \lambda_{At}\mu_{XX} + \left[(1-\beta_B)\lambda_{Bf}\nu'_B - \lambda_B + (1-\beta_A)\lambda_{Af}\nu'_A - \lambda_A\right]\mu_{XY} \\
\frac{\partial}{\partial t}\mu_{YY} &= 2\left[-\lambda_B + (1-\beta_B)\lambda_{Bf}\nu'_B\right]\mu_{YY} + 2\lambda_{At}\mu_{XY} + 2\lambda_{Bc}\mu_{YW} + S_B r''_B + \lambda_{Bf}\nu_{Bpp}\bar{N}_B \\
\frac{\partial}{\partial t}\mu_{XV} &= ((1-\beta_A)\nu'_A\lambda_{Af} - \lambda_A - \lambda_{Ac})\mu_{XV} + \beta_A\nu'_A\lambda_{Af}\mu_{XX} + \lambda_{Ac}\mu_{VV} + \lambda_{Bt}\mu_{YV} + \lambda_{Af}\nu_{Apd}\bar{N}_A \\
\frac{\partial}{\partial t}\mu_{YV} &= \lambda_{Bc}\mu_{VW} + \left[(1-\beta_B)\lambda_{Bf}\nu'_B - \lambda_B - \lambda_{Ac}\right]\mu_{YV} + \lambda_{At}\mu_{XV} + \beta_A\lambda_{Af}\nu'_A\mu_{XY} \\
\frac{\partial}{\partial t}\mu_{VV} &= -2\lambda_{Ac}\mu_{VV} + 2\beta_A\lambda_{Af}\nu'_A\mu_{XV} + \lambda_{Af}\nu_{Add}\bar{N}_A \\
\frac{\partial}{\partial t}\mu_{XW} &= \lambda_{Ac}\mu_{VW} + \lambda_{Bt}\mu_{YW} + \left[(1-\beta_A)\lambda_{Af}\nu'_A - \lambda_A - \lambda_{Bc}\right]\mu_{XW} + \beta_B\lambda_{Bf}\nu'_B\mu_{XY} \\
\frac{\partial}{\partial t}\mu_{YW} &= \lambda_{Bc}\mu_{WW} + \left[(1-\beta_B)\lambda_{Bf}\nu'_B - \lambda_B - \lambda_{Bc}\right]\mu_{YW} + \beta_B\lambda_{Bf}\nu'_B\mu_{YY} + \lambda_{At}\mu_{XW} + \lambda_{Bf}\nu_{Bpd}\bar{N}_B \\
\frac{\partial}{\partial t}\mu_{VW} &= \left(-\lambda_{Ac} - \lambda_{Bc}\right)\mu_{VW} + \beta_B\lambda_{Bf}\nu'_B\mu_{YV} + \beta_A\lambda_{Af}\nu'_A\mu_{XW} \\
\frac{\partial}{\partial t}\mu_{WW} &= -2\lambda_{Bc}\mu_{WW} + 2\beta_B\lambda_{Bf}\nu'_B\mu_{YW} + \lambda_{Bf}\nu_{Bdd}\bar{N}_B ,
\end{aligned}
$$

where

$$
\begin{aligned}
\left.\frac{\partial^2}{\partial X^2} q_A(X,V)\right|_{X=1,V=1} &= \sum_k \sum_l k(k-1) f_{Af}(k,l) \\
&= \nu_{App} \\
\left.\frac{\partial^2}{\partial Y^2} q_B(Y,W)\right|_{Y=1,W=1} &= \sum_k \sum_m k(k-1) f_{Bf}(k,m) \\
&= \nu_{Bpp} \\
\left.\frac{\partial^2}{\partial V^2} q_A(X,V)\right|_{X=1,V=1} &= \sum_k \sum_l l(l-1) f_{Af}(k,l) \\
&= \nu_{Add} \\
\left.\frac{\partial^2}{\partial V^2} q_B(Y,W)\right|_{Y=1,W=1} &= \sum_k \sum_l l(l-1) f_{Bf}(k,l) \\
&= \nu_{Bdd} \\
\left.\frac{\partial^2}{\partial V \partial X} q_A(X,V)\right|_{X=1,V=1} &= \sum_k \sum_l kl f_{Af}(k,l) \\
&= \nu_{Apd} \\
\left.\frac{\partial^2}{\partial W \partial Y} q_B(Y,W)\right|_{Y=1,W=1} &= \sum_k \sum_l kl f_{Bf}(k,l) \\
&= \nu_{Bpd}.
\end{aligned}
$$

The system above is solved for the stationary case when $\frac{\partial}{\partial t} = 0$. Four roots $\omega_1$, $\omega_2$, $\omega_3$ and $\omega_4$ can be obtained by solving the forth order equation with coefficients $a$, $b$, $c$, $d$ specified as below:

$$\omega^4 + a \cdot \omega^3 + b \cdot \omega^2 + c \cdot \omega + d = 0,$$

where the coefficients $a$, $b$, $c$ and $d$ are given in appendix B.

The stationary modified variance of the particle detections in region A can be obtained from the coupled equation system by using the Laplace transform technique:

$$
\begin{aligned}
\frac{\partial}{\partial t}\mu_{XM} &= \lambda_{Ac}\mu_{VM} + \lambda_{Bt}\mu_{YM} + \left[(1-\beta_A)\lambda_{Af}\nu'_A - \lambda_A\right]\mu_{XM} + \lambda_{Ad}\mu_{XX} \\
\frac{\partial}{\partial t}\mu_{YM} &= \lambda_{Bc}\mu_{WM} + \left[(1-\beta_B)\lambda_{Bf}\nu'_B - \lambda_B\right]\mu_{YM} + \lambda_{At}\mu_{XM} + \lambda_{Ad}\mu_{XY} \\
\frac{\partial}{\partial t}\mu_{VM} &= -\lambda_{Ac}\mu_{VM} + \beta_A\lambda_{Af}\nu'_A\mu_{XM} + \lambda_{Ad}\mu_{XV} \\
\frac{\partial}{\partial t}\mu_{WM} &= -\lambda_{Bc}\mu_{WM} + \beta_B\lambda_{Bf}\nu'_B\mu_{YM} + \lambda_{Ad}\mu_{XW} \\
\frac{\partial}{\partial t}\mu_{MM} &= 2\lambda_{Ad}\mu_{XM}.
\end{aligned}
$$

A similar coupled equation system can be derived for the particle detections in region B:

$$
\begin{aligned}
\frac{\partial}{\partial t}\mu_{XN} &= \lambda_{Ac}\mu_{VN} + \lambda_{Bt}\mu_{YN} + \left[(1-\beta_A)\lambda_{Af}\nu'_A - \lambda_A\right]\mu_{XN} + \lambda_{Bd}\mu_{XY} \\
\frac{\partial}{\partial t}\mu_{YN} &= \lambda_{Bc}\mu_{WN} + \left[(1-\beta_B)\lambda_{Bf}\nu'_B - \lambda_B\right]\mu_{YN} + \lambda_{At}\mu_{XN} + \lambda_{Bd}\mu_{YY} \\
\frac{\partial}{\partial t}\mu_{VN} &= -\lambda_{Ac}\mu_{VN} + \beta_A\lambda_{Af}\nu'_A\mu_{XN} + \lambda_{Bd}\mu_{YV} \\
\frac{\partial}{\partial t}\mu_{WN} &= -\lambda_{Bc}\mu_{WN} + \beta_B\lambda_{Bf}\nu'_B\mu_{YN} + \lambda_{Bd}\mu_{YW} \\
\frac{\partial}{\partial t}\mu_{NN} &= 2\lambda_{Bd}\mu_{YN}.
\end{aligned}
$$

Thus, a final expression for the two-point one-group Feynman-alpha formula for region A and B is written below:

$$\frac{\sigma^2_{ZZ}(t)}{\bar{Z}_A/\bar{Z}_B} = 1 + Y(t) + 1 + \sum_{i=1}^{4} Y_i \left(1 - \frac{1 - e^{-\omega_i t}}{\omega_i t}\right).$$

If the detector is placed in region A, the following expressions for the functions $Y_i$ should be used:

$$-Y_1 = \frac{2\lambda_{Ad}\left(K_0 - \omega_1\left(\omega_1\left(K_3\omega_1 - K_2\right) + K_1\right)\right)}{\bar{N}_A\omega_1\left(\omega_1 - \omega_2\right)\left(\omega_1 - \omega_3\right)\left(\omega_1 - \omega_4\right)}$$

$$-Y_2 = \frac{2\lambda_{Ad}\left(K_0 - \omega_2\left(\omega_2\left(K_3\omega_2 - K_2\right) + K_1\right)\right)}{\bar{N}_A\omega_2\left(\omega_2 - \omega_1\right)\left(\omega_2 - \omega_3\right)\left(\omega_2 - \omega_4\right)}$$

$$-Y_3 = \frac{2\lambda_{Ad}\left(K_0 - \omega_3\left(\omega_3\left(K_3\omega_3 - K_2\right) + K_1\right)\right)}{\bar{N}_A\omega_3\left(\omega_3 - \omega_1\right)\left(\omega_3 - \omega_2\right)\left(\omega_3 - \omega_4\right)}$$

$$-Y_4 = \frac{2\lambda_{Ad}\left(K_0 - \omega_4\left(\omega_4\left(K_3\omega_4 - K_2\right) + K_1\right)\right)}{\bar{N}_A\omega_4\left(\omega_4 - \omega_1\right)\left(\omega_4 - \omega_2\right)\left(\omega_4 - \omega_3\right)}$$

and it can be proved that

$$\begin{aligned} Y_0 &= Y_1 + Y_2 + Y_3 + Y_4 \\ &= \frac{2K_0\lambda_{Ad}}{\bar{N}_A\omega_1\omega_2\omega_3\omega_4}, \end{aligned}$$

where the coefficients $K$ are given in appendix B.

If the detector is placed in region B, the following expressions for the functions $Y_i$ should be used:

$$-Y_1 = \frac{2\lambda_{Bd}\left(L_0 - \omega_1\left(\omega_1\left(L_3\omega_1 - L_2\right) + L_1\right)\right)}{\omega_1\left(\omega_1 - \omega_2\right)\left(\omega_1 - \omega_3\right)\left(\omega_1 - \omega_4\right)\bar{N}_B}$$

$$-Y_2 = \frac{2\lambda_{Bd}\left(L_0 - \omega_2\left(\omega_2\left(L_3\omega_2 - L_2\right) + L_1\right)\right)}{\omega_2\left(\omega_2 - \omega_1\right)\left(\omega_2 - \omega_3\right)\left(\omega_2 - \omega_4\right)\bar{N}_B}$$

$$-Y_3 = \frac{2\lambda_{Bd}\left(L_0 - \omega_3\left(\omega_3\left(L_3\omega_3 - L_2\right) + L_1\right)\right)}{\omega_3\left(\omega_3 - \omega_1\right)\left(\omega_3 - \omega_2\right)\left(\omega_3 - \omega_4\right)\bar{N}_B}$$

$$-Y_4 = \frac{2\lambda_{Bd}\left(L_0 - \omega_4\left(\omega_4\left(L_3\omega_4 - L_2\right) + L_1\right)\right)}{\omega_4\left(\omega_4 - \omega_1\right)\left(\omega_4 - \omega_2\right)\left(\omega_4 - \omega_3\right)\bar{N}_B}$$

and it can be proved that

$$\begin{aligned} Y_0 &= Y_1 + Y_2 + Y_3 + Y_4 \\ &= \frac{2L_0\lambda_{Bd}}{\omega_1\omega_2\omega_3\omega_4\bar{N}_B}, \end{aligned}$$

where the coefficients $L$ are given in appendix B.

## 4 Discussion and quantitative analysis

In the following, we shall perform a comparison of the two-point two-group version of the Feynman-alpha theoretical formula to the two-point one-group, the one-point two-group and the one-point one-group (*i.e.* traditional) versions.

### 4.1 The simulation set-up

In order to compare the four different versions of the Feynman-alpha theory, quantitative values of the transition probabilities and reaction intensities were obtained by using Monte Carlo simulations in a way similar to that described in [31, 35, 38, 39]. The simulation setup consists of two regions, region A and region B, as shown in fig. 4. Region A represents nuclear material (radius 4.46 cm), in particular a mixture of 2.5% $^{235}$U and 97.5% $^{238}$U. Region B consists

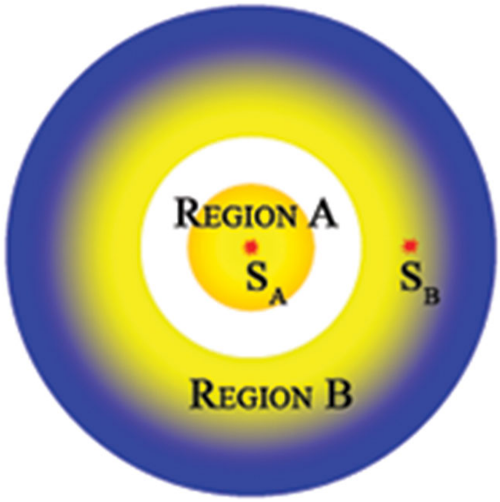


**Fig. 4.** Geometry used for the Monte Carlo simulations.

of a moderating material with a thickness of 21 cm. The neutron source emits the neutrons with an energy of 2.5 MeV. Two cases are considered in the simulations, one when the neutron source is in the center of region A and another when the neutron source is at a distance of 15 cm from the center of the nuclear material, in region B. Two point detectors, in region A and in region B, are included in the simulation setup. Delayed neutron precursors are not included in the simulations[8].

### 4.2 Coefficients

Initially, the transition probabilities and reaction intensities are obtained in MCNPX simulations [40] for the two-point two-group case by merging the information from neutron weight balance table ("print table 130" in MCNPX) with simulated reaction rates (normalized to one starting neutron). Alternatively, one can obtain similar information by processing MCNPX PTRAC file. Then, the values of reaction intensities of the two-point two-group case are condensed in order to get the reaction intensities which correspond to the two-point one-group, the one-point two-group and one-point one-group cases. Afterwards, these values are used in order to obtain the values of the $Y$ and $\omega$ coefficients in the Feynman-alpha formulas for the cases when the source is in region A, and in region B for fast neutron detections and thermal neutron detections, as shown in tables 1–4, respectively.

Since there is only one region considered in the two-group one-point and the one-point one-group Feynman-alpha formulas, the coefficients are the same for the detection in the different regions of the initial system used for the simulations. The same is true for the energy-dependent factor in the two-point one-group and the one-point one-group Feynman-alpha formulas, the coefficients are the same for the fast and thermal neutron detection.

In the studies described below we assume that the two-point two-group version of Feynman-alpha formulas gives the most accurate predictions as the most involved one among the four various versions, *i.e.* the two-point two-group, the two-point one-group, the two-group one-point and one-point one-group theories.

### 4.3 Comparison of the four versions of the Feynman-alpha theoretical formulas for the case of fast neutrons detections

Figures 5 and 6 show a quantitative illustration of the dependence of the variance to mean of the number of fast neutron detections on the detection time for four versions of Feynman-alpha theories when the source is in region A. Different curves in figs. 5 and 6 are created based on the parameter values from tables 1 and 2.

As it is shown in fig. 5, when fast neutrons are detected in region A, the two-point two-group, the two-point one-group and one-point one-group versions of the Feynman-alpha theoretical formulas give very similar results. However, the one-point two-group version of the formulas overestimates the asymptotic ratio of the variance to mean. Thus, we can conclude that the region dependence of the model plays a more important role than the energy dependence for the

[8] A Mathematica notebook for visualization of two-point two-group, two-point one-group and one-point two-group Feynman-alpha formulas can be downloaded from `dx.doi.org/10.13140/2.1.3251.5209`.

**Table 1.** The values of the $Y_i$ and $\omega_i$ calculated for four various versions of Feynman-alpha formulas (the source is in region A, fast neutron detector is used either in region A or B).

| | Source in region A | | | |
|---|---|---|---|---|
| | 2-point 2-group | 2-point 1-group | 1-point 2-group | 1-point 1-group |
| $\omega_1$ | 1.52001 | 1.8083 | 1.45471 | 0.923611 |
| $\omega_2$ | 1.12141 | 0.743335 | 0.350087 | |
| $\omega_3$ | 0.759289 | | | |
| $\omega_4$ | 0.0983484 | | | |
| | Fast neutron detections in region A | | | |
| $Y_1$ | 0.00239792 | 0.00154341 | 0.000823734 | 0.0126651 |
| $Y_2$ | 0.00354816 | 0.00938149 | 0.0544565 | |
| $Y_3$ | 0.00061659 | | | |
| $Y_4$ | 0.0056358 | | | |
| | Fast neutron detections in region B | | | |
| $Y_1$ | −0.0000243761 | −0.000134373 | 0.000823734 | 0.0126651 |
| $Y_2$ | −0.000094874 | 0.000795215 | 0.0544565 | |
| $Y_3$ | 0.000301926 | | | |
| $Y_4$ | 0.000161625 | | | |

**Table 2.** The values of the $Y_i$ and $\omega_i$ calculated for four various versions of Feynman-alpha formulas (the source is in region B, fast neutron detector is used either in region A or B).

| | Source in region B | | | |
|---|---|---|---|---|
| | 2-point 2-group | 2-point 1-group | 1-point 2-group | 1-point 1-group |
| $\omega_1$ | 1.00891 | 1.06295 | 1.12123 | 0.905974 |
| $\omega_2$ | 0.721378 | 0.0660208 | 0.516779 | |
| $\omega_3$ | 0.28402 | | | |
| $\omega_4$ | 0.00211384 | | | |
| | Fast neutron detections in region A | | | |
| $Y_1$ | 5.90179E-7 | 1.29166E-6 | 9.406E-6 | 0.943027E-4 |
| $Y_2$ | 4.20869E-6 | 0.014281 | 0.122662E-3 | |
| $Y_3$ | 0.159498E-2 | | | |
| $Y_4$ | 1.07716 | | | |
| | Fast neutron detections in region B | | | |
| $Y_1$ | −2.65992E-9 | −2.18905E-7 | 9.406E-6 | 0.943027E-4 |
| $Y_2$ | −1.33593E-6 | 0.567443E-4 | 0.122662E-3 | |
| $Y_3$ | 8.38725E-6 | | | |
| $Y_4$ | 0.00477401 | | | |

case when the source and the fast neutron detector are both placed in the region of the nuclear material. Therefore, in this situation all three versions of the Feynman-alpha theory, the two-point two-group, the two-point one-group and one-point one-group, can be used, although it is more time-efficient to use the one-group one-point version of the Feynman-alpha theory. As an example, in reality this case may be related to the measurements performed in the spent fuel pool when the detector is placed in the control tube of fuel assembly.

In the case when the fast neutron detector is in region B (fig. 6), a slight difference is observed between the two-point two-group and the two-point one-group versions of Feynman-alpha theories. At the same time, the one-point two-group

**Table 3.** The values of the $Y_i$ and $\omega_i$ calculated for four various versions of Feynman-alpha formulas (the source is in region A, thermal neutron detector is used either in region A or B).

| | Source in region A | | | |
|---|---|---|---|---|
| | 2-point 2-group | 2-point 1-group | 1-point 2-group | 1-point 1-group |
| $\omega_1$ | 1.52001 | 1.8083 | 1.45471 | 0.923611 |
| $\omega_2$ | 1.12141 | 0.743335 | 0.350087 | |
| $\omega_3$ | 0.759289 | | | |
| $\omega_4$ | 0.0983484 | | | |
| | Thermal neutron detections in region A | | | |
| $Y_1$ | −1.31014E-7 | 0.00154341 | −0.806821E-4 | 0.0126651 |
| $Y_2$ | 9.33539E-7 | 0.00938149 | 0.00139308 | |
| $Y_3$ | −2.39735E-6 | | | |
| $Y_4$ | 0.000052813 | | | |
| | Thermal neutron detections in region B | | | |
| $Y_1$ | 0.108441E-3 | −0.134373E-3 | −0.806821E-4 | 0.0126651 |
| $Y_2$ | −0.505571E-3 | 0.795215E-3 | 0.00139308 | |
| $Y_3$ | 0.663882E-3 | | | |
| $Y_4$ | 0.258841E-3 | | | |

**Table 4.** The values of the $Y_i$ and $\omega_i$ calculated for four various versions of Feynman-alpha formulas (the source is in region B, thermal neutron detector is used either in region A or B).

| | Source in region B | | | |
|---|---|---|---|---|
| | 2-point 2-group | 2-point 1-group | 1-point 2-group | 1-point 1-group |
| $\omega_1$ | 1.00891 | 1.06295 | 1.12123 | 0.905974 |
| $\omega_2$ | 0.721378 | 0.0660208 | 0.516779 | |
| $\omega_3$ | 0.28402 | | | |
| $\omega_4$ | 0.00211384 | | | |
| | Thermal neutron detections in region A | | | |
| $Y_1$ | −8.84973E-9 | 1.29166E-6 | −4.82586E-6 | 0.943027E-4 |
| $Y_2$ | 4.87895E-8 | 0.014281 | 0.227173E-4 | |
| $Y_3$ | −5.6035E-6 | | | |
| $Y_4$ | 0.0974942 | | | |
| | Thermal neutron detections in region B | | | |
| $Y_1$ | 1.71005E-6 | −2.18905E-7 | −4.82586E-6 | 0.943027E-4 |
| $Y_2$ | −7.10267E-6 | 0.567443E-4 | 0.227173E-4 | |
| $Y_3$ | 0.232948E-4 | | | |
| $Y_4$ | 0.0170835 | | | |

and one-point one-group versions of Feynman-alpha theory significantly overestimate the values of variance to mean ratio obtained with the two-point two-group version of the formulas. Thus, in this case two versions of Feynman-alpha theory, the two-point two-group and the two-point one-group can be used, although it is more time-efficient to use the two-point one-group version for quantitative estimates.

The differences between the various versions of Feynman-alpha theory are significantly higher when the neutron source is placed in region B and the fast neutron detector is in either region A or B, see figs. 7 and 8.

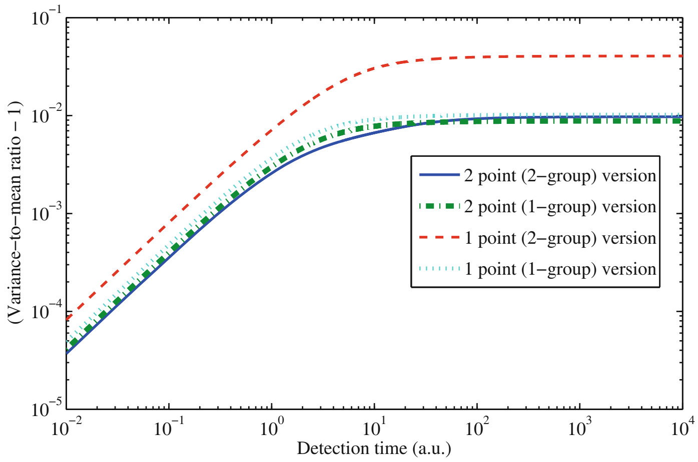


**Fig. 5.** The dependence of the ratio of the variance to mean of the number of fast neutron detections on the detection time for four versions of Feynman-alpha theory (the source is in region A, detector is in region A).

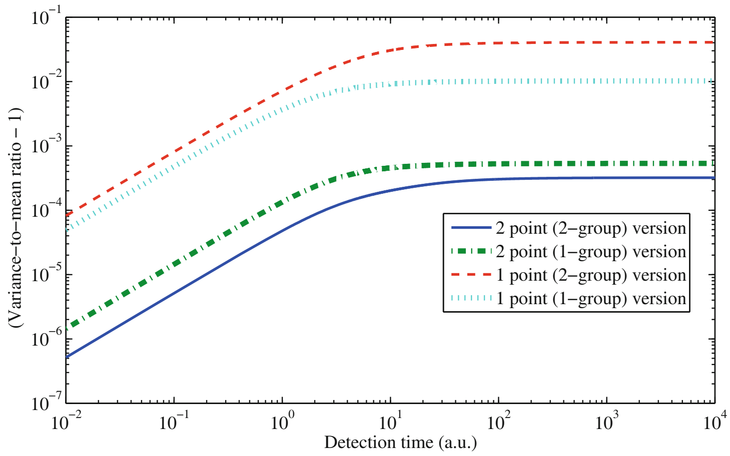


**Fig. 6.** The dependence of the ratio of the variance to mean of the number of fast neutron detections on the detection time for four versions of Feynman-alpha theory (the source is in region A, detector is in region B).

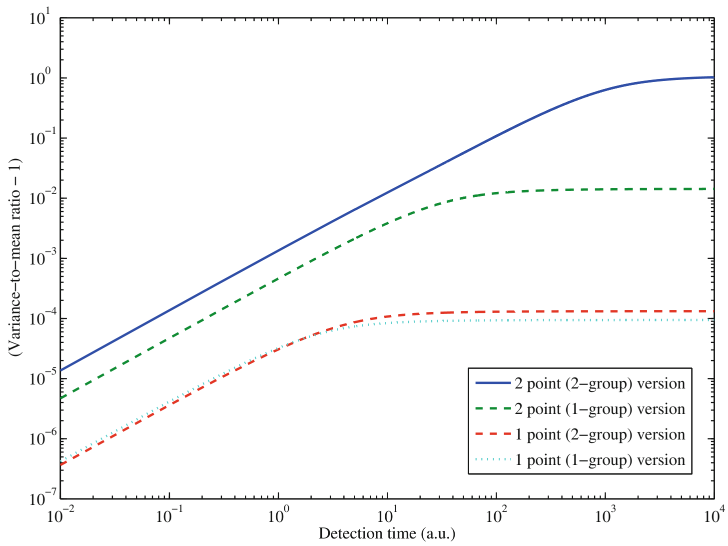


**Fig. 7.** The dependence of the ratio of the variance to mean of the number of fast neutron detections on the detection time for four versions of Feynman-alpha theory (the source is in region B, detector is in region A).

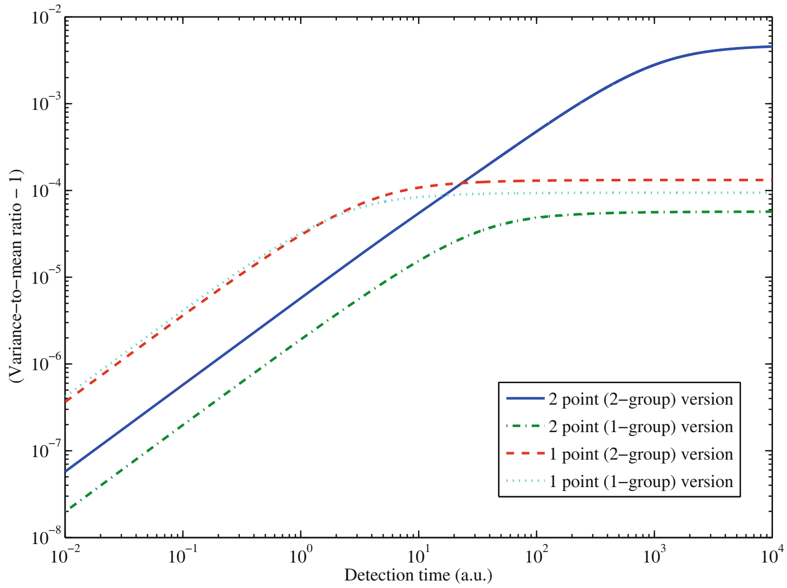


**Fig. 8.** Dependence of the ratio of the variance to mean of the number of fast neutron detections on the detection time for four versions of Feynman-alpha theory (the source is in region B, detector is in region B).

In all cases when the fast neutron detector is used, the two-point two-group version of Feynman-alpha formulas produce high values of the asymptotic variance-to-mean ratio compared to results obtained with other versions, *i.e.* the two-point one-group, one-point two-group and one-point one-group versions of the theory.

### 4.4 Comparison of four versions of the Feynman-alpha theoretical formulas for the case of thermal neutron detections

Regarding thermal neutron detection, when the source and the detector are in region A (fig. 9), the three special versions of the Feynman-alpha theory, the two-point one-group, the one-point two-group and the one-point one-group, all deviate significantly from the two-point two-group version. However, the one-point one-group theory gives very similar predictions of the ratio of the variance to mean as the two-point one-group theory. At the same time, the two-group one-point theory provides somewhat more accurate results. Thus, the impact of the energy dependence appears to be somewhat higher than the impact of the space dependence.

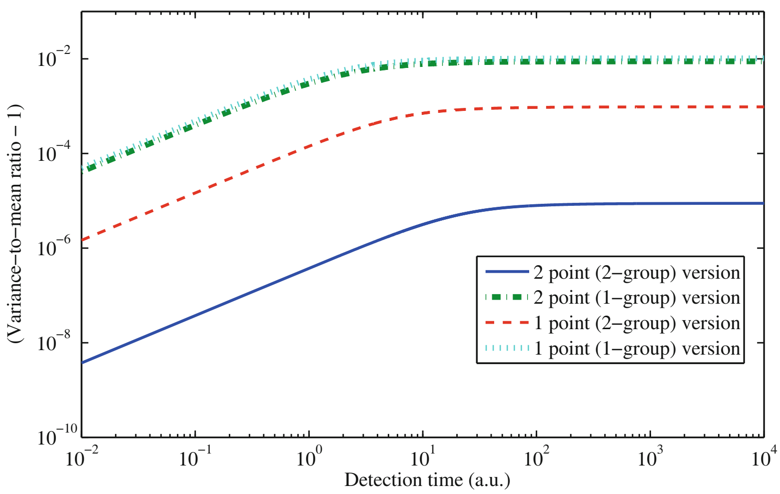


**Fig. 9.** The dependence of the ratio of the variance to mean of the number of thermal neutron detections on the detection time for four versions of Feynman-alpha theory (the source is in region A, detector is in region A).

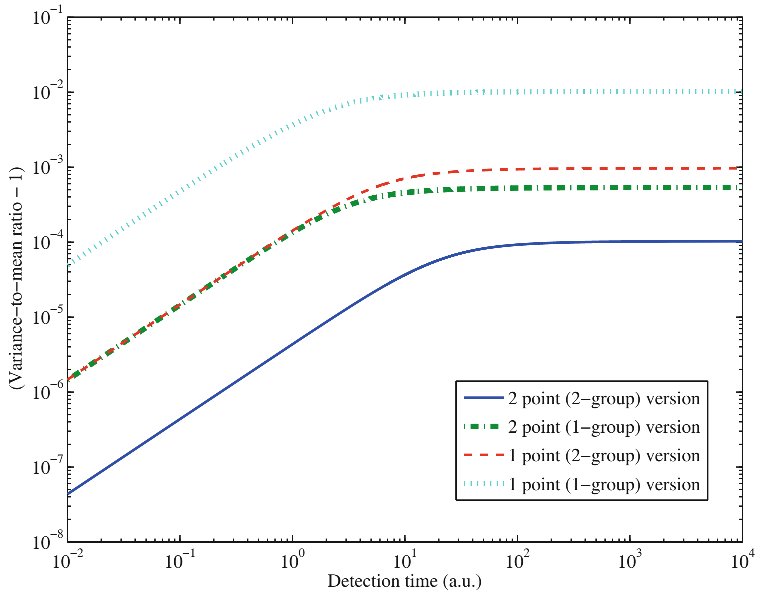


**Fig. 10.** The dependence of the ratio of the variance to mean of the number of thermal neutron detections on the detection time for four versions of Feynman-alpha theory (the source is in region A, detector is in region B).

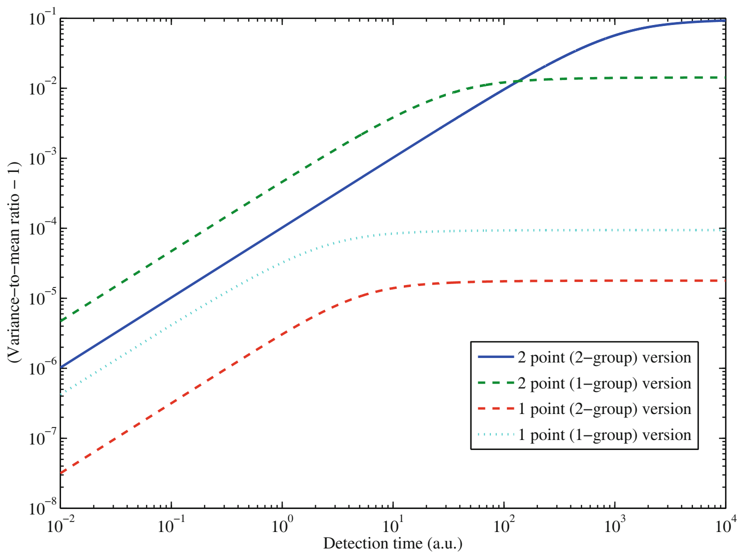


**Fig. 11.** Dependence of the ratio of the variance to mean of the number of thermal neutron detections on the detection time for four versions of Feynman-alpha theory (the source is in region B, detector is in region A).

When the thermal neutron detection is performed in region B (fig. 10), we may conclude that both the space-dependent and energy-dependent aspects play important role for this case.

If the source is in region B and detection is performed in region A (fig. 11), the two-point one-group version of the Feynman-alpha theory gives results which are closer to the one obtained with two-point two-group theory. Thus, the impact of the space dependence to the final results is higher than the impact of energy dependence.

Although, for a case of detector and source being placed in region B (fig. 12), the two-point one-group, two-group one-point and one-group one-point versions provide results of the variance-to-mean ratio that are significantly deviating from the ratio obtained by using the two-point two-group theory. Thus, the space-dependent and energy-dependent aspects, both play the important role in this situation.

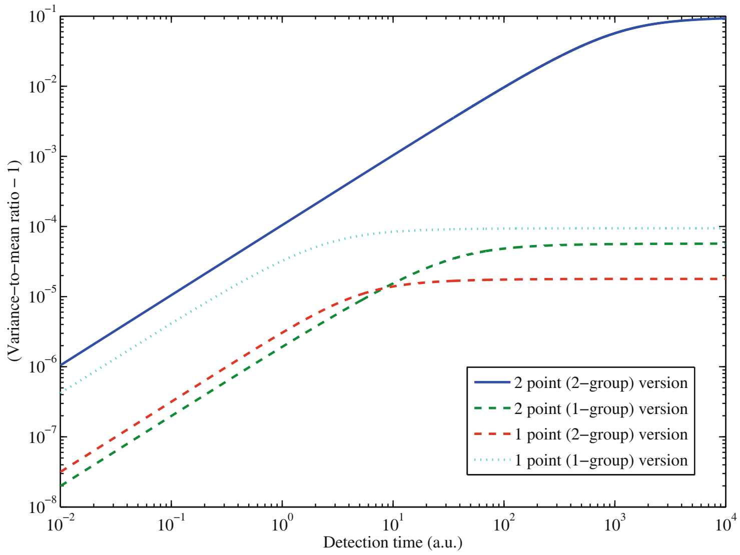


**Fig. 12.** Dependence of the ratio of the variance to mean of the number of thermal neutron detections on the detection time for four versions of Feynman-alpha theory (the source is in region B, detector is in region B).

In general, one can say that for the thermal neutron detections when the detection is made in region A, the energy-dependence has a higher impact to the ratio of the variance to mean than the space-dependent factor. On the other hand, for detection in region B both factors should be equally taken into account.

## 5 Conclusions

The two-group two-point version of Feynman-alpha theory was derived with a use of the forward master equation technique. The two-group one-point Feynman-alpha theory (with delayed neutrons) is extended by including fast neutron detections and fast fissions. The two-point one-group variance-to-mean formula (with delayed neutrons) is enhanced as well, by including detection and source terms in both regions. Thus, this gives the possibility of treating fast reflected systems in a more accurate way, by treating the counts separately in the fast and the thermal groups (or in the nuclear material (fissile region) and reflector regions).

The comparative study of the two-group two-point, the two-group one-point, the one-group two-point and the one-group one-point Feynman-alpha models is made by using the specific reaction intensities obtained in Monte Carlo simulations. It is shown that for all cases when the fast neutron detector is used in measurements, the space-dependent aspect has a higher impact on the final results than the energy-dependent aspect. In particular, when the source and the fast neutron detector are both placed in the region of nuclear material, three versions of the Feynman-alpha theory provide similar accuracy in the determination of the variance to mean ratio. Namely, the two-point two-group, the two-point one-group and one-point one-group can be used, although it is more time-efficient to use the one-group one-point version of the Feynman-alpha theory. The situation is not so optimistic for the case when the fast neutron detector is in region B, because a slight difference is observed between all versions of the theories. The one-point two-group and one-point one-group versions of Feynman-alpha theory significantly overestimate the values of variance to mean ratio obtained with the two-point two-group version of the formulas. Therefore, in this case two versions of Feynman-alpha theory, the two-point two-group, the two-point one-group can be considered as the accurate qualitative estimates. Regarding the use of the thermal neutron detections, both energy- and space-dependent factors are important to take into account. However, when the detection is done in region A, the space-dependence has a higher impact to the ratio of the variance to mean than the energy-dependence, while, for detection in region B both factors should be equally considered.

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## Appendix A.

In appendix A the analytical expressions for some of the coefficients used in two-point two-group Feynman-alpha theory are given as follows:

$$
\begin{aligned}
a &= \lambda_{A1} + \lambda_{A2} + \lambda_{B1} + \lambda_{B2} - \lambda_{A1f}\nu_{A1}{}' - \lambda_{B1f}\nu_{B1}{}',\\
b &= -\lambda_{A1t}\lambda_{B1t} + \lambda_{A2}\lambda_{B2} - \lambda_{A2t}\lambda_{B2t} - \lambda_{A2}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right) - \lambda_{B2}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right) - \lambda_{A2}\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right)\\
&\quad - \lambda_{B2}\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right) + \left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right)\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right) - \lambda_{A2f}\lambda_{Ar}\nu_{A2}{}' - \lambda_{B2f}\lambda_{Br}\nu_{B2}{}',\\
c &= -\lambda_{A1t}\lambda_{A2}\lambda_{B1t} - \lambda_{A1t}\lambda_{B1t}\lambda_{B2} - \lambda_{A2}\lambda_{B2}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right) + \lambda_{A2t}\lambda_{B2t}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right)\\
&\quad - \lambda_{A2}\lambda_{B2}\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right) + \lambda_{A2t}\lambda_{B2t}\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right) + \lambda_{A2}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right)\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right)\\
&\quad + \lambda_{B2}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right)\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right) - \lambda_{A2f}\lambda_{Ar}\lambda_{B2}\nu_{A2}{}' + \lambda_{A2f}\lambda_{Ar}\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right)\nu_{A2}{}'\\
&\quad - \lambda_{A2}\lambda_{B2f}\lambda_{Br}\nu_{B2}{}' + \lambda_{B2f}\lambda_{Br}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right)\nu_{B2}{}',\\
d &= -\lambda_{A1t}\lambda_{A2}\lambda_{B1t}\lambda_{B2} + \lambda_{A1t}\lambda_{A2t}\lambda_{B1t}\lambda_{B2t} + \lambda_{A2}\lambda_{B2}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right)\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right)\\
&\quad - \lambda_{A2t}\lambda_{B2t}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right)\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right) - \lambda_{A1t}\lambda_{A2f}\lambda_{B2t}\lambda_{Br}\nu_{A2}{}' + \lambda_{A2f}\lambda_{Ar}\lambda_{B2}\left(\lambda_{B1f}\nu_{B1}{}' - \lambda_{B1}\right)\nu_{A2}{}'\\
&\quad - \lambda_{A2t}\lambda_{Ar}\lambda_{B1t}\lambda_{B2f}\nu_{B2}{}' + \lambda_{A2}\lambda_{B2f}\lambda_{Br}\left(\lambda_{A1f}\nu_{A1}{}' - \lambda_{A1}\right)\nu_{B2}{}' + \lambda_{A2f}\lambda_{Ar}\lambda_{B2f}\lambda_{Br}\nu_{A2}{}'\nu_{B2}{}',\\
&= \omega_1\omega_2\omega_3\omega_4,\\
K_3 &= \mu_{X_AX_A},\\
K_2 &= \lambda_{A2f}\mu_{X_AY_A}q'_{A2}\left(X_A\right) + \lambda_{A2}\mu_{X_AX_A} - \lambda_{B1f}\mu_{X_AX_A}q'_{B1}\left(X_B\right) + \lambda_{B1}\mu_{X_AX_A} + \lambda_{B1t}\mu_{X_AX_B} + \lambda_{B2}\mu_{X_AX_A},\\
K_1 &= \lambda_{A2f}\lambda_{B1f}\mu_{X_AY_A}q'_{A2}\left(X_A\right)q'_{B1}\left(X_B\right) + \lambda_{A2f}\lambda_{B1}\mu_{X_AY_A}q'_{A2}\left(X_A\right) + \lambda_{A2f}\lambda_{B2}\mu_{X_AY_A}q'_{A2}\left(X_A\right)\\
&\quad + \lambda_{A2f}\lambda_{B2t}\mu_{X_AY_B}q'_{A2}\left(X_A\right) - \lambda_{A2}\lambda_{B1f}\mu_{X_AX_A}q'_{B1}\left(X_B\right) + \lambda_{A2}\lambda_{B1}\mu_{X_AX_A} + \lambda_{A2}\lambda_{B1t}\mu_{X_AX_B}\\
&\quad + \lambda_{A2}\lambda_{B2}\mu_{X_AX_A} - \lambda_{A2t}\lambda_{B2t}\mu_{X_AX_A} - \lambda_{B1f}\lambda_{B2}\mu_{X_AX_A}q'_{B1}\left(X_B\right) + \lambda_{B1t}\lambda_{B2f}\mu_{X_AY_B}q'_{B2}\left(X_B\right)\\
&\quad - \lambda_{B2f}\lambda_{Br}\mu_{X_AX_A}q'_{B2}\left(X_B\right) + \lambda_{B1}\lambda_{B2}\mu_{X_AX_A} + \lambda_{B1t}\lambda_{B2}\mu_{X_AX_B},\\
K_0 &= \lambda_{A2f}\lambda_{B1f}\lambda_{B2}\mu_{X_AY_A}q'_{A2}\left(X_A\right)q'_{B1}\left(X_B\right) - \lambda_{A2f}\lambda_{B1f}\lambda_{B2t}\mu_{X_AY_B}q'_{A2}\left(X_A\right)q'_{B1}\left(X_B\right)\\
&\quad - \lambda_{A2f}\lambda_{B2f}\lambda_{Br}\mu_{X_AY_A}q'_{A2}\left(X_A\right)q'_{B2}\left(X_B\right) + \lambda_{A2f}\lambda_{B1}\lambda_{B2}\mu_{X_AY_A}q'_{A2}\left(X_A\right) + \lambda_{A2f}\lambda_{B1}\lambda_{B2t}\mu_{X_AY_B}q'_{A2}\left(X_A\right)\\
&\quad + \lambda_{A2f}\lambda_{B2t}\lambda_{Br}\mu_{X_AX_B}q'_{A2}\left(X_A\right) - \lambda_{A2}\lambda_{B1f}\lambda_{B2}\mu_{X_AX_A}q'_{B1}\left(X_B\right) + \lambda_{A2}\lambda_{B1t}\lambda_{B2f}\mu_{X_AY_B}q'_{B2}\left(X_B\right)\\
&\quad - \lambda_{A2}\lambda_{B2f}\lambda_{Br}\mu_{X_AX_A}q'_{B2}\left(X_B\right) + \lambda_{A2}\lambda_{B1}\lambda_{B2}\mu_{X_AX_A} + \lambda_{A2}\lambda_{B1t}\lambda_{B2}\mu_{X_AX_B}\\
&\quad + \lambda_{A2t}\lambda_{B1f}\lambda_{B2t}\mu_{X_AX_A}q'_{B1}\left(X_B\right) + \lambda_{A2t}\lambda_{B1t}\lambda_{B2f}\mu_{X_AY_A}q'_{B2}\left(X_B\right) - \lambda_{A2t}\lambda_{B1}\lambda_{B2t}\mu_{X_AX_A} - \lambda_{A2t}\lambda_{B1t}\lambda_{B2t}\mu_{X_AX_B},\\
L_3 &= \mu_{Y_AY_A},\\
L_2 &= -\lambda_{A1f}\mu_{Y_AY_A}q'_{A1}\left(X_A\right) + \lambda_{A1}\mu_{Y_AY_A} + \lambda_{Ar}\mu_{X_AY_A} - \lambda_{B1f}\mu_{Y_AY_A}q'_{B1}\left(X_B\right) + \lambda_{B1}\mu_{Y_AY_A} + \lambda_{B2}\mu_{Y_AY_A} + \lambda_{B2t}\mu_{Y_BY_A},\\
L_1 &= \lambda_{A1f}\lambda_{B1f}\mu_{Y_AY_A}q'_{A1}\left(X_A\right)q'_{B1}\left(X_B\right) - \lambda_{A1f}\lambda_{B1}\mu_{Y_AY_A}q'_{A1}\left(X_A\right) - \lambda_{A1f}\lambda_{B2}\mu_{Y_AY_A}q'_{A1}\left(X_A\right)\\
&\quad - \lambda_{A1f}\lambda_{B2t}\mu_{Y_BY_A}q'_{A1}\left(X_A\right) - \lambda_{A1}\lambda_{B1f}\mu_{Y_AY_A}q'_{B1}\left(X_B\right) + \lambda_{A1}\lambda_{B1}\mu_{Y_AY_A} + \lambda_{A1}\lambda_{B2}\mu_{Y_AY_A}\\
&\quad + \lambda_{A1}\lambda_{B2t}\mu_{Y_BY_A} - \lambda_{A1t}\lambda_{B1t}\mu_{Y_AY_A} - \lambda_{Ar}\lambda_{B1f}\mu_{X_AY_A}q'_{B1}\left(X_B\right) + \lambda_{Ar}\lambda_{B1}\mu_{X_AY_A}\\
&\quad + \lambda_{Ar}\lambda_{B1t}\mu_{X_BY_A} + \lambda_{Ar}\lambda_{B2}\mu_{X_AY_A} - \lambda_{B1f}\lambda_{B2}\mu_{Y_AY_A}q'_{B1}\left(X_B\right) - \lambda_{B1f}\lambda_{B2t}\mu_{Y_BY_A}q'_{B1}\left(X_B\right)\\
&\quad - \lambda_{B2f}\lambda_{Br}\mu_{Y_AY_A}q'_{B2}\left(X_B\right) + \lambda_{B1}\lambda_{B2}\mu_{Y_AY_A} + \lambda_{B1}\lambda_{B2t}\mu_{Y_BY_A} + \lambda_{B2t}\lambda_{Br}\mu_{X_BY_A},\\
L_0 &= \lambda_{A1f}\lambda_{B1f}\lambda_{B2}\mu_{Y_AY_A}q'_{A1}\left(X_A\right)q'_{B1}\left(X_B\right) + \lambda_{A1f}\lambda_{B1f}\lambda_{B2t}\mu_{Y_BY_A}q'_{A1}\left(X_A\right)q'_{B1}\left(X_B\right)\\
&\quad + \lambda_{A1f}\lambda_{B2f}\lambda_{Br}\mu_{Y_AY_A}q'_{A1}\left(X_A\right)q'_{B2}\left(X_B\right) - \lambda_{A1f}\lambda_{B1}\lambda_{B2}\mu_{Y_AY_A}q'_{A1}\left(X_A\right) - \lambda_{A1f}\lambda_{B1}\lambda_{B2t}\mu_{Y_BY_A}q'_{A1}\left(X_A\right)\\
&\quad - \lambda_{A1f}\lambda_{B2t}\lambda_{Br}\mu_{X_BY_A}q'_{A1}\left(X_A\right) - \lambda_{A1}\lambda_{B1f}\lambda_{B2}\mu_{Y_AY_A}q'_{B1}\left(X_B\right) - \lambda_{A1}\lambda_{B1f}\lambda_{B2t}\mu_{Y_BY_A}q'_{B1}\left(X_B\right)\\
&\quad - \lambda_{A1}\lambda_{B2f}\lambda_{Br}\mu_{Y_AY_A}q'_{B2}\left(X_B\right) + \lambda_{A1}\lambda_{B1}\lambda_{B2}\mu_{Y_AY_A} + \lambda_{A1}\lambda_{B1}\lambda_{B2t}\mu_{Y_BY_A} + \lambda_{A1}\lambda_{B2t}\lambda_{Br}\mu_{X_BY_A}\\
&\quad - \lambda_{A1t}\lambda_{B1t}\lambda_{B2}\mu_{Y_AY_A} - \lambda_{A1t}\lambda_{B1t}\lambda_{B2t}\mu_{Y_BY_A} + \lambda_{A1t}\lambda_{B2t}\lambda_{Br}\mu_{X_AY_A} - \lambda_{Ar}\lambda_{B1f}\lambda_{B2}\mu_{X_AY_A}q'_{B1}\left(X_B\right)\\
&\quad + \lambda_{Ar}\lambda_{B1t}\lambda_{B2f}\mu_{Y_BY_A}q'_{B2}\left(X_B\right) - \lambda_{Ar}\lambda_{B2f}\lambda_{Br}\mu_{X_AY_A}q'_{B2}\left(X_B\right) + \lambda_{Ar}\lambda_{B1}\lambda_{B2}\mu_{X_AY_A} + \lambda_{Ar}\lambda_{B1t}\lambda_{B2}\mu_{X_BY_A},\\
M_3 &= \mu_{X_BX_B},\\
M_2 &= -\lambda_{A1f}\mu_{X_BX_B}q'_{A1}\left(X_A\right) + \lambda_{A1}\mu_{X_BX_B} + \lambda_{A1t}\mu_{X_AX_B} + \lambda_{A2}\mu_{X_BX_B} + \lambda_{B2f}\mu_{X_BY_B}q'_{B2}\left(X_B\right) + \lambda_{B2}\mu_{X_BX_B},\\
M_1 &= -\lambda_{A1f}\lambda_{A2}\mu_{X_BX_B}q'_{A1}\left(X_A\right) - \lambda_{A1f}\lambda_{B2f}\mu_{X_BY_B}q'_{A1}\left(X_A\right)q'_{B2}\left(X_B\right) - \lambda_{A1f}\lambda_{B2}\mu_{X_BX_B}q'_{A1}\left(X_A\right)\\
&\quad + \lambda_{A1t}\lambda_{A2f}\mu_{X_BY_A}q'_{A2}\left(X_A\right) - \lambda_{A2f}\lambda_{Ar}\mu_{X_BX_B}q'_{A2}\left(X_A\right) + \lambda_{A1}\lambda_{A2}\mu_{X_BX_B} + \lambda_{A1}\lambda_{B2f}\mu_{X_BY_B}q'_{B2}\left(X_B\right)\\
&\quad + \lambda_{A1}\lambda_{B2}\mu_{X_BX_B} + \lambda_{A1t}\lambda_{A2}\mu_{X_AX_B} + \lambda_{A1t}\lambda_{B2}\mu_{X_AX_B} + \lambda_{A2}\lambda_{B2f}\mu_{X_BY_B}q'_{B2}\left(X_B\right) + \lambda_{A2}\lambda_{B2}\mu_{X_BX_B}\\
&\quad + \lambda_{A2t}\lambda_{B2f}\mu_{X_BY_A}q'_{B2}\left(X_B\right) - \lambda_{A2t}\lambda_{B2t}\mu_{X_BX_B},
\end{aligned}
$$

$$
\begin{aligned}
M_0 = &-\lambda_{A1f}\lambda_{A2}\lambda_{B2f}\mu_{X_BY_B}q'_{A1}(X_A)\,q'_{B2}(X_B) - \lambda_{A1f}\lambda_{A2}\lambda_{B2}\mu_{X_BX_B}q'_{A1}(X_A)\\
&- \lambda_{A1f}\lambda_{A2t}\lambda_{B2f}\mu_{X_BY_A}q'_{A1}(X_A)\,q'_{B2}(X_B) + \lambda_{A1f}\lambda_{A2t}\lambda_{B2t}\mu_{X_BX_B}q'_{A1}(X_A)\\
&+ \lambda_{A1t}\lambda_{A2f}\lambda_{B2}\mu_{X_BY_A}q'_{A2}(X_A) + \lambda_{A1t}\lambda_{A2f}\lambda_{B2t}\mu_{X_BY_B}q'_{A2}(X_A)\\
&- \lambda_{A2f}\lambda_{Ar}\lambda_{B2f}\mu_{X_BY_B}q'_{A2}(X_A)\,q'_{B2}(X_B) - \lambda_{A2f}\lambda_{Ar}\lambda_{B2}\mu_{X_BX_B}q'_{A2}(X_A)\\
&+ \lambda_{A1}\lambda_{A2}\lambda_{B2f}\mu_{X_BY_B}q'_{B2}(X_B) + \lambda_{A1}\lambda_{A2}\lambda_{B2}\mu_{X_BX_B} + \lambda_{A1}\lambda_{A2t}\lambda_{B2f}\mu_{X_BY_A}q'_{B2}(X_B)\\
&- \lambda_{A1}\lambda_{A2t}\lambda_{B2t}\mu_{X_BX_B} + \lambda_{A1t}\lambda_{A2}\lambda_{B2}\mu_{X_AX_B} - \lambda_{A1t}\lambda_{A2t}\lambda_{B2t}\mu_{X_AX_B} + \lambda_{A2t}\lambda_{Ar}\lambda_{B2f}\mu_{X_AX_B}q'_{B2}(X_B),\\
P_3 = &\ \mu_{Y_BY_B},\\
P_2 = &-\lambda_{A1f}\mu_{Y_BY_B}q'_{A1}(X_A) + \lambda_{A1}\mu_{Y_BY_B} + \lambda_{A2}\mu_{Y_BY_B} + \lambda_{A2t}\mu_{Y_BY_A} - \lambda_{B1f}\mu_{Y_BY_B}q'_{B1}(X_B) + \lambda_{B1}\mu_{Y_BY_B} + \lambda_{Br}\mu_{X_BY_B},\\
P_1 = &-\lambda_{A1f}\lambda_{A2}\mu_{Y_BY_B}q'_{A1}(X_A) - \lambda_{A1f}\lambda_{A2t}\mu_{Y_BY_A}q'_{A1}(X_A) + \lambda_{A1f}\lambda_{B1f}\mu_{Y_BY_B}q'_{A1}(X_A)\,q'_{B1}(X_B)\\
&- \lambda_{A1f}\lambda_{B1}\mu_{Y_BY_B}q'_{A1}(X_A) - \lambda_{A1f}\lambda_{Br}\mu_{X_BY_B}q'_{A1}(X_A) - \lambda_{A2f}\lambda_{Ar}\mu_{Y_BY_B}q'_{A2}(X_A) + \lambda_{A1}\lambda_{A2}\mu_{Y_BY_B}\\
&+ \lambda_{A1}\lambda_{A2t}\mu_{Y_BY_A} - \lambda_{A1}\lambda_{B1f}\mu_{Y_BY_B}q'_{B1}(X_B) + \lambda_{A1}\lambda_{B1}\mu_{Y_BY_B} + \lambda_{A1}\lambda_{Br}\mu_{X_BY_B} - \lambda_{A1t}\lambda_{B1t}\mu_{Y_BY_B}\\
&+ \lambda_{A1t}\lambda_{Br}\mu_{X_AY_B} - \lambda_{A2}\lambda_{B1f}\mu_{Y_BY_B}q'_{B1}(X_B) + \lambda_{A2}\lambda_{B1}\mu_{Y_BY_B} + \lambda_{A2}\lambda_{Br}\mu_{X_BY_B} + \lambda_{A2t}\lambda_{Ar}\mu_{X_AY_B}\\
&- \lambda_{A2t}\lambda_{B1f}\mu_{Y_BY_A}q'_{B1}(X_B) + \lambda_{A2t}\lambda_{B1}\mu_{Y_BY_A},\\
P_0 = &\ \lambda_{A1f}\lambda_{A2}\lambda_{B1f}\mu_{Y_BY_B}q'_{A1}(X_A)\,q'_{B1}(X_B) - \lambda_{A1f}\lambda_{A2}\lambda_{B1}\mu_{Y_BY_B}q'_{A1}(X_A) - \lambda_{A1f}\lambda_{A2}\lambda_{Br}\mu_{X_BY_B}q'_{A1}(X_A)\\
&+ \lambda_{A1f}\lambda_{A2t}\lambda_{B1f}\mu_{Y_BY_A}q'_{A1}(X_A)\,q'_{B1}(X_B) - \lambda_{A1f}\lambda_{A2t}\lambda_{B1}\mu_{Y_BY_A}q'_{A1}(X_A) + \lambda_{A1t}\lambda_{A2f}\lambda_{Br}\mu_{Y_BY_A}q'_{A2}(X_A)\\
&+ \lambda_{A2f}\lambda_{Ar}\lambda_{B1f}\mu_{Y_BY_B}q'_{A2}(X_A)\,q'_{B1}(X_B) - \lambda_{A2f}\lambda_{Ar}\lambda_{B1}\mu_{Y_BY_B}q'_{A2}(X_A) - \lambda_{A2f}\lambda_{Ar}\lambda_{Br}\mu_{X_BY_B}q'_{A2}(X_A)\\
&- \lambda_{A1}\lambda_{A2}\lambda_{B1f}\mu_{Y_BY_B}q'_{B1}(X_B) + \lambda_{A1}\lambda_{A2}\lambda_{B1}\mu_{Y_BY_B} + \lambda_{A1}\lambda_{A2}\lambda_{Br}\mu_{X_BY_B} - \lambda_{A1}\lambda_{A2t}\lambda_{B1f}\mu_{Y_BY_A}q'_{B1}(X_B)\\
&+ \lambda_{A1}\lambda_{A2t}\lambda_{B1}\mu_{Y_BY_A} - \lambda_{A1t}\lambda_{A2}\lambda_{B1t}\mu_{Y_BY_B} + \lambda_{A1t}\lambda_{A2}\lambda_{Br}\mu_{X_AY_B} - \lambda_{A1t}\lambda_{A2t}\lambda_{B1t}\mu_{Y_BY_A}\\
&- \lambda_{A2t}\lambda_{Ar}\lambda_{B1f}\mu_{X_AY_B}q'_{B1}(X_B) + \lambda_{A2t}\lambda_{Ar}\lambda_{B1}\mu_{X_AY_B} + \lambda_{A2t}\lambda_{Ar}\lambda_{B1t}\mu_{X_BY_B}.
\end{aligned}
$$

## Appendix B.

In appendix B the analytical expressions for some of the coefficients used in one-group two-point Feynman-alpha theory (with delayed neutrons) are given as follows:

$$
\begin{aligned}
a = &\ \beta_A\lambda_{Af}\nu'_A - \lambda_{Af}\nu'_A + \lambda_A + \lambda_{Ac} + \beta_B\lambda_{Bf}\nu'_B - \lambda_{Bf}\nu'_B + \lambda_B + \lambda_{Bc},\\
b = &-\lambda_{Ac}\lambda_{Af}\nu'_A + \lambda_A\lambda_{Ac} + \beta_A\lambda_{Af}\lambda_B\nu'_A - \beta_A\lambda_{Af}\lambda_{Bf}\nu'_A\nu'_B + \beta_A\lambda_{Af}\beta_B\lambda_{Bf}\nu'_A\nu'_B - \lambda_{Af}\beta_B\lambda_{Bf}\nu'_A\nu'_B\\
&+ \lambda_{Af}\lambda_{Bf}\nu'_A\nu'_B - \lambda_{Af}\lambda_B\nu'_A + \beta_A\lambda_{Af}\lambda_{Bc}\nu'_A - \lambda_{Af}\lambda_{Bc}\nu'_A + \lambda_A\beta_B\lambda_{Bf}\nu'_B - \lambda_A\lambda_{Bf}\nu'_B + \lambda_A\lambda_B + \lambda_A\lambda_{Bc}\\
&+ \lambda_{Ac}\beta_B\lambda_{Bf}\nu'_B - \lambda_{Ac}\lambda_{Bf}\nu'_B + \lambda_{Ac}\lambda_B + \lambda_{Ac}\lambda_{Bc} - \lambda_{At}\lambda_{Bt} - \lambda_{Bc}\lambda_{Bf}\nu'_B + \lambda_B\lambda_{Bc},\\
c = &-\lambda_{Ac}\lambda_{Af}\beta_B\lambda_{Bf}\nu'_A\nu'_B + \lambda_{Ac}\lambda_{Af}\lambda_{Bf}\nu'_A\nu'_B - \lambda_{Ac}\lambda_{Af}\lambda_B\nu'_A - \lambda_{Ac}\lambda_{Af}\lambda_{Bc}\nu'_A + \lambda_A\lambda_{Ac}\beta_B\lambda_{Bf}\nu'_B - \lambda_A\lambda_{Ac}\lambda_{Bf}\nu'_B\\
&+ \lambda_A\lambda_{Ac}\lambda_B + \lambda_A\lambda_{Ac}\lambda_{Bc} + \beta_A\lambda_{Af}\lambda_B\lambda_{Bc}\nu'_A - \beta_A\lambda_{Af}\lambda_{Bc}\lambda_{Bf}\nu'_A\nu'_B + \lambda_{Af}\lambda_{Bc}\lambda_{Bf}\nu'_A\nu'_B - \lambda_{Af}\lambda_B\lambda_{Bc}\nu'_A\\
&- \lambda_A\lambda_{Bc}\lambda_{Bf}\nu'_B + \lambda_A\lambda_B\lambda_{Bc} - \lambda_{Ac}\lambda_{At}\lambda_{Bt} - \lambda_{Ac}\lambda_{Bc}\lambda_{Bf}\nu'_B + \lambda_{Ac}\lambda_B\lambda_{Bc} - \lambda_{At}\lambda_{Bc}\lambda_{Bt},\\
d = &\ \lambda_{Ac}\lambda_{Af}\lambda_{Bc}\lambda_{Bf}\nu'_A\nu'_B - \lambda_{Ac}\lambda_{Af}\lambda_B\lambda_{Bc}\nu'_A - \lambda_A\lambda_{Ac}\lambda_{Bc}\lambda_{Bf}\nu'_B + \lambda_A\lambda_{Ac}\lambda_B\lambda_{Bc} - \lambda_{Ac}\lambda_{At}\lambda_{Bc}\lambda_{Bt},\\
K_3 = &\ \mu_{XX},\\
K_2 = &\ \lambda_{Ac}\mu_{XV} + \lambda_{Ac}\mu_{XX} + \beta_B\lambda_{Bf}\mu_{XX}\nu'_B - \lambda_{Bf}\mu_{XX}\nu'_B + \lambda_B\mu_{XX} + \lambda_{Bc}\mu_{XX} + \lambda_{Bt}\mu_{XY},\\
K_1 = &\ \lambda_{Ac}\beta_B\lambda_{Bf}\mu_{XV}\nu'_B - \lambda_{Ac}\lambda_{Bf}\mu_{XV}\nu'_B + \lambda_{Ac}\beta_B\lambda_{Bf}\mu_{XX}\nu'_B - \lambda_{Ac}\lambda_{Bf}\mu_{XX}\nu'_B + \lambda_{Ac}\lambda_B\mu_{XV} + \lambda_{Ac}\lambda_B\mu_{XX}\\
&+ \lambda_{Ac}\lambda_{Bc}\mu_{XV} + \lambda_{Ac}\lambda_{Bc}\mu_{XX} + \lambda_{Ac}\lambda_{Bt}\mu_{XY} - \lambda_{Bc}\lambda_{Bf}\mu_{XX}\nu'_B + \lambda_B\lambda_{Bc}\mu_{XX} + \lambda_{Bc}\lambda_{Bt}\mu_{XW} + \lambda_{Bc}\lambda_{Bt}\mu_{XY},\\
K_0 = &-\lambda_{Ac}\lambda_{Bc}\lambda_{Bf}\mu_{XV}\nu'_B - \lambda_{Ac}\lambda_{Bc}\lambda_{Bf}\mu_{XX}\nu'_B + \lambda_{Ac}\lambda_B\lambda_{Bc}\mu_{XV} + \lambda_{Ac}\lambda_B\lambda_{Bc}\mu_{XX}\\
&+ \lambda_{Ac}\lambda_{Bc}\lambda_{Bt}\mu_{XW} + \lambda_{Ac}\lambda_{Bc}\lambda_{Bt}\mu_{XY},\\
L_3 = &\ \mu_{YY},\\
L_2 = &\ \beta_A\lambda_{Af}\mu_{YY}\nu'_A - \lambda_{Af}\mu_{YY}\nu'_A + \lambda_A\mu_{YY} + \lambda_{Ac}\mu_{YY} + \lambda_{At}\mu_{XY} + \lambda_{Bc}\mu_{YW} + \lambda_{Bc}\mu_{YY},\\
L_1 = &-\lambda_{Ac}\lambda_{Af}\mu_{YY}\nu'_A + \lambda_A\lambda_{Ac}\mu_{YY} + \beta_A\lambda_{Af}\lambda_{Bc}\mu_{YW}\nu'_A - \lambda_{Af}\lambda_{Bc}\mu_{YW}\nu'_A + \beta_A\lambda_{Af}\lambda_{Bc}\mu_{YY}\nu'_A\\
&- \lambda_{Af}\lambda_{Bc}\mu_{YY}\nu'_A + \lambda_A\lambda_{Bc}\mu_{YW} + \lambda_A\lambda_{Bc}\mu_{YY} + \lambda_{Ac}\lambda_{At}\mu_{XY} + \lambda_{Ac}\lambda_{At}\mu_{YV} + \lambda_{Ac}\lambda_{Bc}\mu_{YW}\\
&+ \lambda_{Ac}\lambda_{Bc}\mu_{YY} + \lambda_{At}\lambda_{Bc}\mu_{XY},\\
L_0 = &-\lambda_{Ac}\lambda_{Af}\lambda_{Bc}\mu_{YW}\nu'_A - \lambda_{Ac}\lambda_{Af}\lambda_{Bc}\mu_{YY}\nu'_A + \lambda_A\lambda_{Ac}\lambda_{Bc}\mu_{YW} + \lambda_A\lambda_{Ac}\lambda_{Bc}\mu_{YY}\\
&+ \lambda_{Ac}\lambda_{At}\lambda_{Bc}\mu_{XY} + \lambda_{Ac}\lambda_{At}\lambda_{Bc}\mu_{YV}.
\end{aligned}
$$

## References


1. E. Fermi, R.P. Feynman, F. de Hoffmann, *Theory of the Criticality of the Water Boiler and the Determination of the Number of Delayed Neutrons*, USAEC Report MDDC-383 (LADC-269) Los Alamos Scientific Laboratory (1944).
2. F. de Hoffmann, *Intensity Fluctuations of a Neutron Chain Reaction*, USAEC Report MDDC-382 (LADC-256) Los Alamos Scientific Laboratory (1946).
3. R.P. Feynman, F. de Hoffmann, R. Serber, J. Nucl. Energy **3**, 64 (1956).
4. I. Pázsit, L. Pál, *Neutron Fluctuations: A Treatise on the Physics of Branching Processes* (Elsevier Science Ltd., London, New York, Tokyo, 2008).
5. H. Kiyavitskaya *et al.*, *YALINA-Booster Benchmark Specifications for the IAEA Coordinated Research Projects on Analytical and Experimental Benchmark Analysis on Accelerator Driven Systems and Low Enriched Uranium Fuel Utilization in Accelerator Driven Sub-Critical Assembly Systems*, (IAEA, Vienna, 2007).
6. Y. Gohar, D.L. Smith, *Report YALINA Facility, A Sub-Critical Accelerator-Driven System (ADS) for Nuclear Energy Research. Facility Description and Overview of the Research Program*, Argonne National Laboratory, ANL-10/05 (2010).
7. A. Talamo, Y. Gohar, *Deterministic and Monte Carlo Modeling of YALINA Thermal Subcritical Assembly*, Argonne National Laboratory, ANL-10/17 (2010).
8. R. Soule, W. Assal, P. Chaussonnet *et al.*, Nucl. Sci. Eng. **148**, 124 (2004).
9. A. Kochetkov *et al.*, *Current progress and future plans of the FREYA Project*, in *Proceedings of the International Conference on Technology and Components of Accelerator Driven Systems (TSADS-2), Nantes, France* (2013).
10. P. Cartemo, A. Nordlund, D. Chernikova, *Sensitivity of the neutronic design of an Accelerator-Driven System (ADS) to the anisotropy of yield of the neutron generator and variation of nuclear data libraries*, ESARDA meeting, Brugge (2013).
11. J.-L. Munoz-Cobo, C. Berglof, J. Pena, D. Villamarin, V. Bournos, Ann. Nucl. Energy **38**, 194 (2011).
12. C. Berglof, M. Fernandez-Ordonez, D. Villamarin *et al.*, Ann. Nucl. Energy **38**, 590 (2011).
13. T. Ye *et al.*, in *Proceedings of the 2006 Symposium on nuclear data, Japan* (2006).
14. G.D. Spriggs, Ann. Nucl. Energy **24**, 205 (1997).
15. Y.S. Rana, S.B. Degweker, Nucl. Sci. Eng. **162**, 117 (2009).
16. Y.S. Rana, S.B. Degweker, Nucl. Sci. Eng **169**, 98 (2011).
17. S.B. Degweker, Ann. Nucl. Energy **27**, 1245 (2000).
18. Y. Yamane, A. Uritani, K. Ishitani, H. Kataoka, S. Shiroya, C. Ichihara, K. Kobayashi, S. Okajima, *Experimental study on reactor dynamics of accelerator driven subcritical system*, *KURRI Progress Report*, Vol. **46** (Research Reactor Institute, Kyoto University, 1999) (2000).
19. Y. Yamane, A. Uritani, H. Kataoka, T. Nomura, M. Matsui, R. Naka, A. Tabuchi, T. Hayashi, H. Yamauchi, Y. Kitamura, S. Shiroya, T. Misawa, K. Kobayashi, C. Ichihara, H. Unesaki, H. Nakamura, *Experimental study on reactor dynamics of accelerator driven subcritical system (II)*, *KURRI Progress Report*, Vol. **53** (Research Reactor Institute, Kyoto University, 2001).
20. Y. Yamane, Y. Kitamura, H. Kataoka, K. Ishitani, S. Shiroya, in *Proceedings of PHYSOR-2002 International Conference on the New Frontiers of Nuclear Technology: Reactor Physics, Safety and High-Performance Computing* (Seoul, Korea, 2002).
21. Y. Kitamura, H. Yamauchi, Y. Yamane, Ann. Nucl. Energy **30**, 897 (2003).
22. Y. Kitamura, H. Yamauchi, Y. Yamane, T. Misawa, C. Ichihara, H. Nakamura, Ann. Nucl. Energy **31**, 163 (2004).
23. Y. Kitamura, K. Taguchi, A. Yamamoto, Y. Yamane, T. Misawa, C. Ichihara, H. Nakamura, H. Oigawa, Int. J. Nucl. Energy Sci. Technol. **2**, 266 (2006).
24. Y. Kitamura, I. Pázsit, J. Wright, A. Yamamoto, Y. Yamane, Ann. Nucl. Energy **32**, 671 (2005).
25. J.L. Munoz-Cobo, Y. Rugama, T.E. Valentine, J.T. Michalzo, R.B. Perez, Ann. Nucl. Energy **28**, 1519 (2001).
26. S.B. Degweker, Ann. Nucl. Energy **30**, 223 (2003).
27. M. Ceder, I. Pázsit, Prog. Nucl. Energy **43**, 429 (2003).
28. I. Pázsit, M. Ceder, Z. Kuang, Nucl. Sci. Eng. **148**, 67 (2004).
29. S. Croft, A. Favalli, D.K. Hauck, D. Henzlova, P.A. Santi, Nucl. Instrum. Methods A **686**, 136 (2012).
30. H.O. Menlove, S.H. Menlove, S.T. Tobin, Nucl. Instrum. Methods A **602**, 588 (2009).
31. J. Anderson, L. Pál, I. Pázsit, D. Chernikova, S. Pozzi, Eur. Phys. J. Plus **127**, 21 (2012).
32. L. Pál, I. Pázsit, Sci. Technol. Nucl. Install. , (2012).
33. T. Yamamoto, Ann. Nucl. Energy **57**, 84 (2013).
34. G. Chapline, in *Proceedings of 15th International Conference on Emerging nuclear Energy systems* (San Francisco, 2011).
35. J. Anderson, D. Chernikova, I. Pázsit, L. Pál, S.A. Pozzi, Eur. Phys. J. Plus **127**, 90 (2012).
36. A. Kolmogorov, *Uber die analytischen Methoden in der Wahrscheinlichkeitsrechnung* (1931).
37. Wolfram Research, Inc., Mathematica, Version 10.0 (Champaign, IL, 2014).
38. D. Chernikova, I. Pázsit, W. Ziguan, in *Proceedings of ESARDA meeting* (2013).
39. D. Chernikova, I. Pázsit, L. Pál, W. Ziguan, in *Proceedings of INMM meeting* (2013).
40. D.B. Pelowitz, MCNPX User's Manual, Version 2.7.0, Los Alamos National Laboratory report LA-CP-11-00438, April (2011).