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Uncertain vibration equation of large membranes

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Abstract. The study of the vibration of large membranes is important due to its well-known applications. There exist various investigations for the above problem where the variables and parameters are given as crisp/exact. In practice, we may not have these parameters exactly but those may be known in some uncertain form. In the present paper, these uncertainties are taken as interval/fuzzy and the authors propose here a new method *viz.* that of the double parametric form of fuzzy numbers to handle the uncertain problem of large membranes. Finally, the problem has been solved using the Homotopy Perturbation Method (HPM). The present method performs very well in terms of computational efficiency. The reliability of the method is shown for obtaining an approximate numerical solution for different cases. Results are given in terms of plots and are also compared in special cases.

1 Introduction

Vibration analysis of large membranes has a great importance in many areas of science and engineering problems. In music and acoustics, membranes constitute major components. In addition, membranes constitute components of microphones, speakers and other devices. In bioengineering, many human tissues are considered as membranes. Vibration characteristics of an eardrum are important in understanding hearing. Designs of hearing-aid devices involve knowledge of vibration behaviour of membranes.

Moreover, membranes may also be used to study two-dimensional wave mechanics and propagation. The fundamental equations of wave propagation in two dimensions are the same as the membrane vibration equations, *i.e.* partial differential equations. Membranes of various shapes are being analyzed throughout the globe following different modelling aspects and corresponding computational techniques. Two spatial dimensions may be represented using the Cartesian coordinate system (usually for rectangular membranes) or using the polar coordinate system (usually for circular membranes). In this paper we have considered vibration of circular membranes. The problem of vibration of circular membranes was first studied by Lord Rayleigh [1].

In particular, the vibration equation of very large membranes has been analysed by very few authors, *viz.* [2,3]. As such, Yildirim *et al.* [2] obtained the solution of the vibration equation of large membranes using the homotopy perturbation method (HPM), whereas Mohyud-Din *et al.* [3] have studied the vibration equation of large membranes of fractional order. Sunny *et al.* [4] applied the Adomian decomposition method to obtain the solution of the nonlinear vibration problem of a prestressed membrane.

In general the parameters and initial condition involved in the vibration equation of large membrane are considered as crisp or defined exactly. But, in actual practice, rather than the particular value, only uncertain or vague estimates of the variables and parameters are known, because those are found in general by some observation, experiment or experience. So, to handle the uncertainties and vagueness, one may use interval/fuzzy parameters and variables in the governing differential equations. This represents a natural way to model physical systems under uncertainty. Since it is too difficult to obtain the exact solution of uncertain (interval/fuzzy) differential equations, one may need a reliable and efficient numerical technique for the solution of interval/fuzzy differential equations. In this respect, the concept of fuzzy derivative was first introduced by Chang and Zadeh [5], where they proposed the concept of a fuzzy derivative.

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Dubois and Prade [6] defined and used the extension principle in their approach. Fuzzy differential equations and fuzzy initial value problems are studied by Kaleva [7,8] and Seikkala [9]. Various numerical methods for solving fuzzy differential equations are also introduced in refs. [10–22].

The above literature reveals that the fuzzy differential equations related to the physical systems are always converted to two crisp differential equations to obtain the solution. In this paper, the fuzzy vibration equation has been converted to a single crisp differential equation using a new concept of double parametric form of fuzzy numbers. Finally, the corresponding differential equation is solved by HPM to obtain interval/fuzzy solution in double parametric form.

Recently, HPM is found to be a powerful tool for the analysis of linear and nonlinear physical problems. The HPM was first developed by He [23,24] and then many authors applied this method to solve various linear and nonlinear functional equations of scientific and engineering problems. In this method, solution is considered as the sum of infinite series, which converges rapidly. In the homotopy technique (in topology), a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered as a "small parameter". Very recently, the homotopy perturbation method has been applied to a wide class of physical problems [25–44] with crisp parameters. In addition to these, He's polynomial has also been used to solve related problems [45–47]. Recently, few researchers have investigated the solution of fuzzy differential equations using the HPM [48–52].

Our aim in this paper is to apply the HPM [23,24] in solving the uncertain vibration equation of large membranes,

$$\frac{\partial^2 \tilde{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{v}}{\partial r} = \frac{1}{\tilde{c}^2}\frac{\partial^2 \tilde{v}}{\partial t^2}, \quad r \geq 0,\ t \geq 0,$$

with uncertain initial conditions in terms of fuzzy numbers (discussed in the next section)

$$\tilde{v}(r,0) = (0.8, 1, 1.2) f(r),$$
$$\tilde{v}'(r,0) = \tilde{c} g(r),$$

where $\tilde{v}(r,t)$ represents the uncertain displacement and $\tilde{c}$ is the uncertain wave velocity of free vibration and $f(r)$ is a function of r *viz.* radius of membrane.

Present paper is organized as follows. In sect. 2, we have given basic preliminaries related to the investigation. The proposed technique is discussed in sect. 3. This section also includes the general solution of the fuzzy vibration equation for large membranes using the HPM. In sect. 4 various cases with respect to the initial conditions are given. Next, numerical results and discussions are presented in sect. 5. Finally in the last section conclusions are drawn.

2 Preliminaries

In this section, we present notations, definitions and preliminaries which are used in this paper [53–56].

Definition 1. Fuzzy number.

A fuzzy number $\tilde{U}$ is a convex normalised fuzzy set $\tilde{U}$ of the real line R such that

$$\{\mu_{\tilde{U}}(x) : R \to [0,1],\ \forall\, x \in R\},$$

where, $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set and it is piecewise continuous.

Definition 2. Triangular fuzzy number.

A triangular fuzzy number $\tilde{U}$ is a convex normalized fuzzy set $\tilde{U}$ of the real line R such that

1) there exists exactly one $x_0 \in R$ with $\mu_{\tilde{U}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{U}$), where $\mu_{\tilde{U}}$ is called the membership function of the fuzzy set;
2) $\mu_{\tilde{U}}(x)$ is piecewise continuous.

We denote an arbitrary triangular fuzzy number as $\tilde{U} = (a,b,c)$. The membership function $\mu_{\tilde{U}}$ of $\tilde{U}$ is then defined as follows:

$$\mu_{\tilde{U}}(x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{b-a}, & a \leq x \leq b, \\ \dfrac{c-x}{c-b}, & b \leq x \leq c, \\ 0, & x \geq c. \end{cases}$$

Definition 3. Single parametric form of fuzzy numbers.

The triangular fuzzy number $\tilde{U} = (a,b,c)$ can be represented with an ordered pair of functions through the α-cut approach *viz.* $[\underline{u}(\alpha), \bar{u}(\alpha)] = [(b-a)\alpha + a, -(c-b)\alpha + c]$, where $\alpha \in [0,1]$.

The α-cut form is known as parametric form or single parametric form of fuzzy numbers. It may be noted that the lower and upper bounds of the fuzzy numbers satisfy the following requirements:

i) $\underline{u}(\alpha)$ is a bounded left continuous nondecreasing function over $[0,1]$;

ii) $\bar{u}(\alpha)$ is a bounded right continuous nonincreasing function over $[0,1]$;

iii) $\underline{u}(\alpha) \le \bar{u}(\alpha)$, $0 \le \alpha \le 1$.

Definition 4. Double parametric form of fuzzy number.

Using the single parametric form as discussed in definition 3 we have $\tilde{U} = [\underline{u}(\alpha), \bar{u}(\alpha)]$.

Now one may write this as a crisp number with double parametric form as $\tilde{U}(\alpha,\beta) = \beta(\bar{u}(\alpha) - \underline{u}(\alpha)) + \underline{u}(\alpha)$, where α and $\beta \in [0,1]$.

Definition 5. Fuzzy arithmetic.

For any two arbitrary fuzzy numbers $\tilde{x} = [\underline{x}(\alpha), \bar{x}(\alpha)]$, $\tilde{y} = [\underline{y}(\alpha), \bar{y}(\alpha)]$ and scalar k, fuzzy arithmetics are defined as follows:

i) $\tilde{x} = \tilde{y}$ if and only if $\underline{x}(\alpha) = \underline{y}(\alpha)$and $\bar{x}(\alpha) = \bar{y}(\alpha)$;

ii) $\tilde{x} + \tilde{y} = [\underline{x}(\alpha) + \underline{y}(\alpha), \bar{x}(\alpha) + \bar{y}(\alpha)]$;

iii) $\tilde{x} \times \tilde{y} = \begin{bmatrix} \min\left(\underline{x}(\alpha)\times\underline{y}(\alpha), \underline{x}(\alpha)\times\bar{y}(\alpha), \bar{x}(\alpha)\times\underline{y}(\alpha), \bar{x}(\alpha)\times\bar{y}(\alpha)\right), \\ \max\left(\underline{x}(\alpha)\times\underline{y}(\alpha), \underline{x}(\alpha)\times\bar{y}(\alpha), \bar{x}(\alpha)\times\underline{y}(\alpha), \bar{x}(\alpha)\times\bar{y}(\alpha)\right) \end{bmatrix}$;

iv) $k\tilde{x} = \begin{cases} [k\bar{x}(\alpha), k\underline{x}(\alpha)], & k < 0 \\ [k\underline{x}(\alpha), k\bar{x}(\alpha)], & k \ge 0 \end{cases}$.

3 Double-parametric–based fuzzy vibration equation

We first convert the fuzzy vibration differential equation to an interval-based fuzzy differential equation using the single parametric form. Then, by using the double parametric form, the interval-based fuzzy differential equation is reduced to a crisp vibration equation. Finally, we apply the HPM to solve the corresponding differential equation to obtain the required solution in terms of interval/fuzzy. Let us now consider the fuzzy vibration equation of large membranes,

$$\frac{\partial^2 \tilde{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{v}}{\partial r} = \frac{1}{\tilde{c}^2}\frac{\partial^2 \tilde{v}}{\partial t^2}, \quad r \ge 0,\ t \ge 0. \tag{1}$$

The above equation may be written as

$$\frac{\partial^2 \tilde{v}}{\partial t^2} = \tilde{c}^2\left(\frac{\partial^2 \tilde{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{v}}{\partial r}\right), \tag{2}$$

with fuzzy initial conditions

$$\tilde{v}(r,0) = (0.8, 1, 1.2) f(r) \tag{3}$$

$$\tilde{v}'(r,0) = \tilde{c} g(r). \tag{4}$$

Let us write eq. (2) as

$$L_{tt}\tilde{v}(r,t) = \tilde{c}^2\left(L_{rr}\tilde{v}(r,t) + \frac{1}{r}L_r\tilde{v}(r,t)\right), \tag{5}$$

where $L_{tt} \equiv \partial^2/\partial t^2$, $L_{rr} \equiv \partial^2/\partial r^2$ and $L_r \equiv \partial/\partial r$.

As per the single parametric form, we may write the above fuzzy vibration equation (eq. (5)) as

$$L_{tt}\tilde{v}(r,t;\alpha) = [L_{tt}\underline{v}(r,t;\alpha), L_{tt}\bar{v}(r,t;\alpha)] = [\underline{c}(\alpha), \bar{c}(\alpha)]^2\left([L_{rr}\underline{v}(r,t), L_{rr}\bar{v}(r,t)] + \frac{1}{r}[L_r\underline{v}(r,t), L_r\bar{v}(r,t)]\right), \tag{6}$$

subject to fuzzy initial conditions

$$[\underline{v}(r,0;\alpha), \bar{v}(r,0;\alpha)] = [0.2\alpha + 0.8, 1.2 - 0.2\alpha] f(r),$$

$$[\underline{v}'(r,0;\alpha), \bar{v}'(r,0;\alpha)] = [\underline{c}(\alpha), \bar{c}(\alpha)] g(r),$$

where $\alpha \in [0,1]$. One may see, here, that the above eq. (6) with fuzzy initial conditions is all in interval form. It may be noted here that one may directly solve this interval differential equation but the interval computation is sometimes difficult to handle. So here the authors propose to use the double parametric form (as discussed in definition 4) in the above eq. (6). Accordingly, the same may be expressed as

$$\{\beta\left(L_{tt}\bar{v}(r,t;\alpha) - L_{tt}\underline{v}(r,t;\alpha)\right) + L_{tt}\underline{v}(r,t;\alpha)\} = \{\beta\left(\bar{c}(\alpha) - \underline{c}(\alpha)\right) + \underline{c}(\alpha)\}^2$$
$$\left(\{\beta\left(L_{rr}\underline{v}(r,t;\alpha) - L_{rr}\bar{v}(r,t;\alpha)\right) + L_{rr}\underline{v}(r,t;\alpha)\} + \frac{1}{r}\{\beta\left(L_{r}\underline{v}(r,t;\alpha) - L_{r}\bar{v}(r,t;\alpha)\right) + L_{r}\underline{v}(r,t;\alpha)\}\right), \tag{7}$$

subject to the initial conditions

$$\{\beta\left(\underline{v}(r,0;\alpha) - \bar{v}(r,0;\alpha)\right) + \underline{v}(r,0;\alpha)\} = \{\beta\left(0.4 - 0.4\alpha\right) + (0.2\alpha + 0.8)\}\, f(r)$$
$$\{\beta\left(\underline{v}'(r,0;\alpha) - \bar{v}'(r,0;\alpha)\right) + \underline{v}'(r,0;\alpha)\} = \{\beta\left(\bar{c}(\alpha) - \underline{c}(\alpha)\right) + \underline{c}(\alpha)\}\, g(r),$$

where $\alpha, \beta \in [0,1]$. It is now worth mentioning that the above eq. (7) with the interval initial conditions are all now converted to crisp form for particular values of α and β.

Let us now denote

$$\{\beta\left(L_{tt}\bar{v}(r,t;\alpha) - L_{tt}\underline{v}(r,t;\alpha)\right) + L_{tt}\underline{v}(r,t;\alpha)\} = L_{tt}\tilde{v}(r,t;\alpha,\beta),$$
$$\{\beta\left(L_{rr}\underline{v}(r,t;\alpha) - L_{rr}\bar{v}(r,t;\alpha)\right) + L_{rr}\underline{v}(r,t;\alpha)\} = L_{rr}\tilde{v}(r,t;\alpha,\beta),$$
$$\{\beta\left(L_{r}\underline{v}(r,t;\alpha) - L_{r}\bar{v}(r,t;\alpha)\right) + L_{r}\underline{v}(r,t;\alpha)\} = L_{r}\tilde{v}(r,t;\alpha,\beta),$$
$$\{\beta\left(\bar{c}(\alpha) - \underline{c}(\alpha)\right) + \underline{c}(\alpha)\} = \tilde{c}(\alpha,\beta),$$
$$\{\beta\left(\underline{v}(r,0;\alpha) - \bar{v}(r,0;\alpha)\right) + \underline{v}(r,0;\alpha)\} = \tilde{v}(r,0;\alpha,\beta),$$
$$\{\beta\left(\underline{v}'(r,0;\alpha) - \bar{v}'(r,0;\alpha)\right) + \underline{v}'(r,0;\alpha)\} = \tilde{v}'(0;\alpha,\beta).$$

Substituting these values in eq. (7) we may write, in compact form,

$$L_{tt}\tilde{v}(r,t;\alpha,\beta) = \left(\tilde{c}(\alpha,\beta)\right)^2\left(L_{rr}\tilde{v}(r,t;\alpha,\beta) + \frac{1}{r}L_{r}\tilde{v}(r,t;\alpha,\beta)\right), \tag{8}$$

with initial conditions

$$\tilde{v}(r,0;\alpha,\beta) = \{\beta(0.4 - 0.4\alpha) + (0.2\alpha + 0.8)\}\, f(r),$$
$$\tilde{v}'(r,0;\alpha,\beta) = \{\beta\left(\bar{c}(\alpha) - \underline{c}(\alpha)\right) + \underline{c}(\alpha)\}\, g(r).$$

Solving the corresponding crisp differential equation, one may get the solution as $\tilde{v}(r,t;\alpha,\beta)$ in terms of α and β. To obtain the lower and upper bound of the solution in single parametric form, we may put $\beta = 0$ and 1, respectively. This may be represented as $\tilde{v}(r,t;\alpha,0) = \underline{v}(r,t,\alpha)$ and $\tilde{v}(r,t,\alpha,1) = \bar{v}(r,t,\alpha)$. Similarly, other results may be obtained by plugging in different values of α and β.

3.1 Solution by the HPM [23,24] using the proposed methodology

According to the HPM, we may now construct a simple homotopy for eq. (8) with an embedding parameter $p \in [0,1]$, as follows:

$$(1-p)L_{tt}\tilde{v}(r,t;\alpha,\beta) + p\left[L_{tt}\tilde{v}(r,t;\alpha,\beta) - \left(\tilde{c}(\alpha,\beta)\right)^2\left(L_{rr}\tilde{v}(r,t;\alpha,\beta) + \frac{1}{r}L_{r}\tilde{v}(r,t;\alpha,\beta)\right)\right] = 0, \tag{9}$$

or

$$L_{tt}\tilde{v}(r,t;\alpha,\beta) - p\left[\left(\tilde{c}(\alpha,\beta)\right)^2\left(L_{rr}\tilde{v}(r,t;\alpha,\beta) + \frac{1}{r}L_{r}\tilde{v}(r,t;\alpha,\beta)\right)\right] = 0. \tag{10}$$

In the changing process from 0 to 1, for $p = 0$, eq. (9) or (10) give $L_{tt}\tilde{v}(r,t;\alpha,\beta) = 0$ and for $p = 1$, we have the original system

$$L_{tt}\tilde{v}(r,t;\alpha,\beta) - (\tilde{c}(\alpha,\beta))^2 \left(L_{rr}\tilde{v}(r,t;\alpha,\beta) + \frac{1}{r}L_r\tilde{v}(r,t;\alpha,\beta) \right) = 0.$$

This is called deformation in topology.

$L_{tt}\tilde{v}(r,t;\alpha,\beta)$ and $-(\tilde{c}(\alpha,\beta))^2(L_{rr}\tilde{v}(r,t;\alpha,\beta) + \frac{1}{r}L_r\tilde{v}(r,t;\alpha,\beta))$ are called homotopic. Next, we can assume the solution of eq. (9) or (10) as a power series expansion in p as

$$\tilde{v}(r,t;\alpha,\beta) = \tilde{v}_0(r,t;\alpha,\beta) + p\tilde{v}_1(r,t;\alpha,\beta) + p^2\tilde{v}_2(r,t;\alpha,\beta) + p^3\tilde{v}_3(r,t;\alpha,\beta) + \dots, \tag{11}$$

where, $\tilde{v}_i(r,t;\alpha,\beta)$ for $i = 0,1,2,3,\dots$ are functions yet to be determined. Substituting eq. (11) into eq. (9) or (10) and equating the terms with the identical powers of p, we have

$$p^0 : L_{tt}\tilde{v}_0(r,t;\alpha,\beta) = 0, \tag{12}$$

$$p^1 : L_{tt}\tilde{v}_1(r,t;\alpha,\beta) - (\tilde{c}(\alpha,\beta))^2 L_{rr}\tilde{v}_0(r,t;\alpha,\beta) - \frac{(\tilde{c}(\alpha,\beta))^2}{r}L_r\tilde{v}_0(r,t;\alpha,\beta) = 0, \tag{13}$$

$$p^2 : L_{tt}\tilde{v}_2(r,t;\alpha,\beta) - (\tilde{c}(\alpha,\beta))^2 L_{rr}\tilde{v}_1(r,t;\alpha,\beta) - \frac{(\tilde{c}(\alpha,\beta))^2}{r}L_r\tilde{v}_1(r,t;\alpha,\beta) = 0, \tag{14}$$

$$p^3 : L_{tt}\tilde{v}_3(r,t;\alpha,\beta) - (\tilde{c}(\alpha,\beta))^2 L_{rr}\tilde{v}_2(r,t;\alpha,\beta) - \frac{(\tilde{c}(\alpha,\beta))^2}{r}L_r\tilde{v}_2(r,t;\alpha,\beta) = 0, \tag{15}$$

and so on.

Choosing the initial approximation $\tilde{v}(r,0;\alpha,\beta)$ and applying the operator L_{tt}^{-1} (the inverse operator of L_{tt}) on both sides of eqs. (12) to (15) we have

$$p^0 : \tilde{v}_0(r,t;\alpha,\beta) = t\dot{\tilde{v}}_0(r,0;\alpha,\beta) + \tilde{v}_0(r,t;\alpha,\beta), \tag{16}$$

$$p^1 : \tilde{v}_1(r,t;\alpha,\beta) = L_{tt}^{-1}\left((\tilde{c}(\alpha,\beta))^2 L_{rr}\tilde{v}_0(r,t;\alpha,\beta) + \frac{(\tilde{c}(\alpha,\beta))^2}{r}L_r\tilde{v}_0(r,t;\alpha,\beta) \right), \tag{17}$$

$$p^2 : \tilde{v}_2(r,t;\alpha,\beta) = L_{tt}^{-1}\left((\tilde{c}(\alpha,\beta))^2 L_{rr}\tilde{v}_1(r,t;\alpha,\beta) + \frac{(\tilde{c}(\alpha,\beta))^2}{r}L_r\tilde{v}_1(r,t;\alpha,\beta) \right), \tag{18}$$

$$p^3 : \tilde{v}_3(r,t;\alpha,\beta) = L_{tt}^{-1}\left((\tilde{c}(\alpha,\beta))^2 L_{rr}\tilde{v}_2(r,t;\alpha,\beta) + \frac{(\tilde{c}(\alpha,\beta))^2}{r}L_r\tilde{v}_2(r,t;\alpha,\beta) \right), \tag{19}$$

and so on.

One may get the approximate solution $\tilde{v}(r,t;\alpha,\beta) = \lim_{p\to 1}\tilde{v}(r,t;\alpha,\beta)$, which can be expressed as

$$\tilde{v}(r,t;\alpha,\beta) = \tilde{v}_0(r,t;\alpha,\beta) + \tilde{v}_1(r,t;\alpha,\beta) + \tilde{v}_2(r,t;\alpha,\beta) + \tilde{v}_3(r,t;\alpha,\beta) + \dots. \tag{20}$$

The series obtained by the HPM converges very rapidly and only few terms are required to get the approximate solutions. The proof may be found in [23,24].

4 Solution bounds for particular cases

In this section we consider fuzzy initial conditions in single parametric form as $\tilde{v}(r,0;\alpha) = [0.2\alpha + 0.8, 1.2 - 0.2\alpha]f(r)$, $\dot{\tilde{v}}(r,0;\alpha) = [\alpha + 5, 7 - \alpha]g(r)$ and the wave velocity as $\tilde{c} = [\alpha + 5, 7 - \alpha]$. Depending upon the functions $f(r)$ and $g(r)$, we will have different cases [1] which are discussed in the following paragraphs for finding the uncertain solution bounds.

Case 1.

Here we have taken $f(r) = r^2$ and $g(r) = r$ in the above fuzzy initial conditions. Hence eq. (5) will become

$$L_{tt}\tilde{v}(r,t) = [\alpha+5, 7-\alpha]^2 \left(L_{rr}\tilde{v}(r,t) + \frac{1}{r} L_r \tilde{v}(r,t) \right).$$

Using the double parametric form, eq. (8) and the corresponding fuzzy initial conditions will become

$$L_{tt}\tilde{v}(r,t;\alpha,\beta) = (\beta(2-2\alpha) + (\alpha+5))^2 \left(L_{rr}\tilde{v}(r,t;\alpha,\beta) + \frac{1}{r} L_r \tilde{v}(r,t;\alpha,\beta) \right)$$

and

$$\tilde{v}(r,0;\alpha,\beta) = \beta(0.4-0.4\alpha) + (0.2\alpha+0.8) f(r), \quad \dot{\tilde{v}}(r,0;\alpha,\beta) = \beta(2-2\alpha) + (\alpha+5) g(r).$$

Let us now denote

$$\begin{aligned} &\beta(0.4-0.4\alpha) + (0.2\alpha+0.8) = \eta, \\ &\beta(2-2\alpha) + (\alpha+5) = \delta. \end{aligned}$$

Applying the HPM, we have

$$\tilde{v}_0(r,t;\alpha,\beta) = \delta r t + \eta r^2, \tag{21}$$

$$\tilde{v}_1(r,t;\alpha,\beta) = 2\eta\delta^2 t^2 + \frac{\delta^3 t^3}{6r}, \tag{22}$$

$$\tilde{v}_2(r,t;\alpha,\beta) = \frac{\delta^5 t^5}{120 r^3}, \tag{23}$$

$$\tilde{v}_3(r,t;\alpha,\beta) = \frac{\delta^7 t^7}{560 r^5}, \tag{24}$$

and so on.

In a similar manner, a higher-order approximation may be obtained as discussed above.

Therefore, the solution can be written as

$$\tilde{v}(r,t;\alpha,\beta) = r^2 \left(\eta + \delta\left(\frac{t}{r}\right) + 2\eta\delta^2 \left(\frac{t}{r}\right)^2 + \frac{\delta^3}{6}\left(\frac{t}{r}\right)^3 + \frac{\delta^5}{120}\left(\frac{t}{r}\right)^5 + \frac{\delta^7}{560}\left(\frac{t}{r}\right)^7 + \cdots \right), \tag{25}$$

or

$$\tilde{v}(r,t;\alpha,\beta) = r^2 \begin{pmatrix} (\beta(0.4-0.4\alpha) + (0.2\alpha+0.8)) + (\beta(2-2\alpha)+(\alpha+5))\left(\frac{t}{r}\right) \\ +2\left(\beta(0.4-0.4\alpha)+(0.2\alpha+0.8)\right)\left(\beta(2-2\alpha)+(\alpha+5)\right)^2\left(\frac{t}{r}\right)^2 \\ +\frac{(\beta(2-2\alpha)+(\alpha+5))^3}{6}\left(\frac{t}{r}\right)^3 + \frac{(\beta(2-2\alpha)+(\alpha+5))^5}{120}\left(\frac{t}{r}\right)^5 \\ +\frac{(\beta(2-2\alpha)+(\alpha+5))^7}{560}\left(\frac{t}{r}\right)^7 + \cdots \end{pmatrix}. \tag{26}$$

To obtain the solution bounds in single parametric form we may put $\beta = 0$ and 1 in eq. (26) for lower and upper bounds of the solution, respectively. So we get

$$\underline{v}(r,t;\alpha,0) = r^2 \begin{pmatrix} (0.2\alpha+0.8) + (\alpha+5)\left(\frac{t}{r}\right) + 2(0.2\alpha+0.8)(\alpha+5)^2\left(\frac{t}{r}\right)^2 + \frac{(\alpha+5)^3}{6}\left(\frac{t}{r}\right)^3 \\ +\frac{(\alpha+5)^5}{120}\left(\frac{t}{r}\right)^5 + \frac{(\alpha+5)^7}{560}\left(\frac{t}{r}\right)^7 + \cdots \end{pmatrix} \tag{27}$$

and

$$\bar{v}(r,t;\alpha,1) = r^2 \begin{pmatrix} (1.2-0.2\alpha) + (7-\alpha)\left(\frac{t}{r}\right) + 2(1.2-0.2\alpha)(7-\alpha)^2\left(\frac{t}{r}\right)^2 + \frac{(7-\alpha)^3}{6}\left(\frac{t}{r}\right)^3 \\ +\frac{(7-\alpha)^5}{120}\left(\frac{t}{r}\right)^5 + \frac{(7-\alpha)^7}{560}\left(\frac{t}{r}\right)^7 + \cdots \end{pmatrix} \tag{28}$$

One may note that in the special case where $\alpha = 1$ and the wave velocity $c = 6$, the crisp results obtained by the proposed method are exactly the same as the solution obtained by Yildirim *et al.* [2]. The above series will be convergent for the values of $|t/r| \le 1$, *i.e.*, for a large membrane and small range of time.

Case 2.

Now we consider $f(r) = r$ and $g(r) = 1$.

Again, by applying the procedure discussed previously, we get the solution

$$\tilde{v}_0(r,t;\alpha,\beta) = \delta t + \eta r, \tag{29}$$

$$\tilde{v}_1(r,t;\alpha,\beta) = \frac{\delta^2\eta t^2}{2r}, \tag{30}$$

$$\tilde{v}_2(r,t;\alpha,\beta) = \frac{\delta^4\eta t^4}{24r^3}, \tag{31}$$

$$\tilde{v}_3(r,t;\alpha,\beta) = \frac{\eta\delta^6 t^6}{80r^5}, \tag{32}$$

and so on.

The solution in general form may be obtained as

$$\tilde{v}(r,t;\alpha,\beta) = r\left(\delta\left(\frac{t}{r}\right) + \eta + \frac{\delta^2\eta}{2}\left(\frac{t}{r}\right)^2 + \frac{\delta^4\eta}{24}\left(\frac{t}{r}\right)^4 + \frac{\eta\delta^6}{80}\left(\frac{t}{r}\right)^6 + \cdots\right), \tag{33}$$

or

$$\tilde{v}(r,t;\alpha,\beta) = r \begin{pmatrix} \beta(0.4-0.4\alpha) + (0.2\alpha+0.8) + (\beta(2-2\alpha) + (\alpha+5))\left(\frac{t}{r}\right) \\ +\frac{(\beta(2-2\alpha)+(\alpha+5))^2(\beta(0.4-0.4\alpha)+(0.2\alpha+0.8))}{2}\left(\frac{t}{r}\right)^2 \\ +\frac{(\beta(2-2\alpha)+(\alpha+5))^4(\beta(0.4-0.4\alpha)+(0.2\alpha+0.8))}{24}\left(\frac{t}{r}\right)^4 \\ +\frac{(\beta(0.4-0.4\alpha)+(0.2\alpha+0.8))(\beta(2-2\alpha)+(\alpha+5))^6}{80}\left(\frac{t}{r}\right)^6 + \cdots \end{pmatrix}. \tag{34}$$

Putting $\beta = 0$ and 1 in $\tilde{v}(r,t;\alpha,\beta)$ we get the lower and upper bounds of the fuzzy solutions, respectively, as

$$\underline{v}(r,t;\alpha,0) = r \begin{pmatrix} (0.2\alpha+0.8) + (\alpha+5)\left(\frac{t}{r}\right) + \frac{((\alpha+5))^2(0.2\alpha+0.8)}{2}\left(\frac{t}{r}\right)^2 \\ +\frac{(\alpha+5)^4(0.2\alpha+0.8)}{24}\left(\frac{t}{r}\right)^4 + \frac{(0.2\alpha+0.8)((\alpha+5))^6}{80}\left(\frac{t}{r}\right)^6 + \cdots \end{pmatrix} \tag{35}$$

and

$$\bar{v}(r,t;\alpha,1) = r \begin{pmatrix} (1.2-0.2\alpha) + (7-\alpha)\left(\frac{t}{r}\right) + \frac{(7-\alpha)^2(1.2-0.2\alpha)}{2}\left(\frac{t}{r}\right)^2 \\ +\frac{(7-\alpha)^4(1.2-0.2\alpha)}{24}\left(\frac{t}{r}\right)^4 + \frac{(1.2-0.2\alpha)(7-\alpha)^6}{80}\left(\frac{t}{r}\right)^6 + \cdots \end{pmatrix}. \tag{36}$$

The solution obtained by the proposed method for $\alpha = 1$ and the wave velocity $c = 6$, is again found to be exactly the same as that of (crisp result) Yıldırım *et al.* [2].

Case 3.

Next we take $f(r) = \sqrt{r}$ and $g(r) = 1/\sqrt{r}$.

By following the proposed method with the HPM, we get the solution in double parametric form as

$$\tilde{v}(r,t;\alpha,\beta) = \sqrt{r}\begin{pmatrix} \eta + \delta\left(\frac{t}{r}\right) + \frac{\delta^2\eta}{8}\left(\frac{t}{r}\right)^2 + \frac{\delta^3}{24}\left(\frac{t}{r}\right)^3 + \frac{3\delta^4\eta}{128}\left(\frac{t}{r}\right)^4 + \frac{\delta^5}{384}\left(\frac{t}{r}\right)^5 \\ +\frac{49\delta^6\eta}{5120}\left(\frac{t}{r}\right)^6 + \frac{9\delta^7}{7168}\left(\frac{t}{r}\right)^7 \cdots \end{pmatrix}. \tag{37}$$

The lower and upper bounds of the fuzzy solutions may again be written as

$$\underline{v}(r,t;\alpha,0) = \sqrt{r}\begin{pmatrix} (0.2\alpha+0.8) + (\alpha+5)\left(\frac{t}{r}\right) + \frac{(\alpha+5)^2(0.2\alpha+0.8)}{8}\left(\frac{t}{r}\right)^2 + \frac{(\alpha+5)^3}{24}\left(\frac{t}{r}\right)^3 \\ +\frac{3(7-\alpha)^4(0.2\alpha+0.8)}{128}\left(\frac{t}{r}\right)^4 + \frac{(\alpha+5)^5}{384}\left(\frac{t}{r}\right)^5 \\ +\frac{49(\alpha+5)^6(0.2\alpha+0.8)}{5120}\left(\frac{t}{r}\right)^6 + \frac{9(\alpha+5)^7}{7168}\left(\frac{t}{r}\right)^7 \cdots \end{pmatrix} \tag{38}$$

and

$$\bar{v}(r,t;\alpha,1) = \sqrt{r}\begin{pmatrix} (1.2-0.2\alpha) + (7-\alpha)\left(\frac{t}{r}\right) + \frac{(7-\alpha)^2(1.2-0.2\alpha)}{8}\left(\frac{t}{r}\right)^2 + \frac{(7-\alpha)^3}{24}\left(\frac{t}{r}\right)^3 \\ +\frac{3(7-\alpha)^4(1.2-0.2\alpha)}{128}\left(\frac{t}{r}\right)^4 + \frac{(7-\alpha)^5}{384}\left(\frac{t}{r}\right)^5 \\ +\frac{49(7-\alpha)^6(1.2-0.2\alpha)}{5120}\left(\frac{t}{r}\right)^6 + \frac{9^7(7-\alpha)^7}{7168}\left(\frac{t}{r}\right)^7 \cdots \end{pmatrix}. \tag{39}$$

Case 4.

$f(r) = r^2$ and $g(r) = 1$.

In this case we have

$$\tilde{v}_0(r,t;\alpha,\beta) = \delta t + \eta r^2, \tag{40}$$
$$\tilde{v}_1(r,t;\alpha,\beta) = 2\delta^2\eta t^2, \tag{41}$$
$$\tilde{v}_2(r,t;\alpha,\beta) = 0, \tag{42}$$
$$\tilde{v}_n(r,t;\alpha,\beta) = 0, \quad \text{for } n \geq 2. \tag{43}$$

Therefore the solution in double parametric form is as follows:

$$\tilde{v}(r,t;\alpha,\beta) = \eta r^2 + \delta t + 2\delta^2\eta t^2. \tag{44}$$

The lower and upper bounds of the fuzzy solutions are obtained as

$$\underline{v}(r,t;\alpha,0) = (0.2\alpha+0.8)r^2 + (\alpha+5)t + 2(\alpha+5)^2(0.2\alpha+0.8)t^2 \tag{45}$$

and

$$\bar{v}(r,t;\alpha,1) = (1.2-0.2\alpha)r^2 + (7-\alpha)t + 2(7-\alpha)^2(1.2-0.2\alpha)t^2. \tag{46}$$

Again, one may see that the solution obtained by the proposed method for $\alpha = 1$ and the wave velocity $c = 6$ exactly agrees with the solution of Yıldırım *et al.* [2].

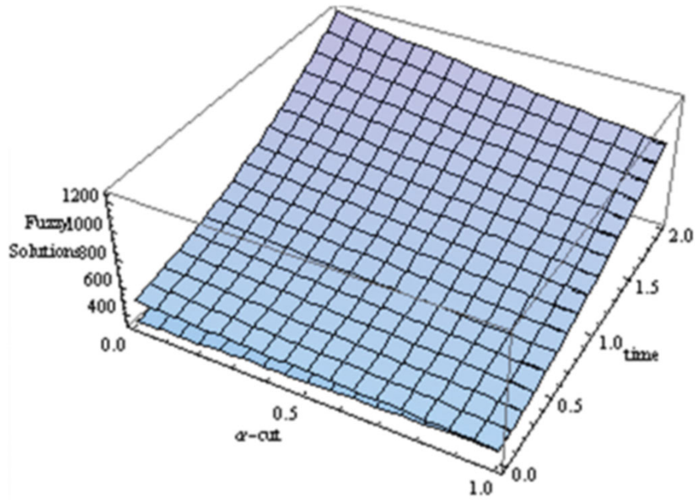

Fig. 1. Fuzzy displacement at $r = 20$ of case 1.

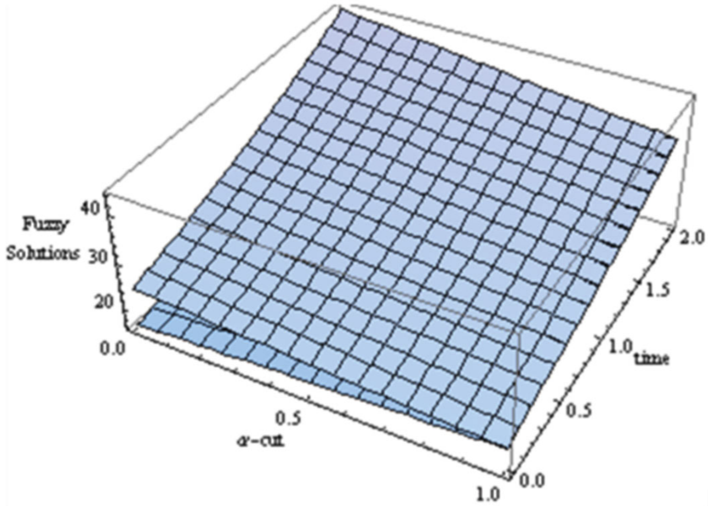

Fig. 2. Fuzzy solution at $r = 20$ of case 2.

Case 5.

Finally we consider $f(r) = r^2$ and $g(r) = r^2$.

We have the solutions in this case as

$$\tilde{v}_0(r,t;\alpha,\beta) = \delta r^2 t + \eta r^2, \tag{47}$$

$$\tilde{v}_1(r,t;\alpha,\beta) = \frac{2}{3}\delta^3 t^3 + 2\delta^2\eta t^2, \tag{48}$$

$$\tilde{v}_2(r,t;\alpha,\beta) = 0, \tag{49}$$

$$\tilde{v}_n(r,t;\alpha,\beta) = 0, \quad \text{for } n \geq 2 \tag{50}$$

and, finally, one may write

$$\tilde{v}(r,t;\alpha,\beta) = \eta r^2 + \delta r^2 t + \frac{2}{3}\delta^3 t^3 + 2\delta^2\eta t^2. \tag{51}$$

The lower and upper bound of the solutions, respectively, are

$$\underline{v}(r,t;\alpha,0) = (0.2\alpha+0.8)r^2 + (\alpha+5)r^2 t + \frac{2}{3}(\alpha+5)^3 t^3 + 2(\alpha+5)^2(0.2\alpha+0.8)t^2 \tag{52}$$

and

$$\bar{v}(r,t;\alpha,1) = (1.2-0.2\alpha)r^2 + (7-\alpha)r^2 t + \frac{2}{3}(7-\alpha)^3 t^3 + 2(7-\alpha)^2(1.2-0.2\alpha)t^2. \tag{53}$$

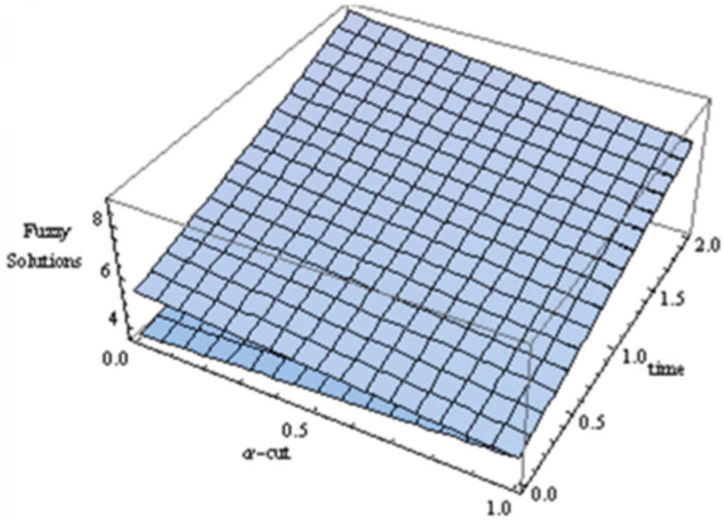

Fig. 3. Fuzzy solution at $r = 20$ of case 3.

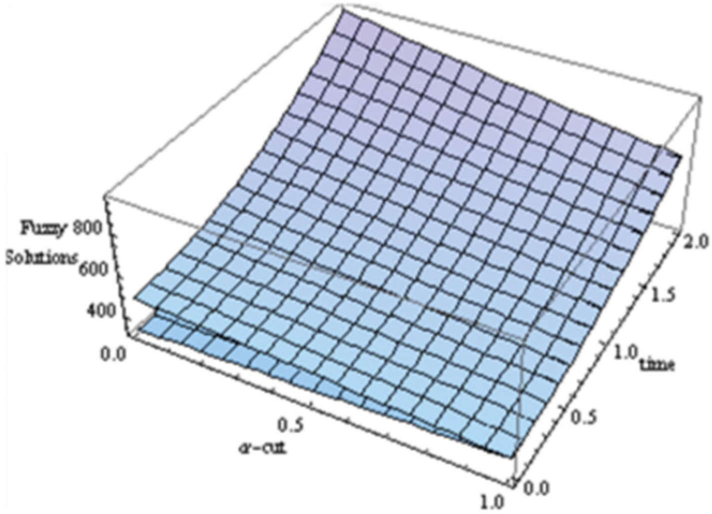

Fig. 4. Fuzzy solution at $r = 20$ of case 4.

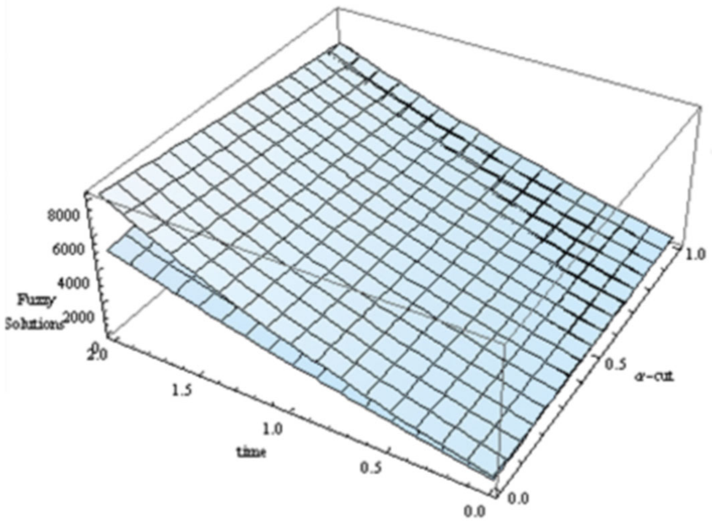

Fig. 5. Fuzzy solution at $r = 20$ of case 5.

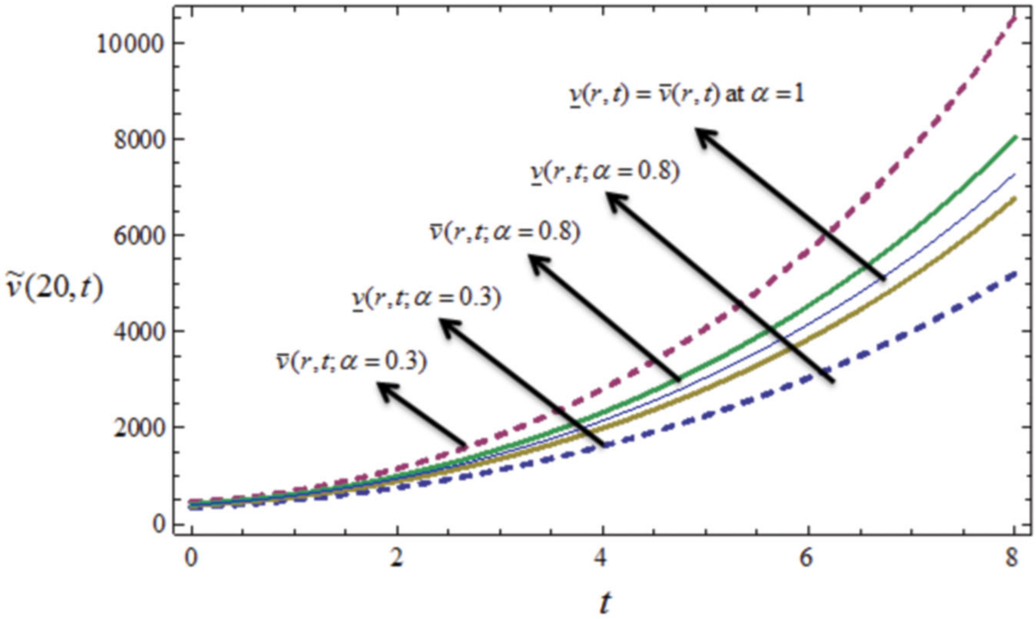

Fig. 6. Interval solution of case 1.

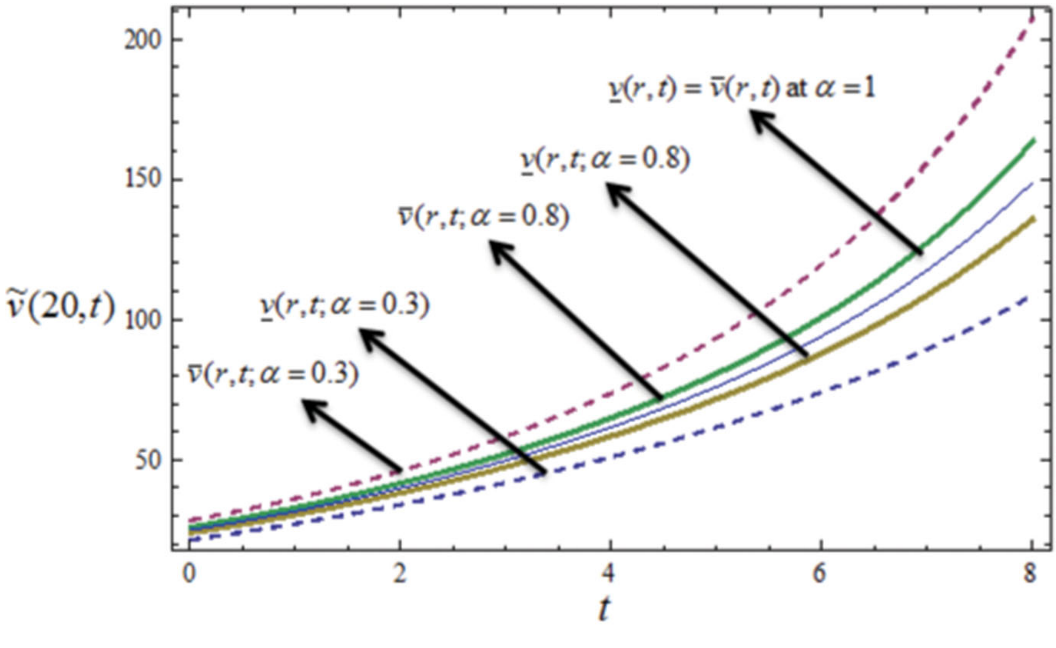

Fig. 7. Interval solution of case 2.

5 Numerical results and discussions

In this section, we present numerical solution of uncertain vibration equation for large membranes using the HPM. It is a gigantic task to include here all the results with respect to various parameters and initial conditions involved in the corresponding fuzzy differential equation. So, some particular values of the parameters are taken to compute the results with the above cases. The obtained results by the present analysis are compared with the existing solutions in [2], in special cases, to show the validation of the proposed analysis. Computed results are depicted in terms of plots.

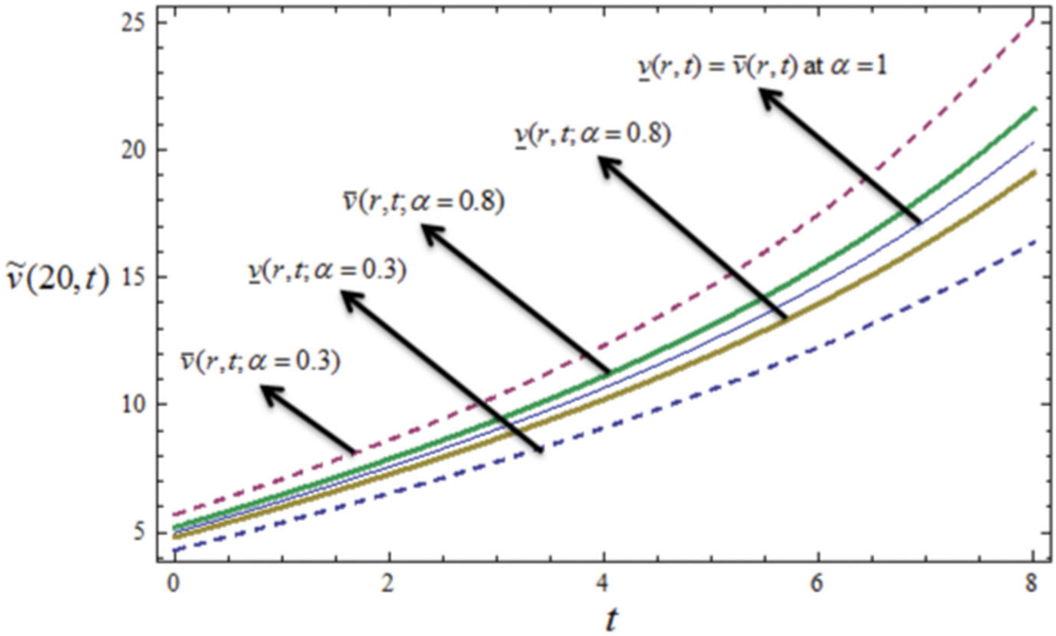

Fig. 8. Interval solution of case 3.

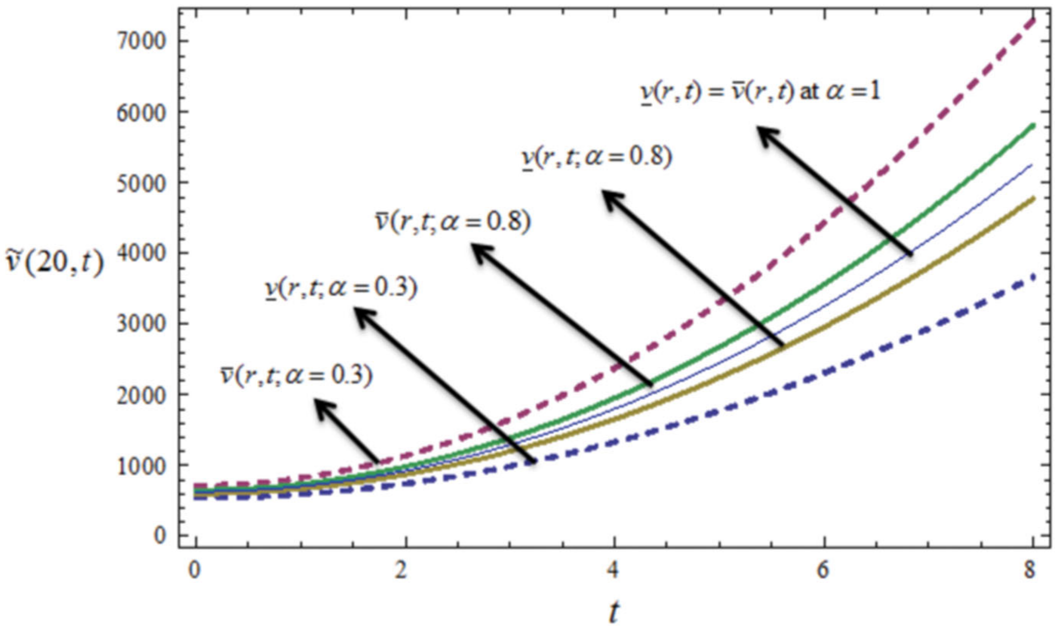

Fig. 9. Interval solution of case 4.

Here, numerical computations have been done by truncating the infinite series (27), (28), (35), (36), (38), (39), (45), (46) and (52), (53) to a finite number ($n = 3$) of terms. Triangular fuzzy solutions for particular cases 1 to 5 are depicted in figs. 1 to 5, by varying time t from 0 to 2 and for a particular value of radius of membrane $r = 20$. Next, interval solutions for α-cut 0.3, 0.8 and 1 and varying t from 0 to 8 for different cases have been given in figs. 6 to 10, respectively with radius of membrane, $r = 20$. One may see, from these figures, that the crisp result ($\alpha = 1$) is the central line and the interval solutions are spread on both sides of the crisp results. Similarly, for $t = 8$ and different values of r (for all the five cases) we plot the interval solutions in figs. 11 to 15. It may be worth mentioning that for all the cases, the present results with $\alpha = 1$ exactly agree with the solution in [2].

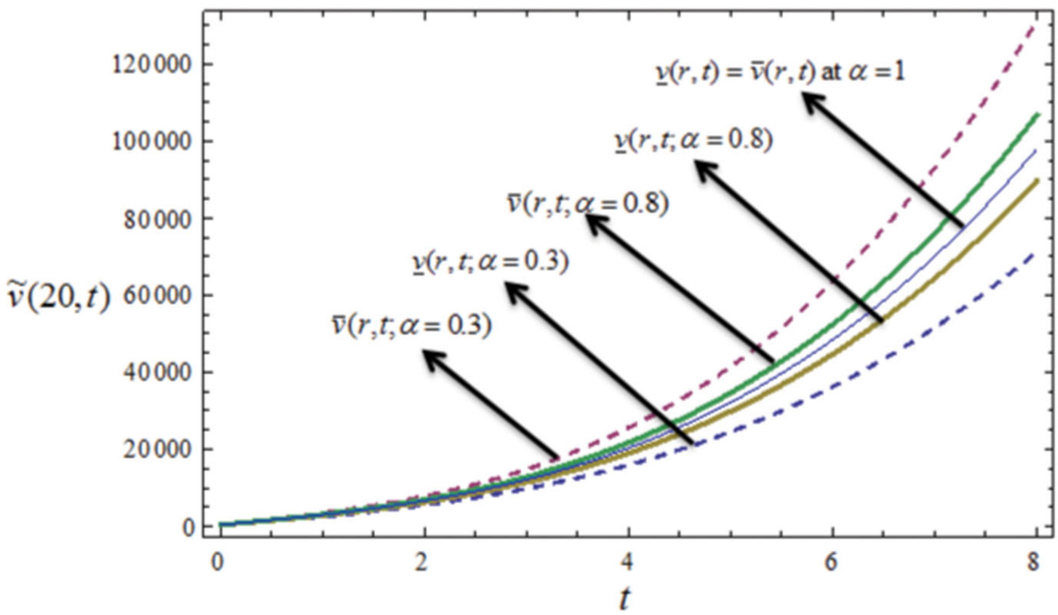

Fig. 10. Interval solution of case 5.

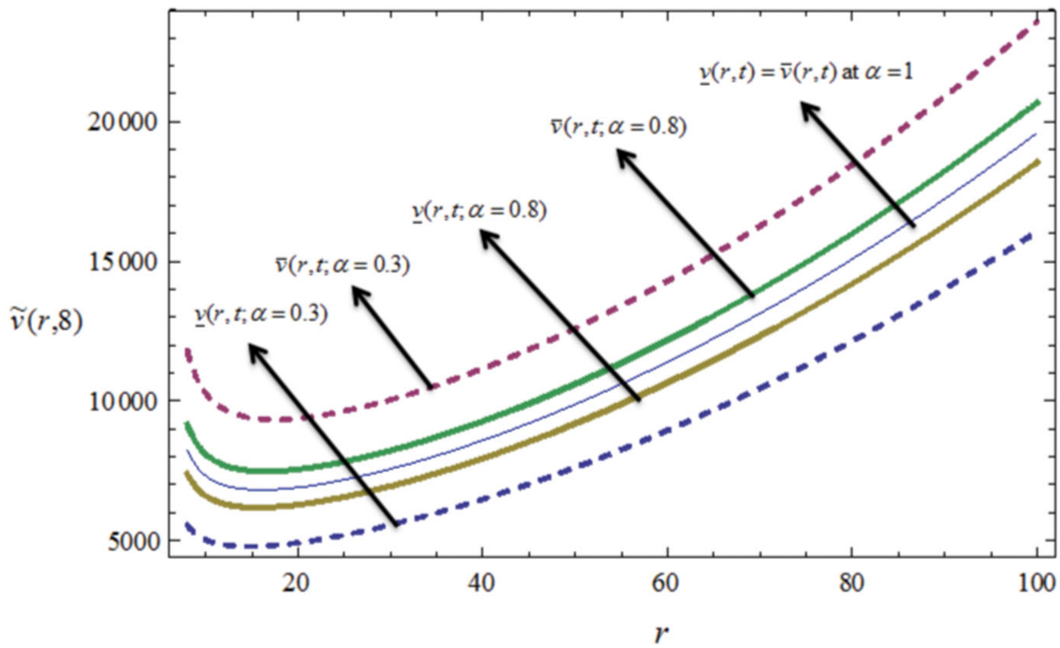

Fig. 11. Interval solution of case 1 at $t = 8$.

Also it is interesting to note, from figs. 6 to 10, that the left and right bounds of the uncertain displacement, *i.e.* $\tilde{v}(r,t)$ (with particular values of α and r), gradually increase with an increase in time. Moreover, in figs. 11 to 13, for particular values of α and t, the uncertain displacement first decreases and then increases with an increase in the radius of the membrane, r, for cases 1 to 3. But figs. 14 and 15, for cases 4 and 5, show that $\tilde{v}(r,t)$ increases with increase in r. The rate of increase in uncertain displacement is faster in case 1 than for cases 2 and 3. The rate of increase in uncertain displacement in case 5 is faster than that of case 4.

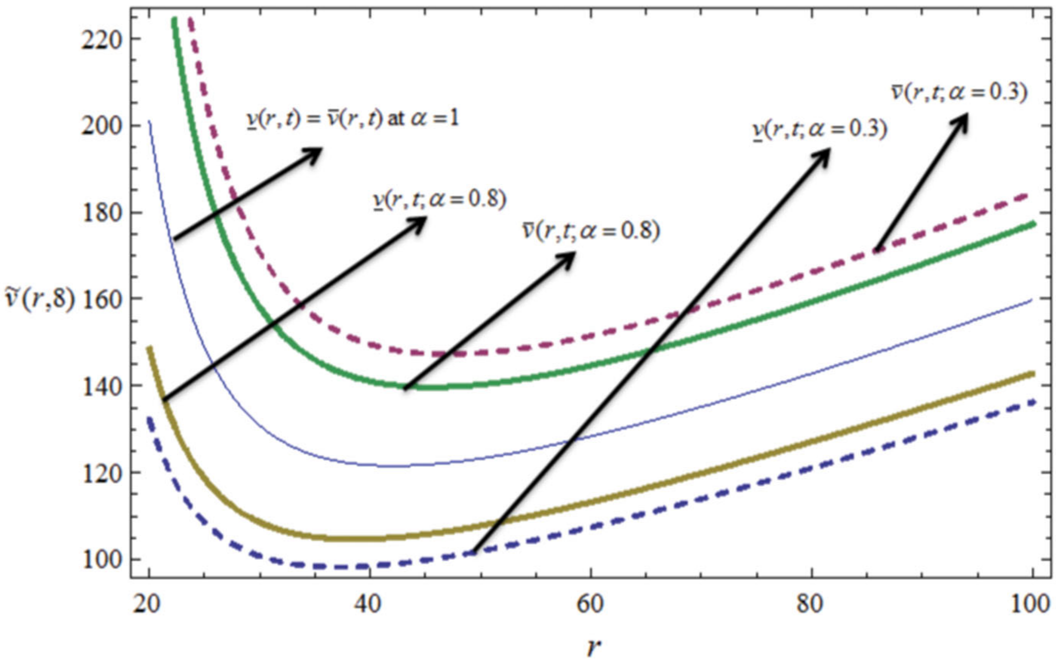

Fig. 12. Interval solution of case 2 at $t = 8$.

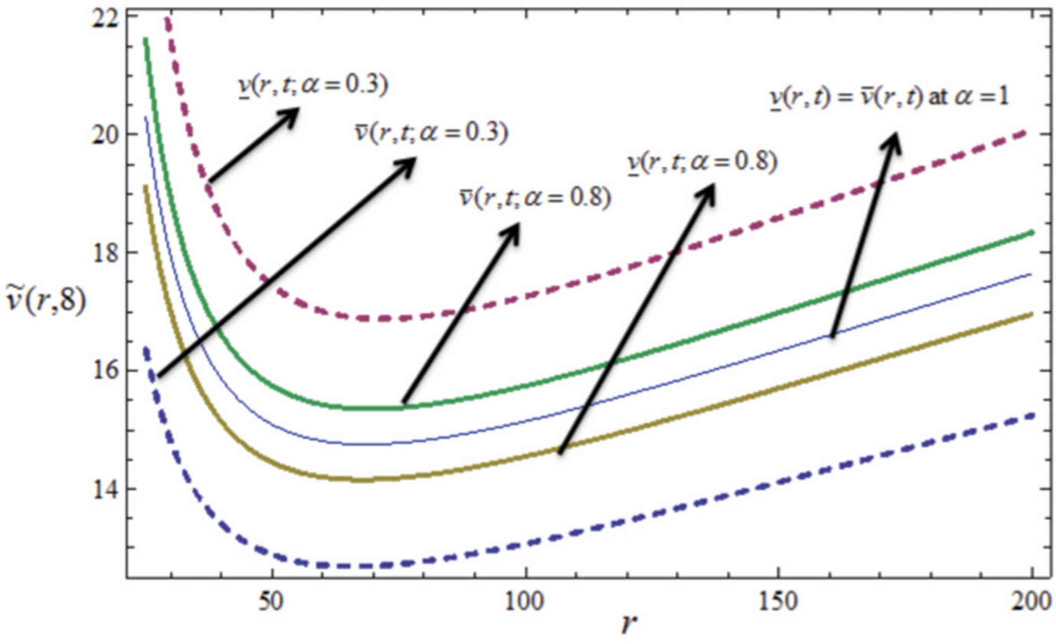

Fig. 13. Interval solution of case 3 at $t = 8$.

6 Conclusions

In this paper, the double parametric form of fuzzy numbers has been successfully applied to the solution of the fuzzy vibration equation of large membranes using the HPM. The double parametric form approach is found to be easy and straightforward. Performance of the method is shown by using triangular fuzzy number. It is interesting to note that the lower bound is equal to the upper bound solution for $\alpha = 1$. Though the solution by the HPM is of the form of an infinite series, it can be written in a closed form. The main advantage of the HPM is the capability to achieve exact solution and rapid convergence with few terms.

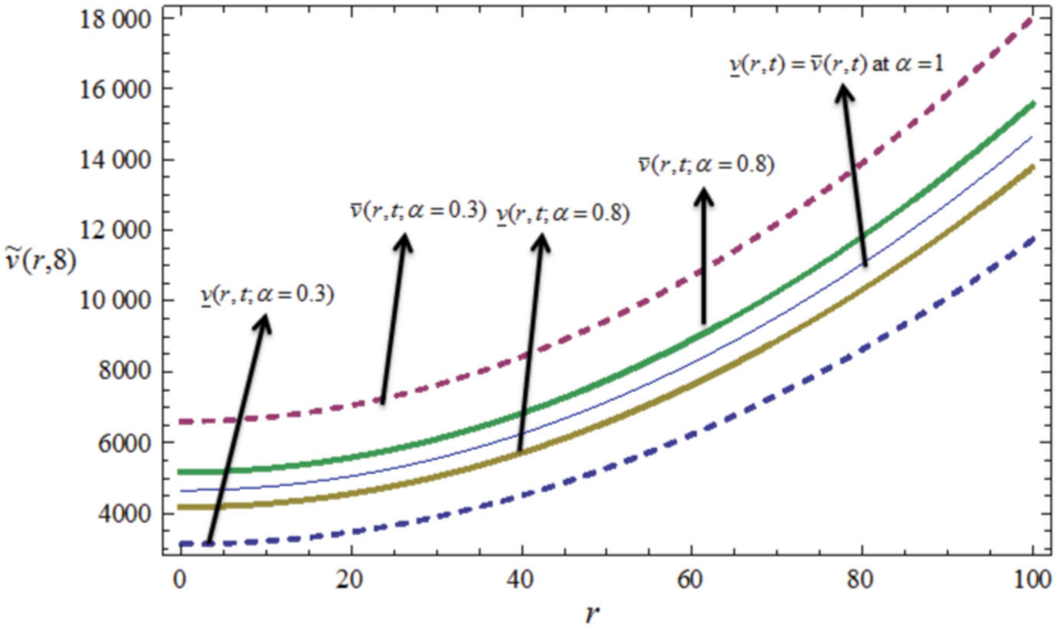

Fig. 14. Interval solution of case 4 at $t = 8$.

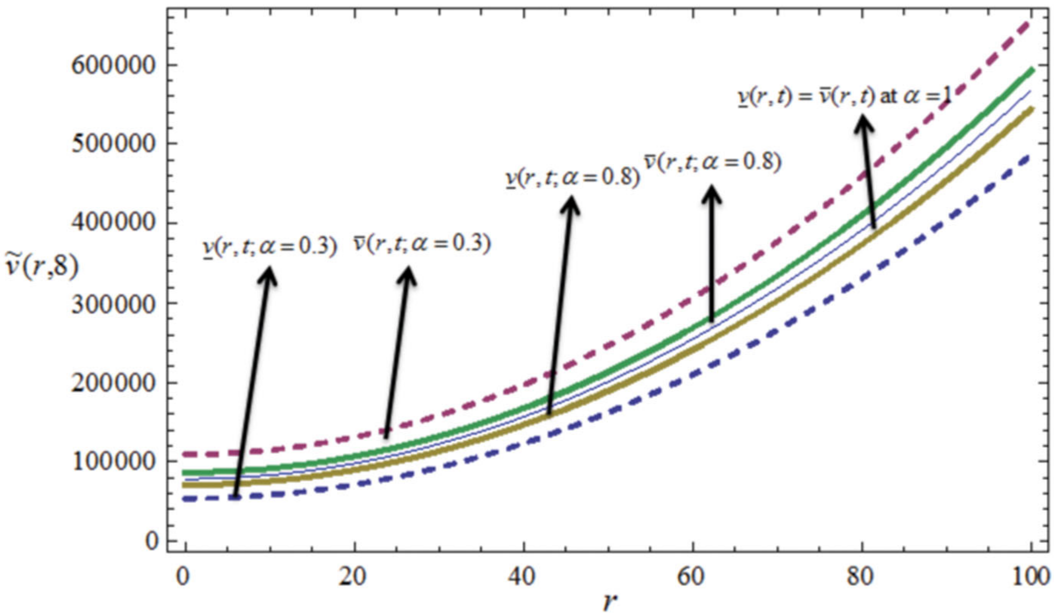

Fig. 15. Interval solution of case 5 at $t = 8$.

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