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Fractals and the Korcak-law: a history and a correction

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Abstract. The Korcak-law – first presented in an empirical form in 1938 to describe the size-distribution of various geographical objects, including lakes and islands by Jaromír Korčák – was one of the examples used by Benoit Mandelbrot to show that fractals are not only mathematical monsters, but that they are applicable to describe many natural objects and phenomena too. In this paper, we would like to give a brief overview about the history of the Korcak-law and its connection to other similar rules. Moreover, we would like to show, that although there are similarities between fractal-related laws and the Korcak-law, the Korcak-exponent is not directly related to fractal dimension. In this sense, the measure introduced by Benoit Mandelbrot based on Korčák's empirical findings is not a fractal measure.

1 Introduction

The introduction of fractals in the late 1960s and 1970s by Benoit Mandelbrot represented a major intellectual breakthrough that not only affected mathematics and physics but penetrated to almost all areas of science and also to art. In addition to many specific advances and applications, the emergence of science of fractals has notably stimulated interdisciplinary dialogue between researchers dealing with complex phenomena in both nature and society. A key inspiration for the Mandelbrot's most popular works on fractal geometry (e.g. Mandelbrot 1967, 1975, 1982) has been various empirical observations of statistical size distributions of diverse environmental phenomena. Similarly, some of his key ideas that led him to the determination of fractal dimension originated in the concept of geographical scale and resolution in maps. There is a rich and partly unknown history behind Mandelbrot's fractal geometry, which provides a nice example how scientific ideas travel between disciplines and across both space and time. As is also common in the history of scientific progress, Mandelbrot's original work is not free of errors waiting to be fixed.

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A physicist and a geographer are engaged in collaboration on the present paper with the twofold objective. As the first objective we would like to communicate an interesting piece of history that contributed to fractal geometry related to what has Mandelbrot himself credited as the Korcak-law (Mandelbrot 1975, 1982). The original scaling law presented by Mandelbrot as Korcak-law stated that

$$N(A > A_0) = cA_0^{-K}, \quad (1)$$

where N is the number of objects with size, mass or other measurable property larger than a limiting value (A_0), c is a form-factor and K is the Korcak-exponent. Thanks to Mandelbrot, the Korcak-law and Korcak exponent has become known among other (more popular) empirically observed scaling laws.

In addition, a particular importance of Korcak-law to fractal geometry stems from the fact that Mandlebrot denoted K as $D/2$, where D is the fractal-dimension of the measured property (Mandelbrot 1975, 1982). As the second objective of this paper, however, we would like to show that this relationship between the Korcak-exponent and fractal dimension is not necessarily true, not even for a set of fractal-samples. Therefore – even when the Korcak-law can be applied – the obtained Korcak-exponent is not necessarily related to the fractal dimension. In this sense, the exponent is not a fractal-measure; the Korcak-law can be considered as a novel non-fractal measure. This will be proved using Koch-snowflakes and some sets of Euclidean objects with well-known fractal dimensions.

2 History behind the Korcak-law

It was on September 13th of 1938 when Jaromír Korčák,[1] a 43 years old Czech geographer and statistician, presented a short paper at the 24th session of International Institute of Statistics held in Prague. His talk entitled "Les deux types fondamentaux de distribution statistique" (The two basic types of statistical distribution) was arguably the last scientific presentation at the congress, which was suspended prematurely on the same day (Bowley 1939). It was due to the escalation of the Sudeten crisis (also known as the Munich crisis), a prelude to the World War II., which led to the annexation of parts of Czechoslovakia by Nazi Germany in October 1938. Not incidentally, no German members of the International Institute of Statistics appeared at the Prague meeting (Bowley 1939, p. 83)[2]. In his Prague talk, Korčák presented some tables, showing the existence of a "hyperbolic" empirical rule concerning the size-distributions of lakes and islands according to their area, rivers according to their length, or Earth surface categories according to the altitude. A short paper based on the Korčák presentation was published two years later among other papers from the Prague meeting in French (Korčák 1940). A considerably more elaborate paper with many additional empirical examples appeared in Czech language in 1941 (Korčák 1941). Similar empirical regularities related to observations of highly right-skewed size distributions dates well back before the Korčák's work (Tab. 1). However, he was

[1] We use Czech characters "č" and "á" when referring to Korčák as a person throughout this paper. However, we don't apply the Czech diacritic marks when referring to the Korcak-law and Korcak-exponent.

[2] In this context, it can be mentioned that in the same year of 1938 Korčák (being both geographer and statistician) published a book entitled "The Geopolitical Foundations of Czechoslovakia. Its Tribal Areas" (Korčák 1938). Also in reaction to intensified geopolitical ambitions of Germany, he sought to justify the territorial integrity of Czechoslovakia on the basis of geo-historical stability of settlement. It undoubtedly was a daring deed at that time.

Table 1. Some older empirical rules of highly right-skewed size distributions.

Newcomb-Benford's-law (Newcomb 1881; Benford 1938)	Distribution of the leading digits of various numerical data-set
Pareto-law (Pareto 1896)	Distribution of income
Auerbach-Zipf-law (Auerbach 1913; Zipf 1941, 1949)	Population size of cities
Estoup-Condon-Zipf-law (Estoup 1916; Condon 1928; Hanley 1937; Zipf 1941, 1949)	Frequency of individual words in a text
Willis-Yule-law (Willis and Yule 1922)	Relative abundance of species
Lotka-law (Lotka 1926)	Frequency of publications of scientists
Gibrat-law (Gibrat 1932)	Size of firms (and proportionate growth of firms)
Kleiber-law (Kleiber Kleiber)	Body mass and basal metabolic rate
Korcak-Frechet-law (Korčák 1940, 1941; Fréchet 1941)	Size of islands (and other geographical phenomena)
Gutenberg-Richter-Law (Gutenberg and Richter 1944)	Magnitude of earthquakes
Wright-Richardson-law (Wright 1942; Richardson 1945, 1948)	Duration and magnitude of wars
Heaps-Herdan-law (Herdan 1960; Heaps 1978)	Number of distinct words in a document as a function of the document length

Source: adapted based on Novotný (2010).

probably the first one who sought to stress more general importance of such size-distributions for many environmental phenomena. In fact, he argued that such a form of differentiation should be similarly considered as fundamental as the so called "normal" distribution derived from the "law of symmetrically distributed errors". Hence he used the notion of the "natural duality of statistical distribution" in order to stress a philosophical or ontological importance of his empirical findings (Hampl 2000).

As a geographer working with maps, Korčák was inspired by the analysis of maps and more specifically by the work of his colleague Láska (1928) who proposed a method for a map-scale determination based on the examination of the frequency distributions of objects shown in the map in question (Novotný and Nosek 2009). Also, perhaps not incidentally, Mandelbrot's work on fractals "was inspired, in no small part, by his childhood love of maps; he began to think about creating "random coastlines from a simple formula", as he put it (Garner 2012). This coincides with what Mandelbrot wrote a year before his death to one of the authors of this paper "More generally, my father loved maps and geography and it had been a pleasure for me to make some of your field's features familiar to a different and broad public" (excerpt from an email communication with Benoit Mandelbrot from July 17th 2009).

It should be clear that Korčák himself never attempted to formalize his empirical findings mathematically by a functional form. This fact and the isolation of Czechoslovak science during the communist regime (1948–1989) contributed to the fact that Korčák's work had never become internationally known. An exception is the abovementioned short conference paper published in 1940 that attracted attention of a famous French mathematician Maurice René Fréchet. He probably didn't meet Korčák personally, but read the paper in the conference proceedings. It directly motivated him for his own publication (Fréchet 1941) in which he (following the method used earlier by Vilfredo Pareto) displayed Korčák's empirical data on double logarithmic plots, calculated their respective slopes of linear fit, and suggested apparent analogies with the distributions studied by Pareto. In addition, it was Fréchet, together with another famous mathematician Lewis Fry Richardson, who mediated virtual interaction between Korčák findings and Mandelbrot. As Mandelbrot explained: "My work on coastlines has been mostly affected by a paper by Lewis Fry Richardson, which I noticed on a pile of office garbage about to be picked by the garbage man! This was one of the main sparks that led me to Fractal Geometry. Richardson reminded me of a talk I heard repeatedly from Maurice Frechet. The talk quoted Korčák as having determined that the number of islands of area above "s" is inversely proportional to the square root of s. I had already shown that, in fact, it is proportional to a power of s different from 1/2" (excerpt from an email communication with Benoit Mandelbrot from July 17th 2009). Unfortunately, due to the existence of the Iron Curtain between East and West Europe, Korčák – who died in 1989 – was almost certainly not aware that his findings contributed to the birth of fractal geometry.

The original scaling law presented by Mandelbrot is expressed above in equation (1). For a limiting value A_0, a set should be chosen as $A_{0,1}$, $A_{0,2}, \ldots$, $A_{0,i}, \ldots$, $A_{0,n}$. The "hyperbolic" statistic, originally proposed by Korčák is valid, when $K = 1$; it is an established empirical relation for example between the surface area and abundance of lakes (see for example in Downing et al. 2006, Hendriks et al. 2012, Seekell et al. 2013). Plotting A_0s vs. Ns and using double logarithmic scale, one can obtain K as the slope of the linear fit; obviously there are several samples, where a linear fit is not appropriate, for those samples the Korcak-law is not valid. Analysis of some data-set used by Korčák (Korčák, 1939) can be seen in the Appendix A of this paper.

Similar power-laws were developed by various researchers, dealing with various fields (see for example Dohnanyi 1969, Lovejoy 1982, Zaninetti et al. 1995, Reed 2001, Jones et al. 2004 or Newman 2005); some of the older ones dated earlier than Mandelbrot's work are referred in Table 1. Although the origin is different, mathematically equation (1) is practically identical (or very similar) to the Pareto-distribution, except that in Pareto-distribution the c is not a form factor, but a coefficient related to the minimal value of A and also for traditional Pareto-distribution (analyzing the distribution of wealth), exponent K (called Pareto-index) has to be bigger than 1 (Pareto 1896). Among the many, another well-known distribution with similar form were developed by Zipf, to describe the size-distribution of cities (Zipf 1949); in that case, even the exponent is the same, $q = 1$, i.e. the distribution is hyperbolic. In spite of this multi-naming, we are going to use "Korcak-law", being the most connected to fractal geometry. It should be noted here, that although in some cases, the power-law description of some property originates from the fractality of the object or objects (and therefore the exponent of the power law might be related to the fractal dimension), but there are several cases, when these scaling laws cannot be related to the fractal geometry or the exponent is not connected to a single fractal dimension (see for example Avnir et al. 1998, Campo Bagatin et al. 2002, Jones et al. 2004 or Ballesteros et al. 2015).

While Richardson's method (Richardson-plot – see e.g. Mandelbrot 1967, Richardson, 1961) is still widely used, Korčák's method never gained so much popularity; partly because the fractal dimension obtained by Korčák was not the expected one. On the other hand, the form, proposed as the mathematical form of the Korcak-law (Eq. (1)) is a widely used one to describe ranking or abundance of various size distributions (Brakman et al. 1999, Newman 2005). The only field where the original Korcak-method was very frequently used was the one related to ecology and a bit widely to biology (Sugihara and May 1990, Seuront 2010); here the Korcak-law was frequently used to describe "patchiness" created by various biological or ecological processes. Unlike some other techniques of fractal analysis such as the perimeter-area method, a potential appeal and practical applicability of the Korcak-analysis stems from the fact it solely requires the knowledge of the areas of the objects in question, which can be determined more accurately than the perimeter (Imre et al. 2012).

Most recently – in the past four to five years – Korcak-method seems to have a "Renaissance" in geography, planetology, biology and ecology (Seuront 2010; Erlandsson et al. 2011; Hayes 2011; Imre et al. 2011; Convertino et al. 2012; Imre et al. 2012; Jang and Jang 2012; Imre 2015). Still, the incomparability of the "Korcak-dimension" with the fractal dimension obtained by other methods (like perimeter-area relation, box-counting, etc.) was always a problem.

3 Korcak-law and ranking

It can be easily shown that the value of the obtained K exponent can be influenced by the choice of the limiting A_0 values, i.e. the observer can influence the result (Imre et al. 2011, Imre et al. 2012). To avoid this problem, there are three ways to choose the A_0-set:

– using a "usual" set like (2, 4, 8, 16,. . .); logarithmically equidistant; etc.;
– using a "natural" set, given by the measurement (for example by using a set of filters to determine particle size distribution, it is natural to use the sizes of the filters for $A_0 s$);
– and finally as the most independent one (i.e. independent from the observer) is the use of a measured property – A_i – of the sample as

$$A_{0,i} = A_i - \delta, \quad i = 1, \ldots, M, \tag{2}$$

where M is the number of particles, etc. within the sample and δ is a small number. In the limiting case of $\delta \to 0$, N will be the so-called rank or abundance.

It can be seen, that the first two methods are not really observer-independent (although one can define a special K exponent, by always using one of them), therefore the third one should be prioritized.

There are numerous signs to show, that the exponent obtained from a Korcak-fit (K) does not have any connection with the fractal dimensions (Imre et al. 2012). It can be often seen, that the obtained exponent in various methods (like perimeter-area or this Korcak-method) differs significantly from the expected $2/D$ value, where D is the fractal dimension obtained for the same set, for example by using the perimeter-area method (Imre 1992, 2006). In some cases (like perimeter-area or perimeter-maximum diameter) this can be explained by the violation of similarity, i.e. although the individual patches are fractals, they are not similar ones; therefore the obtained exponent is not related to the fractal dimension (Imre 1992). In those examples, it is often anticipated, that for using a set of similar patches, the Korcak-exponent would be surely equal to $2/D$. Here we can show, that for the Korcak method, even by holding the strictest condition of similarity and fractality of the individual patches (using "classical" fractals, namely Koch-snowflakes), and generating the samples with well-known fractal-generating methods recommended by Mandelbrot (see later, Fig. 2), the Korcak exponent is not related to the fractal dimension of the individual patches.

4 Individual vs. group properties

Fractal dimension is an individual property of ONE object. For "theoretical" Koch-snowflakes (i.e. for a snow-flake constructed by infinite step) the fractal dimension of the perimeter is log(4)/log(3), while for the area it is 2. For "real" Koch-snowflakes (constructed by N finite steps) there is a size-range, where the objects show properties similar to the ideal Koch-snowflake and in that range, the object termed as "physical fractal", having a fractal dimension (equal to the theoretical one) in a limited size-range.

There are methods (like the perimeter-area method (Mandelbrot 1982, see below) where instead of having one object, we have a set of objects created during a single physical, chemical, geological, biological, etc. process. In several cases, various size-distributions of the set can be described by a power function in a limited size-range, represented by a single exponent. That exponent is a group-property (or collective property) of the set. Assuming that the objects represent various generations of a fractal-generation (like various steps for Koch-snowflakes), one can expect that the exponent of the size-distribution is related to the fractal dimension. In this way, one can use the size-distribution of the set to reveal the fractal dimension of the "arche-type fractal" of the set.

It is tempting to assume that a straightforward relationship between the power-law size distributions and fractal phenomena exists. Such an assumption is invoked by the notion of self-similarity inherently associated both with power laws and fractals. The sets of objects that follow a power-law size distribution are scale-invariant in some sense, so these objects are considered to be self-similar with respect to the measured quantity. The notion of self-similarity used in fractal geometry is broader and more complex and many fractals are not exactly self-similar but rather self-affine in some sense. One should be aware that even when the size-distribution can be described by a single exponent power-law, it is not necessary that the exponent is related to the fractal dimension. Only in some cases (like perimeter-area and perimeter-maximal diameter distributions) the conditions for the relation is more or less known (see for example Imre 1992, 1995; Cheng 1995). In addition, even when a universal applicability of the Mandelbrot's relationship between the fractal dimension and Korcak-exponent is questioned, it still might be assumed that it has a weaker validity, e.g. for exactly self-similar fractals which are structurally similar at different scales. In the following, we will nevertheless show that using the Korcak-method proposed by Mandelbrot one can obtain a "false" fractal dimension even when the analyzed set consists of strictly self-similar fractal objects.

5 The fractality of Koch-snowflakes

Four different data-sets will be analyzed in this part; all of them are strongly related to the arche-type of fractals, the so-called Koch-snowflake. Koch-snowflakes are iterated from a triangle as the zeroth iteration. In the first iteration, each side (with "a" length) of the triangle will be divided into three equal parts (with $a/3$ length) and the middle one will be replaced by two line segments (see Fig. 1). For traditional Koch-curves, the length of the newly inserted segments are $a/3$ (Fig. 1a), but it can be smaller (Figs. 1b–1d), down to $a/6$ (Fig. 1e); in the latter case, the "new" side will not differ from the "original" one. Then this process can be repeated again and again, in infinite steps. The obtained object will be a Koch-snowflake (also called Koch-island). The fractal dimension of the perimeter of the original Koch-snowflake – when the newly inserted segments are $a/3$ long – is $\log(4)/\log(3) = 1.26186$; it can go down to $\log(3)/\log(3) = 1$, by using shorter and shorter replacements. The area-dimension is always 2 (Mandelbrot 1982).

On Figures 2a and 2b, two different ways can be seen to construct Koch-snowflakes. In both cases, the dividing-method, described in the previous paragraph is used and the side-length of the initial triangle was taken as unity. On Figure 2a, an additional step will be applied in each iteration, namely the size of the object will be increased by a factor of 3; in this way, size-length – the linear building-blocks – will remain

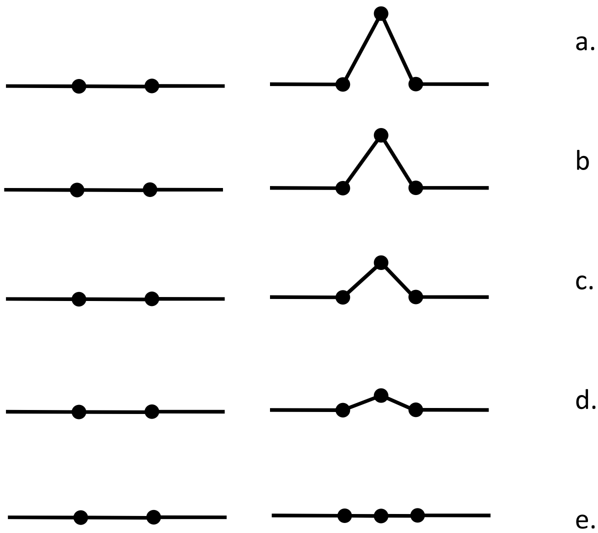

Fig. 1. The iteration of the sides of Koch-snowflakes using various replacement-length for the middle part. (a) Represents the traditional Koch-curve, when the length of the replacements are equal to with the replaced part; in this case – starting from a triangle – the perimeter-dimension of the Koch-snowflake will be log(4)/log(3) = 1.26186. (e) Represents an extreme case, where the length of the replacements are only half of the replaced part, i.e. the two replacements will give back the replaced part, keeping the original line. In this case starting from a triangle – the perimeter-dimension of the Koch-snowflake will be log(3)/log(3) = 1.

constant, giving a set of snowflakes growing three times bigger in each step. In the following part, this set will be referred to as growing Koch-snowflakes. In Figure 2b, the simple dividing-method is used. The lateral size (the radius or the diameter of the circumscribed circle) is kept constant, giving a set of uniformly sized snowflakes with more and more fine-structured edge. The diameter of the circumscribed circle (except for the initial triangle) is equal to the so-called maximal Feret's diameter or caliper length). In the following part, this set will be referred to as constant-size Koch-snowflakes. In both cases, the perimeter- and area-dimension of the final object will be log(4)/log(3) = 1.26186 and 2, respectively.

Two other sets will be also used; because they are very simple, there no need to display them graphically. The elements of the third set are theoretically also Koch-snowflakes, but in this case the middle replacement is only $a/6$ – i.e. it will remain the same triangle – but the object is growing to three times the size with each step. Eventually, the elements of this third set will be triangles, the smaller one with unit-size sides, the second one is three times bigger, the third one is nine-times bigger, etc. In the following part, this set will be referred as tripled triangles. The fourth set will be also a set of triangles with the smallest one equal to the initial triangle of the previous set; then each following one is double in lateral size. In the following part, this set will be referred to as doubled triangles. In both cases of the third and fourth sets, the perimeter- and area-dimension of the final object will be 1 and 2, respectively.

For the sake of better handling, each sample-set considered in the subsequent analysis contains only 20 samples; the zeroth generation original triangle and the 1st, 2nd, … 19th generations. In case of doubling, the set will cover more than six order of magnitudes (from the original to original $\times$ 2^{19}), while in the case of tripling, it will be more than ten orders of magnitude. The "physical fractal" behavior can be

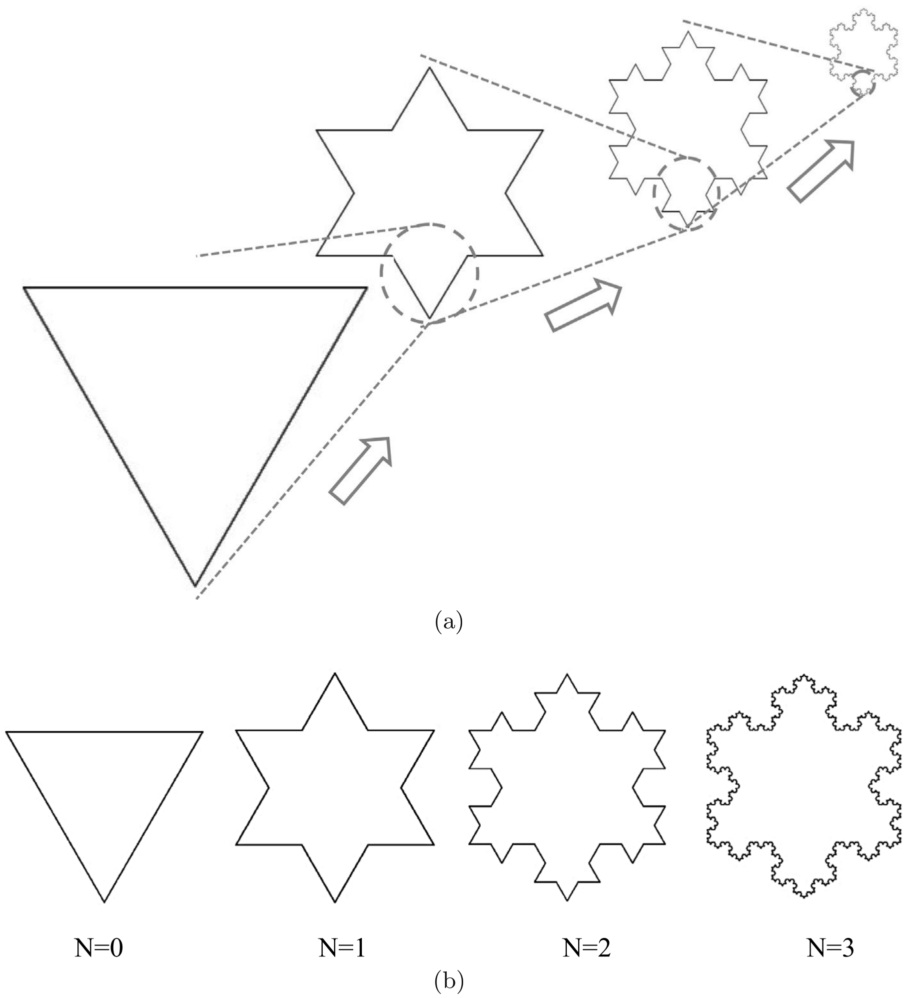

Fig. 2. Two various ways to create Koch-snowflakes; (a) by keeping the length-size fixed in each iteration or (b) by keeping the lateral size fixed in each iteration. The first set is referred to as growing Koch-snowflakes, while the second size is referred to as constant size Koch-snowflakes.

expected from the 1st to the 19th generation; the 6 and 10 orders of magnitudes are much larger that the 2–3 orders expected to justify the use of fractal description for physical objects (Avnir et al. 1998).

Being these objects – within each of the sets – similar, perimeter-area and perimeter-maximal diameter relations (Mandelbrot et al. 1984; Imre 1992, 1995; Mu et al. 1993), can be used to determine the fractal dimension of perimeter (D_P), the fractal dimension of the area (D_A) and their ratio, using the following equations:

$$P = C_P L^{D_p} \tag{3}$$

$$A = C_A L^{D_A} \tag{4}$$

$$A = C_{PA} L^{D_A/D_p} \tag{5}$$

where L is the lateral size, while C_A, C_P and C_{PA} are various constants (form-factors).

In Figures 3a and 3b one can see a double-logarithmic perimeter-maximal diameter plot and the double-logarithmic area-maximal diameter plot for a set of growing Koch-snowflakes. The slope of the fit for perimeter-maximum diameter (without the first triangle) gives back the exact dimension of the Koch-curves, $\log(4)/\log(3) = 1.26186$. For the area-maximum diameter, the slope is exactly 2, showing the Euclidean nature of the area for this object. It is an often overlooked fact, that Koch-snowflakes are perimeter-fractals (i.e. their perimeter shows fractal properties), while their area shows Euclidean nature, being exactly two-dimensional.

On Figure 3c, the double-logarithmic perimeter-area plot for the growing Korcak-snowflakes can be seen; the slope is equal to the ratio of the two (perimeter and area) fractal dimensions ($D_A/D_P = 1.585$), as it was expected from equation (5).

In Figures 4a and 4b one can see a double-logarithmic perimeter-maximal diameter plot and the double-logarithmic area-maximal diameter plot for a set of constant-size Koch-snowflakes. Being the lateral size (maximal diameter) is constant, while the perimeter goes to infinity and the area goes to an upper limit (given by a circle with the same diameter), the slopes for these two plots are infinite, i.e. in this case the examination of the sets containing all iterations does not provide information about the fractality of the individual objects.

In Figure 4c the double-logarithmic perimeter-area plot for the constant-size Korcak-snowflakes can be seen; in this case the points cannot be approximated properly with a linear fit.

In Figures 5a and 5b one can see a double-logarithmic perimeter-maximal diameter plot and the double-logarithmic area-maximal diameter plot for a set of tripled triangles. The slope of the fit for perimeter-maximum diameter (without the first triangle) gives back the exact dimension of the an Euclidean triangle (1), which is the same as the fractal dimension of a Koch-curve with $a/6$ replacements (i.e. zero-angle between the original and replacement lines), $\log(3)/\log(3) = 1$. For the area-maximum diameter, the slope is exactly 2, which is not surprising from a Euclidean object.

In Figure 5c, the double-logarithmic perimeter-area plot for the tripled triangle set can be seen; the slope is equal to the ratio of the two (perimeter and area) dimensions ($D_A/D_P = 2$), as expected from equation (5).

Finally, in Figures 6a and 6b one can see double-logarithmic perimeter-maximal diameter plot and the double-logarithmic area-maximal diameter plot for a set of doubled triangles. Just like in the tripled triangle case, the slope of the fit for perimeter-maximum diameter (without the first triangle) gives back the exact dimension of an Euclidean triangle (1), while for the area-maximum diameter, the slope is exactly 2.

In Figure 6c, the double-logarithmic perimeter-area plot for the doubled triangle set can be seen; the slope is equal to the ratio of the two (perimeter and area) dimensions ($D_A/D_P = 2$), as expected from equation (5), just like in the tripled triangle case.

Exponents obtained from the fits (slopes of the fits) are listed in Table 2 (column 3), together with the theoretically expected values (column 4).

6 Applicability of the Korcak-law on the two sets of Koch-snowflakes and on the two sets of multiplied triangles

In Figures 7a–7d, the Korcak-plots of maximal diameters, perimeters and area of growing Koch-snowflakes (Fig. 7a), constant-size Koch-snowflakes (Fig. 7b), tripled triangles (Fig. 7c) and doubled triangles (Fig. 7d) can be seen. It is clear, that a linear fit of the log-log plotted data is not justified for any of these sets. On the other hand, for natural sample sets it is quite common that only a part (small-size; high rank) is fitted. In this case, that part seems to obey a linear relationship, therefore points ranked from 11 to 19 are fitted (20th is the initial triangle, excluded from all fits).

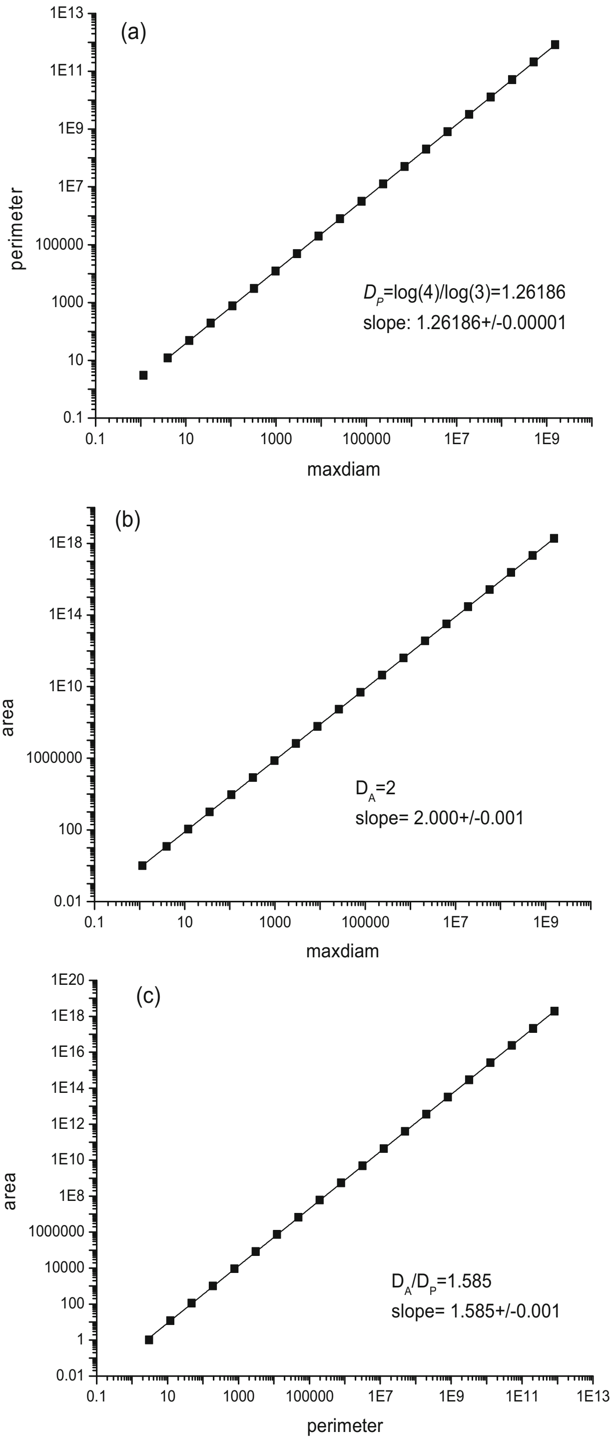

Fig. 3. Double-logarithmic perimeter-maximal diameter (a), area- maximal diameter (b) and perimeter-area (c) plot for a set of growing Koch-snowflakes. For the first two plots, the slope of the fit (without the first triangle) gives back the exact perimeter- and area-dimension of the Koch-curves, while for the third one, the slope is equal with ratio of these two values.

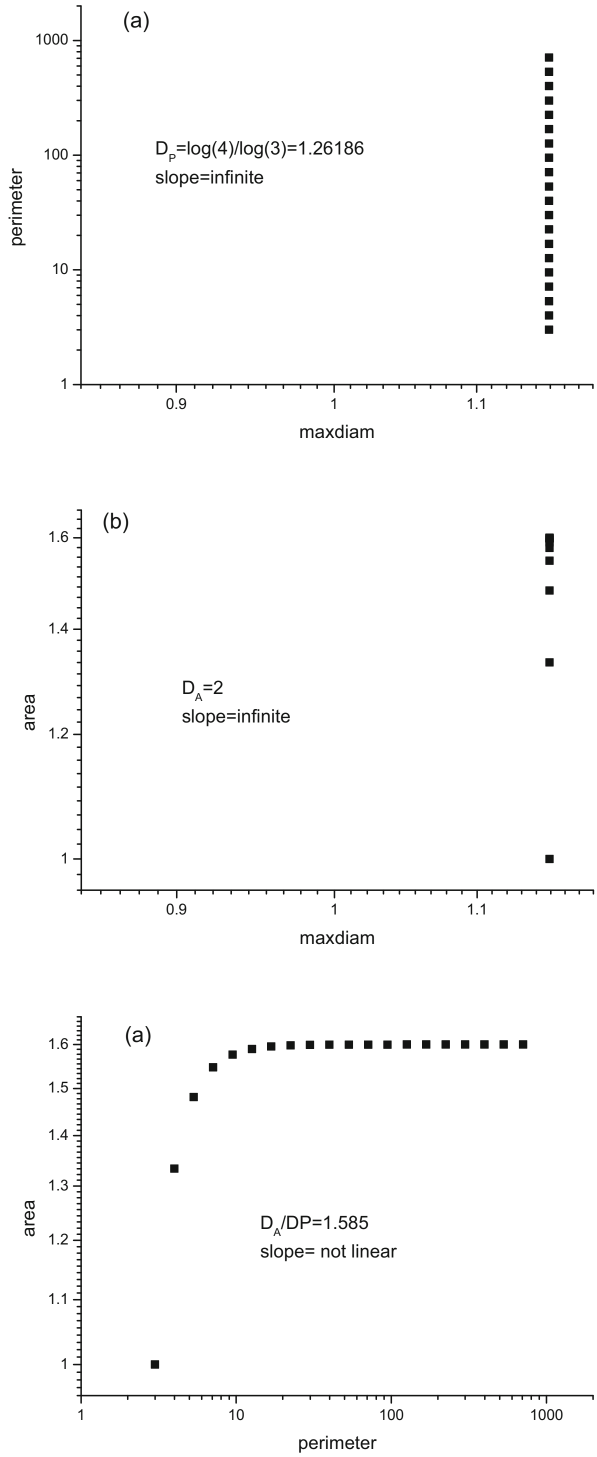

Fig. 4. Double-logarithmic perimeter-maximal diameter (a), area- maximal diameter (b) and perimeter-area (a) plot for a set of constant-size Koch-snowflakes. For the first two plots, the slope of the fit is infinite, there is no agreement with the theoretical fractal dimension, while for the third one the data-set cannot be fitted by a linear fit.

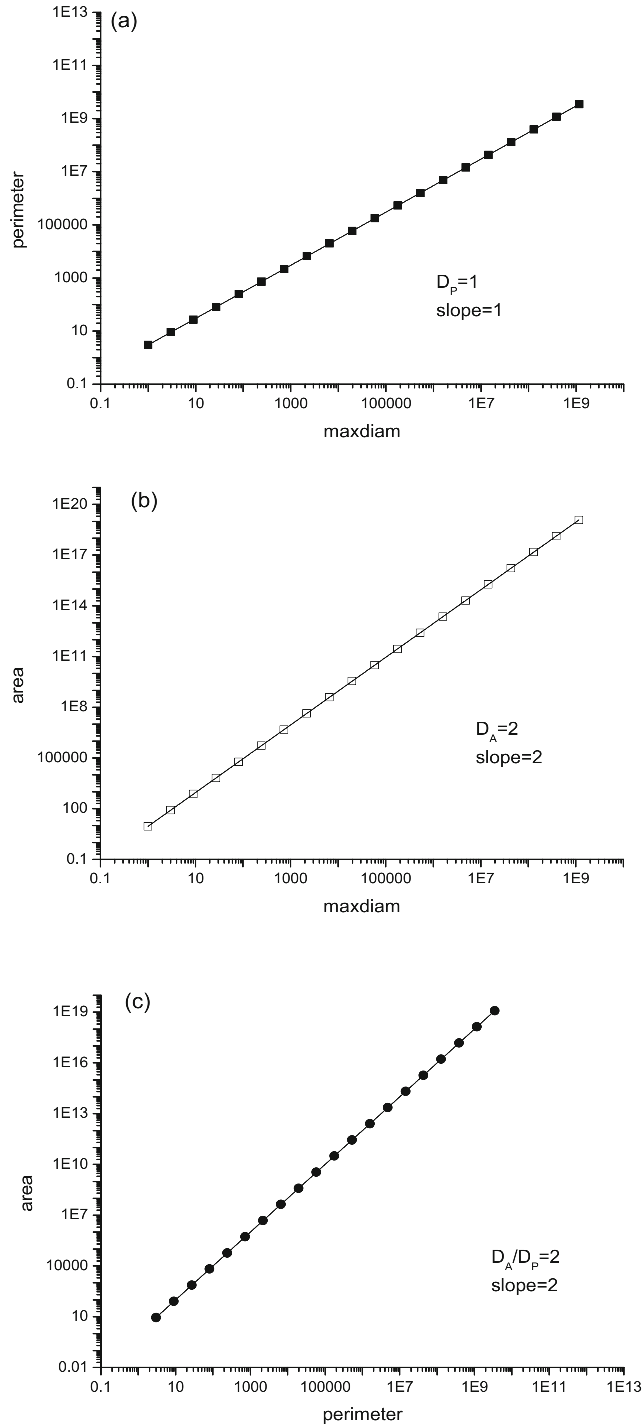

Fig. 5. Double-logarithmic perimeter-maximal diameter (a), area-maximal diameter (b) and perimeter-area (c) plot for a set of tripled triangles. For the first two plots, the slope of the fit (without the first triangle) gives back the exact perimeter- and area-dimension of the Euclidean objects, while for the third one, the slope is equal with ratio of these two values.

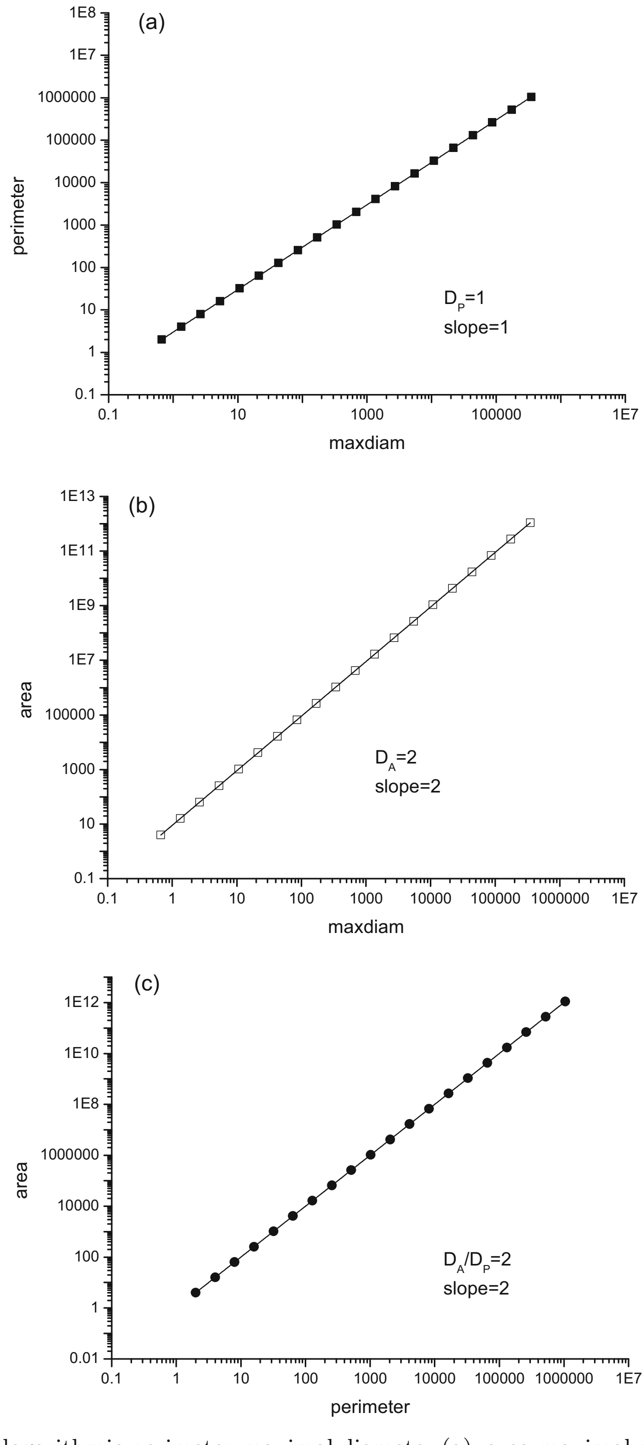

Fig. 6. Double-logarithmic perimeter-maximal diameter (a), area-maximal diameter (b) and perimeter-area (c) plot for a set of doubled triangles. For the first two plots, the slope of the fit (without the first triangle) gives back the exact perimeter- and area-dimension of the Euclidean objects, while for the third one, the slope is equal to the ratio of these two values.

Table 2. Obtained slopes for perimeter-area and ranking plots for the various datasets, compared to the theoretically expected slopes.

Type of the set	Property	$D/2$ (from PA-method)	$D/2$ (exact)	Initial Korcak-slope	Final Korcak-slope	Expected Korcak-slope ($-D/2$)
Growing Koch	max. diameter	1/2	1/2	-0.064 ± 0.002	–0.63	–1/2
	perimeter	0.63	0.63	-0.053 ± 0.002	–0.5	–0.63
	area	1	1	-0.033 ± 0.001	–0.315	–1
Constant-size Koch	max. diameter	infinite	1/2	not applicable/ infinite	–infinite	-1/2
	perimeter	infinite	0.63	-0.236 ± 0.007	–2.41	–0.63
	area	not applicable	1	not applicable/ infinite	–infinite	–1
Tripled triangle	max. diameter	1/2	1/2	-0.062 ± 0.002	–0.63	–1/2
	perimeter	1/2	1/2	-0.062 ± 0.002	–0.63	–1/2
	area	1	1	-0.032 ± 0.001	–0.315	–1
Doubled triangle	max. diameter	1/2	1/2	-0.098 ± 0.003	–1	–1/2
	perimeter	1/2	1/2	-0.098 ± 0.003	–1	–1/2
	area	1	1	-0.048 ± 0.001	–0.5	–1

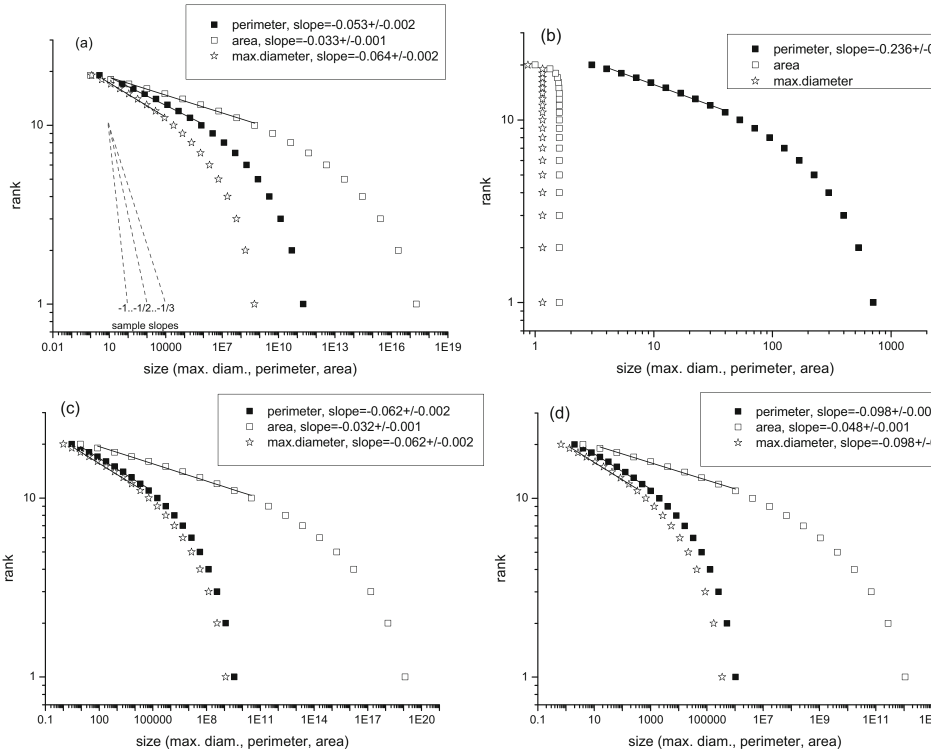

Fig. 7. (a) Double logarithmic Korcak-plots (size vs. ranking) for the set of growing Koch-snowflakes for perimeters (full squares), areas (empty squares) and maximal diameters (stars). Linear fit for the data representing the objects with iteration number 2 to 10 are plotted and the slopes for the lines are listed in the insert. For comparison, dashed lines with various slopes (–1, –1/2 and –1/3) are also plotted. (b)–(d) the same plots for constant-size Koch-snowflakes (b), for tripled triangles (c) and for doubled triangles (d). For (d), (c) and (d) figures, lines with sample-slopes are not included.

The obtained initial slopes are listed in Table 2, column 5; the theoretical values are in column 7. It can be clearly seen, that theoretical ($-D/2$, see Mandelbrot 1982) values are very far from the experimentally obtained ones, the difference is around one order of magnitude. On the other hand, some – presumably linear – relationship can be seen between the initial Korcak-slopes and the expected ones. This is most striking in the tripled and doubled triangle sets, where – within the error of fit – the obtained slopes for perimeter and maximal diameter are twice that of the slope for area-data, reflecting the ratio of fractal dimensions (1 for perimeter and diameter, 2 for area). For the growing Koch-snowflakes set, the same inverse relation can be obtained (0.5/0.63/1 and 0.033/0.053/0.064), therefore it seems to be a plausible conclusion, that although the Korcak-slope (K) does not obey the simple relationship proposed by Mandelbrot (1982), a $K(D)$ function seems to exist, where the D-dependence is probably inverse linear ($1/D$).

One might say that picking the high-rank end of the distribution was an error and one would rather check the final slope (the part between rank 1 and 2). In Figure 7a one can see three dashed lines, demonstrating linear dependences with –1. –1/2 and –1/3 slopes. These slopes (close to the expected Korcak-exponents) seem to be similar to the slope of the last segments. This last segment slope can be easily calculated, even for $N \rightarrow \infty$ iterations. The final slope (K_{f}) between ranks 1 and 2 on the double logarithmic plot is:

$$\lim_{n\to\infty}\left(\frac{\log(2)-\log(1)}{\log\left(size(n)\right)-\log(size(n-1))}\right)=K_f$$

where "size" is the corresponding size (maximal diameter, perimeter or area). The numerator is log(2), while the denominator is the logarithm of the size ratio between the objects from iteration n and $n-1$. For the growing Koch-snowflakes, this ratio is 3 for the diameter, 4 for the perimeter and and 3^2 for the area, therefore the final values should ALWAYS be (independently from the step of iterations) $\log(2)/\log(3) = 0.63$ for the diameter, $\log(2)/\log(4) = 0.5$ for the perimeter and $\log(2)/\log(9) = 0.315$ for the area. For tripled triangles – where the linear size-ratio in each iteration is three - these values will be (also independently from the step of iterations) $\log(2)/\log(3) = 0.63$ for the diameter and for the perimeter, while $\log(2)/\log(9) = 0.315$ for the area. For doubled triangles – where the linear size-ratio in each iteration is two – these values will be always $\log(2)/\log(2) = 1$ for the diameter and for the perimeter, while $\log(2)/\log(4) = 0.5$ for the area. Finally, for constant-size Koch-snowflakes, these values (which depend on the step of iteration now, therefore $n \to \infty$ cases are given) is $\log(2)/\log(1) = \infty$ for the diameter and for the area (maximal diameter is constant in each iteration, while the area should converge to the area of circumscribed circle, giving a smaller and smaller ratio in each iteration), while for the diameter and for the perimeter, it should be $\log(2)/\log(4/3) = 2.41$. The relation shown in previous cases (i.e. that the slopes for perimeter and area is twice as high as for the area) seems to be valid here too (except for the set of constant-size Koch-snowflakes), also suggesting a $K(D)$ function, where the dependence of D should be in the form of $1/D$. On the other hand, it is definitely not a simple inverse linear relationship (like the $2/D$ suggested by Mandelbrot). The most striking example is the comparison of the two sets of triangles, where D-s should be equal, but the obtained slopes differ significantly!

In Appendix B, we are demonstrating that the problems shown here are independent from the choices of limiting sizes and therefore also appear in the traditional Korcak-plot.

7 The effect of sampling

While as a result of a process (like aggregation, fragmentation) a complete set, containing each iteration can be produced, during sample collection, some of the samples can be lost. In that case, classical fractality-based size-distributions (like maximal diameter vs perimeter, maximal diameter vs. area or perimeter- area) will not change. Although some points representing the missing elements of the set will be missing, the other points remain in the same x-y coordinates, providing the same slope. Therefore these obtained results are not sensitive for the sampling. It can be seen on Figure 8a, demonstrated by the perimeter-maximal diameter dependence of growing Koch-snowflakes.

The situation is completely different in ranking. Leaving out even one sample, one can artificially induce a "break" in the virtual fit (Imre 2015). While the x-coordinates (size) remain for the remaining samples, the y-coordinates for samples that are smaller than the omitted one must change. With some systematically biased sampling – like taking every second sample – one might completely falsify the distribution. In Figure 8b one can see an example where the objects corresponding to every odd-numbered iterations are omitted. For the sake of simplicity, only the perimeter-ranking (original with all objects and biased after omitting the odd-numbered) of growing Koch-snowflakes can be seen. While for the complete sets, the initial slopes are –0.053,

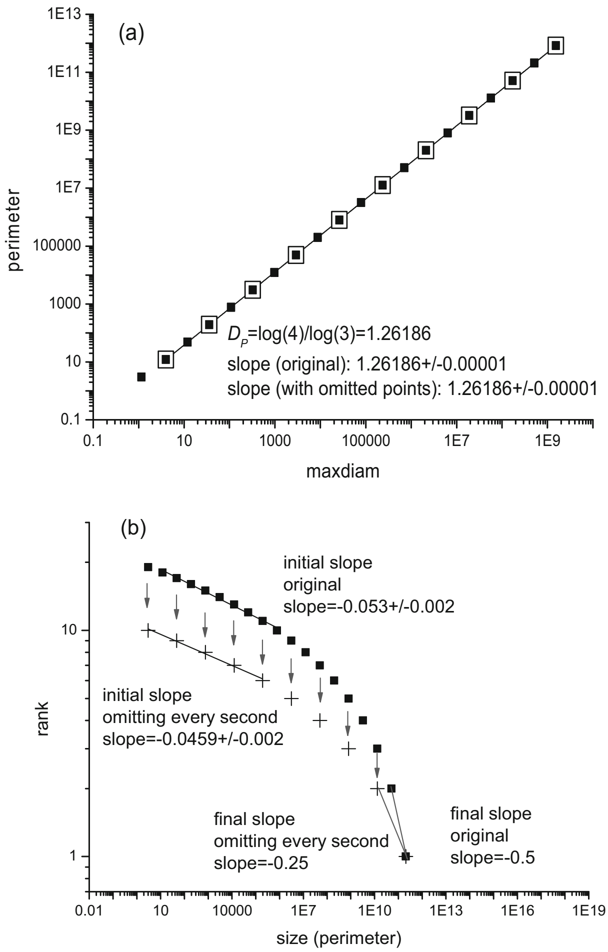

Fig. 8. Effect of sampling. (a) For perimeter-area plot, omitting every second point does not change the run of the points; the remaining point (bigger empty squares) remain on the same coordinates. (b) For the ranking (Korcak) plot, by omitting every second points, the new points (+signs) keeps the x-coordinates (size), but change the y-coordinates (rank). Therefore the initial and final slopes change significantly.

keeping the even-numbered and fitting only the first five (the same size-range for the first ten in the full sets) the slopes will change to –0.046. The final slopes also change, even more drastically, from –0.5 to $-(\log(2)/\log(16)) = -0.25$, because due to the missing "iteration", the perimeter change is not 4, but $4 \times 4 = 16$.

From these results it can be seen, that even sampling can influence this ranking distribution, while the "traditional" size-distributions to provide fractal dimensions are insensitive for that. Therefore the obtained K exponent of the set-distribution cannot be a function only of the individual fractal dimension of the samples.

8 Discussion and conclusions

The first part of the paper described the history behind what is known as the Korcak-law. It outlined the context and the way in which one piece of inspiration for Benoit Mandelbrot's fractal geometry had travelled already before he published his most famous works on fractals in nature. The main second part then addressed the relationship established by Mandelbrot between the Korcak-exponent and fractal dimension. Its universal applicability was contested by the means of the calculation of fractal dimensions using the Korcak-method and other popular techniques for several examples of well-defined sets of fractal objects. The remainder of this section provides some concluding comments on the findings that were obtained.

It was stressed that fractal dimension – or rather fractal dimensions, in plural - are properties of individual objects. Sometimes various properties for a set of objects can reflect the individual fractal dimensions of the elements of the set. Therefore some size distributions describing various properties (like diameter, area, perimeter, etc.) for a set can be used to determine the individual fractal dimensions. For example perimeter-area and perimeter-maximal diameter method used for a set of objects can obtain the individual perimeter fractal dimension of the object, but if and only if the objects are strictly similar (Imre 1992, 1995; Cheng 1995). The application of the Korcak-law on a set was thought to have a similar power to give back the individual fractal dimension of the ranked property (perimeter, area or even volume).

Four different sets were analyzed in this paper. Two sets of equilateral triangles with different sides (doubled or tripled in each iteration step), started from a triangle with unit-side and containing the iterative elements from the zeroth to 19th element, referred as doubled- and tripled-triangles. One set of nineteen Koch-snowflakes (also started from an equilateral, unit-sided triangle as zeroth element); in each iteration the lateral sizes were tripled, referred to as growing Koch-snowflakes. Finally, a set of similarly prepared Koch-snowflakes, shrunk in each iteration to keep the diameter of the circumscribed circle constant, referred to as constant-size Koch-snowflakes.

For the set of growing Koch-snowflakes, individual fractal dimensions can be obtained by applying traditional lateral-size vs. perimeter and lateral-size vs. area studies. For perimeter-area study, the ratio of area and perimeter fractals can be obtained.

For the two sets of triangles, the lateral-size vs. perimeter or area studies also applicable to obtain the individual fractal dimensions and the perimeter-area relation also gave back the ratio of the area and the perimeter dimension.

For the constant-size Koch-snowflakes – although they were strictly similar fractals – neither of these methods were able to provide any information about the individual dimensions.

Because three of the four sets scored properly using the previous methods, one might expect them to score properly by applying the Korcak-law on them. However, the results for Korcak-method provided unsatisfactory results in several respects:

1. Korcak-law cannot be applied (i.e. the ranked properties cannot be described with a single power law exponent) even for sets consisting of strictly similar fractal objects.
2. Forcing Korcak-law on some part of the distribution, the obtained slope are related to the inverse fractal dimension ($1/D$), but the relation is definitely not what Mandelbrot expected ($K = 2/D$). The relationship seems to be more complex, even the sampling ratio can affect the value of K, among other possible influences.

Therefore one can conclude, that Korcak-type ranking in not a simple fractal-type description, although under strict conditions (for a set of strictly similar fractal-like objects) it might be related to the individual fractality of the described objects.

In general, we can conclude that although the Korcak-law can be a useful tool to describe ranking distributions, its applicability is not guaranteed even for the sets of strictly similar fractal objects. The obtained Korcak-exponent – although probably related to the fractal dimension – cannot straightforwardly give back the fractal dimensions of the individual objects. In this sense, Korcak-law is not one of the

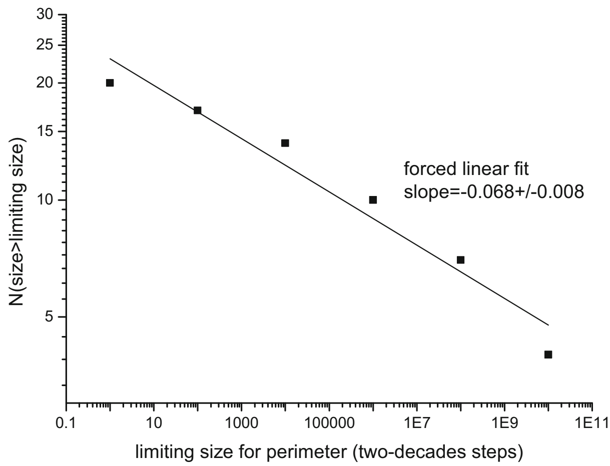

Fig. A.1. Traditional Korcak-plot of growing Koch-snowflakes with logarithmically equidistant limiting sizes (1, 10, 100, 1000, etc.). The run of the points are significantly curved; although one can force a linear fit on the data, but the slope does not coincide with $D/2$.

fractal-geometry related laws, but a separately existing, although mathematically similar scaling-law.

Appendix A

The traditional Korcak-law is using log-equidistant limiting sizes with a step-size large enough to fit several samples into one category (Mandelbrot 1982). Just to demonstrate that the problems described here are not caused by our choice of limiting sizes, we are showing here the analysis of the perimeters of the growing Koch-snowflake set, using limiting sizes (in unity) with two-decade width, as 1, 100, 10 000, 1 000 000, etc. The Korcak-plot can be seen in Figure A.1. The points cannot be fitted linearly; forcing a linear fit, the apparent slope is around –0.07, still an order of magnitude smaller than the one ($-2/D = -0.63$) predicted by Mandelbrot.

Obviously the slope can be influenced by the choice of limiting sizes (one can shrink or expand the x-axis). By using logarithmically not equidistant limiting sizes, even the tendency of the curve can be changed (for example it can be "smoothed" a bit) and in this way one might rather accidentally construct the plot with the proper $/2/D$) slope – but that can be done by sheer luck or deliberate bias.

Appendix B

It is remarkable, that the Korcak-law is not strictly valid for the sets originally presented by Korčák. For example he presented the size statistic of almost two thousand European lakes; on a linear scale it looks more or less hyperbolic (see the insert of Figs. B.1 and B.2; solid lines represent exact hyperbolic fit), but the distribution cannot be described by equation (1), yielding a single linear fit with slope $=-1$ in the log-log diagrams (Figs. B.1 and B.2, main figures). On the other hand, the "full" lake-size distribution seems to obey to equation (1) (Downing et al. 2012), at least for bigger lakes. Even for the original set of lakes presented by Korčák (using Halbfass data – Halbfass, 1922), the "hyperbolic" ($K = 1$) distribution seems to be valid after

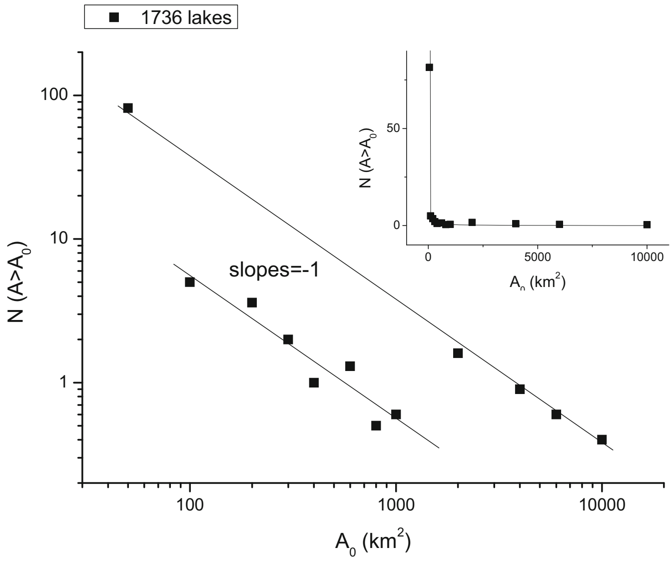

Fig. B.1. The traditional Korcak-plot (using the data reported by Korcak 1936) of lake-size distribution on linear (insert) and double-logarithmic plot. The straight lines with slopes = −1 represent the real hyperbolic distribution. While on linear scale (insert) the distribution seems to be hyperbolic, on double logarithmic plot one can see that medium-size lakes and small&big lakes have two, well distinguishable distributions.

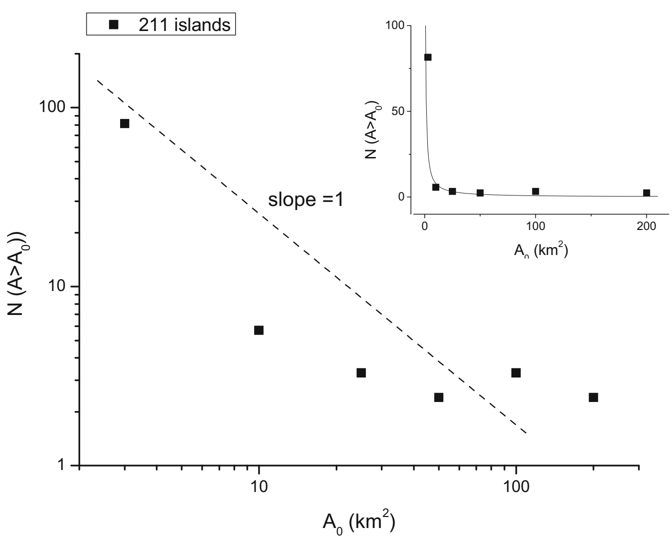

Fig. B.2. The traditional Korcak-plot (using the data reported by Korcak 1936) of island-size distribution on linear (insert) and double-logarithmic plot. The straight lines with slope = −1 represent the real hyperbolic distribution. While on linear scale (insert) the distribution seems to be hyperbolic, on double logarithmic plot one can see that the points cannot be fitted with linear fit, therefore neither the hyperbolic distribution (slope = −1) nor any other single power-law distribution (slope = constant) are valid.

dividing the set into two sub-sets. Using it for islands (for example for the Cyclades), hyperbolicity is not valid, not even in the extended ($K \neq 1$) version, although for using bigger sample sets (all landmasses above 100 km^2 area) islands and continents obey separately to equation (1), with different exponents (Imre 2015).

Since lakes are usually compact bodies of water with irregular shores, as well as islands are compact bodies of lands, also with irregular shores, one can expect that the shore-line will be fractal, but the area will be more-or-less Euclidian. Therefore applying equation (1) for an area-distribution, the slope should be around –1 ($K = 1$). For an object embedded into a 2-dimensional space, the fractal dimension should be between 1 and 2, therefore – if and only if K would be related to the fractal dimension – K would be between 0.5 and 1. This is a characteristic difference from the Pareto-law, where the expected K can be above 1 (although usually scatters around 1).

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