

Local quantum uncertainty for multipartite quantum systems

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Abstract. Local quantum uncertainty captures purely quantum correlations excluding their classical counterpart. This measure is quantum discord type, however with the advantage that there is no need to carry out the complicated optimization procedure over measurements. This measure is initially defined for bipartite quantum systems and a closed formula exists only for $2 \otimes d$ systems. We extend the idea of local quantum uncertainty to multi-qubit systems and provide the similar closed formula to compute this measure. We explicitly calculate local quantum uncertainty for various quantum states of three and four qubits, like GHZ state, W state, Dicke state, Cluster state, Singlet state, and Chi state all mixed with white noise. We compute this measure for some other well known three qubit quantum states as well. We show that for all such symmetric states, it is sufficient to apply measurements on any single qubit to compute this measure, whereas in general one has to apply measurements on all parties as local quantum uncertainties for each bipartition can be different for an arbitrary quantum state.

Quantum states are fundamentally different than classical states in such a way that any local measurements on one part of either bipartite or multipartite states necessarily give rise to uncertainty in results. This randomness is not a fault of measuring device but an integral nature of quantum states. Quantum entanglement, quantum non-locality, and quantum discord are few quantitative manifestation of this randomness. The only states which are invariant under such local measurements are those states which can be described by classical probability distribution. Such states have zero quantum discord [1–4]. Quantum states for two or more parties may be entangled, however entanglement is not the only quantum correlation present among quantum states. There are quantum states which are separable, nevertheless quantum correlated (nonzero quantum discord). Quantum discord may be defined as the difference between quantum mutual information and classical correlations [1–8]. Due to complicated minimization process, the computation of quantum discord is not an easy task and analytical results are known only for some restricted families of states [9,10]. For $2 \otimes d$ quantum systems, analytical results for quantum discord are known for a specific family of states [9] and the general procedure to calculate discord is also worked out [10,11]. Some authors have proposed quantum discord for multipartite systems [12–21]. Some other measures of such non-classical correlations include quantum work deficit [22–24], quantum deficit [25,26], measurement-induced non-locality [27], etc (see Ref. in [28]). The quantum correlations have utilization in potential applications, includ-

ing remote state preparation [29], entanglement distribution [30,31], transmission of correlations [32], and quantum metrology [33] to name a few. It is in general a hard task to characterize and quantify quantum correlations. Several authors have proposed different techniques to compute quantum correlations. The theory of quantum correlations has attracted lot of interest and considerable efforts have been devoted to it [34–36].

Recently, a discord-like measure has been proposed, known as local quantum uncertainty [37]. This measure is quantified via skew information which is achievable on a single local measurement [38–40]. This measure has a closed formula calculated for $2 \otimes d$ bipartite quantum systems. Later on, some authors tried to study local quantum uncertainty for orthogonally invariant class of states [41]. This measure was also studied for quantum phase transitions [42,43]. The relationship between local quantum uncertainty and quantum Fisher information under non-Markovian environment was also discussed [44]. Recently, some authors have studied local quantum uncertainty under various decoherence models and also worked out some preliminary results for three qubits [45–48]. We extend local quantum uncertainty for multi-qubit quantum system. As there are several bipartition for multi-qubit system, we can define local quantum uncertainty for each bipartition. After calculating all such local quantum uncertainties, we suggest an arithmetic mean to calculate the average local quantum uncertainty for a given multi-qubit state. Nevertheless, we find that for all specific quantum states which we study here, each local quantum uncertainty for every bipartition is exactly same due to symmetry of these quantum states. However, by taking a

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random state, we explicitly demonstrate that local quantum uncertainty can have a different value for each bipartition, so the average value gives local quantum uncertainty for given quantum state. We calculate this measure for various well known families of quantum states for three and four qubits and obtain analytical results. The benefit of this measure and its extension to multi-qubit has the advantage that we do not need any complicated maximization or minimization over parameters related with measurements as one has to do to calculate quantum discord. Interestingly, for four qubits, we find that except W -states mixed with white noise, all other specific quantum states have same expressions for local quantum uncertainty.

Local quantum uncertainty is a measure of quantum correlations which captures purely quantum part in a given quantum state by applying local measurements on one part of quantum state. This measure has been defined recently for $2 \otimes d$ quantum systems [37]. It is a quantum discord-type measure and for certain quantum states, quantum discord and local quantum uncertainty captures precisely same correlations and are equal to each other, whereas for some other states, they are different measures. The advantage of local quantum uncertainty over quantum discord is the fact that to compute local quantum uncertainty we only need to find the maximum eigenvalue of a symmetric 3×3 matrix. This is quite an easy task as compared with complicated minimization procedure over parameters related with measurements. Local quantum uncertainty is defined as the minimum skew information which is obtained via local measurement on qubit part only, that is,

$$\mathcal{Q}(\rho) \equiv \min_{K_A} \mathcal{J}(\rho, K_A \otimes \mathbb{I}_B), \quad (1)$$

where K_A is a hermitian operator (local observable) on subsystem A , and \mathcal{J} is the skew information [38–40] of the density operator ρ , defined as

$$\mathcal{J}(\rho, K_A \otimes \mathbb{I}_B) = -\frac{1}{2} \text{Tr}([\sqrt{\rho}, K_A \otimes \mathbb{I}_B]^2). \quad (2)$$

The skew information is nonnegative, and non-increasing under classical mixing. It has been shown [37] that for $2 \otimes d$ quantum systems, the compact formula for local quantum uncertainty is given as

$$\mathcal{Q}(\rho) = 1 - \max \{ \lambda_1, \lambda_2, \lambda_3 \}, \quad (3)$$

where λ_i are the eigenvalues of 3×3 symmetric matrix \mathcal{M} . The matrix elements of symmetric matrix \mathcal{M} are calculated by the relationship

$$m_{ij} \equiv \text{Tr} \{ \sqrt{\rho} (\sigma_i \otimes \mathbb{I}_B) \sqrt{\rho} (\sigma_j \otimes \mathbb{I}_B) \}, \quad (4)$$

where $i, j = 1, 2, 3$ and σ_i are the standard Pauli matrices.

We generalize this definition of local quantum uncertainty for multi-qubit quantum systems as follows. First, we observe that the definition of local quantum uncertainty for $2 \otimes d$ systems can be applied to multi-qubit systems without any technical consequences because we can always regard multi-qubit system as $2 \otimes d$ systems, where $d = 2 \otimes 2 \otimes \dots \otimes 2$ may represent the remaining

$N - 1$ qubits as d dimensional quantum system. However, we note that multi-qubit systems have richer structure as compared with bipartite quantum systems. It might be the case that some bipartition are quantum correlated and some may be classically correlated. So we need to apply the local measurements across each bipartition in order to capture quantum correlations. To this aim, let ρ be an arbitrary density matrix for N qubits. We can apply the local measurements on each qubit A, B, \dots, N . When we apply measurements on qubit A , we regard all rest of the qubits as d -dimensional system. Thus we obtain N symmetric matrices. For each bipartition, the matrix elements belonging to these N symmetric matrices are calculated according to relations

$$\begin{aligned} \tilde{m}_{ij}^A &= \text{Tr} \{ \sqrt{\rho} (\sigma_i \otimes \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_2) \\ &\quad \times \sqrt{\rho} (\sigma_j \otimes \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_2) \}, \\ \tilde{m}_{ij}^B &= \text{Tr} \{ \sqrt{\rho} (\mathbb{I}_2 \otimes \sigma_i \otimes \dots \otimes \mathbb{I}_2) \\ &\quad \times \sqrt{\rho} (\mathbb{I}_2 \otimes \sigma_j \otimes \dots \otimes \mathbb{I}_2) \}, \\ &\quad \vdots \\ &\quad \vdots \\ \tilde{m}_{ij}^N &= \text{Tr} \{ \sqrt{\rho} (\mathbb{I}_2 \otimes \mathbb{I}_2 \dots \otimes \sigma_i) \\ &\quad \times \sqrt{\rho} (\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \dots \otimes \sigma_j) \}. \end{aligned} \quad (5)$$

The corresponding eigenvalues of such 3×3 symmetric matrices $\tilde{\mathcal{M}}_i$ can be determined easily. The local quantum uncertainties related with each bipartition are defined as follows

$$\begin{aligned} \mathcal{Q}_{A/BC\dots N}(\rho) &= 1 - \max \{ \text{Spectrum of } \tilde{\mathcal{M}}_A \} \\ \mathcal{Q}_{B/AC\dots N}(\rho) &= 1 - \max \{ \text{Spectrum of } \tilde{\mathcal{M}}_B \} \\ &\quad \vdots \\ &\quad \vdots \\ \mathcal{Q}_{N/ABC\dots N-1}(\rho) &= 1 - \max \{ \text{Spectrum of } \tilde{\mathcal{M}}_N \}. \end{aligned} \quad (6)$$

Finally we propose the mean value of local quantum uncertainty for a given N -qubits quantum state to be calculated as

$$\mathcal{Q}(\rho_N) = \frac{\sum_{i=A}^N \mathcal{Q}_{i/N_i}}{N}, \quad (7)$$

where N_i are the remaining $N - 1$ qubits except i . (After the submission of this work, I have further refined the calculation of local quantum uncertainty with an alternative definition, which gives the same results for all quantum states discussed in this work [61].)

As a concrete example, let us consider the case of three qubits. Following the procedure mentioned above, we can find $\tilde{\mathcal{M}}_A$ for bipartition A/BC , $\tilde{\mathcal{M}}_B$ for bipartition B/CA , and $\tilde{\mathcal{M}}_C$ for bipartition C/AB . The respective matrix elements are calculated using relations

$$\tilde{m}_{ij}^A = \text{Tr} \{ \sqrt{\rho_{ABC}} (\sigma_i \otimes \mathbb{I}_2 \otimes \mathbb{I}_2) \sqrt{\rho_{ABC}} (\sigma_j \otimes \mathbb{I}_2 \otimes \mathbb{I}_2) \}, \quad (8)$$

$$\tilde{m}_{ij}^B = \text{Tr} \{ \sqrt{\rho_{ABC}} (\mathbb{I}_2 \otimes \sigma_i \otimes \mathbb{I}_2) \sqrt{\rho_{ABC}} (\mathbb{I}_2 \otimes \sigma_j \otimes \mathbb{I}_2) \}, \quad (9)$$

$$\tilde{m}_{ij}^C = \text{Tr} \{ \sqrt{\rho_{ABC}} (\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma_i) \sqrt{\rho_{ABC}} (\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \sigma_j) \}, \quad (10)$$

where $\tilde{m}_{ij}^A \neq \tilde{m}_{ij}^B \neq \tilde{m}_{ij}^C$ in general, however they may be equal to each other for some special cases. We mention here that the number of these symmetric matrices are same as the number of qubits. The local quantum uncertainty in this situation would be defined as

$$\mathcal{Q}(\rho_{ABC}) = \frac{(\mathcal{Q}_{A/BC} + \mathcal{Q}_{B/CA} + \mathcal{Q}_{C/AB})}{3}, \quad (11)$$

where

$$\mathcal{Q}_{A/BC} = 1 - \max \{ \{ \tilde{\mathcal{M}}_A \} \}, \quad (12)$$

$$\mathcal{Q}_{B/CA} = 1 - \max \{ \{ \tilde{\mathcal{M}}_B \} \}, \quad (13)$$

$$\mathcal{Q}_{C/AB} = 1 - \max \{ \{ \tilde{\mathcal{M}}_C \} \}, \quad (14)$$

where $\{ \tilde{\mathcal{M}}_i \}$ denote the spectrum (eigenvalues) of the corresponding 3×3 matrix $\tilde{\mathcal{M}}_i$. For the special case when all three matrices have the same set of eigenvalues then $\mathcal{Q}_{A/BC} = \mathcal{Q}_{B/CA} = \mathcal{Q}_{C/AB}$ and $\mathcal{Q}(\rho_{ABC}) = \mathcal{Q}_{A/BC}$. In this case measurements need to be applied to any qubit.

We will now present some examples computing local quantum uncertainty for various families of three qubits and four qubits quantum states. An important family of quantum states is GHZ states [54] mixed with white noise. The states for three qubits are defined as

$$\rho_{\text{GHZ}_3} = (1 - \alpha) |\text{GHZ}_3\rangle\langle\text{GHZ}_3| + \frac{\alpha}{8} \mathbb{I}_8, \quad (15)$$

where $0 \leq \alpha \leq 1$, $\mathbb{I}_8/8$ is maximally mixed state, and maximally entangled pure state is given as

$$|\text{GHZ}_3\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle). \quad (16)$$

Entanglement properties of these states equation (15) are well known [49–53]. It is known that these states are fully separable for $0.8 \leq \alpha \leq 1$, bi-separable for $0.571 \leq \alpha < 0.8$, and genuine entangled for $0 \leq \alpha < 0.571$ [49–53].

We have calculated all three symmetric matrices $\tilde{\mathcal{M}}_i$ for measurements on qubit A , B , and C . It turns out that all three matrices are same and therefore have the same set of three eigenvalues. In addition, all three eigenvalues are also same, so the problem to pick the maximum eigenvalue is even trivial. The maximum eigenvalue is given as

$$\lambda = \frac{3\alpha + \sqrt{\alpha(8-7\alpha)}}{4}. \quad (17)$$

Therefore, local quantum uncertainty for states equation (15) is simply

$$\mathcal{Q}(\rho_{\text{GHZ}_3}) = 1 - \frac{3\alpha + \sqrt{\alpha(8-7\alpha)}}{4}. \quad (18)$$

We observe that for $\alpha = 0$, $\mathcal{Q}(\rho_{\text{GHZ}_3}) = 1$ which is expected as pure maximally entangled state has maximum correlations. We note that for $\alpha = 1$, we have

$\mathcal{Q}(\rho_{\text{GHZ}_3}) = 0$, which is also expected result because maximally mixed state is classically correlated and has no quantum correlations in it. For other values of $\alpha < 1$, local quantum uncertainty $\mathcal{Q}(\rho_{\text{GHZ}_3}) > 0$. We have seen that local quantum uncertainty precisely captures quantum correlations just like quantum discord.

Second important class of states for three qubits is W state [55] mixed with white noise. These states are defined as

$$\rho_{W_3} = (1 - \beta) |W_3\rangle\langle W_3| + \frac{\beta}{8} \mathbb{I}_8, \quad (19)$$

where $0 \leq \beta \leq 1$ and W_3 state is given as

$$|W_3\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle). \quad (20)$$

The entanglement properties of equation (19) are also well known. These states are fully separable or bi-separable for $0.521 \leq \beta \leq 1$, whereas genuine tripartite entangled for $0 \leq \beta < 0.521$ [49–53]. We now carry out the same procedure as mentioned earlier to compute local quantum uncertainty. We calculated all three symmetric matrices and found them to be exactly equal to each other as it was the case for ρ_{GHZ_3} states. Therefore, we get same set of three eigenvalues for all three bipartition. Two of the eigenvalues are equal to each other, whereas third eigenvalue is different. These eigenvalues are given as

$$\begin{aligned} w_1 = w_2 &= \frac{3\beta + \sqrt{\beta(8-7\beta)}}{4}, \\ w_3 &= \frac{1 + 6\beta + 2\sqrt{\beta(8-7\beta)}}{9}. \end{aligned} \quad (21)$$

It is not difficult to check that $w_3 > w_1$, for all values of parameter β . The local quantum uncertainty for states equation (19) is simply given as

$$\mathcal{Q}(\rho_{W_3}) = \frac{8 - 6\beta - 2\sqrt{\beta(8-7\beta)}}{9}. \quad (22)$$

We can readily check that for $\beta = 0$, we get $\mathcal{Q}(\rho_{W_3}) = 8/9$. This means that for pure W_3 state, local quantum uncertainty does not have a maximum value of 1. The genuine negativity for W_3 state is also not maximum, whereas GHZ state is regarded as maximally entangled as measured by genuine negativity [49–53]. We can also check that for $\beta = 1$, we have $\mathcal{Q}(\rho_{W_3}) = 0$ as it should be.

Let us take another example of three qubits quantum states defined as

$$\rho_{AK} = \frac{1}{8+8\gamma} \begin{pmatrix} 4+\gamma & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & \gamma & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & \gamma & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & \gamma & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & \gamma & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & \gamma & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & \gamma \\ 2 & 0 & 0 & 0 & 0 & 0 & 4+\gamma \end{pmatrix}. \quad (23)$$

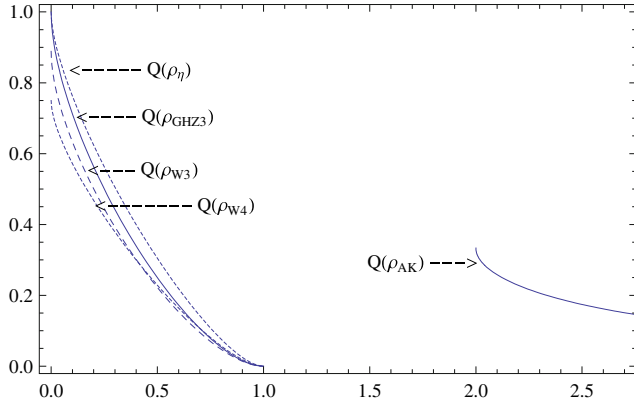


Fig. 1. Local quantum uncertainty is plotted against single parameter for ρ_{GHZ_3} , ρ_{W_3} , ρ_{AK} , ρ_η , and ρ_{W_4} states. See text for explanations.

This matrix is a valid quantum state for $\gamma \geq 2$. This family of states may be called Kay states as they were introduced by Kay [56]. The states have positive partial transpose (PPT) with respect of all bipartition. It is known that for $2 \leq \gamma < 2\sqrt{2}$, this density matrix is bound entangled and for $\gamma \geq 2\sqrt{2}$, the state is separable. We calculate all three symmetric matrices \tilde{M}_i for this state and find that once again, they are equal to each other. There are two eigenvalues which are same whereas the third eigenvalue is different. These eigenvalues are given as

$$k_1 = k_2 = \frac{1}{4} \sqrt{\frac{\gamma+2}{\gamma+1}} \left(3\sqrt{\frac{-2+\gamma}{1+\gamma}} + \sqrt{\frac{6+\gamma}{1+\gamma}} \right),$$

$$k_3 = \frac{3\gamma+2 + \sqrt{(\gamma-2)(6+\gamma)}}{4(\gamma+1)}. \quad (24)$$

It is not difficult to find that $k_3 > k_1$, therefore local quantum uncertainty for Kay-states is given as

$$\mathcal{Q}(\rho_{AK}) = \frac{2+\gamma - \sqrt{(\gamma-2)(6+\gamma)}}{4(1+\gamma)}. \quad (25)$$

This expression is not real for $\gamma < 2$, so local quantum uncertainty also reflects this restriction on parameter in quantum states.

Figure 1 shows local quantum uncertainty $\mathcal{Q}(\rho)$ plotted against corresponding single parameter for equations (15), (19), (23), and similar states for GHZ and W states of four qubits equation (29). Quantum uncertainty for ρ_η equation (31) turns out to be highest. Quantum uncertainty for W_3 mixed with white noise is higher initially but as single parameter increases from 0 to 1, it becomes smaller. In all cases as mixing increases, LQU decrease and finally become zero for maximally mixed state. For Kay-states the quantum correlations are largest for $\gamma = 2$, which is a bound entangled state. As we increase the value of parameter, quantum states move towards separable states and quantum correlations are smaller than the bound entangled state. We have checked local quantum uncertainty even for very large values of parameter γ

and found local quantum uncertainty still strictly greater than zero.

In all above examples, we have seen that local quantum uncertainty for each bipartition turns out to be same. However, it is not true for the set of all quantum states as there exist other states for which each bipartition may have different local quantum uncertainty. We demonstrate this difference simply by taking a random state and calculating local quantum uncertainty for each bipartition. To this aim, first we generate a random pure state and then mix it with white noise such that white noise fraction is 0.2 and random state fraction is 0.8. The corresponding symmetric matrix with measurements on qubit A is given as

$$\tilde{M}_A \approx \begin{pmatrix} 0.65 & 0.014 & 0.115 \\ 0.014 & 0.594 & -0.015 \\ 0.115 & -0.015 & 0.757 \end{pmatrix}, \quad (26)$$

with the eigenvalues (0.83, 0.61, 0.56). For measurements on qubit B , we get

$$\tilde{M}_B \approx \begin{pmatrix} 0.59 & 0.01 & -0.05 \\ 0.01 & 0.651 & 0.107 \\ -0.05 & 0.107 & 0.687 \end{pmatrix}, \quad (27)$$

with eigenvalues (0.78, 0.61, 0.53), and finally for qubit C , we have

$$\tilde{M}_C \approx \begin{pmatrix} 0.63 & -0.112 & -0.032 \\ -0.112 & 0.83 & 0.12 \\ -0.032 & 0.12 & 0.711 \end{pmatrix}, \quad (28)$$

with eigenvalues (0.94, 0.65, 0.57). The respective local quantum uncertainties are $\mathcal{Q}(\rho_{A/BC}) \approx 0.17$, $\mathcal{Q}(\rho_{B/CA}) \approx 0.22$, and $\mathcal{Q}(\rho_{C/AB}) \approx 0.06$. The average value is $\mathcal{Q}(\rho) \approx 0.15$.

Let us consider few examples of four qubit quantum states. Two important quantum states for four qubits are the GHZ state and W state given as

$$|\text{GHZ}_4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),$$

$$|W_4\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle). \quad (29)$$

For the GHZ state, the entanglement monotone has a value of $E(|\text{GHZ}_4\rangle\langle\text{GHZ}_4|) = 1$, while for the W state, its value is $E(|W_3\rangle\langle W_3|) \approx 0.886$ and $E(|W_4\rangle\langle W_4|) \approx 0.732$.

Several other four qubit quantum states are interesting and have been discussed in the literature. These states are the Dicke state $|D_{2,4}\rangle$ [57], the four-qubit singlet state $|\Psi_{S,4}\rangle$ [58], the cluster state $|CL\rangle$ [59] and the so-called χ -state $|\chi_4\rangle$ [60]. These quantum states are explicitly given

as

$$\begin{aligned}
 |D_{2,4}\rangle &= \frac{1}{\sqrt{6}} [|0011\rangle + |1100\rangle + |0101\rangle + |0110\rangle \\
 &\quad + |1001\rangle + |1010\rangle], \\
 |\Psi_{S,4}\rangle &= \frac{1}{\sqrt{3}} [|0011\rangle + |1100\rangle - \frac{1}{2} \{ |0101\rangle + |0110\rangle \\
 &\quad + |1001\rangle + |1010\rangle \}], \\
 |CL\rangle &= \frac{1}{2} [|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle], \\
 |\chi_4\rangle &= \frac{1}{\sqrt{6}} \{ \sqrt{2} |1111\rangle + |0001\rangle + |0010\rangle + |0100\rangle \\
 &\quad + |1000\rangle \}. \tag{30}
 \end{aligned}$$

Note that all of these states have the maximum value of entanglement $E(|D_{2,4}\rangle\langle D_{2,4}|) = E(|\Psi_{S,4}\rangle\langle \Psi_{S,4}|) = E(|CL\rangle\langle CL|) = E(|\chi_4\rangle\langle \chi_4|) = 1$. Further entanglement properties of these states are reviewed in reference [35].

To find local quantum uncertainty, we first mix all of these states with white noise as follows

$$\rho_\eta = (1 - \eta) |\psi\rangle\langle \psi| + \frac{\eta}{16} \mathbb{I}_{16}, \tag{31}$$

where $0 \leq \eta \leq 1$, and $|\psi\rangle$ is any of the above defined four qubit pure states. Next we calculate the four symmetric matrices for each one of these states and find out that they are all equal for every bipartition, that is, $\tilde{M}_A = \tilde{M}_B = \tilde{M}_C = \tilde{M}_D$. This implies that local quantum uncertainty for each bipartition is same. Another interesting observation is that except W_4 state mixed with white noise, all other remaining five mixtures have exactly the same eigenvalues and consequently exactly the same expressions for local quantum uncertainty as well, that is,

$$\begin{aligned}
 \mathcal{Q}(\rho_{GHZ_4}) &= \mathcal{Q}(\rho_{D_{2,4}}) = \mathcal{Q}(\rho_{\Psi_{S,4}}) \\
 &= \mathcal{Q}(\rho_{CL}) = \mathcal{Q}(\rho_{\chi_4}) = \mathcal{Q}(\rho_\eta), \tag{32}
 \end{aligned}$$

where local quantum uncertainty for any of such states equation (31) is given as

$$\mathcal{Q}(\rho_\eta) = 1 - \frac{7\eta + \sqrt{\eta(16 - 15\eta)}}{8}. \tag{33}$$

We note that $\mathcal{Q}(\rho_\eta) = 1$ for $\eta = 0$, which means that GHZ state, Dicke State, Singlet state, Cluster state and Chi state all have maximum amount of quantum correlations. We also note that $\mathcal{Q}(\rho_\eta) = 0$ for $\eta = 1$. For W_4 state mixed with white noise, local quantum uncertainty is given as

$$\mathcal{Q}(\rho_{W_4}) = 1 - \frac{8 + 21\eta + 3\sqrt{\eta(16 - 15\eta)}}{32}. \tag{34}$$

This value is $3/4 = 0.75$ for $\eta = 0$ and zero for $\eta = 1$. We have seen that for both W_3 and W_4 state, the numerical value of local quantum uncertainty is slightly larger than numerical value of genuine entanglement.

We can easily demonstrate by generating a random state of four qubits that in general $\mathcal{Q}(\rho_{A/BCD}) \neq$

$\mathcal{Q}(\rho_{B/CDA}) \neq \mathcal{Q}(\rho_{C/DBA}) \neq \mathcal{Q}(\rho_{D/ABC})$ as we have seen for three qubits.

In summary, we have extended the idea of local quantum uncertainty for multi-qubit quantum systems. We have analytically calculated this measure for several important families of quantum states of three and four qubits mixed with white noise. We find that all specific quantum states mixtures are symmetric as they all give the same value of local quantum uncertainty for measurements on each bipartition. Therefore for such states, measurements on any single qubit is sufficient to compute local quantum uncertainty. We have explicitly shown by taking a random state of three qubits that symmetric matrices resulting from measurements on each partition are not the same and hence the corresponding eigenvalues and local quantum uncertainties are also not equal to each other. Hence we get a different numerical value of local quantum uncertainty for each bipartition. Similar matrices should also be different for an arbitrary quantum state of four or higher number of qubits. This method is applicable to any arbitrary initial quantum state of N qubits.

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References

1. H. Ollivier, W.H. Zurek, Phys. Rev. Lett. **88**, 017901 (2001)
2. L. Henderson, V. Vedral, J. Phys. A **34**, 6899 (2001)
3. V. Vedral, Phys. Rev. Lett. **90**, 050401 (2003)
4. K. Modi, A. Brodutch, H. Cable, T. Paterek, V. Vedral, Rev. Mod. Phys. **84**, 1655 (2012)
5. S. Luo, Phys. Rev. A **77**, 042303 (2008)
6. M. Ali, A.R.P. Rau, G. Alber, Phys. Rev A **81**, 042105 (2010)
7. M. Ali, A.R.P. Rau, G. Alber, Phys. Rev. A **82**, 069902 (E) (2010)
8. Y. Huang, Phys. Rev. A **88**, 014302 (2013)
9. M. Ali, J. Phys. A: Math. Gen. **43**, 495303 (2010)
10. S. Vinjanampathy, A.R.P. Rau, J. Phys. A: Math. Gen. **45**, 095303 (2012)
11. A.R.P. Rau, Quant. Info. Proc. **17**, 216 (2018)
12. M. Mahdian, R. Yousefjani, S. Salimi, Eur. Phys. J. D **66**, 133 (2012)
13. J. Xu, Phys. Lett. A **377**, 238 (2013)
14. Y-K. Bai, N. Zhang, M-Y. Ye, Z.D. Wang, Phys. Rev. A **88**, 012123 (2013)
15. H.C. Braga, C.C. Rulli, T.R. de Oliveira, M.S. Sarandy, Phys. Rev. A **86**, 062106 (2012)
16. M. Daoud, R. Ahl Laamara, Phys. Lett. A **376**, 2361 (2012)
17. M. Daoud, R. Ahl Laamara, Int. J. Qunt. Info. **10**, 1250060 (2012)

18. F. Buscemi, P. Bordone, Phys. Rev. A **87**, 042310 (2013)
19. A. Beggi, F. Buscemi, P. Bordone, Quan. Inf. Proc. **14**, 573 (2015)
20. D.P. Chi, J. S. Kim, K. Lee, Phys. Rev. A **87**, 062339 (2013)
21. A.Y. Chernyavskiy, S.I. Doronin, E.B. Feldman, Phys. Scr. T **160**, 014007 (2014)
22. J. Oppenheim, M. Horodecki, P. Horodecki, R. Horodecki, Phys. Rev. Lett. **89**, 180402 (2002)
23. M. Horodecki, K. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen (De), U. Sen, Phys. Rev. Lett. **90**, 100402 (2003)
24. M. Horodecki, P. Horodecki, R. Horodecki, J. Oppenheim, A. Sen (De), U. Sen, B. Synak-Radtke, Phys. Rev. A **71**, 062307 (2005)
25. A.K. Rajagopal, R.W. Rendell, Phys. Rev. A **66**, 022104 (2002)
26. S. Luo, Phys. Rev. A **77**, 022301 (2008)
27. S. Luo, S. Fu, Phys. Rev. Lett. **106**, 120401 (2011)
28. A. Bera, T. Das, D. Sadhukhan, S.S. Roy, A. Sen De, U. Sen, Rep. Prog. Phys. **81**, 024001 (2018)
29. B. Dakic, et al., Nat. Phys. **8**, 666 (2012)
30. A. Streltsov, H. Kempermann, D. Bruß, Phys. Rev. Lett. **108**, 250501 (2012)
31. T.K. Chuan, et al., Phys. Rev. Lett. **109**, 070501 (2012)
32. A. Streltsov, W.H. Zurek, Phys. Rev. Lett. **111**, 040401 (2013)
33. K. Modi, H. Cable, M. Williamson, V. Vedral, Phys. Rev. X **1**, 021022 (2011)
34. R. Horodecki et al., Rev. Mod. Phys. **81**, 865 (2009)
35. O. Gühne, G. Tóth, Phys. Rep. **474**, 1 (2009)
36. G.D. Chiara, A. Sanpera, Rep. Prog. Phys. **81**, 074002 (2018)
37. D. Girolami, T. Tufarelli, G. Adesso, Phys. Rev. Lett. **110**, 240402 (2013)
38. S. Luo, Phys. Rev. Lett. **91**, 180403 (2003)
39. E.P. Wigner, M.M. Yanase, Proc. Natl. Acad. Sci. U.S.A. **49**, 910 (1963)
40. S. Luo, S. Fu, C.H. Oh, Phys. Rev. A **85**, 032117 (2012)
41. A. Sen, A. Bhar, D. Sarkar, Quan. Info. Proc. **14**, 269 (2015)
42. G. Karpat, B. Cakmak, F.F. Fanchini, Phys. Rev. B **90**, 104431 (2014)
43. I.B. Coulamy, J.H. Warnes, M.S. Sarandy, A. Saguia, Phys. Lett. A **380**, 1724 (2016)
44. S-X. Wu, Y. Zhang, C-S. Yu, Ann. Phys. **390**, 71 (2018)
45. A. Slaoui, M.I. Shaukat, M. Daoud, R. Ahl Laamara, Eur. Phys. J. Plus **133**, 413 (2018)
46. A. Slaoui, M. Daoud, R. Ahl Laamara, Quan. Info. Proc. **17**, 178 (2018)
47. A. Slaoui, L. Bakmou, M. Daoud, R. Ahl Laamara, Phys. Lett. A **383**, 2241 (2019)
48. A. Slaoui, M. Daoud, R. Ahl Laamara, Quan. Info. Proc. **18**, 250 (2019)
49. O. Gühne, M. Seevinck, New. J. Phys. **12**, 053002 (2010)
50. O. Gühne, Phys. Lett. A **375**, 406 (2011)
51. B. Jungnitsch, T. Moroder, O. Gühne, Phys. Rev. Lett. **106**, 190502 (2011)
52. L. Novo, T. Moroder, O. Gühne, Phys. Rev. A **88**, 012305 (2013)
53. M. Hofmann, T. Moroder, O. Gühne, J. Phys. A: Math. Gen. **47**, 155301 (2014)
54. D.M. Greenberger, M.A. Horne, A. Zeilinger, Going beyond Bell's theorem, in *Bell's Theorem, Quantum Theory, and Conceptions of the Universe*, edited by M. Kafatos (Kluwer, Dordrecht, 1989), p. 69
55. W. Dür, G. Vidal, J.I. Cirac, Phys. Rev. A **62**, 062314 (2000)
56. A. Kay, Phys. Rev. A **83**, 020303 (2011)
57. R.H. Dicke, Phys. Rev. **93**, 99 (1954)
58. H. Weinfurter, M. Żukowski, Phys. Rev. A **64**, 010102 (2001)
59. W. Dür, H. J. Briegel, Phys. Rev. Lett. **92**, 180403 (2004)
60. B. Kraus, Entanglement properties of quantum states and quantum operations, Ph.D. thesis, University of Innsbruck (2003)
61. M. Ali, AXiv preprint [arXiv: [2005.05013](https://arxiv.org/abs/2005.05013)]