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Non-Gaussian states with strong positive partial transpose

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Abstract. We consider the problem of entanglement criteria in non-Gaussian systems, providing a necessary and sufficient condition of separability of non-Gaussian states based on strong positive partial transpose (SPPT) of the corresponding density matrices. The separability conditions of states generated by photon subtraction from vacuum squeezed states are given.

Introduction

Quantum entanglement is considered as a key resource for many quantum technologies including quantum computing, quantum communications and quantum cryptography [1,2]. On the other hand, Quantum systems described by Gaussian states play a crucial role in Quantum information theory [3,4] and have attracted a lot of attention. Further, the non-Gaussian states have also attracted a lot of attention due to their applicabilities in quantum teleportation and quantum information tasks [5–7]. In addition, the non-Gaussian states are necessary for quantum computing with continuous variable systems [8].

Theoretically, in continuous variable systems, the most studies of entanglement and separability have been restricted to Gaussian systems [9–12], where the separability problem reduces to the study of their covariance matrices. For general non-Gaussian states, the separability is extremely complicated and there are almost non-rigorous results allowing us to quantify their entanglement.

In general, a state $\hat{\rho}$ is separable if and only if it can be written as a classical mixture of tensor product of two modes, i.e., $\hat{\rho} = \sum_i p_i \hat{\rho}_i^A \otimes \hat{\rho}_i^B$. The practical criterion which allows us to detect the separability in discrete systems is the positivity of partial transpose (PPT) [13]. The PPT criteria is a necessary and sufficient condition of separability of $2\otimes 2$ and $2\otimes 3$ systems [2]. For $N\otimes M$ systems, Chruscinsk et al. [14] constructed a subclass of PPT states namely strong PPT states (SPPT), and it was conjectured that SPPT states are separable when $NM < 9$. Recently, it has been shown that a pure state is separable if and only if it is SPPT [15]. In this work, we consider that non-Gaussian states can be written as a decomposition of N bi-mode Fock states (N is very large). We give the condition under which these states are SPPT.

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The paper is organized as follows. In the next section, we give a short description of the generation protocol of non-Gaussian state in question. In Section 2 we provide the necessary and sufficient condition under which these classes of non-Gaussian states are separable. Finally, our conclusions are given in Section 3.

1 Measurement-induced non-Gaussian operation

In this section, we briefly recall the statement on the measurement-induced non-Gaussian operation on the two-mode squeezed vacuum state and its mathematical framework. In Figure 1 we describe schematically a basic protocol of the generation of non-Gaussian states with vacuum squeezed state, where the two primary sources are identical. The vacuum state after going through any squeezing operation at path i characterized by the squeezing operator $\hat{S}_i(r)$ becomes a single-mode vacuum squeezed state, which can be mathematically represented by

$$|\phi_r\rangle_i = \hat{S}_i(r)|0\rangle_i, \tag{1}$$

where $\hat{S}_i(r) = \exp\left[-\frac{r}{2}\left((a_i^\dagger)^2 - a_i^2\right)\right]$ and r is the squeezing parameter. To generate the two mode squeezed vacuum state, we combined state 1 via a balanced beam splitter ($\theta = \frac{\pi}{4}$) as appears in Figure 1. Thus, we have

$$\begin{aligned}|\psi_r\rangle_{12} &= \hat{\Theta}_{12}\left(\frac{\pi}{4}\right)|\phi_r\rangle_1|\phi_{-r}\rangle_2\\ &= \hat{S}_{12}(-r)|0\rangle_{12}\\ &= \sum_{n=0}^{\infty}\alpha_n|n\rangle_1|n\rangle_2\end{aligned} \tag{2}$$

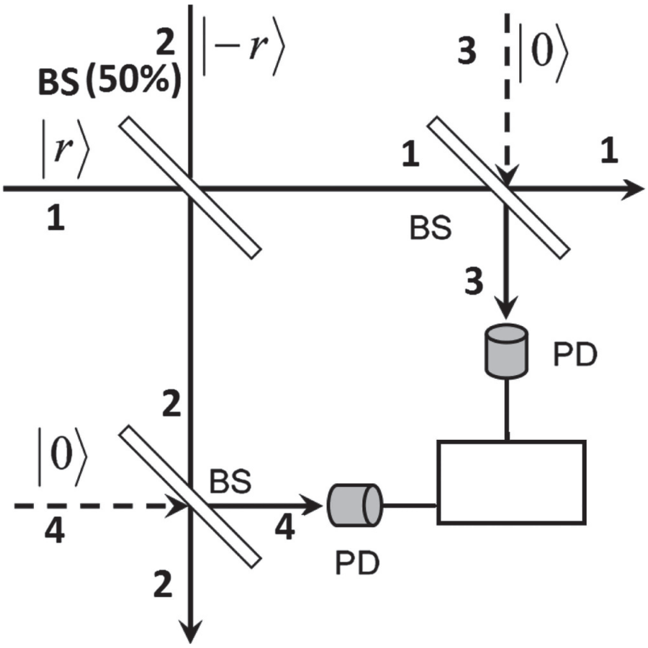

Fig. 1. Measurement-induced non-Gaussian operation on the two-mode squeezed vacuum state. BS, PD are beam splitter, photon detector, respectively.

where $\hat{\Theta}_{ij}(\theta) = \exp\left[\theta\left(a_i^\dagger a_j - a_i a_j^\dagger\right)\right]$ is the beam splitter operator, $\hat{S}_{ij}(r) = \exp\left[-r\left(a_i^\dagger a_j^\dagger - a_i a_j\right)\right]$ and parameter θ is related to transmittance T by

$$\tanh(\theta) = \sqrt{\frac{1-T}{T}} \tag{3}$$

and the Schmidt coefficients α_n are given by

$$\alpha_n = \tanh^n(r)\sqrt{1-\tanh^2(r)}. \tag{4}$$

By a beam splitter of transmittance T, the beam at path 3 (4) is then tapped off from path 1 (2), respectively. The resulting state after the second beam splitter is a state of four modes:

$$\begin{aligned} |\psi\rangle_{1234} &= \hat{\Theta}_{24}(\theta)\,\hat{\Theta}_{13}(\theta)\,|\phi_r\rangle_{12}|0\rangle_{34} \\ &= \sum_{n=0}^{\infty}\alpha_n\sum_{i,j=0}^{n}\gamma_{n,i}\gamma_{n,j}|n-i\rangle_1|n-j\rangle_2|i\rangle_3|j\rangle_4 \end{aligned} \tag{5}$$

where $\gamma_{n,\mu} = (-1)^\mu\sqrt{\binom{n}{\mu}}\left(\sqrt{T}\right)^{n-\mu}\left(\sqrt{1-T}\right)^\mu$. In the next step, when i photons are detected at path 3 and j photons are detected at path 4, the conditional state is given by

$$\begin{aligned} \left|\psi_{NG}^{ij}\right\rangle_{12} &= {}_4\langle j|_3\langle i|\psi\rangle_{1234} \\ &= \sum_{n=\max\{i,j\}}^{\infty}\alpha_n\gamma_{n,i}\gamma_{n,j}|n-i\rangle_1|n-j\rangle_2. \end{aligned} \tag{6}$$

In the following, we will consider that ψ_{NG}^{ij} can be written as the finite sum

$$\left|\psi_{NG}^{ij}\right\rangle_{12} = \sum_{n=\max\{i,j\}}^{N} c_{i,j}|n-i\rangle|n-j\rangle \tag{7}$$

where $c_{i,j} = \alpha_n\gamma_{n,i}\gamma_{n,j}$ and N is a large positive integer. In fact, (7) is just an approximation and state $\left|\psi_{NG}^{ij}\right\rangle_{12}$ may be considered as it belongs to an $N\times N$-dimensional Hilbert space. We adopt this approximation to apply the separability criteria on $\left|\psi_{NG}^{ij}\right\rangle_{12}$ and provide the conditions under which it is separable.

2 Separability of N + N dimensional systems

In this section, we provide the condition of separability of state (7) using SPPT criteria. First, we expand state (7) in its form of density matrix

$$\hat{\rho}_{NG} = \sum_{i,j,k,l}\rho_{ijkl}|n-i,n-j\rangle\langle m-k,m-l|, \tag{8}$$

where

$$\rho_{i,j,k,l} = {}_2\langle n-j|_1\langle n-i|\hat{\rho}_{NG}|m-k\rangle_1|m-l\rangle_2. \tag{9}$$

From (7), elements $\rho_{i,j,k,l}$ can be written as $\rho_{i,j,k,l} = c_{i,j}^* c_{k,l}$. In the following, we consider the state $\hat{\rho}_{NG}$ as an $NN\times NN$ matrix where N is a large enough number. Positivity of $\hat{\rho}_{NG}$ implies that it can be written as $\hat{\rho}_{NG} = X^\dagger X$, where X is some $NN\times NN$ matrix. Again, this matrix may be considered as a block $N\times N$ matrix with $N\times N$ blocks

$$X = \left(\begin{array}{c|c|c|c} S_{11}X_1 & S_{12}X_1 & \dots & S_{1N}X_1 \\ \hline S_{21}X_2 & S_{22}X_2 & \dots & S_{2N}X_2 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline S_{N1}X_N & S_{N2}X_N & \dots & S_{NN}X_N \end{array}\right), \tag{10}$$

where X_i and S_{ij} are both $N\times N$ matrices with

$$S_{ij} = \begin{cases} \mathbb{1}_N, & \text{if } i=j, \\ 0, & \text{if } i>j, \end{cases} \tag{11}$$

where $\mathbb{1}_N$ denotes the identity operator acting on space $\mathbb{C}^N$. Therefore, $\mathbf{X}$ is an upper triangular matrix

$$X = \left(\begin{array}{c|c|c|c|c} X_1 & S_{12}X_1 & S_{13}X_1 & \dots & S_{1N}X_1 \\ \hline 0 & X_2 & S_{23}X_1 & \dots & S_{2N}X_2 \\ \hline 0 & 0 & X_3 & \dots & S_{2N}X_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & \dots & X_N \end{array}\right). \tag{12}$$

State $\hat{\rho}_{NG}$ has SPPT if and only if there exists an upper triangular matrix Y such that $\hat{\rho}_{NG}^{T_A} = Y^\dagger Y$, where Y is expressed in terms of different S_{ij} and X_i as

$$Y = \begin{pmatrix} X_1 & S_{12}^\dagger X_1 & S_{13}^\dagger X_1 & \ldots & S_{1N}^\dagger X_1 \\ 0 & X_2 & S_{23}^\dagger X_1 & \ldots & S_{2N}^\dagger X_2 \\ 0 & 0 & X_3 & \ldots & S_{2N}^\dagger X_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & X_N \end{pmatrix}. \quad (13)$$

Then, state $\hat{\rho}_{NG}$ is SPPT if and only if [14,15]

$$\sum_{k=1}^{N} X_k^\dagger [S_{kj}^\dagger, S_{ki}] X_k = 0, \quad (14)$$

for $1 \leq i \leq j \leq N$. From equations (8) and (12), it is easy to see that X has the following form

$$X = \begin{pmatrix} c_{11}^* & c_{12}^* \ldots & c_{1N}^* & c_{21}^* & c_{22}^* \ldots & c_{2N}^* \ldots \ldots c_{NN}^* \\ 0 & 0 \ldots & 0 & 0 & 0 \ldots & 0 \ldots \ldots 0 \\ \vdots & \vdots \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 \ldots & 0 & 0 & 0 \ldots & 0 \ldots \ldots 0 \end{pmatrix}, \quad (15)$$

and by using (12) and (15), one can conclude that

$$\begin{pmatrix} c_{i1}^* & c_{i2}^* \ldots & c_{iN}^* \\ 0 & 0 \ldots & 0 \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \ldots & 0 \end{pmatrix} = \begin{pmatrix} s_{11}^i & s_{12}^i \ldots & s_{1N}^i \\ s_{21}^i & s_{22}^i \ldots & s_{2N}^i \\ \vdots & \vdots \ddots & \vdots \\ s_{N1}^i & s_{N2}^i \ldots & s_{NN}^i \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \ldots & c_{1N}^* \\ 0 & 0 \ldots & 0 \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \ldots & 0 \end{pmatrix}, \quad (16)$$

where $S_{1i} = \begin{pmatrix} s_{11}^i & s_{12}^i \ldots & s_{1N}^i \\ s_{21}^i & s_{22}^i \ldots & s_{2N}^i \\ \vdots & \vdots \ddots & \vdots \\ s_{N1}^i & s_{N1}^i \ldots & s_{NN}^i \end{pmatrix}$ and $s_{jk}^i = 0$ for $\{i, k, j \neq 1\} = \{1, 2, \ldots, N\}$, consequently,

$$\begin{cases} s_{j1}^i = 0 \quad \text{for} \quad \{i, j \neq 1\} = \{1, 2, \ldots, N\} \\ \frac{c_{ij}^*}{c_{1j}^*} = s_{11}^i \quad \text{for} \quad \{i, j\} = \{1, 2, \ldots, N\}. \end{cases} \quad (17)$$

From (14), state $\hat{\rho}_{NG}$ is SPPT if and only if

$$s_{1n}^{j*} s_{11}^i - \sum_{k=1}^{N} s_{nk}^i s_{1k}^{j*} = 0, \quad (18)$$

for $\{i, j, n\} = \{1, 2, \ldots, N\}$ and $1 \leq i \leq j \leq N$. Since, $\hat{\rho}_{NG}$ is a pure state, it is separable if and only if condition (18) is satisfied.

Example 1. Let $\hat{\rho}_{NG}^1$ be a particular "non-Gaussian state" of form (8), with $i = k = 1$, state $\hat{\rho}_{NG}^1$ can be expressed as

$$\hat{\rho}_{NG}^1 = \sum_{j,l} c_{1j}^* c_{1l} |1j\rangle\langle 1l|. \quad (19)$$

Matrices S_{ij} and in X_i recalled in (10)–(11) have, in this case, the following form

$$S_{ij} = \begin{cases} \mathbb{1}_N, & \text{if} \quad i = j = 1, \\ 0, & \text{if not}, \end{cases} \quad (20)$$

and $X_i = 0$ for $i \neq 1$. Since $s_{nk}^i = 0$ for $\{n, k\} = \{1, 2, \ldots, N\}$ and $i = 2, \ldots, N$, it is easy to see that (18) is verified, then state $\hat{\rho}_{NG}^1$ is an SPPT state, then it is separable.

Example 2. Let us consider a particular "non-Gaussian" state with $\frac{c_{ij}}{c_{i1}} \in \mathbb{R}$ for $i, j = \{1, 2, \ldots, N\}$, it follows then from (17) that for any $i = \{1, 2, \ldots, N\}$, $s_{1i} \in \mathbb{R}$ and from (18) that the condition of separability can now be written as

$$s_{1n}^{j*} s_{11}^i - s_{11}^i s_{11}^{j*} = \sum_{k=2}^{N} s_{nk}^i s_{1k}^{j*}, \quad (21)$$

for $1 \leq i \leq j \leq N$. Let these coefficients satisfy

$$s_{nk}^{j*} = s_{kn}^j, \quad \forall \quad \{n, k, j\} = \{1, 2, \ldots, N\}, \quad (22)$$

which means that S_{ij} are Hermitian. Then, by using (17) condition (21) is reduced to

$$s_{11}^{j*} s_{11}^i - s_{11}^{i*} s_{11}^j = 0, \quad \forall \quad i \leq j \leq N. \quad (23)$$

Consequently, condition (18) is satisfied, then the state is separable. Finally, we can conclude that if matrices S_{ij} are Hermitian and $\frac{c_{ij}}{c_{i1}} \in \mathbb{R}$, $\forall$ $\{i, j\} = \{1, 2, \ldots, N\}$, state (7) is separable.

Example 3. If we take $i = k$ and $j = l$ in (8), state $\hat{\rho}_{NG}$ becomes a diagonal matrix, we note it $\hat{\rho}_{NG}^2$ and it can be written as

$$\hat{\rho}_{NG}^2 = \sum_{i,j=1}^{N} c_{ij}^* c_{ij} |ij\rangle\langle ij|, \quad (24)$$

in this case

$$S_{ij} = \begin{cases} \mathbb{1}_N, & \text{if} \quad i = j, \\ 0, & \text{if not}. \end{cases} \quad (25)$$

Thus, by using (14), the state has SPPT, then it is separable. The separability of $\hat{\rho}_{NG}^2$ is trivial as it is a diagonal matrix.

Example 4. Using (14) it is easy to check that if the particular state having the form

$$\hat{\rho}_{NG}^3 = \sum_{i,i} c_{ii}^* c_{jj} |ii\rangle\langle jj|, \quad (26)$$

does not have SPPT, then it is entangled.

3 Conclusion

We have introduced a technique to investigate the separability of non-Gaussian states. We have given a necessary and sufficient condition of separability of non-Gaussian states generated by photon-subtracted from squeezed state based on SPPT criteria. This work comes up with many interesting perspectives, the interesting one concerns to find a new construction of the negativity of partial transpose systems.

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Author contribution statement

All authors contributed equally to this work.

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