

Non-Hermitian Hamiltonians and stability of pure states

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Abstract. We demonstrate that quantum fluctuations can cause, under certain conditions, the dynamical instability of pure states that can result in their evolution into mixed states. It is shown that the degree and type of such an instability are controlled by the environment-induced anti-Hermitian terms in Hamiltonians. Using the quantum-statistical approach for non-Hermitian Hamiltonians and related non-linear master equation, we derive the equations that are necessary to study the stability properties of any model described by a non-Hermitian Hamiltonian. It turns out that the instability of pure states is not preassigned in the evolution equation but arises as the emergent phenomenon in its solutions. In order to illustrate the general formalism and different types of instability that may occur, we perform the local stability analysis of some exactly solvable two-state models, which are being used in the theories of open quantum-optical and spin systems.

1 Introduction

It is well-known that purity is exactly preserved during unitary evolution driven by Hermitian Hamiltonians. This property is natural for describing isolated quantum systems but in the case of open ones it is no longer compulsory [1,2]. From the viewpoint of the theory of open quantum systems, the isolated system is a mere theoretical idealization – since in the real world all quantum systems are embedded into a background of some kind [3,4]. Even the process of quantum measurement itself fits into this framework, because it involves interaction of the quantum system, which is being measured, with an external apparatus. Correspondingly, once the system is brought into interaction with its environment, such as a heat bath, dissipation usually increases its entropy and pure states are converted into mixed ones [5,6].

Recently, the dynamical behaviour of quantum purity and pure states has become of considerable research interest when studied within the framework of the non-Hermitian (NH) formalism [7,8]. Non-Hermitian Hamiltonians find numerous applications in many areas of physics including studies of Feshbach resonances and decaying states, quantum transport and scattering by complex potentials, multiphoton ionization, free-electron lasers and optical resonators and waveguides. But the biggest area of application is the theory of open quantum systems where the anti-Hermitian part appears as a result of the interaction of systems with their environment [9–21]. In this paper we demonstrate the profound connection between NH Hamiltonians and dynamical stability of pure states for the quantum systems of general type. We show that quantum fluctuations can

cause the instability of pure states whose degree and type are governed by the environment-induced anti-Hermitian terms in Hamiltonians.

The contents of this paper are as follows. In Section 2, we provide a brief account of the density operator approach for NH dynamics and formulate the essence of the stability problem for pure states. In Section 3, we derive the general equations which are needed for stability analysis of pure states for NH-driven systems and discuss their generic features. In Section 4, we study the linearized limit of the stability equations, which can be used for the analysis of local stability. In Section 5, we consider some two-level systems in order to illustrate the general formalism. Discussions and conclusions are given in Section 6.

2 Non-Hermitian dynamics

If the Hamiltonian of a quantum system is a non-Hermitian operator, then it can be decomposed into its Hermitian and anti-Hermitian parts, respectively:

$$\hat{H} = \hat{H}_+ + \hat{H}_- = \hat{H}_+ - i\hat{\Gamma}, \quad (1)$$

where $\hat{H}_\pm = \pm\hat{H}_\pm^\dagger$, and $\hat{\Gamma} = \hat{\Gamma}^\dagger$ is usually dubbed the decay operator. The probability-conserved time evolution of such a system is described by the normalized density operator $\hat{\rho}$, which can be cast in the form

$$\hat{\rho} = \hat{\Omega}/\text{tr } \hat{\Omega}, \quad (2)$$

where $\hat{\Omega}$ is called the non-normalized density operator. This operator is defined as a solution of the evolution equation

$$\frac{d}{dt}\hat{\Omega} = -\frac{i}{\hbar} [\hat{H}_+, \hat{\Omega}] - \frac{1}{\hbar} \{ \hat{\Gamma}, \hat{\Omega} \}, \quad (3)$$

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where the square brackets denote the commutator and the curly ones denote the anti-commutator. This evolution equation can be directly derived from the Schrödinger equation (see, for instance, Ref. [7]). Further, in this equation one can change from $\hat{\Omega}$ to $\hat{\rho}$, and obtain the evolution equation for the normalized density operator itself

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar} [\hat{H}_+, \hat{\rho}] - \frac{1}{\hbar} \{\hat{\Gamma}, \hat{\rho}\} + \frac{2}{\hbar} \langle \Gamma \rangle \hat{\rho}, \quad (4)$$

where we imply the standard definition for mean values, $\langle A \rangle = \text{tr}(\hat{\rho} A)$.

From the mathematical point of view, equations (3) or (4), together with the definition for computing mean values, represent the map that allows to describe the time evolution of system (1) in terms of the matrix differential equation which is defined on similar axiomatic foundations as the conventional master equations of the Lindblad kind [22,23]. According to this map, the Hermitian operator $\hat{H}_+ = (\hat{H} + \hat{H}^\dagger)/2$ takes over a role of the system's Hamiltonian (cf. the commutator in Eqs. (3) or (4)) whereas the Hermitian operator $\hat{\Gamma} = i(\hat{H} - \hat{H}^\dagger)/2$ induces the additional terms in the evolution equation that are supposed to account for new effects. In other words, a theory with the non-Hermitian Hamiltonian \hat{H} is dual to a theory with the Hermitian Hamiltonian $(\hat{H} + \hat{H}^\dagger)/2$ but with the modified evolution equation which thus becomes the master equation of a special kind. This mapping not only reveals new features of the dynamics driven by non-Hermitian Hamiltonians but also facilitates their application for open quantum systems [18].

From the viewpoint of theory of open quantum systems, the evolution equation for the non-normalized density operator $\hat{\Omega}$ effectively describes the original subsystem (with Hamiltonian \hat{H}_+) and the effect of environment (represented by $\hat{\Gamma}$). Consequently, the evolution equation for the normalized density operator $\hat{\rho}$ effectively describes the original subsystem \hat{H}_+ together with the effect of environment $\hat{\Gamma}$ and the probability flow between the subsystem and reservoir. From the viewpoint of the subsystem alone, this flow is an essentially non-Hamiltonian process since it is described not by means of any kind of Hamiltonian but through the last term in the evolution equation (4). This makes NH models somewhat similar to the Lindblad-type ones, where the effect of the environment is encoded in the evolution equation through the additional term often dubbed the dissipator. However, the important difference is that the Lindblad dissipator, which is a traceless operator by construction, does not affect the conservation of probability of the system whereas the last term in equation (4) restores the probability's conservation which is otherwise broken by the anticommutator term. Besides, one can see that the last term in equation (4) is nonlinear with respect to $\hat{\rho}$, unlike its Lindblad analogue. It is interesting that the appearance of nonlinearities in NH-related theories has been also suggested some time ago, although on different grounds of the Feshbach-Fano projection formalism [24].

Further, if one introduces the quantum purity $\mathcal{P} = \text{tr}(\hat{\rho}^2)$ then one can show that its time evolution is governed by the equation

$$\frac{d}{dt}\mathcal{P} = \mathcal{R}(\hat{\rho}) \equiv \frac{4}{\hbar} [\langle \Gamma \rangle \mathcal{P} - \text{tr}(\hat{\rho}^2 \hat{\Gamma})], \quad (5)$$

where $\mathcal{R}(\hat{\rho})$ is the purity rate function [7]. It is easy to see that the rate function vanishes identically in the case of Hermitian evolution ($\hat{\Gamma} = 0$), but is otherwise an essentially non-trivial function of both the density operator and the anti-Hermitian part of Hamiltonian. As long as the density matrix $\hat{\rho}_p$ of a pure state, which is defined via the property of projectivity $\hat{\rho}_p^2 = \hat{\rho}_p$, is the equilibrium point in both the Hermitian and non-Hermitian cases (i.e., $\mathcal{R}(\hat{\rho}_p) = 0$), one might expect that any pure state is always preserved during the NH evolution.

However, this could only be possible if one disregards the other important player in the quantum realm – quantum fluctuations. Indeed, if an (initially) pure state is not protected against the fluctuations that can alter its purity, then during time evolution it will be driven away from being pure, no matter how small these fluctuations initially were. This phenomenon of dynamical instability is not directly seen in the evolution equation for the density operator, but emerges via solutions thereof. It is somewhat analogous to the spontaneous symmetry breaking in field theory – except that here one deals not with the actual field potential, but with some kind of the fictitious-particle potential function (usually dubbed as the Lyapunov function candidate) that determines whether the equilibrium point $\hat{\rho}_p$ is stable or not. This also makes NH models different from the Lindblad-type ones, in which the non-conservation of purity and pure states directly follows from the underlying evolution equation. Besides, the Lindblad master equation approach has a different range of applicability because it implies a number of certain approximations, such as the Markovian, Born and rotating-wave ones, whereas NH Hamiltonians often appear in theories in a more direct way, one example to be the Feshbach-Fano projections [25].

It is worth mentioning also that the above-mentioned kind of chaotic behaviour should not be confused with the notion of “quantum chaos”, which is currently being reserved in the field of research devoted to how classical chaotic dynamical systems can be described by means of methods and concepts of quantum mechanics [26–28].

3 General stability approach

In order to develop the general approach for performing the stability analysis of pure states, let us introduce the non-purity (mixedness) operator

$$\hat{M} = \hat{\rho} - \hat{\rho}^2, \quad (6)$$

whose trace is known as the linear entropy [29]

$$S_L = \text{tr} \hat{M} = 1 - \mathcal{P}. \quad (7)$$

In order to consider variations of the density operator around some pure state $\hat{\rho}_p$, we perform the decomposition

$$\hat{\rho} = \hat{\rho}_p + \hat{\Delta}, \quad (8)$$

where $\hat{\Delta}$ is the variation operator. As long as the main properties of the density matrix should be left intact by such decomposition, $\hat{\Delta}$ must be Hermitian and traceless; this automatically ensures the well-defined probability and real mean values of operators. One obtains that

$$\hat{\rho}^2 = \hat{\rho}_p + \left\{ \hat{\rho}_p, \hat{\Delta} \right\} + \hat{\Delta}^2, \quad \hat{M} = \hat{\Delta} - \left\{ \hat{\rho}_p, \hat{\Delta} \right\} - \hat{\Delta}^2, \quad (9)$$

such that the operator \hat{M} is clearly a measure of deviation of a state from being pure. Consequently, equations (4) and (5) yield the equations that are more suitable for stability analysis,

$$\begin{aligned} \frac{d}{dt} \hat{\Delta} = & -\frac{i}{\hbar} \left[\hat{H}_+, \hat{\Delta} \right] - \frac{1}{\hbar} \left\{ \hat{\Gamma}, \hat{\Delta} \right\} + \frac{2}{\hbar} \hat{\rho}_p \langle \Gamma \rangle_{\Delta} \\ & + \frac{2}{\hbar} (\langle \Gamma \rangle_p + \langle \Gamma \rangle_{\Delta}) \hat{\Delta}, \end{aligned} \quad (10)$$

$$\begin{aligned} \frac{d}{dt} \hat{M} = & -\frac{i}{\hbar} \left[\hat{H}_+, \hat{M} \right] - \frac{1}{\hbar} \left\{ \hat{\Gamma}, \hat{M} \right\} + \frac{4}{\hbar} (\langle \Gamma \rangle_p + \langle \Gamma \rangle_{\Delta}) \hat{M} \\ & + \frac{2}{\hbar} \left[(\hat{\rho}_p + \hat{\Delta}) \hat{\Delta}^{(\Gamma)} + \hat{\Delta} \hat{\rho}_p^{(\Gamma)} \right], \end{aligned} \quad (11)$$

$$\frac{d}{dt} S_L = \frac{4}{\hbar} \left[(\langle \Gamma \rangle_p + \langle \Gamma \rangle_{\Delta}) S_L - \text{tr}(\hat{\Gamma} \hat{M}) \right], \quad (12)$$

where $\langle \Gamma \rangle_p = \text{tr}(\hat{\rho}_p \hat{\Gamma})$ is the average of the operator Γ with respect to the unperturbed density operator $\hat{\rho}_p$, $\langle \Gamma \rangle_{\Delta} = \text{tr}(\hat{\Delta} \hat{\Gamma})$ is the difference between the averages $\langle \Gamma \rangle$ and $\langle \Gamma \rangle_p$, and we have used the notation $\hat{A}^{(\Gamma)} = \hat{\Gamma} \hat{A} - \text{tr}(\hat{\Gamma} \hat{A}) \hat{I}$ with \hat{I} being the identity operator. One can easily verify that both the tracelessness and hermiticity of the variation operator $\hat{\Delta}$ are preserved during time evolution.

Equations (10)–(12) are matrix differential equations that can all be used for the stability study, but it must be emphasized that equations (11) and (12) describe only those variations that can potentially lead to the transition of a pure state $\hat{\rho}_p$ into a mixed one (dubbed as the “mixing” fluctuations in what follows), whereas equation (10) alone governs the variations of a general type, regardless on whether they alter the purity of $\hat{\rho}_p$ or not. Therefore, equations (11) and (12) will be of special interest here. It is easy to see that one can apply to them the standard methods of dynamical stability analysis, such as the Lyapunov or Vakhitov-Kolokolov criteria [30].

However, some generic features can be noticed straight away. In particular, equation (12) reveals that stability against the “mixing” fluctuations essentially depends on the result of competition between the terms $\langle \Gamma \rangle$ and $\text{tr}(\hat{\Gamma} \hat{M})/S_L = (\langle \Gamma \rangle - \text{tr}(\hat{\rho}^2 \hat{\Gamma}))/S_L$. If their difference is non-negative at any time, then during evolution the fluctuations will keep the initial state away from being pure. If the difference is negative then the fluctuations will get suppressed.

Yet another property, which can be immediately seen from equation (12), is that, in the absence of the anti-Hermitian part of the Hamiltonian, random fluctuations of a given pure state would never get suppressed but their magnitude remains at much the same level. This means that in order to fully suppress fluctuations, suitably chosen anti-Hermitian terms must be present to the Hamiltonian. In those cases the anti-Hermitian part would ensure the full stability of pure states even if it is negligibly small compared to the Hermitian part.

4 Local stability

It is clear that the local instability of a pure state against fluctuations leading to the mixing of a state does yet not imply global instability (i.e., when the purity of a perturbed pure state never goes back to its original value 1). However, this case is still interesting from a physical point of view: quantum systems might exist in which the purification time (i.e., the time of return back into a pure state) can be larger than the lifetime of the system itself or the ultimately possible time of measurement/observation of the system. Yet another application area of local instability is that it can point to the presence of singular points at which the density matrix components diverge; thus indicating that the underlying system becomes critically unstable. In any of these cases, local instability of a pure state might become the dominating chaotic-type phenomenon, which can strongly affect the physical properties of systems’ evolution. From the technical point of view, the advantage of the study of local instability is that one does not have to know the whole solution for the density operator, but only a few characteristic exponents.

The analysis of local stability for a given state of the NH quantum system can be made based on linearized equations, which are easier to solve or analyze using well-known methods from the theory of stability and dynamical chaos. Indeed, once we assume that the variation operator is small, $\hat{\Delta} = \delta\hat{\rho}$, we can perform the linearization with respect to it. Then equations (10)–(12) yield, respectively:

$$\begin{aligned} \frac{d}{dt} \delta\hat{\rho} = & -\frac{i}{\hbar} \left[\hat{H}_+, \delta\hat{\rho} \right] - \frac{1}{\hbar} \left\{ \hat{\Gamma}, \delta\hat{\rho} \right\} \\ & + \frac{2}{\hbar} \left[\langle \Gamma \rangle_p \delta\hat{\rho} + \hat{\rho}_p \text{tr}(\delta\hat{\rho} \hat{\Gamma}) \right] + \mathcal{O}(\delta\hat{\rho}^2), \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{d}{dt} \delta\hat{M} = & -\frac{i}{\hbar} \left[\hat{H}_+, \delta\hat{M} \right] - \frac{1}{\hbar} \left\{ \hat{\Gamma}, \delta\hat{M} \right\} \\ & + \frac{2}{\hbar} \left[2\langle \Gamma \rangle_p \delta\hat{M} + \hat{\rho}_p \hat{\Delta}^{(\Gamma)} + \hat{\rho}_p^{(\Gamma)} \delta\hat{\rho} \right] + \mathcal{O}(\delta\hat{\rho}^2), \end{aligned} \quad (14)$$

$$\frac{d}{dt} \delta S_L = \frac{4}{\hbar} \left[\langle \Gamma \rangle_p \delta S_L - \text{tr}(\hat{\Gamma} \delta\hat{M}) \right] + \mathcal{O}(\delta\hat{\rho}^2), \quad (15)$$

where

$$\delta\hat{M} = \hat{M}|_{\hat{\Delta} \rightarrow \delta\hat{\rho}} = \delta\hat{\rho} - \left\{ \hat{\rho}_p, \delta\hat{\rho} \right\}$$

and

$$\delta S_L = \text{tr}(\delta \hat{M}) = -2\text{tr}(\hat{\rho}_p \delta \hat{\rho}) = -\delta \mathcal{P}$$

are the variations of the non-purity operator and its trace, respectively; \mathcal{O} denotes the terms that can be neglected in the linear order of approximation.

As discussed above, in order to study the stability of a system against the “mixing” fluctuations, one has to consider primarily equations (14) and (15). In the leading approximation (14) can be rewritten in the form $\frac{d}{dt} \mathbf{X} = \mathbf{A} \mathbf{X}$, where \mathbf{X} is a column vector constructed out of the linearly independent components of the matrix $\delta \hat{M}$, and $\mathbf{A} = \mathbf{A}(\hat{\rho}_p, \hat{H}_+, \hat{\Gamma}, t)$ is a characteristic matrix. If all real parts of its eigenvalues are negative then the pure state $\hat{\rho}_p$ is locally stable against small “mixing” fluctuations. This condition is equivalent to the matrix $\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A}$ being negative definite for some positive definite matrix $\mathbf{P} = \mathbf{P}^T$. The corresponding Lyapunov function candidate can be determined then as $V(\mathbf{X}) = (\mathbf{X})^T \mathbf{P} \mathbf{X}$. Further classification of instability types (“center”, “node”, “saddle”, “spiral”, etc.) can be done for a specific system by analysis of the characteristic matrix’s eigenvalues [30].

5 Example: two-level systems

In order to illustrate the formalism, let us consider some models which exhibit, under certain conditions, the dynamical instability of pure states.

The NH two-level systems (TLS) are simple yet very useful physical models [31], which can provide a clear visualization of different kinds of stability that may occur. One can check that for two-level systems the non-purity matrix has only one independent component and can be written in the form

$$\hat{M} = \frac{1}{2} S_L \hat{I}, \quad (16)$$

then one obtains

$$\delta \hat{M} = \frac{1}{2} \delta S_L \hat{I}, \quad (17)$$

i.e., the scalar δS_L provides complete information about the local stability of a given pure state $\hat{\rho}_p$, whereas the characteristic matrix \mathbf{A} can be reduced to a 1×1 matrix. The linearized equation (15) takes the simple form

$$\frac{1}{\delta S_L} \frac{d}{dt} \delta S_L = \Lambda(\hat{\rho}_p, \hat{\Gamma}) \equiv \frac{2}{\hbar} \left(2\langle \Gamma \rangle_p - \text{tr} \hat{\Gamma} \right), \quad (18)$$

where $\Lambda = \Lambda(\hat{\rho}_p, \hat{\Gamma})$ is the characteristic exponent. According to the approach, the state $\hat{\rho}_p$ is locally stable against the “mixing” fluctuations if $\Lambda(\hat{\rho}_p, \hat{\Gamma}) < 0$, and locally unstable otherwise. To illustrate this, let us consider the following exactly-solvable models.

5.1 Tunneling models with non-Hermitian detuning

This is a set of NH models whose Hermitian and anti-Hermitian parts of Hamiltonian are, respectively:

$$\hat{H}_+ = -\hbar \omega \hat{\sigma}_x, \quad \hat{\Gamma} = \hbar \lambda \hat{\sigma}_z, \quad (19)$$

where $\hat{\sigma}_i$ ’s are Pauli matrices, positive parameter ω is related to the matrix element for tunneling between two wells, and λ is a constant parameter, real-valued but otherwise free, that can be viewed as the imaginary counterpart of the detuning parameter [32]. Such models are popular in many areas of quantum physics, including the theory of open quantum-optical systems (in particular, when describing the direct photodetection of a driven two-level atom interacting with the electromagnetic field – in which case λ would be related to the atomic damping rate) [2,18] and non-Hermitian quantum mechanics with real energy spectra [33,34].

Here we would like to determine for which values of the parameters ω and λ a pure state, say, the one defined by:

$$\hat{\rho}_p^{(1)} = |e\rangle\langle e| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (20)$$

is locally stable against the “mixing” fluctuations. In case of quantum-optical or spin systems, this information could be instrumental for determining whether a system can undergo, respectively, the spontaneous emission or spin-flip transition.

The characteristic exponent Λ , which can be immediately computed from equation (18), turns out to be equal to λ , up to a positive factor. Therefore, we expect the state $\hat{\rho}_p^{(1)}$ to be locally stable against small fluctuations for the models with negative $\tilde{\lambda} = \lambda/\omega$, and locally unstable otherwise.

To verify this, one needs to solve the evolution equation assuming the perturbed initial conditions:

$$\hat{\rho}(0) = \hat{\rho}_p^{(1)} + \delta \hat{\rho}(0), \quad (21)$$

where, according to the approach, the variation matrix can be chosen in the form

$$\hat{\Delta}_{(0)} = \delta \hat{\rho}(0) = \begin{pmatrix} \delta_2 & \delta_1 \\ \delta_1 & -\delta_2 \end{pmatrix}, \quad (22)$$

with δ_i ’s being arbitrary real-valued numbers (for simplicity we have omitted the off-diagonal imaginary components of $\hat{\Delta}$). The positivity of the total density operator $\hat{\rho}(0)$ can be always ensured by imposing constraints on the values of variation matrix’s components. However, here we assume the fluctuations to be arbitrary enough in a sense that we do not postulate them to exactly preserve the positivity of the density operator at the initial moment of time, similarly to the approach [19]. Instead, one might be interested to see whether this property can be dynamically restored during evolution if it has been initially broken by fluctuations.

Solving the evolution equation (4) with the initial condition (21), we obtain the following expression for the normalized density operator:

$$\hat{\rho}(t) = \frac{f_x(t)}{F(t)} \hat{\sigma}_x + \frac{f_y(t)}{F(t)} \hat{\sigma}_y + \frac{f_z(t)}{2F(t)} \hat{\sigma}_z + \frac{1}{2} \hat{I}, \quad (23)$$

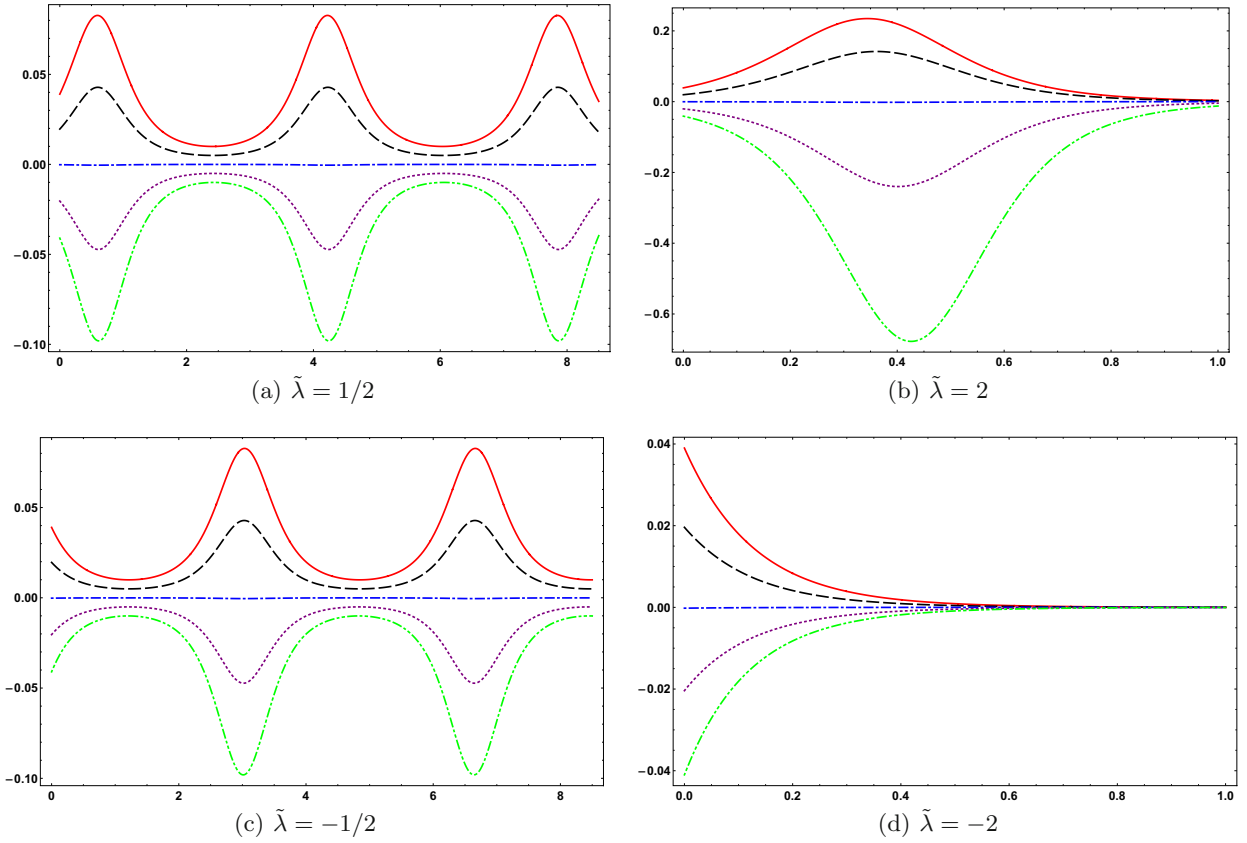


Fig. 1. Linear entropy S_L versus time (in units of ω^{-1}) for the family of models (19), at different values of $\tilde{\lambda}$ and initial perturbations of the state $\hat{\rho}_p^{(1)}$. The value of δ_1 is 0.01 for all curves, the values of δ_2 are: -0.02 (solid curves), -0.01 (dashed curves), 0 (dash-dotted curves), 0.01 (dotted curves) and 0.02 (dash-double-dotted curves).

where we have denoted:

$$f_x(t) = \delta_1 \mu^2, \quad (24)$$

$$f_y(t) = \sinh(\mu\tau) \left[p_2 \mu \cosh(\mu\tau) - \tilde{\lambda} \sinh(\mu\tau) \right], \quad (25)$$

$$f_z(t) = \mu \left[p_2 \mu \cosh(2\mu\tau) - \tilde{\lambda} \sinh(2\mu\tau) \right], \quad (26)$$

$$F(t) = \tilde{\lambda}^2 \cosh(2\mu\tau) - p_2 \mu \tilde{\lambda} \sinh(2\mu\tau) - 1, \quad (27)$$

where $\tau = \omega t$, $\tilde{\lambda} = \lambda/\omega$, $p_2 = 2\delta_2 + 1$ and $\mu = \sqrt{\tilde{\lambda}^2 - 1}$.

The evolution of purity for this solution is presented in Figure 1. The top (bottom) row of the figure shows models for which the initial state $\hat{\rho}_p^{(1)}$ is unstable (stable) locally, as predicted by theory. Indeed, one can see that for positive values of $\tilde{\lambda}$, which has the same sign as λ , the magnitude of the linear entropy initially increases from its initial value, whereas for negative $\tilde{\lambda}$'s the magnitude of the linear entropy decreases straight from the beginning. We reiterate here that the local stability or instability does not yet imply the global one. However, if the measurement time or the lifetime of a system is limited and smaller than the purification time (which is of the scale ω^{-1} here), then an observer would not be able to empirically distinguish between the local and global (asymptotical) types of instability.

Further, the left (right) column of Figure 1 shows models for which the initial state $\hat{\rho}_p^{(1)}$ is unstable (stable) asymptotically. Asymptotical stability cannot be assessed based on the linearized equation (18) but requires the usage of a general approach (10)–(12), which is quite bulky and thus would bring us beyond our purposes here. Instead, in Figures 1b and 1d we demonstrate that corresponding models contain the mechanism that asymptotically suppresses quantum fluctuations, whereas Figures 1a and 1c reveal that in some models fluctuations may keep the state away from being pure in an oscillatory way. As long as both the conventional von Neumann entropy and the NH-adapted entropy (defined in Ref. [21], see for details) would also oscillate in those cases, one can regard this behaviour as the quantum non-Hermitian analogue of the self-organization phenomenon.

5.2 Tunneling models with analytically continued matrix element

This is a family of models where

$$\hat{H}_+ = -\hbar\omega\hat{\sigma}_x, \quad \hat{\Gamma} = \hbar\eta\hat{\sigma}_x, \quad (28)$$

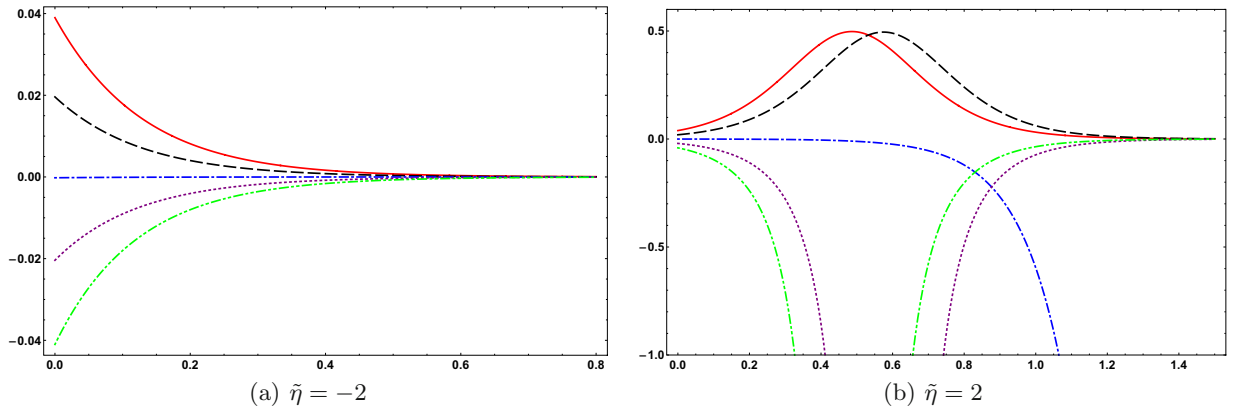


Fig. 2. Linear entropy S_L versus time (in units of ω^{-1}) for the family of models (28), at different values of $\tilde{\eta}$ and initial perturbations of the state $\hat{\rho}_p^{(2)}$. The value of δ_2 is 0.01 for all curves, the values of δ_1 are: -0.02 (solid curves), -0.01 (dashed curves), 0 (dash-dotted curves), 0.01 (dotted curves) and 0.02 (dash-double-dotted curves).

with η being a constant parameter, real-valued but otherwise free. Such models describe physical cases when a quantum environment of some kind (external fields or vacuum oscillations) effectively shifts the value of the tunneling parameter into a complex domain: $\omega \rightarrow \omega + i\eta$.

Here we would like to determine for which values of the parameters ω and η the pure state

$$\hat{\rho}_p^{(2)} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (29)$$

is locally stable against the “mixing” fluctuations. From the physical point of view, it might be used to determine whether a given quantum superposition of states is protected against the spontaneous decay into a mixed state.

The characteristic exponent (18), turns out to be equal to η , up to a positive factor, hence, we expect state $\hat{\rho}_p^{(2)}$ to be locally stable against small fluctuations in the models with negative $\tilde{\eta} = \eta/\omega$, and locally unstable otherwise.

To verify this, one needs to solve the evolution equation (4) assuming the perturbed initial conditions

$$\hat{\rho}(0) = \hat{\rho}_p^{(2)} + \delta\hat{\rho}_{(0)}, \quad (30)$$

where $\delta\hat{\rho}_{(0)}$ is given by (22). We obtain the following expression for the normalized density operator:

$$\hat{\rho}(t) = \frac{g_x(t)}{2G(t)}\hat{\sigma}_x + \frac{g_y(t)}{G(t)}\hat{\sigma}_y + \frac{g_z(t)}{G(t)}\hat{\sigma}_z + \frac{1}{2}\hat{I}, \quad (31)$$

where we have denoted:

$$g_x(t) = p_1 \cosh(2\tilde{\eta}\tau) - \sinh(2\tilde{\eta}\tau), \quad (32)$$

$$g_y(t) = \delta_2 \sin(2\tau), \quad (33)$$

$$g_z(t) = \delta_2 \cos(2\tau), \quad (34)$$

$$G(t) = \cosh(2\tilde{\eta}\tau) - p_1 \sinh(2\tilde{\eta}\tau), \quad (35)$$

where $\tau = \omega t$, $\tilde{\eta} = \eta/\omega$, and $p_1 = 2\delta_1 + 1$.

The evolution of purity for this solution is presented in Figure 2. Figure 2a illustrates the model for which the initial state $\hat{\rho}_p^{(2)}$ is stable locally, as predicted by the linearized approach. It also turns out to be stable globally,

which makes this case similar to the one discussed earlier, regarding Figure 1d.

In Figure 2b we show the model for which some fluctuation modes of the initial state $\hat{\rho}_p^{(2)}$ grow indefinitely, so that the system evolves into a singularity. This is another kind of dynamical instability, which is different from those discussed in the previous set of models. It illustrates that for some models, quantum fluctuations either get significantly amplified during certain interval of time (solid and dashed curves in Fig. 2b) or destabilize the original state up to the full destruction (dotted, dash-dotted and dash-double-dotted curves).

6 Conclusion

Using the density operator approach for non-Hermitian Hamiltonians, we have demonstrated that quantum fluctuations can cause, under certain conditions, the instability of pure states, which is controlled by the environment-induced anti-Hermitian parts of Hamiltonians. This is drastically different from the Hermitian case where both the purity's value and pure states are preserved during time evolution.

It is shown that the instability of pure states is not preassigned in the evolution equation but arises as the emergent phenomenon in its solutions. We have derived the equations that are necessary to study the stability issues of any quantum system, regardless of the number of its degrees of freedom, dimensionality of Hilbert space, etc. Thus, the formalism and main results are applicable for systems described by non-Hermitian Hamiltonians of general type.

Finally, in order to illustrate the different types of instability that may occur, we have considered some exactly solvable two-state models. Apart from being instructive examples on their own, these models can be used in a theory of open quantum-optical and spin systems where our stability analysis might be helpful in achieving a better understanding of such phenomena as the spontaneous emission, decay or spin flip (which, as a matter of fact,

could be one of the directions for future work). By means of those models we have visualized the main types of stability of a pure state against small “mixing” fluctuations that may occur in the open quantum system: fluctuations get permanently suppressed with time (the state is locally and globally stable), fluctuations amplify during a finite period of time but eventually get suppressed (the state is locally unstable but asymptotically stable), fluctuations never get suppressed with time but stay bound by a finite value (the state is globally unstable), and, finally, fluctuations indefinitely grow thus leading to the singularity and critical instability of the system (the state and system are globally unstable). It should be also noticed that for the two-state models studied above the fluctuations, which cause the purity to acquire unphysical values, either get suppressed or lead to the overall instability of the system.

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