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Remark on neutrino oscillations

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Abstract The oscillations of ultra-relativistic neutrinos are realized by the propagation of assumed zero-mass on-shell neutrinos with the speed of light in vacuum combined with the phase modulation by the small mass term $\exp[-i(m_{\nu_k}^2/2|\vec{p}|)\tau]$ with a time parameter τ. This picture is realized in the first quantization by the mass expansion and in field theory by the use of $\delta(x^0-y^0-\tau)\langle 0|T^\star \nu_{Lk}(x)\overline{\nu_{Lk}(y)}|0\rangle$ with the neutrino mass eigenstates ν_{Lk} and a finite positive τ after the contour integral of the propagating neutrino energies. By noting that the conventional detectors are insensitive to neutrino masses, the measured energy-momenta of the initial and final states with assumed zero-mass neutrinos are conserved. The propagating neutrinos preserve the three-momentum in this sense but the energies of the massive neutrinos are conserved up to uncertainty relations and thus leading to oscillations. Conceptual complications in the case of Majorana neutrinos due to the charge conjugation in $d = 4$ are also discussed.

1 Neutrino oscillations and mass expansion

The phenomenon of neutrino oscillations [1–3] is fundamental to measure the small neutrino masses, and it would be disastrous if the different formulations should lead to different neutrino masses. If one writes the neutrino mixing with the PMNS matrices $U^{\alpha k}$

$$|\nu_\alpha\rangle = \sum_k U^{\alpha k\star}|\nu_k\rangle, \tag{1}$$

where $|\nu_k\rangle$ are the mass eigenstates which diagonalize the neutrino mass matrix, and the flavor eigenstates $|\nu_\alpha\rangle$, ($\alpha = e, \mu, \tau$), are related to each other by the above mixing formula. We define the charged lepton flavor eigenstates by the mass eigenstates. One may start with the production of the flavor eigenstate neutrino ν_μ in the energetic pion decay

$$\pi^+ \to \mu^+ + \nu_\mu, \tag{2}$$

for example, by measuring π^+ and μ^+, and the neutrinos thus produced propagate toward the detector in the oscillation experiment; the direction of each mixed neutrinos may not necessarily be in the exact specific direction considering the accuracy of the measurements of π^+ and μ^+. The oscillation is observed in each direction of the mixed neutrinos. We analyze the Dirac neutrinos in the main part of the present paper for simplicity, and the case of the Majorana neutrinos, which are constrained by the complications of the Majorana fermions in $d = 4$, shall be discussed in Appendix. It is known that the relation (1), if interpreted as a superposition of on-shell mass eigenstates with *identical* three-momentum, leads to the standard oscillation formula [4,5]

$$\begin{aligned}|\langle\nu_\beta(0)|\nu_\alpha(t)\rangle|^2 &= \left|\sum_k U^{\beta k}\exp[i\vec{p}\cdot\vec{x} - i\sqrt{\vec{p}^2+m_{\nu_k}^2}\,t](U^\dagger)^{k\alpha}\right|^2\\ &= \left|\sum_k U^{\beta k}\exp[-i\frac{m_{\nu_k}^2}{2|\vec{p}|}t](U^\dagger)^{k\alpha}\right|^2,\end{aligned} \tag{3}$$

where $t = L$, the neutrino propagation distance, is assumed together with $|\vec{p}|^2 \gg m_{\nu_k}^2$. It is known also that the identical energy assumption of neutrinos, instead of the identical three-momentum assumption in the above derivation, gives essentially the same formula [6]. In this paper we want to understand how the oscillation formula is robust against various ways to derive it.

We first mention the idea of the wave packet of neutrinos in the first quantization formalism [7]. If one of the mass eigenstates in the neutrino ν_μ, for example ν_1, should be identified immediately after the pion decay (2) such a mass eigenstate due to the reduction of quantum states would propagate without oscillations, although the flavor change $\beta \to \alpha$ would be induced by the (inverse) mixing in (1). The actual values of neutrino masses are, however, very small and thus the

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specification of the mass of one of neutrinos is practically impossible; in fact, as is explained later, the neutrino masses need to be not measured by the conventional detectors to measure the neutrino oscillations. To treat the un-identified neutrino mass eigenstates consistently, Kayser [7] suggested the idea of the wave packet of particles involved, such as ν_μ in (2). The wave packet is more generally understood as a means to incorporate the semi-classical aspects of neutrino oscillations into quantum mechanics in a consistent manner, and it has been successfully incorporated in the field theoretical formulations [6,9–13] and a related quantum mechanical formulation [14]. It has been shown also that the kinematics of the neutrino production generally implies the mass dependence of the neutrino momentum such as $\vec{p} = \vec{p}(m^2_{\nu_k})$ and that the mass-dependence of the momentum depends how they are produced [15,16]; for example, the two-body decay $\pi^+ \to \mu^+ + \nu$ or other neutrino production processes.

One may consider the propagating phase of a flavor eigenstate in vacuum determined by the phase of the mass eigenstates initially located at $(t, \vec{x}) = (0, 0)$

$$|\nu_\alpha(t, \vec{x})\rangle = \sum_k U^{\alpha k\star} \exp[i\phi(t, \vec{x}; m_{\nu_k})]|\nu_k(0, 0)\rangle \tag{4}$$

by assuming that the neutrino masses are very small and thus the measured neutrinos are ultra-relativistic, in accord with experimental facts which imply the mass differences Δm^2 on the order of $(10^{-2}eV)^2$. One then obtains the Lorentz invariant

$$\begin{aligned}\phi(t, \vec{x}; m_{\nu_k}) &= \vec{p}(m^2_{\nu_k})\vec{x} - E(\vec{p}(m^2_{\nu_k}), m^2_{\nu_k})t \\ &= \vec{p}(0)\vec{x} - E(\vec{p}(0), 0)t + m^2_{\nu_k}\vec{p}\,'(0)\cdot(\vec{x} - \vec{v}_g t) \\ &\quad -\frac{1}{2}\frac{m^2_{\nu_k}}{|\vec{p}(0)|}t + O(m^4_{\nu_k})\end{aligned} \tag{5}$$

where

$$\begin{aligned}\vec{p}(m^2_{\nu_k}) &= \vec{p}(0) + m^2_{\nu_k}\vec{p}\,'(0) + O(m^4_{\nu_k}), \\ E(\vec{p}(m^2_{\nu_k}), m^2_{\nu_k}) &= \sqrt{\vec{p}(m^2_{\nu_k})^2 + m^2_{\nu_k}}, \\ &= E(\vec{p}(0), 0) + m^2_{\nu_k}\vec{p}\,'(0)\cdot\vec{v}_g \\ &\quad +\frac{1}{2}\frac{m^2_{\nu_k}}{|\vec{p}(0)|} + O(m^4_{\nu_k})\end{aligned} \tag{6}$$

with

$$E(\vec{p}(0), 0) = \sqrt{\vec{p}(0)^2}, \quad \vec{v}_g = \frac{\partial E(\vec{p}(0), 0)}{\partial \vec{p}(0)} = \frac{\vec{p}(0)}{|\vec{p}(0)|}. \tag{7}$$

We assume that the momentum $\vec{p}(0)$ is common to all the mass eigenstates of neutrinos. The velocity $\vec{v}_g$ is the group velocity of the propagating (now regarded as massless) neutrinos. The basic assumption of the wave packet (although we do not write an explicit form of the wave packet following the analysis by Giunti and Kim [16]) is that the neutrinos are concentrated at the center of the wave packet (see also [6])

$$\vec{x} - \vec{v}_g t \simeq 0. \tag{8}$$

The neutrino wave packets are essentially the spreading of the initial and final state weak vertices, since the neutrinos rarely interact with surrounding materials. We assume that the geometrical spreads of the weak vertices, which are large in the microscopic sense so that the spread of the neutrino momentum is negligibly small by the uncertainty principle, but still the geometrical spreads are very small compared with the macroscopic distance between the two weak vertices. In other words, all the neutrino mass eigenstates are assumed to be measured at $\vec{x} - \vec{v}_g t \simeq 0$ or at a finite distance away from 0, then the term $m^2_{\nu_k}\vec{p}\,'(0)\cdot(\vec{x} - \vec{v}_g t)$ is much smaller than $-\frac{1}{2}\frac{m^2_{\nu_k}}{|\vec{p}(0)|}t$ in (5) for large t; this is also ensured by the fact that $|m^2_{\nu_k}\vec{p}\,'(0)|$ is of about equal magnitude to $\frac{1}{2}\frac{m^2_{\nu_k}}{|\vec{p}(0)|}$, which is confirmed to be the case. The small quantities with $O(m^4_{\nu_k})$ or higher powers in the neutrino mass are neglected.

One thus measures the oscillations caused by next to the last term of (5) with the common momentum factor

$$\sum_k U^{\beta k} \exp\left[-i\frac{m^2_{\nu_k}}{2|\vec{p}(0)|}t\right](U^\dagger)^{k\alpha}. \tag{9}$$

When the exponential factor is written in the form

$$-(\sqrt{\vec{p}(0)^2 + m^2_{\nu_k}} - \sqrt{\vec{p}(0)^2})t, \tag{10}$$

(9) is universal, i.e., depends on the intrinsic properties of the neutrinos independently of how the neutrinos were produced. The common factor $\vec{p}(0)\vec{x} - E(0)t$ for each mass eigenstate in (5) does not contribute to the oscillation. We thus recover the standard oscillation formula (3) using $t \simeq L$ which arises from $\vec{x} - \vec{v}_g t \simeq 0$ in (8) with $|\vec{v}_g| = 1$. Physically, the semi-classical relation $t \simeq L$ with a small error, which is determined by the conventional detectors, is not influenced by the neutrino masses. In the above analysis, we chose the vanishing masses of propagating neutrinos as the fiducial values and the observed neutrinos are assumed to be essentially massless.

Alternatively, if one assumes the time-to-distance conversion $t = L$ to be a valid ansatz for massless on-shell neutrinos,[1] one would obtain the formula (9) directly since the term $m^2_{\nu_k}\vec{p}\,'(0)\cdot(\vec{x} - \vec{v}_g t)$ then vanishes. Also this picture is consistent with the exact three-momentum conservation induced by the three dimensional integration at both of the initial and final weak vertices, which are implicitly assumed. We derived the formula (5) by the mass expansion, which

[1] This idea of the time-to-distance conversion was criticized in [6]. To avoid the criticism, we use this idea only for the on-shell massless neutrinos in (5) and also in the field theoretical amplitude discussed in the next section. See also [17].

is an expansion in terms of a Lorentz scalar quantity of the Lorentz invariant phase $\vec{p}(m_{\nu_k}^2)\vec{x} - E(m_{\nu_k}^2)t$; this may imply the Lorentz invariance of the oscillation formula (9) [6]. The present picture may agree with our intuitive understanding of neutrino oscillations; the ultra-relativistic neutrinos propagate with the speed of light $\vec{x} - \vec{v}_g t = 0$ in vacuum for a measured momentum $\vec{p}(0)$, and the effects of the small mass differences $-\frac{1}{2}\frac{m_{\nu_k}^2}{|\vec{p}(0)|}t$ are measured by oscillations in vacuum.[2] The ratio $-\frac{1}{2}\frac{m_{\nu_k}^2}{|\vec{p}(0)|}$ provides an important quantity in this analysis, namely, it needs to be very small and not measured by the conventional detectors; this constraint, namely, not measurable by conventional detectors, generally arises because of the energy non-conservation in neutrino oscillations (or by an analysis of energy-time uncertainty relations). In the next section on the Feynman amplitude approach to neutrino oscillations we discuss how the same criterion arises.

2 Feynman amplitude approach

To understand the oscillation phenomena in a field theoretical formulation, one may start with an extension of the Standard Model. The leptonic sector is given by

$$\begin{aligned}
\mathcal{L}_{leptons}(x) &= (\overline{e}, \overline{\mu}, \overline{\tau})\,[i\gamma^\alpha\partial_\alpha - \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}] \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \\
&+ (\overline{\nu}_1, \overline{\nu}_2, \overline{\nu}_3)\,[i\gamma^\alpha\partial_\alpha - \begin{pmatrix} m_{\nu_1} & 0 & 0 \\ 0 & m_{\nu_2} & 0 \\ 0 & 0 & m_{\nu_3} \end{pmatrix}] \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \\
&- \frac{g}{\sqrt{2}}\{(\overline{\nu}_1, \overline{\nu}_2, \overline{\nu}_3)_L [U^\dagger \gamma^\alpha W_\alpha] \frac{(1-\gamma_5)}{2} \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} + h.c.\}
\end{aligned} \tag{11}$$

with a 3×3 unitary mixing matrix U in (1). We ignore the neutral current and electromagnetic interactions. All the particles belong to respective mass eigenstates and the lowest order Feynman amplitudes (using the Fermi approximation) are well-defined without infrared singularities. One can confirm that the conventional tree-level Feynman amplitude, which consists of the production and detection weak vertices connected by the Feynman propagator of massive neutrinos, does not give rise to the neutrino oscillations when one integrates over all the values of the weak interaction points x^μ and y^μ with $x^\mu > y^\mu$ as well as $y^\mu > x^\mu$ in the Fermi approximation of weak interactions we work. This integration over interaction points, which incorporates backward moving off-shell anti-neutrinos as well as forward moving off-shell neutrinos, preserves the energy-momentum precisely at each interaction point. We thus have no time scale to measure oscillations which are related to the time translation non-invariance. We shall demonstrate below that the Feynman amplitude with only a part of the forward on-shell neutrino propagator reproduces the neutrino oscillation amplitude (3) by preserving the measured overall energy-momentum up to uncertainty relations. Feynman rules are used to specify the quantum mechanically allowed couplings.

The Feynman amplitude approach to neutrino oscillations has been discussed in the field theoretical formulation by Kobzarev et al. [8], Grimus and Stockinger [10], Giunti, Kim and Lee [11] who emphasized the wave packets, and using plane waves by Egorov and Volobuev [17–19], among others. In the latter approach [17–19], the (effective) limit $x^0 - y^0 \to \infty$ was considered in the propagating neutrinos of the form $\delta(x^0 - y^0 - \tau)\langle T^\star \nu_L^l(x)\overline{\nu_L^k(y)}\rangle$ using a generalization of the Grimus and Stockinger theorem [10] and thus achieving the on-shell condition of all the propagating neutrinos. They emphasized that the momentum space Feynman-like amplitude thus defined produces the oscillation amplitude [17–19] and the probability interpretation of the oscillation amplitude is justified based on the probability interpretation of the conventional Feynman amplitude. They note the simplicity of their formulation compared to those of the past formulations such as [10]. We follow the basic ideas of [10,11,17–19], and we shall simplify the derivation of oscillation amplitudes and add several remarks on the robustness of the amplitude thus derived.

We write the effective Lagrangian of neutrino processes as

$$\begin{aligned}
\mathcal{L}_{eff} &= \sum_k \{\overline{\nu^k(x)}[i\gamma^\mu\partial_\mu - M_{\nu_k}]\nu^k(x) \\
&+ \overline{J_R^\alpha(x)}U^{\alpha k}\nu_L^k(x) + \overline{\nu_L^k(x)}(U^\dagger)^{k\alpha}J_R^\alpha(x)\}
\end{aligned} \tag{12}$$

by incorporating the neutrino production and detection processes in the sources $J_R^\alpha(y)$ and $\overline{J_R^\beta(x)}$, respectively, which are chosen generally not to be Hermitian conjugate to each other. We have a generalization of Schwinger's source functions[3]

$$\begin{aligned}
\overline{J_R^\beta(x)} &= \int \frac{d^4P_f}{(2\pi)^4} e^{iP_f x}\,\overline{J_R^\beta(P_f)}, \\
J_R^\alpha(y) &= \int \frac{d^4P_i}{(2\pi)^4} e^{-iP_i y}\,J_R^\alpha(P_i).
\end{aligned} \tag{13}$$

For example, $\sum_k \{\overline{\nu_L^k(x)}(U^\dagger)^{k\alpha}J_R^\alpha(x)\}$ describes the decay $\pi^+ \to \mu^+ + \nu_\mu$ and $\sum_k \{\overline{J_R^\alpha(x)}U^{\alpha k}\nu_L^k(x)\}$ describes the electron production of

[2] The oscillations in the dense medium are not considered here.

[3] Schwinger's source functions in the conventional sense stand for the c-number quantities, whereas we include parts of weak vertices in them. We thus generalize the notations of Schwinger's source functions in the present use and ours are regarded as short hand notations of the conventional Feynman diagram.

$\nu_e + n \rightarrow p + e$. The source functions (in a generalized context as above) are expressed in terms of plane waves as in the conventional Feynman amplitudes. In the neutrino oscillation experiments, the macroscopic distance between the production vertex and the detection vertex is one of the main observables. Following [17–19] (see also [16]), we work in the framework of plane waves and consider the configurations in a 4-dimensional sense where the neutrino production and absorption points, which are denoted by y^μ and x^μ, respectively, are correlated by a fixed time difference

$$x^0 = y^0 + \tau \tag{14}$$

with a very large macroscopic $\tau > 0$.

A suitable choice of $\overline{J_R^\beta(P_f)}$ and $J_R^\alpha(P_i)$ specifies the initial and final systems of the neutrino oscillation experiments as described above. We then have the oscillation amplitude for the initial (production) vertex such as

$$\pi^+ \rightarrow \mu^+ + \nu_\mu \tag{15}$$

and the final (absorption) vertex such as

$$\nu_e + n \rightarrow p + e \tag{16}$$

with the neutrino oscillations communicating $\nu_\mu(\alpha = \mu)$ to $\nu_e(\beta = e)$. Our proposal is to analyze the conventional Feynman amplitude for the weak process, where the neutrino is exchanged,

$$\pi^+ + n \rightarrow \mu^+ + p + e \tag{17}$$

which is written by the prescriptions of source functions described above with an extra δ-function as

$$\begin{aligned}
&\int d^4x d^4y e^{iP_f x} \overline{J_R^\beta(P_f)} U^{\beta k} \delta(x^0 - y^0 - \tau) \\
&\quad \times \langle T^\star \nu_{Lk}(x) \overline{\nu_{Ll}(y)} \rangle (U^\dagger)^{l\alpha} J_R^\alpha(P_i) e^{-iP_i y} \\
&= \int d^4x d^4y \delta(x^0 - y^0 - \tau) e^{iP_f x} \overline{J_R^\beta(P_f)} (\frac{1-\gamma_5}{2}) \\
&\quad \times U^{\beta k} \int \frac{d^4p}{(2\pi)^4} \left(\frac{i \not{p}}{p^2 - m_{\nu_k}^2 + i\epsilon} \right) \delta_{kl} \\
&\quad \times e^{-ip(x-y)} (U^\dagger)^{l\alpha} J_R^\alpha(P_i) e^{-iP_i y} \\
&= \int dy^0 (2\pi)^3 \delta^3(P_f - P_i) \overline{J_R^\beta(P_f)} (\frac{1-\gamma_5}{2}) U^{\beta k} \\
&\quad \times \int \frac{d^4p}{(2\pi)^4} \left(\frac{i \not{p}}{p^2 - m_{\nu_k}^2 + i\epsilon} \right) \delta_{kl} \\
&\quad \times (2\pi)^3 \delta^3(p - P_i) e^{-ip^0\tau + iP_f^0\tau} (U^\dagger)^{l\alpha} J_R^\alpha(P_i) e^{i(P_f^0 - P_i^0)y^0} \\
&= (2\pi)^4 \delta^4(P_f - P_i) \overline{J_R^\beta(P_f)} (\frac{1-\gamma_5}{2}) \\
&\quad \times U^{\beta k} \int \frac{dp_0}{2\pi} \left(\frac{i \not{p}}{p^2 - m_{\nu_k}^2 + i\epsilon} \right) \delta_{kl} e^{-ip^0\tau + iP_f^0\tau} (U^\dagger)^{l\alpha}|_{\vec{p}=\vec{P}_i} J_R^\alpha(P_i)
\end{aligned} \tag{18}$$

where $P_i = q_\pi - p_\mu$ is the entering four-momentum in the case of the pion decay, and $P_f = p_p + p_e - p_n$ is the four-momentum of the outgoing final system; $m_{\nu_k}^2$ stands for the neutrino mass eigenvalue squared. Up to this point, the formula is faithful to what defined by the first line in (18).

We now make an approximation. Since the energy resolution of conventional detectors cannot detect neutrino masses, we neglect the possible neutrino mass dependence in P_i and P_f; those four-momenta are written as if all the propagating neutrinos are massless. The summation $\sum_k$ over the neutrino masses then operates only on the neutrino propagators, and the formula (18) is written after the contour integral over the neutrino energy as

$$\begin{aligned}
&(2\pi)^4 \delta^4(P_f - P_i) \overline{J_R^\beta(P_f)} \left(\frac{1-\gamma_5}{2} \right) \\
&\times \left\{ \sum_k U^{\beta k} \frac{\not{p}}{2p^0} e^{-ip^0\tau + iP_i^0\tau} (U^\dagger)^{k\alpha}|_{p^0 = \sqrt{\vec{p}^2 + m_{\nu_k}^2},\ \vec{p} = \vec{P}_i} \right\} J_R^\alpha(P_i).
\end{aligned} \tag{19}$$

The effect of the energy non-conservation induced by $\delta(x^0 - y^0 - \tau)$ is still seen by the presence of the time parameter τ in the formula.

We would like to add several comments on the above derivation of the formula (19). It implies that the Feynman amplitude with the modified Feynman propagator [17–20] describing only a part of the forward propagating neutrinos

$$\delta(x^0 - y^0 - \tau) \langle T^\star \nu_{Lk}(x) \overline{\nu_{Ll}(y)} \rangle \tag{20}$$

gives rise to the neutrino oscillation probability for large fixed τ, as is seen in (24) later; the formula (19) gives the conventional result (3) if one assumes the time-to-distance conversion $\tau = L$. In the present paper, we adopted the propagator of neutrinos (20) which is the same as in [17–19] with a fixed large τ, but we obtain the *on-shell* condition of neutrinos by performing the contour integral with respect to the neutrino energy [20], instead of taking the (effective) limit $x^0 - y^0 \rightarrow \infty$ in [17–19] with the help of the Grimus and Stockinger theorem. By this way we performed the above calculations of the amplitude (19) with an approximation stated above, in the lowest order of perturbation. This simplified evaluation was possible since we assumed that the energy-momentum of the initial and final systems, represented by P_i and P_f, respectively, are independent of the neutrino masses because of the limited accuracy of conventional detectors of weak interactions. The neutrino momentum $\vec{p} = \vec{P}_i$ also becomes mass independent. By this assumption we were able to take the summation over the massive neutrinos $\sum_k$ outside the δ-function as in the final formula (19).[4] We are assuming that the neutrinos and other particles contained in the source functions are expressed by plane waves, and the on-shell condition of propagating neutrinos $p^0 = \sqrt{\vec{p}^2 + m_{\nu_k}^2}$

[4] If one takes the summation $\sum_k$ including $\delta^4(P_f - P_i)$, one would be able to describe the sum of three-independent Feynman amplitudes of mass eigenstates with an extra constraint $\delta(x^0 - y^0 - \tau)$.

arises from the contour integral $\int dp^0$ containing the factor $e^{-ip^0\tau}$ with $\tau > 0$.

The phase $e^{iP_i^0\tau}$ in (19) is common to all the massive neutrinos and thus neglected in the absolute square of the amplitude. We have the essential part of the amplitude characteristic to oscillations from (19) [17–20]

$$\sum_k U^{\beta k} \frac{i\ \not{p}}{2p^0} e^{-ip^0\tau} (U^\dagger)^{k\alpha}|_{p^0=\sqrt{\vec{p}^2+m_{\nu_k}^2},\ \ \vec{p}=\vec{P}_i} \tag{21}$$

with $\tau = L$. Namely, we recognize the mass differences of the massive neutrinos only by the neutrino oscillations which are caused by the second term on the right-hand side of

$$e^{-ip^0\tau} = e^{-i\sqrt{\vec{p}^2}\tau - i(m_{\nu_k}^2/2|\vec{p}|)\tau - iO(m_{\nu_k}^4)\tau}, \tag{22}$$

leading to the flavor-changed oscillating final states recognized by weak interactions as in (3).

For the ultra-relativistic neutrinos, the spin factor

$$\frac{\not{p}}{2p_0} = \frac{1}{2}\left[\gamma^0 + \gamma^l \frac{p_l}{p_0}\right] \tag{23}$$

in (21) is regarded to be independent of the neutrino masses since $p_l/p_0 = [p_l/|\vec{p}|](1-(1/2)m_{\nu_k}^2/|\vec{p}|^2+...) \simeq p_l/|\vec{p}|$, and thus $/p/(2p_0) \simeq [\gamma^0 + \gamma^l p_l/|\vec{p}|]/2$ is regarded as independent of the neutrino masses. The amplitude (19) then contains the well-known oscillating factor in the quantum mechanical formulation (5) [4,5] with the spin factor $\frac{(1-\gamma_5)}{2}\frac{\not{p}}{2|\vec{p}|}$, which does not influence the τ or L-dependence of the oscillation probability and thus may be absorbed in the initial and final states. This appearance of the spin factor is a new aspect of the Feynman diagram approach, although it does not influence oscillations. We thus have the essential part of the oscillating amplitude

$$\begin{aligned}&\left|\sum_k U^{\beta k} e^{-ip_0\tau} (U^\dagger)^{k\alpha}|_{p_0=\sqrt{\vec{p}^2+m_{\nu_k}^2},\ \ \vec{p}=\vec{P}_i}\right|^2\\ &\quad = \left|\sum_k U^{\beta k} e^{-i\frac{m_{\nu_k}^2}{2|\vec{p}|}\tau} (U^\dagger)^{k\alpha}|_{\vec{p}=\vec{P}_i}\right|^2\end{aligned} \tag{24}$$

with $\tau = L$.

The analysis of an explicit connection between τ and the distance $L = |\vec{x} - \vec{y}|$, namely, the time-to-distance conversion, and the specification of the appropriate extensions of the initial $\vec{y}$ and the final $\vec{x}$ around the points fixed by $\tau = L = |\vec{x} - \vec{y}|$ do not appear in the above formulation (19), although we used $\tau = L$ at several places already. This absence of the analysis of the time-to-distance conversion is analogous to the case of (3) and thus we have to remedy the shortcomings, although the direction from $\vec{y}$ to $\vec{x}$ in the present case is specified by the given common momentum $\vec{p} = \vec{P}_i$. The simplest idea may be to assume that the time-to-distance conversion is a valid ansatz in the analysis of neutrino oscillations for effectively massless on-shell neutrinos, since the conventional detectors cannot recognize the neutrino masses. One would then obtain the desired result from (24) (or (5)) directly. This abstract picture is consistent also with the integration over y^0, namely, the arbitrariness of the origin of initial time in obtaining (19), and also with the exact conservation of the three-momentum at the two weak vertices following from the integration over $\vec{x}$ and $\vec{y}$. Further discussions on this matter shall be given later.

For the specific *two-flavor* case and the non-diagonal process $\mu \to e$, for example, the formula (24) gives the well-known oscillation probability

$$|\langle e|\mu\rangle|^2 = (\sin 2\theta)^2 \frac{1}{2}\left\{1 - \cos\left[\left(\frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2|\vec{p}|}\right)\tau\right]\right\}. \tag{25}$$

The interval of $L(=\tau)$ to measure the oscillations is then specified by the standard

$$\left|\frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2|\vec{p}|}\right| L = 2\pi \tag{26}$$

depending on the mass difference of neutrinos, and the momentum $\vec{p}$ carried by the (massless) neutrinos which is determined by the measured $\vec{p} = \vec{P}_f$; this value is assumed to be independent of neutrino masses. The precise energy-conservation (i.e., time-independence) in (25) is given by

$$m_{\nu_1}^2 - m_{\nu_2}^2 = 0 \tag{27}$$

namely, the vanishing oscillation $1-\cos[\left(\frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2|\vec{p}|}\right)\tau] = 0$. The persistent probability is

$$|\langle\mu|\mu\rangle|^2 = 1 - (\sin 2\theta)^2 \frac{1}{2}\left\{1 - \cos\left[\left(\frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2|\vec{p}|}\right)\tau\right]\right\} \tag{28}$$

which satisfies $|\langle\mu|\mu\rangle|^2 + |\langle e|\mu\rangle|^2 = 1$ in the present case of two-flavor neutrinos. In general, the absolute normalization of the oscillation probability is not well-specified [6], in particular, in the present case multiplying the Feynman amplitude by $\delta(x^0 - y^0 - \tau)$ and thus using only a part of the Feynman amplitude, but the specific oscillation probabilities are well-normalized as above.

3 Discussion and conclusion

The observed oscillation of ultra-relativistic neutrinos is based on the two basic conditions. The first is that the massless *on-shell* neutrinos (with various originally small mass eigenstates) propagate for a given momentum $\vec{p}(=\vec{P}_i)$ from the position $\vec{y}$ to another $\vec{x}$ with the speed of light in vacuum

$$\vec{v}\tau = (\vec{p}/\sqrt{\vec{p}^2})\tau = \vec{x} - \vec{y} \tag{29}$$

when measured with the conventional detectors which do not recognize the neutrino masses; the present consideration is thus limited to length scales of the neutrino propagation approximately within those of the atmospheric neutrino oscillations. This may be called the semi-classical (particle) aspect. The second is that the oscillations are caused by the small phase modulation with the common momentum factor

$$\sum_k U^{\beta k} \exp\left[-i\frac{m_{\nu_k}^2}{2|\vec{p}|}\tau\right](U^\dagger)^{k\alpha} \tag{30}$$

such as in (24) and (5). This phase modulation is the quantum mechanical (wave) effect and supplemented by the subsidiary condition $\tau = L = |\vec{x} - \vec{y}|$ following from the semi-classical consideration (29). Any formulation satisfying these two conditions (29) and (30) gives the oscillation formula.

We assumed in Sect. 2 that the oscillation formula is, in principle, applicable to any experiments without referring to the specific positions $\vec{y}$ and $\vec{x}$ of weak vertices, as the abstract formulations of (5) and (24) suggest. The assumption of the time-to-distance conversion was accepted as a valid ansatz for semi-classical on-shell massless neutrinos with the (common) given momentum $\vec{p}$. In this understanding, the oscillation formulas are valid for any (4-dimensional) configurations parallel transported from each other with fixed $\vec{p}$, independently of the specification of the origin of time y^0. The neutrino oscillations are universal phenomena and applicable to any chosen $\vec{x}$ and $\vec{y}$ in the direction of $\vec{p}$ with $\tau = L = |\vec{x} - \vec{y}|$. The simple Feynman diagram approach in [17–19] and the present derivation of the oscillation formula (19) may be counted among the schemes based on these assumptions.

Alternatively, one may follow the elaborate wave packet analyses in the past [6,9–14]. These analyses may be regarded as clarifying the mechanisms how to satisfy these conditions (29) and (30) including the quantum coherence of neutrinos and the specification of positions $\vec{y}$ and $\vec{x}$. The weak vertices at $\vec{x}$ and $\vec{y}$ are treated naturally in these wave packet pictures, while the treatment of weak vertices are less transparent in the present field theoretical treatment with Lorentz invariant plane waves. In the wave packet picture, one may assume that the semi-classical relation (29) is valid for the points $\vec{y}$ and $\vec{x}$ that are by themselves spreading over the three-dimensional domains so that the three-momentum conservation is ensured within the constraints of uncertainty relations. The spreads of the points $\vec{y}$ and $\vec{x}$ of two weak vertices are still assumed to be much smaller than the semi-classical distance $L = |\vec{x} - \vec{y}|$ between them; as for an explicit wave packet realization of these conditions see, for example, [11]. In fact, the wave packet formalism is regarded as an attempt to incorporate semi-classical constraints such as (29) in the framework of quantum mechanics consistently. In this sense, the justification of the present Feynman amplitude approach is also given by the idea of wave packets.

The on-shell neutrinos with the same three-momentum and different masses mean that the energy conservation is *not* satisfied in the intermediate states of the oscillation in (19) (also in (5)), as is well known [6]. The time-dependent neutrino oscillation in the case of two flavors in (25), for example, may be regarded as measuring the effective energy-nonconservation in the intermediate states (with the notation with explicit $\hbar$)

$$\Delta E\tau = 2\pi\hbar \tag{31}$$

where

$$\Delta E = |\sqrt{\vec{p}^2 + m_{\nu_1}^2} - \sqrt{\vec{p}^2 + m_{\nu_2}^2}| = |\frac{m_{\nu_1}^2 - m_{\nu_2}^2}{2|\vec{p}|}| \tag{32}$$

standing for the propagating neutrino energy splitting. The energy non-conservation is manifested as *time-dependent oscillations*, i.e., the breaking of time translation invariance. But one may regard that the energy non-conservation is superficial since the energy-time uncertainty relation $\Delta E\tau \geq \hbar/2$ is satisfied naturally although rather marginally by (31) with macroscopic τ. This shows that ΔE is actually very small and thus measurable only with oscillations, of which effects vanish on average, but not measurable with the conventional detectors. If ΔE should be measured by conventional detectors, the energy-nonconservation would be confirmed by oscillations; this would be a contradiction.[5]

Without the delta-functional constraint $\delta(x^0 - y^0 - \tau)$, one would have the conventional Feynman amplitude with the propagating neutrinos

$$\left(\frac{1-\gamma_5}{2}\right)\left\{\sum_k U^{\beta k}\left(\frac{i\not{p}}{p^2 - M_{\nu_k}^2 + i\epsilon}\right)\delta_{kl}(U^\dagger)^{l\alpha}\right\} \tag{33}$$

where $p = P_i$(now exact), and because of Feynman's $i\epsilon$ both the forward propagating neutrinos and the backward propagating anti-neutrinos contribute. In contrast, in our formula (19) only a part of the neutrinos propagating forward in time appear and those anti-neutrinos propagating backward in time with negative energy are neglected.[6] With the presence of $\delta(x^0 - y^0 - \tau)$ in the measurement of oscillations, the neutrino propagation may be regarded as a large scale quantum effect. One may regard that the microscopic CP symmetry of the oscillation amplitude is an indication of the CP symmetry of the original Feynman amplitude; the

[5] This contradiction would also be understood as the conventional detectors measuring the energies below the uncertainty limits. The oscillation (or the non-invariance of time translation) would not occur if ΔE should be detected by conventional detectors.

[6] We are assuming that the neutrino is propagating in the oscillation experiment, for simplicity. The anti-neutrino oscillation is measured in addition to examine the CP violation.

microscopic CP violation is described by the phase of the PMNS matrix in the case of three or more leptonic flavors (except for an extra CP violating $U(1)$ phase even for two flavors in the case of Majorana neutrinos, which is however not measured by oscillation experiments [21–23]).

In conclusion, we first re-formulated the ultra-relativistic neutrino oscillations using the neutrino mass expansion, which simplifies the formulation substantially in the framework of the first quantization. The characteristic property of neutrino oscillations is that the conventional detectors cannot recognize the finite neutrino masses, and thus the absence of energy-momentum conservation in a precise sense. These ideas were then applied to the Feynman amplitude approach to neutrino oscillations with plane waves by constraining the amplitude to a part of forward propagating massive neutrinos. The time-to-distance conversion, which may be justified for effectively massless on-shell neutrinos, was assumed to simplify the analyses of ultra-relativistic neutrino oscillations. A unified picture of ultra-relativistic neutrino oscillations was thus presented.

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A Majorana neutrinos and charge conjugation

In this appendix we briefly summarize the conceptual complications in the case of Majorana neutrinos using firstly Weinberg's model [24] and later the seesaw models [25]. The Weinberg's model is defined by an effective Hermitian Lagrangian [24]

$$\begin{aligned}\mathcal{L}_\nu &= \overline{\nu_L}(x) i\gamma^\mu\partial_\mu \nu_L(x) - (1/2)\{\nu_L^T(x) C M_\nu \nu_L(x) + h.c.\}\\ &= (1/2)\{\bar{\psi} i\gamma^\mu\partial_\mu\psi(x) - \bar{\psi}(x) M_\nu \psi(x)\}\end{aligned} \tag{34}$$

where M_ν stands for the 3×3 diagonalized neutrino mass matrix and we defined

$$\psi(x) \equiv \nu_L(x) + C\overline{\nu_L}^T(x), \tag{35}$$

with $C = i\gamma^2\gamma^0$. The field $\psi(x)$ satisfies the classical Majorana condition

$$\psi(x) = C\overline{\psi(x)}^T \tag{36}$$

identically regardless of the choice of ν_L. One may define the Majorana fermion by (35) together with the Dirac equation $[i\gamma^\mu\partial_\mu - M_\nu]\psi(x) = 0$. This is the conventional procedure.

In general one cannot define simultaneously a Majorana fermion with well-defined C and P and a Weyl fermion for which C nor P are defined in $d = 4$. The conceptual complications are how to define an isolated free Majorana neutrino with well-defined C and P, while weak interactions are described by a chiral fermion. Starting with a Dirac fermion, one may obtain under the charge conjugation

$$\nu_L(x) \to C\overline{\nu_R(x)}^T \tag{37}$$

instead of the *pseudo-charge conjugation* symmetry $\nu_L(x) \to C\overline{\nu_L(x)}^T$ [26] *implicit* in the fermion (35). If one adopts the pseudo-C symmetry $\nu_L(x) \to C\overline{\nu_L(x)}^T$ together with the representation of a Majorana fermion (35), one would encounter various puzzling aspects. For example, the first expression of the effective Lagrangian (34) is not invariant under the P operation; also the mass term $\nu_L^T(x) C M_\nu \nu_L(x) = \nu_L^T(x) C M_\nu (\frac{1-\gamma_5}{2})\nu_L(x)$ vanishes under the pseudo-C symmetry. Apparently an idea of the pseudo-C symmetry needs to be understood better [26].

A way to deal with Majorana neutrinos in a consistent manner may be to use a general class of *seesaw models* [25], which contain the equal number of left-handed and right-handed fermions. One may use a Bogoliubov-type transformation to change the definition of the vacuum of the Weyl fermion to the vacuum of the Majorana fermion [27]. The conventional approach gives rise to

$$\begin{aligned}\psi_+(x) &= \nu_R + C\overline{\nu_R}^T\\ \psi_-(x) &= \nu_L - C\overline{\nu_L}^T\end{aligned} \tag{38}$$

in the seesaw model, which are Majorana fermions with masses $M_+ \neq M_-$ if one uses the pseudo-C symmetry. But the conventional parity is not well-defined.

After a suitable Pauli–Gursey transformation, which is equivalent to the Bogoliubov transformation but extended easily to three generations, one can rewrite the Majorana fermions as solutions of the seesaw model in the form [27]

$$\psi_{\pm}(x) = \frac{1}{\sqrt{2}}[N(x) \pm C\overline{N}^{T}(x)] \tag{39}$$

using Dirac-type massive fermions $N(x)$, satisfying $[i\gamma^{\mu}\partial_{\mu} - M_{\pm}]\psi_{\pm}(x) = 0$ with $\psi_{\pm}(x)^{C} = \pm\psi_{\pm}(x)$ and the same masses $M_{+} \neq M_{-}$ as in (38). The parity is defined by [27]

$$N(x) \to i\gamma^{0}N(t, -\vec{x}). \tag{40}$$

Those fields $\psi_{\pm}(x)$ in (39) are the Majorana fermions with well-defined C and P in the conventional sense.

The formal left-handed components $\psi_L(x) = \frac{(1-\gamma_5)}{2}\psi_{-}(x)$ with a $U^{\alpha k}$ *modified* by the Pauli–Gursey transformation [27] may be used in a model of neutrino oscillations (19). The modification of $U^{\alpha k}$ is cancelled by a modification of $\frac{(1-\gamma_5)}{2}\psi_{-}(x)$ induced by the Pauli–Gursey transformation when used in the oscillation formula (19). If $\frac{(1-\gamma_5)}{2}\psi_{-}(x)$ is measured, $\frac{(1+\gamma_5)}{2}\psi_{-}(x)$ is recovered by the parity operation.

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