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Thermodynamics of Taub-NUT-AdS spacetimes

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Abstract We apply the generalised Komar method proposed in [arXiv:2208.05494] to Taub-NUT-AdS spacetime in the theory of Einstein gravity plus a cosmological constant. Based on a generalised closed 2-form, we derive the total mass and NUT charge of the Taub-NUT-AdS spacetime. Together with other thermodynamic quantities calculated through standard method, we conform the first law and Smarr relation. Then, we consider charged AdS NUT spacetimes in Einstein–Maxwell theory with a cosmological constant, and show that the generalised Komar method works, too. We obtain all the thermodynamic quantities, and the first law and Smarr relation are checked to be satisfied automatically.

1 Introduction

Taub-NUT spacetime, constructed in the middle of last century [1,2], is one of the the simplest generalization of the Schwarzschild metric. It adds a time bundle, parameterized by the NUT parameter n, over the round two-sphere, therefore preserving the $SU(2) \sim SO(3)$ isometry group of the Schwarzschild metric. Despite its simplicity, (much simpler than the Kerr metric,) the physical interpretation of the Taub-NUT metric has been controversial. The metric is not asymptotic to Minkowski, but to locally flat spacetime instead. In addition, it contains two more Killing horizons at the south and north poles respectively, giving rise to a global string-like singularity, which is called Misner string [3]. For these reasons, people's interest in the Taub-NUT metric is largely in the Euclidean signature, where, for appropriate parameter choice, it belongs to a class of Gibbons–Hawking gravitational instantons [4], which play an important role of Wheeler's spacetime foam proposal of quantum gravity [5,6].

However, the Taub-NUT spacetime in Lorentzian signature has remained tantalizing. The NUT parameter is a continuous parameter, and hence its deviation from the Minkowski spacetime in the asymptotic infinity region can be arbitrarily small. Therefore astrophysical observations can only constrain its upper bound, but cannot completely rule it out. The Misner string singularity can be rendered by imposing a periodic condition on the time coordinate such that the Misner string becomes a pure gauge and hence nonphysical [3], but naked closed time-like curve is certainly undesirable. It was shown that it is not necessary to impose the time periodic condition, since the Taub-NUT spacetime is geodesically complete even at the presence of the misner string [7,8]. In this work, we shall therefore not impose the periodic condition on time, but treat the time coordinate as real line, and thus the Misner strings remain physical. However, the absence of Minkowski asymptotic region and the existence of Misner string however imply that we cannot use the ADM formalism to calculate the mass (or energy), which is an important parameter of any spacetime.

It is thus an important topic to determine the energy and understand the physical meaning of the NUT parameter. The NUT parameter n is sometimes controversially viewed as a magnetic dual of the mass parameter m [9–11]. The Taub-NUT spacetime has an event horizon, as in the case of the Schwarzschild black hole; it is therefore called Taub-NUT black hole. One expects there should be a black hole thermodynamic first law, which involves not only the mass, temperature and entropy, but also the NUT charge and its thermodynamic potential. The Wald formalism implies that there must be a first-order differential relation involving TdS, but it does not tell how to read off independently the mass, NUT charge and its thermodynamic potential [12,13]. Various results have been proposed, but the answer is not unique [14–21,28]. In [22–27], more than one conserved charges

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associated with NUT parameter n were introduced. Many proposals treated the mass parameter m as the total mass of the spacetime [29–32]; however, for non-vanishing n, the value of m can be arbitrarily negative and yet event horizon exists. Even for the AdS spacetime, it is well known that the mass can be negative in the asymptotic AdS spacetime, but it must satisfy the Breitenlohner-Freedman bound [33]. A definition of mass that allows an arbitrarily negative value is clearly unsatisfactory.

These two problems were resolved by a new proposal for Taub-NUT spacetime in Einstein gravity theories without cosmological constant [34]. All the thermodynamic quantities can be calculated independently from the the first law, and yet the first law is satisfied. In this paper, we generalize the proposal of [34] to include a negative cosmological constant, so that the resulting spacetime is asymptotic to locally (anti-de Sitter) spacetime with Misner string. The spacetime can be called Taub-NUT-AdS black hole. The motivation is threefold. One is that the spacetime metric exists and is sufficiently simple and hence it is worth some effort to understand it. Another motivation is to test whether the proposal of [34] is applicable for the asymptotically locally AdS spacetime. The third is related to the AdS/CFT correspondence. The CFT dual of the Misner string that exists in the bulk is certainly an intriguing, but it requires us to determine the mass of the Taub-NUT-AdS black hole, which is related to the conformal dimension of the dual physical operator.

In Sect. 2, we consider Taub-NUT-AdS solutions in Einstein gravity plus a cosmological constant. We generalised the closed Komar form of pure Einstein gravity, and then find that the method works for Einstein gravity with a cosmological constant. All the thermodynamic variables are derived and the first law is checked to be satisfied. In Sect. 3, we go a step further to consider charged Taub-NUT-AdS spacetime in Einstein–Maxwell theory with a cosmological constant, and not surprisingly, we find the method works, too. We conclude our results in Sect. 4.

2 Taub-NUT-AdS

We consider Taub-NUT solution in Einstein gravity theory with cosmological constant in four dimensional spacetime, the Lagrangian is simple

$$\mathcal{L} = \sqrt{g}(R - 2\Lambda), \tag{1}$$

and the Taub-NUT-AdS solution is given by [7,35]

$$\begin{aligned} ds^2_{(4)} &= -f(dt + 2n\cos\theta d\phi)^2 \\ &\quad + \frac{dr^2}{f} + (r^2 + n^2)(d\theta^2 + \sin\theta^2 d\phi^2), \end{aligned} \tag{2}$$

with

$$f = \frac{r^2 - 2mr - n^2}{r^2 + n^2} - \frac{3n^4 - 6n^2r^2 - r^4}{l^2(r^2 + n^2)}. \tag{3}$$

Hereafter, we set $\Lambda = -\frac{3}{l^2}$. The solution has two integration constants, mass parameter m and NUT parameter n, l is a constant which is related to the cosmological constant Λ. When the NUT parameter n vanishes, the solution goes back to Schwarzschild-AdS black hole.

The Taub-NUT-AdS black hole has an event horizon at r_+, which is the largest root of $f(r) = 0$. Note that such a positive root exists for arbitrarily negative m, when the parameter n is non-vanishing. The temperature and entropy can be obtained through the standard method, given by

$$T = \frac{1}{4\pi r_+}\left(1 + \frac{3(r_+^2 + n^2)}{l^2}\right), \quad S = \pi(r_+^2 + n^2). \tag{4}$$

However, it is far trickier to determine the rest of the thermodynamic variables. The metric is not asymptotic to the AdS spacetime owing to the NUT parameter n. Therefore, we cannot use the ADM procedure to determine the mass. The Wald formalism provides a first-order relation among the integration constants and the horizon data, but by itself without additional input, it cannot parse the parameters and define definitely the thermodynamic valuables.

We therefore adopt the Komar integration method associated with a Killing vector to derive the conserved quantities. For example, for the time like Killing vector $\xi = \partial_t$, we can define Komar 2-form $\mathbf{Q}[\partial_t] = *d\xi$. For Einstein pure gravity, $\mathbf{Q}[\partial_t]$ is a close 2-form, satisfying $d\mathbf{Q}[\partial_t] = 0$. It therefore contains a "conserved charge" that can be read off by integrating over any closed surface encircling the source, analogous to read off the electric charge from the Gauss theorem using the Stoke's theorem, namely namely

$$M \sim \int_\Sigma \mathbf{Q}[\partial_t], \tag{5}$$

When matter is involved, we no longer have simply $d\mathbf{Q}[\partial_t] = 0$. (Note the similarity of this to vacuum Maxwell equation $d{*}dA = 0$ and the corresponding charge formula $Q \sim \int_\Sigma *dA$.) The closure of the Komar 2-form requires us to generalize to include the matter contribution. In many examples of Einstein gravity, with minimally coupled matter, the matter's back reaction to the metric falls off faster than the graviton mode, we can therefore still use the unmodified Komar 2-form $\mathbf{Q}[\partial_t] = *d\xi$ to calculate the mass, as long as we push the surface Σ in (5) to asymptotic infinity, i.e. $r \to \infty$. This provide the most convenient way to calculate the mass in for asymptotic Minkowsky geometry.

When the spacetime involves Misner geometry, the application of the Stoke's theorem becomes subtler, since the boundaries of regular spacetime now include the infinitesimal tubes that encompass the Misner string that extend from north (south) pole to asymptotic infinity, in addition to the spherically boundary at $r \to \infty$. Thus Σ in (5) consists not only the asymptotic 2-sphere, but also the tubes. Since the tubes are non local, extending from the north (south) poles to asymptotic infinity, the matter contribution that ensure the closure, $d\mathbf{Q}[\partial_t] = 0$, cannot be ignore. Thus when we apply the Komar integration over Taub-NUT geometry, we have to generalize the usual Komar integration: (1) Contribution from the matter fields, including the cosmological constant, should be carefully included. (2) The contribution from the infinitesimal tubes associated with the Misner string has to be included. The technique detail was developed in [34], and we shall not repeat here.

One important property of the Taub-NUT geometry is that there is a symmetry or correspondence between the radial and latitude coordinates $(r, u = \cos\theta)$. It was shown that the mass from (5) can be understood that for appropriate gauge choice, the Komar 2-form can be written as $\mathbb{Q}[\partial_t] \sim M\Sigma_{(2)}$, where $\Sigma_{(2)}$ is the radially-independent surface 2-form that depends only on θ and longitudinal ϕ. This leads to consider an alternative gauge choice such that $\mathbb{Q}[\partial_t] \sim Q_N \widetilde{\Sigma}_{(2)}$, where $\widetilde{\Sigma}_{(2)}$ is θ independent surface that depends only on ϕ and r, running from horizon to asymptotic infinity. This naturally leads to an additional charge, called NUT charge Q_N by

$$Q_N \sim \int_{\widetilde{\Sigma}} \mathbf{Q}[\partial_t], \tag{6}$$

where $\widetilde{\Sigma}$ can be any latitudinal θ-independent surface. Thus both mass and NUT charge (M, Q) are derived from the same closed Komar 2-form $\mathbf{Q}[\xi]$ associated with the timelike Killing vector $\xi = \partial_t$. Note that in the case of black holes with no Misner strings, one simply gets $Q_N = 0$.

An additional justification for this approach was by the comparison between the null Killing vector on the horizon and degenerate Killing vectors at the north and south poles in Kerr-Taub-NUT metrics. They are

$$\xi = \partial_t + \Omega\partial_\phi, \quad l_\pm = \partial_\phi \mp 2n\partial_t. \tag{7}$$

This, together with the symmetry of the coordinates $r \leftrightarrow \cos\theta$ in the Kerr-Taub-NUT metric strongly suggests the correspondence:

$$t \leftrightarrow \phi, \quad \Omega \leftrightarrow n. \tag{8}$$

In other words, the NUT charge is analogous to the angular momentum, but derived from timelike Killing vector. Specifically, if we view the angular momentum J as the radially-conserved quantity from the generalized Komar integration of $\mathbf{Q}[\partial_\phi]$ at constant t-slice, then Q_N is the θ-conserved quantity from $\mathbf{Q}[\partial_t]$ at constant ϕ-slice. It is important to note that, *a priori*, the thermodynamic quantities we derived through Komar integration and the degenerate Killing vectors may not necessarily satisfy the first law, but it turns out that they do.

We now apply the above method to the Taub-NUT-AdS spacetime, the key task is to find the closed form. Owing the cosmological constant Λ, for $\xi = \partial_t$, we have

$$-d{*}d\xi = -2\Lambda{*}\xi. \tag{9}$$

We can see $d{*}d\xi$ is proportional to cosmological constant, and vanishes for pure Einstein gravity where $\Lambda = 0$. Thus when there are cosmological constant, the closed 2-form $\mathbf{Q}[\xi]$ is not simply $-{*}d\xi$, but needs to be modified. To find this modification, we note

$$-{*}d\xi = V(r)\Omega_{(2)} + U(r)dr \wedge (dt + 2n\cos\theta d\phi),$$
$$\Omega_{(2)} = \sin\theta d\theta \wedge d\phi, \tag{10}$$

with U, V are

$$V = (r^2 + n^2)f', \quad U = \frac{2nf}{r^2 + n^2}. \tag{11}$$

Thus

$$\begin{aligned} -d * d\xi &= (V' + 2nU)\sin\theta dr \wedge d\theta \wedge d\phi \\ &= -2\Lambda\left(n^2 + r^2\right)\sin\theta dr \wedge d\theta \wedge d\phi. \end{aligned} \tag{12}$$

Assuming there exists such a 2-form ω, whose Hodge dual takes the similar form as ${*}d\xi$, the symmetry dictates that the modification must take the form

$${*}\omega = \tilde{V}(r)\Omega_{(2)} + \tilde{U}(r)dr \wedge (dt + 2n\cos\theta d\phi), \tag{13}$$

so that the combination of ${*}(-d\xi + \omega)$ is closed, namely

$$d{*}(-d\xi + \omega) = 0. \tag{14}$$

Therefore, we must have

$$(\tilde{V} + V)' + 2n(\tilde{U} + U) = \tilde{V}' + 2n\tilde{U} - 2\Lambda(r^2 + n^2) = 0. \tag{15}$$

We can now define the generalized Komar 2-form associated with $\xi = \partial_t$:

$$\begin{aligned} \mathbf{Q}[\partial_t] &= {*}(-d\xi + \omega) \\ &= (V + \tilde{V})\Omega_{(2)} + (U + \tilde{U})dr \wedge (dt + 2n\cos\theta d\phi). \end{aligned} \tag{16}$$

It is worth pointing out that the $(\tilde{U}, \tilde{V})$ can not be uniquely fixed through the above condition. It is easily seen that a 2-form λ, which satisfies $d{*}\lambda = 0$, can be added into $\mathbf{Q}$, without changing the closure condition. As is shown in [21,37], a proper gauge choice is required to produce the consistent mass.

To read off the conserved quantities from the closed Komar 2-form $\mathbf{Q}[\partial_t]$, we note that $\mathbf{Q}[\partial_t] = \mathbf{Q}_1 + \mathbf{Q}_2$, namely

$$\mathbf{Q}_1 = (U + \tilde{U}) dr \wedge dt,$$
$$\mathbf{Q}_2 = (V + \tilde{V})\Omega_{(2)} + 2n(U + \tilde{U}) dr \wedge \cos\theta d\phi, \tag{17}$$

with $d\mathbf{Q}_1 = 0 = d\mathbf{Q}_2$. In other words, the constant t-slice of $\mathbf{Q}[\partial_t]$ gives rise to $\mathbf{Q}_2$ and constant ϕ-slice gives rise to $\mathbf{Q}_1$. It is now clear that $\mathbf{Q}_1$ part gives rise to the θ-conserved NUT charge (6). By choosing an appropriate gauge, $\mathbf{Q}_2$ part gives the r-independent mass (5), since we have

$$\mathbf{Q}_2 = \Big(V + \tilde{V} - 2n \int_{r_+}^{r} (U + \tilde{U})\Big)\Omega_2 + d\lambda, \tag{18}$$

for appropriate λ. The closure of the 2-form ensures that the coefficient of Ω_2 is independent of the radial variable r, giving rise to an r-conserved quantity.

In the above integration over the 2-form, the integration over Killing directions of (ϕ, t) is trivial. The nontrivial part is the integration over the 1-form associated with the coordinate (r, θ).

Now, back to the specific Taub-NUT-AdS case, the integration of U is divergent, thus, our strategy for the choice of $\tilde{U}$ is that the divergent term of U can be cancelled by $\tilde{U}$ and furthermore, no additional contributions from $\tilde{U}$ emerges. This leads to

$$\tilde{U} = (-2n\frac{f(r)}{r})'. \tag{19}$$

Note that the function $f(r)/r$ vanishes on the horizon. The $\tilde{V}$ is thus given by

$$\tilde{V} = 4n^2\frac{f}{r} - \frac{6}{l^2}(\frac{r^3}{3} + n^2 r). \tag{20}$$

The NUT charge, as the θ-conserved quantity, can be calculated from the constant ϕ-slice. We find

$$Q_N = \int_{r_+}^{\infty} (U + \tilde{U}) dr = \frac{n}{r_+}(1 + \frac{3(n^2 - r_+^2)}{l^2}). \tag{21}$$

Note that for the metric to be absent from Misner strings, one needs to impose a periodic condition and the integration $\int dt$ gives a finite value. For Taub-NUT-AdS black holes, the time is taken to be a real number; therefore, Q_N can be interpreted as a growth rate quantity.

For constant t slice, the Komar 2-form reduces to $\mathbf{Q}_2$, which yields the r-conserved quantity, namely the total mass of the Taub-NUT-AdS spacetime

$$\begin{aligned} M &= \frac{\int d\phi}{8\pi}\Big(\int_0^{\pi} (V(r) + \tilde{V}(r)) \sin\theta d\theta \\ &\quad - \int_{r_+}^{r} 2n\cos\theta(U(r') + \tilde{U}(r'))\Big|_{\theta=0}^{\theta=\pi} dr'\Big) \\ &= m + \frac{n^2}{r_+}(1 + \frac{3(n^2 - r_+^2)}{l^2}). \end{aligned} \tag{22}$$

We therefore have the mass and NUT-charge relation:

$$M = m + 2\Phi_N Q_N. \tag{23}$$

When letting $n \to 0$, the mass returns to the result of Schwarzschild mass, which gives a strong support for the correctness of our method.

In the literature, some people advocate that the mass is $M = m$ via the conformal completion or holographic tensor, but in this situation the mass can be negative due to the mass parameter m can be arbitrarily negative while still maintaining the event horizon. At the first looking, the mass we derived through generalised Komar integration (22) can be negative for large r_+ due to the $-r_+^2$ term. However, if we eliminate the m by using $f(r_+) = 0$, we get

$$M = \frac{3n^4 + r_+^4}{2l^2 r_+} + \frac{n^2 + r_+^2}{2r_+}, \tag{24}$$

which is obviously nonnegative.

In the AdS case, the form of the Killing vector on the south and north pole are not changed, thus the NUT potential remains the same as that of Einstein gravity

$$\Phi_N = \frac{n}{2}. \tag{25}$$

At this stage, the first law can be checked and turns out to be satisfied

$$\delta M = T\delta S + \Phi_N \delta Q_N. \tag{26}$$

The first law can also be generalised by taking the cosmological constant Λ as a thermodynamic quantity $P = \frac{3}{8\pi l^2}$, and the corresponding thermodynamic volume is

$$V = \frac{4\pi}{3} r_+^3 (1 + \frac{3n^2}{r_+^2}). \tag{27}$$

Interestingly, the thermodynamical volume is the same as that in [36].

The first law is then given by

$$\delta M = T\delta S + \Phi_N \delta Q_N + V\delta P. \tag{28}$$

As shown in [34], the Smarr relation can be obtained by using $d\mathbf{Q} = 0$

$$\frac{1}{8\pi}\int_\Sigma \mathbf{Q}[\xi] = \frac{1}{4}\big[(V+\tilde{V})|_{r_+}^\infty + \int_{r_+}^\infty 2n(U(r') + \tilde{U}(r'))dr'\big] \\ = \frac{1}{4}\big[V|_{r_+}^\infty + \int_{r_+}^\infty (2nU(r') + 2\Lambda(r'^2+n^2))dr'\big] = 0. \tag{29}$$

The second line is obtained by using the integrability condition (15), and it is worth pointing out that it is not necessary to make a definite choice for $(\tilde{U}, \tilde{V})$ to get the Smarr formula from $d\mathbf{Q} = 0$. Explicitly, in the infinity we get

$$V + \int 2n(U(r) + 2\Lambda(r^2+n^2))dr)|_{r\to\infty} = 2m, \tag{30}$$

whilst, on the horizon

$$(V + \int 2n(U(r') + 2\Lambda(r'^2+n^2))dr')|_{r_+} \\ = 4TS - 4PV - 4\Phi_N Q_N. \tag{31}$$

Then putting them together, we get the exact form of Smarr relation of Taub-NUT-AdS spacetime

$$M = 2(TS - PV), \tag{32}$$

the NUT charge and potential don't contribute directly to the Smarr formula as the same as that in Einstein gravity. The Gibbs free energy is

$$F = M - TS - \Phi_N Q_N = \frac{m}{2} - \frac{1}{2l^2}(3n^2 r_+ + r_+^3), \tag{33}$$

which is consistent with Euclidean action [36].

So far, we consider the Taub-NUT solution with symmetrically distributed Misner strings. Our method can easily be generalised to the asymmetric case. The solution with asymmetric Misner strings can be obtained through a linear coordinate transformation

$$t \to t - 2n\alpha\phi, \quad \phi \to \phi, \tag{34}$$

where the parameter α is a real constant. Then the Killing vectors at the north and south pole change to

$$l_\pm = \partial_\phi \mp 4\Phi_N^\pm \partial_t, \quad \Phi_n^\pm = \frac{1}{2}n(1\pm\alpha). \tag{35}$$

And thus the distribution of Misner strings affect the NUT potential, too. When $\alpha = 0$, the Misner strings are symmetric and the north and south poles are in the equal foot. When $\alpha = 1$, the Misner string only emerges at the south pole, whilst $\alpha = -1$ disappears at the south pole.

Though, the NUT potential has a direct contribution from α, the rest thermodynamical quantities, mass, NUT charge, temperature and entropy are not modified by α. The first law becomes

$$\delta M = T\delta S + \Phi_N^+ \delta Q_N^+ + \Phi_N^- \delta Q_N^-, \tag{36}$$

where the NUT charge at south and north poles are the same

$$Q_N^\pm = \frac{n}{2r_+}(1 + \frac{3(n^2 - r_+^2)}{l^2}). \tag{37}$$

Since $Q_N^\pm$ are the same, the parameter α in the last two terms $\Phi_N^+\delta Q_N^+ + \Phi_N^-\delta Q_N^-$ of the first law will cancel out, and α will not appear in the first law. Note that there is a factor 2 in the denominator of NUT charges. When $\alpha = 0$, the south and north poles have the same NUT potential $\Phi_N^\pm = \Phi_N$, thus the south and north pole NUT charges can be recognized as the same class, and they can be summed as a whole NUT charge $Q_N^+ + Q_N^- = Q_N$, then we recover results of the symmetric distributed Misner strings.

3 Dyonic Taub-NUT-AdS

In this section we generalise the method to charged case in theory of Einstein-Maxwell gravity with a cosmological constant

$$L = \sqrt{g}(R - 2\Lambda - F^2), \tag{38}$$

where $F = dA$. The solution is

$$ds_{(4)}^2 = -f(dt + 2n\cos\theta d\phi)^2 + \frac{dr^2}{f} + (r^2+n^2)(d\theta^2 + \sin\theta d\phi^2), \tag{39}$$

with

$$f = \frac{r^2 - 2mr - n^2 + e^2 + g^2}{r^2+n^2} - \frac{3n^4 - 6n^2r^2 - r^4}{l^2(r^2+n^2)}, \tag{40}$$

and the Maxwell field is

$$A = -g\cos\theta d\phi + \frac{(gn + er)(dt + 2n\cos\theta d\phi)}{r^2+n^2}. \tag{41}$$

The corresponding dual filed is $\tilde{F} = dB = *F$ with gauge potential B given by

$$B = e\cos\theta d\phi - \frac{(en - gr)(dt + 2n\cos\theta d\phi)}{r^2+n^2}. \tag{42}$$

Beyond the mass and NUT parameters (m, n), there exist two additional integration parameters (e, g) which correspond to the electric and magnetic charges. The spacetime has an event horizon at $r = r_+$, with r_+ as the largest root of $f(r) = 0$. There exist two other Killing horizons, too, the associated Killing vectors are the same as the neutral case (7). Thus, we have the same NUT potential

$$\Phi_N = \frac{n}{2}. \tag{43}$$

The temperature and entropy of the charged NUTy spacetime can be obtained through the standard method as before, and they are given by

$$T = \frac{1}{4\pi r_+}(1+\frac{3(r_+^2+n^2)}{l^2}-\frac{e^2+g^2}{r_+^2+n^2}), \quad S = \pi(r_+^2+n^2). \tag{44}$$

Turn to the thermodynamics of the charged NUTy spacetime, we first need to obtain the closed 2-form $\mathbf{Q}$. Without cosmological constant, the combination in Einstein-Maxwell theory,

$$-*d\xi - *FA_\lambda A^\lambda - *\tilde{F}B_\lambda B^\lambda \tag{45}$$

is closed [34], where ξ is a Killing vector. Again, due to the inclusion of the cosmological constant, the above combination is not closed anymore, and thus an additional term is required as before. Fortunately, the recipe is the same as the case in the previous section, too. We find that

$$\mathbf{Q}[\xi] = -*d\xi - *FA_\lambda A^\lambda - *\tilde{F}B_\lambda B^\lambda + *\omega \tag{46}$$

is closed, with

$$*\omega = \tilde{V}(r)\Omega_{(2)} + \tilde{U}(r)dr\wedge(dt+2n\cos\theta d\phi), \tag{47}$$

and $\tilde{U}$,$\tilde{V}$ are under constraint

$$\tilde{V}' + 2n\tilde{U} - 2\Lambda(r^2+n^2) = 0. \tag{48}$$

It is worth mentioning that the above expression is derived by using equation of motions and the constraint $(\tilde{U},\tilde{V})$ is the same as that of the previous section. Therefore, they are again given by (19) and (20), since the metric of the dyonic solution takes the same form as the neutral one.

With the generalised Komar 2-form $\mathbf{Q}$ of (46), we follow the same procedure and obtain the mass and NUT charge

$$\begin{aligned} M &= \frac{\int d\phi}{8\pi}\Big(\int_0^\pi (V(r,\theta)+\tilde{V}(r,\theta))d\theta \\ &\quad - \int_{r_+}^r 2n\cos\theta(U(r',\theta)+\tilde{U}(r',\theta))\Big|_{\theta=0}^{\theta=\pi}dr'\Big) \\ &= m + \frac{n^2}{r_+}(1-\frac{3(r_+^2-n^2)}{l^2}-\frac{e^2+g^2}{r_+^2+n^2}), \end{aligned}$$

$$Q_N = \frac{n}{r_+}(1-\frac{3(r^2-n^2)}{l^2}-\frac{e^2+g^2}{r_+^2+n^2}). \tag{49}$$

It can be verified that the relation (23) between the mass and NUT charge continues to holds.

The equation of motion of the 2-form Maxwell field strength and its Bianchi identity are $d*F = 0$ and $dF = 0$ respectively. We can therefore extract the conserved quantities exactly the same one as we have done with the generalized Komar 2-form $\mathbf{Q}[\xi]$. This leads to the standard electromagnetic charges, as well as NUT-induced charges, following the prescription of [34]. We find that the electromagnetic charges are

$$\begin{aligned} Q_e &= -\frac{1}{2}\Big(\int_0^\pi \tilde{F}_{\theta\phi}(r,\theta')d\theta' - \int_{r_+}^r \tilde{F}_{r\phi}(r',\theta)\big|_{\theta=0}^{\theta=\pi}dr'\Big) \\ &= -\frac{1}{2}B_\phi(r_+)\big|_{\theta=0}^{\theta=\pi} \\ &= e + 2n\frac{gr_+-en}{r_+^2+n^2}, \\ Q_g &= \frac{1}{2}\Big(\int_0^\pi F_{\theta\phi}(r,\theta')d\theta' - \int_{r_+}^r F_{r\phi}(r',\theta)\big|_{\theta=0}^{\theta=\pi}dr'\Big) \\ &= \frac{1}{2}A_\phi(r_+)\big|_{\theta=0}^{\theta=\pi} \\ &= g - 2n\frac{er_++gn}{r_+^2+n^2}. \end{aligned} \tag{50}$$

Their electric and magnetic potential are defined by

$$\Phi_e = \xi^\mu A_\mu\big|_\infty^{r_+} = \frac{er_++gn}{r_+^2+n^2}, \quad \Phi_g = \xi^\mu B_\mu\big|_\infty^{r_+} = \frac{gr_+-en}{r_+^2+n^2}. \tag{51}$$

Similar to NUT charge, we can also define NUT-induced electric and magnetic charges through $d*F = 0$ and $dF = 0$ at the south and north poles

$$\begin{aligned} Q_{eN} &= \int_{r_+}^\infty \tilde{F}_{tr}dr = \frac{1}{2}B_t\big|_\infty^{r_+} = \frac{gr_+-en}{n^2+r_+^2}, \\ Q_{gN} &= \int_{r_+}^\infty F_{tr}dr = \frac{1}{2}A_t(r_+)\big|_\infty^{r_+} = \frac{er_++gn}{n^2+r_+^2}, \end{aligned} \tag{52}$$

and the corresponding potential are defined through the analogous method of defining the electric and magnetic potentials

$$\begin{aligned} \Phi_{eN} &= \frac{1}{4}l^\mu(A_\mu(\theta=0)+A_\mu(\theta=\pi))\big|_\infty^{r_+} = -\frac{n(er_++gn)}{n^2+r_+^2}, \\ \Phi_{gN} &= -\frac{1}{4}l^\mu(B_\mu(\theta=0)+B_\mu(\theta=\pi))\big|_\infty^{r_+} = \frac{n(gr_+-en)}{n^2+r_+^2}. \end{aligned} \tag{53}$$

At this stage, we derived all the thermodynamical quantities and they are summarized as follow.

$$T = \frac{1}{4\pi r_+}(1+\frac{3(r_+^2+n^2)}{l^2}-\frac{e^2+g^2}{r_+^2+n^2}), \quad S = \pi(r_+^2+n^2),$$

$$
\begin{aligned}
M &= m + \frac{n^2}{r_+}(1 - \frac{3(r_+^2 - n^2)}{l^2} - \frac{e^2+g^2}{r_+^2+n^2}),\\
Q_N &= \frac{n}{r_+}(1 - \frac{3(r_+^2 - n^2)}{l^2} - \frac{e^2+g^2}{r_+^2+n^2}), \quad \Phi_N = \frac{n}{2},\\
Q_{eN} &= \frac{gr_+ - en}{r_+^2+n^2}, \quad \Phi_{eN} = -\frac{n(er_+ + gn)}{r_+^2+n^2},\\
Q_{gN} &= \frac{er_+ + gn}{r_+^2+n^2}, \quad \Phi_{gN} = \frac{n(gr_+ - en)}{r_+^2+n^2},\\
Q_e &= e + 2nQ_{eN}, \quad \Phi_e = \frac{(er_+ + gn)}{r_+^2+n^2},\\
Q_g &= g - 2nQ_{gN}, \quad \Phi_g = \frac{(gr_+ - en)}{r_+^2+n^2}.
\end{aligned}
\tag{54}
$$

When setting $l \to \infty$, these quantities turn back to the result of Einstein–Maxwell case without cosmological constant as expected.

It is worth pointing out that all the thermodynamic quantities were obtained independently without making a referencing to the first law. Therefore, a priori, there is no obvious reason that they would satisfy the first law. However, it can be verified that the first law is nevertheless satisfied, namely

$$
\begin{aligned}
&\delta M = T\delta S + \Phi_N \delta Q_N + \Phi_e \delta Q_e + \Phi_g \delta Q_g\\
&+\Phi_{eN}\delta Q_{eN} + \Phi_{gN}\delta Q_{gN}.
\end{aligned}
\tag{55}
$$

Again, when we treat the cosmological constant as a thermodynamical variable, the corresponding thermodynamical volume can also be derived,

$$
P = \frac{3}{8\pi l^2}, \quad V = \frac{4\pi}{3} r_+^3 (1 + \frac{3n^2}{r^2}).
\tag{56}
$$

And the generalised first law is

$$
\begin{aligned}
&\delta M = T\delta S + V\delta P + \Phi_N \delta Q_N + \Phi_e \delta Q_e\\
&+\Phi_g \delta Q_g + \Phi_{eN}\delta Q_{eN} + \Phi_{gN}\delta Q_{gN}.
\end{aligned}
\tag{57}
$$

From $d\mathbf{Q} = 0$, we can derive the Smarr relation

$$
M = 2(TS - PV) + \Phi_e Q_e + \Phi_g Q_g.
\tag{58}
$$

The Free energy can be evaluated through the Euclidean action $G = I/\beta$, with β is the inverse of the temperature, $\beta = 1/T$. The full action is given by

$$
\begin{aligned}
I &= \frac{1}{16\pi}\int_M d^4x\sqrt{g}\big(R + \frac{6}{l^2} - F^2\big)\\
&+\frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}\big(\mathcal{K} - \frac{2}{l} - \frac{l}{2}\mathcal{R}(h)\big),
\end{aligned}
\tag{59}
$$

where, h is determinant of the induced metric, $\mathcal{K}$ is the extrinsic curvature and $\mathcal{R}(h)$ is the boundary Ricci scalar. Substituting the solution into the whole action, we can obtain the free energy

$$
\begin{aligned}
G &= \frac{m}{2} - \frac{r_+(3n^2 + r_+^2)}{2l^2}\\
&-\frac{r_+\big((e^2-g^2)(r_+^2-n^2) + 4egnr_+\big)}{2(r_+^2+n^2)^2}.
\end{aligned}
\tag{60}
$$

This result is consistent with our newly derived thermodynamical quantities

$$
F = M - TS - \Phi_N Q_N - \Phi_e Q_e - \Phi_{eN} Q_{eN}.
\tag{61}
$$

Finally, it can be easily checked that the first law has electromagnetic duality under

$$
e \to g, \quad g \to -e.
\tag{62}
$$

As in the previous section, the method can be easily generalized to asymmetric distributed Misner strings, and the results are similar.

4 Conclusions

Though the Taub-NUT-AdS solution exists for years, the thermodynamical properties has not been totally understood. Many works has been done on the first law of Taub-NUT spacetimes, but there still lack in uniquely deriving the NUT charge. Recently, a systematic way of defining and calculating the NUT charge and the total mass of the Taub-NUT spacetime has been proposed. In this paper, we apply this method to Taub-NUT-AdS spacetime in Einstein gravity plus cosmological constants. A key ingredient of this method is to get a closed 2-form $\mathbf{Q}$. For, pure Einstein gravity, the closed 2-form is just the derivative of Killing vector $- * d\xi$, however, it is no longer closed for Einstein gravity plus cosmological constant. To solve this problem, we construct a generalised closed 2-form by introducing an additional term. Then starting with this generalised 2-form, we obtain the NUT charge and total mass of Taub-NUT-AdS spacetime. From our approach to calculate the NUT charge Q_N, we would like to think that in the (r, θ, t, ϕ) coordinates, the Misner strings carry the NUT charges, distributed along the Minser strings at north and south poles from r_+ to asymptotic infinity. They also contribute to the mass by $M = m + 2\Phi_N Q_N$, such that it is non-negative even though the quantity m can be arbitrarily negative. Note that the event horizon $r_+ > 0$ can exist for arbitrarily negative m, associated with the condensation of the massless graviton, it is the Misner string's additional contribution to the energy that makes the total mass or energy positive. Together with the entropy and temperature which can be derived through standard method, the first law is checked to be satisfied. The Smarr relation can be obtained through the closure of the generalised 2-form as usual, and it is consistent with the first law. All these indicate that our global analysis that leads to the independent derivation of all the thermodynamic variables are correct.

Then, we turn to the Einstein–Maxwell theory with a cosmological constant. The same problem, that the usual closed 2-form in Einstein–Maxwell without cosmological constant is no longer closed, emerges. Fortunately, the recipe for this problem is similar, too. Especially, the expression of the additional term needed to construct the new generalised 2-form has the same expression, when written in terms of metric functions. With this generalised closed 2-form, we derive the NUT charge and total mass of the spacetime. Since Maxwell's equation of motion and Bianchi identity, $*F_{(2)}$ and $F_{(2)}$ are closed, too. We can calculate electric and magnetic charges through the 2-forms, analogous to the NUT charge, the NUT-induced charges can also be derived. Finally, we present all the thermodynamic quantities, the first law and Smarr relation of the spacetime are indeed satisfied. We calculate the free energy of the system by using the derived thermodynamic variables and the results are consistent with that of Euclidean action.

We mainly studied the thermodynamics of the Taub-NUT-AdS spacetimes, it will have fruitful applications in holography, since although the asymptotic region is locally AdS, we expect that the AdS/CFT correspondence still applies. As we mentioned in the introduction, the mass of the bulk spacetime is dual to conformal dimension of the boundary operator. To treat m as mass, which can be arbitrarily negative, will certainly not be satisfactory. Our properly derived mass, which is positive definite, resolve this issue. In the holographic approach [38,39] the Lloyd's complexity bound, the mass of the bulk spacetime gives this upper bound, therefore, it must be positive. Our proper definition of NUT charge is the first step towards understanding its physical dual in the corresponding CFT.

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