Quantization of Weyl invariant unimodular gravity with antisymmetric ghost fields

David García-López[a], Carmelo P. Martin[b]

Departamento de Física Teórica and IPARCOS, Facultad de Ciencias Físicas, Universidad Complutense de Madrid (UCM), 28040 Madrid, Spain

Abstract The enforcement of the unimodularity condition in a gravity theory by means of a Lagrange multiplier leads, in general, to inconsistencies upon quantization. This is so, in particular, when the classic linear splitting of the metric between the background and quantum fields is used. To avoid the need of introducing such a Lagrange multiplier while using the classic linear splitting, we carry out the quantization of unimodular gravity with extra Weyl symmetry by using Becchi–Rouet–Stora–Tyutin (BRST) techniques. Here, two gauge symmetries are to be gauge-fixed: transverse diffeomorphisms and Weyl transformations. We perform the gauge-fixing of the transverse diffeomorphism invariance by using BRST transformations that involve antisymmetric ghost fields. We show that these BRST transformations are compatible with the BRST transformations needed to gauge-fix the Weyl symmetry, so that they can be combined in a set of transformations generated by a single BRST operator. Newton's law of gravitation is derived within the BRST formalism we put forward as well as the Slavnov–Taylor equation.

1 Introduction

Unimodular gravity – see Refs. [1] and [2], for introductions – solves [3–7] a part of the so-called Cosmological Constant problem, for in that gravity theory the vacuum energy does not gravitate. Indeed, all the field configurations in unimodular gravity have got determinant equal to -1.

There exists sundry approaches to the quantization of unimodular gravity, e.g., Refs. [8–16]. It is not known whether they give rise to the same quantum theory on a Minkowski background, for they involve different sets of ghosts. This is

an open problem, as it is their equivalence to quantum General Relativity when the Cosmological Constant is not set to zero.

It has been shown recently – see Ref. [17] – that the unimodularity condition cannot be implemented in the path integral by using a Lagrange multiplier unless the unimodularity condition be equivalent to imposing a linear constraint on the graviton field. This fact precludes the use of the classic linear splitting of the gravity field into background metric and graviton field if the unimodularity condition is to be enforced by using a Lagrange multiplier. As also shown in Ref. [17] the problem goes away if the so-called exponential splitting [18] is used. We think that it is quite a drawback to have a formalism where the classic linear splitting – which is standardly used in General Relativity – cannot be employed. This is so even sidestepping the issue that even in General Relativity the exponential parametrization may not yield the same quantum theory as the linear splitting: the exponential parametrization is not a field redefinition of the linear splitting – see Ref. [19] and Refs. therein.

In Refs. [9,11,14] and [16] the unimodularity condition is imposed by using a Lagrange multiplier and thus those formalisms do not make sense for the linear splitting of the metric. In Refs. [8,12,13] and [15] the exponential parametrization is implemented already at classical level and prior to quantization. The BRST formalism put forward in Ref. [10] admits any kind of parametrization in terms of background and quantum fields – the linear and exponential parametrizations, in particular – for no Lagrange multiplier is introduced. Indeed, in Ref. [10], building on Refs. [20] and [21], the unimodularity condition is solved, prior to quantization, by expressing the unimodular metric, $\hat{g}_{\mu\nu}$, in terms of an unconstrained tensor field $g_{\mu\nu}$ as follows

$$\hat{g}_{\mu\nu}(x) = \frac{g_{\mu\nu}(x)}{|g|^{1/D}(x)}. \tag{1.1}$$

[a] e-mail: davgar20@ucm.es

[b] e-mail: carmelop@fis.ucm.es (corresponding author)

D is the space-time dimension. Several properties of the formalism set up in Ref. [10] have been analyzed leading to sound results – see Refs. [22,23] and [24], for recent publications. Let us recall that because of Eq. (1.1) the classical action of unimodular gravity has two gauge symmetries: transverse diffeomorphisms and Weyl transformations of $g_{\mu\nu}$. This is why the gravity theory in question is called Weyl invariant unimodular gravity.

As we have pointed out above, the BRST formalism put forward in Ref. [16] breaksdown at the quantum level if the classic linear splitting of the gravity field is employed. This is the parametrization originally considered in Ref. [16]. Of course, the inconsistency disappears if the exponential parametrization is used. The question arises as to whether this formalism can be modified so that the classic linear splitting can be utilized consistently.

The purpose of this paper is to show that, by using Eq. (1.1), the framework of Ref. [16] can be combined with part of the formalism of Ref. [10] so that the linear splitting of the gravity field works at the quantum level: no Lagrange multiplier is needed here to enforce the unimodularity condition. That this combination works is non trivial since the implementation of Eq. (1.1) introduces an additional gauge symmetry – a Weyl symmetry – in the theory. This gauge symmetry leads to additional BRST transformations which might clash with BRST transformations arising from the transverse gauge transformations as defined in Ref. [16]. Let us point out that in the unimodular gravity with Weyl invariance formalism -see [2] and references therein – the gravity field entering the path integral measure is $g_{\mu\nu}$ in (1.1), not the unimodular metric $\hat{g}_{\mu\nu}(x)$. Hence, the linear splitting must be applied to $g_{\mu\nu}$, not to $\hat{g}_{\mu\nu}(x)$.

The layout of this paper is as follows. In Sect. 2 we discuss the Becchi–Rouet–Stora–Tyutin (BRST) quantization of our unimodular theory. This second section contains several subsections since our unimodular theory has two gauge symmetries, namely, transverse diffeomorphisms and Weyl transformations. In Sect. 2.1, we carry out the gauge-fixing of the invariance under transverse diffeomorhisms by adapting to our case the BRST formalism of Ref. [16]. Section 2.2 is devoted to the analysis of the gauge-fixing of the Weyl symmetry by using BRST methods. In this latter subsection, we show the compatibility of the BRST transformations associated to transverse diffeomorphism and the BRST transformations coming from the Weyl transformations. We prove that the two BRST operators, each generating one of those two BRST transformations, can be combined into a single BRST operator. This single operator gives rise to the full BRST invariance of the theory. In Sect. 2.3, we give the full BRST invariant action of the theory. In Sect. 2, we compute the free graviton propagator and show that it propagates physical free gravitons into physical free gravitons as befits a quantum theory of gravitons. We make a consistency check of the BRST formalism we have put forward by obtaining, in Sect. 4, Newton's potential. In Sect. 5, we derive the Slavnov–Taylor equation, and the corresponding linearized Slavnov–Taylor operator, for cohomological techniques will be needed to obtain sensible renormalized radiative corrections: as we shall discuss, dimensional regularization does not preserve the BRST symmetry of the theory.

2 BRST quantization

In 4 space-time dimensions, the classical action of the unimodular theory with Weyl invariance reads

$$S_{class} = 2 \int d^4x \; R\left[\hat{g}\right],\tag{2.1}$$

where $\hat{g}_{\mu\nu}$ is defined in terms of the unconstrained field $g_{\mu\nu}$ in Eq. (1.1), $g_{\mu\nu}$ being the dynamical field variable. $R\left[\hat{g}\right]$ above denotes the Ricci scalar for unimodular metric $\hat{g}_{\mu\nu}$.

The action in Eq. (2.1) has two gauge symmetries acting on $g_{\mu\nu}$: it is invariant under transverse diffeomorphisms and under Weyl transformations. The infinitesimal transverse diffeomorphisms[1] $x^\mu \to x^\mu + \xi_{\mathrm{T}}^\mu$, with $\partial_\mu \xi_{\mathrm{T}}^\mu = 0$, are defined as follows:

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \quad \delta g_{\mu\nu} = \nabla_\mu \xi_\nu^{\mathrm{T}} + \nabla_\nu \xi_\mu^{\mathrm{T}},\tag{2.2}$$

where $\xi_\mu^{\mathrm{T}} \equiv g_{\mu\nu}\xi_{\mathrm{T}}^\nu$ and ∇_μ is the covariant derivative for $g_{\mu\nu}$. The infinitesimal Weyl transformations read

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}, \quad \delta g_{\mu\nu} = 2\theta g_{\mu\nu},$$

θ being an infinitesimal scalar field. Notice that these transformations leave $\hat{g}_{\mu\nu}$, as defined in Eq. (1.1), invariant by construction.

Before going ahead, let us show that the transformation in (2.2) generates an infinitesimal transverse diffeomorphism of $\hat{g}_{\mu\nu}$:

$$\begin{aligned}\delta\hat{g}_{\mu\nu} &= \delta\left(\frac{g_{\mu\nu}}{(-g)^{1/4}}\right) = \frac{1}{(-g)^{1/4}}\left(\delta g_{\mu\nu} - \frac{1}{4g}g_{\mu\nu}\delta g\right)\\&= \frac{1}{(-g)^{1/4}}\left(\nabla_\mu\xi_\nu^{\mathrm{T}} + \nabla_\nu\xi_\mu^{\mathrm{T}} - \frac{1}{2}g_{\mu\nu}\nabla_\mu\xi_{\mathrm{T}}^\mu\right)\\&= \hat{\nabla}_\mu\hat{\xi}_\nu^{\mathrm{T}} + \hat{\nabla}_\nu\hat{\xi}_\mu^{\mathrm{T}},\end{aligned}\tag{2.3}$$

where $\hat{\xi}_\mu^{\mathrm{T}} = \hat{g}_{\mu\nu}\xi_{\mathrm{T}}^\nu$ and $\hat{\nabla}_\mu$ is the covariant derivative with regard to $\hat{g}_{\mu\nu}$. The previous variation of $\hat{g}_{\mu\nu}$ leaves its deter-

---

[1] Bear in mind that infinitesimal transverse diffeomorphims are the infinitesimal version of coordinate transformations with determinant equal to 1. Under these coordinate transformations the determinant, $g(x)$, of $g_{\mu\nu}$ is a scalar so that $\hat{g}_{\mu\nu}$ and $g_{\mu\nu}$ are tensors – making S_{class} an invariant – under such changes of coordinates.

minant invariant, since $\hat{\nabla}_\mu \xi_T^\mu = \partial_\mu \xi_T^\mu$ for a unimodular metric. Indeed

$$\delta \hat{g} = \hat{g} \hat{g}^{\mu\nu} \delta \hat{g}_{\mu\nu} = -\hat{g}^{\mu\nu} \left(\hat{\nabla}_\mu \hat{\xi}_\nu^T + \hat{\nabla}_\nu \hat{\xi}_\mu^T \right)$$
$$= -2\hat{\nabla}_\mu \xi_T^\mu = -2\partial_\mu \xi_T^\mu = 0. \quad (2.4)$$

Below we shall use

$$\delta g^{\mu\nu} = -\nabla^\mu \xi_T^\nu - \nabla^\nu \xi_T^\mu, \quad \delta \hat{g}^{\mu\nu} = -\hat{\nabla}^\mu \xi_T^\nu - \hat{\nabla}^\nu \xi_T^\mu, \quad (2.5)$$

which are equivalent to Eqs. (2.2) and (2.3), respectively

2.1 BRST formalism for the transverse diffeomorphisms

In this subsection we shall adapt the formalism put forward in Ref. [16] to case at hand, i.e., the metric, $\hat{g}_{\mu\nu}$, will be unimodular from the very beginning, its transverse diffeomorphisms being generated by the transverse diffeomorphisms of $g_{\mu\nu}$ – see Eqs. (2.2), (2.3) and (2.4) – and the transversality of ξ_T^μ is not with regard to the covariant derivative of $g_{\mu\nu}$, but with regard to the partial derivative. This entails a modification, though minor, of the formalism in Ref. [16], for transversality in Ref. [16] – see its Eq. (1.8) – is with regard to the covariant derivative of the metric; this metric being unimodular only after having solved for the Lagrange multiplier.

Although only a minor modification of the formalism in Ref. [16] is needed for our transverse diffeomorphisms, we shall give all the relevant details to make this subsection self-consistent and to set the notation.

Let $\mathfrak{B}_D$ denote the BRST operator with ghost number 1 and let us introduce the transverse ghost fields c_T^μ ($N_{\text{ghost}}(c) = 1$) and $\bar{c}_T^\nu$ ($N_{\text{ghost}}(\bar{c}) = -1$), where N_{ghost} stands for the ghost number. The action of $\mathfrak{B}_D$ on $g^{\mu\nu}$ reads

$$\mathfrak{B}_D g^{\mu\nu} = -\nabla^\mu c_T^\nu - \nabla^\nu c_T^\mu, \quad (2.6)$$

which in turn – see Eq. (2.3) – induces the following BRST transformation of $\hat{g}^{\mu\nu}$:

$$\mathfrak{B}_D \hat{g}^{\mu\nu} = -\hat{\nabla}^\mu c_T^\nu - \hat{\nabla}^\nu c_T^\mu. \quad (2.7)$$

Equations (2.6) and (2.7) are the BRST counterparts of the transformations in Eq. (2.5), respectively. Note that ∇_μ denotes the covariant derivative with regard to $g_{\mu\nu}$.

We should keep in mind that when a linear splitting of $g_{\mu\nu}$ into background, $\overline{g}_{\mu\nu}$, and quantum, $h_{\mu\nu}$, fields is carried out,

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu},$$

the BRST transformation in Eq. (2.6) boils down, by definition, to

$$\mathfrak{B}_D \overline{g}_{\mu\nu} = 0,$$
$$\mathfrak{B}_D h_{\mu\nu} = \mathfrak{B}_D g_{\mu\nu} = \nabla_\mu c_\nu^T + \nabla_\nu c_\mu^T$$
$$= \overline{\nabla}_\mu (\overline{g}_{\nu\rho} c_T^\rho) + \overline{\nabla}_\nu (\overline{g}_{\mu\rho} c_T^\rho) + c_T^\rho \overline{\nabla}_\rho h_{\mu\nu}$$
$$+ (\overline{\nabla}_\mu c_T^\rho) h_{\rho\nu} + (\overline{\nabla}_\nu c_T^\rho) h_{\rho\mu}. \quad (2.8)$$

$\overline{\nabla}_\mu$ is the covariant derivative with regard to $\overline{g}_{\mu\nu}$.

The action of the BRST operator on the ghost fields is defined to be

$$\mathfrak{B}_D c_T^\mu = \hat{\nabla}_\nu (c_T^\nu c_T^\mu), \quad (2.9)$$
$$\mathfrak{B}_D \bar{c}_T^\mu = i b_T^\mu, \quad (2.10)$$
$$\mathfrak{B}_D b_T^\mu = 0. \quad (2.11)$$

b_T^μ is the transverse *Nakanishi-Lautrup* auxiliary field. Let us recall that

$$\hat{\nabla}_\mu c_T^\mu = \partial_\mu c_T^\mu = 0, \quad \hat{\nabla}_\mu \bar{c}_T^\mu = \partial_\mu \bar{c}_T^\mu = 0 \quad \text{and that}$$
$$\hat{\nabla}_\mu b_T^\mu = \partial_\mu b_T^\mu = 0. \quad (2.12)$$

The reader should always bear in mind that, for any vector field V^μ,

$$\hat{\nabla}_\mu V^\mu = \partial_\mu V^\mu,$$

since $\hat{g}_{\mu\nu}$ has determinant equal to -1.

Taking into account the previous definitions and that

$$\hat{\nabla}_\nu (c_T^\nu c_T^\mu) = c_T^\nu \hat{\nabla}_\nu c_T^\mu = c_T^\nu \partial_\nu c_T^\mu, \quad (2.13)$$

it is easy to show that $\mathfrak{B}_D$ is nilpotent:

$$\mathfrak{B}_D^2 = 0.$$

Once we have defined the action of the BRST operator on the fields, we may try to use, in a naïve way, the BRST quantization procedure and add to the classical lagrangian the following BRST-exact object

$$\mathcal{L}_{\text{ghost}}^{(D)} = -i\mathfrak{B}_D \left[\hat{g}_{\mu\nu} \bar{c}_T^\nu \partial_\lambda \hat{g}^{\lambda\mu} \right], \quad (2.14)$$

to define the path integral. But this will not do for the ghost fields we have introduced are constrained fields and we do not know its integration measure. To overcome this difficulty we shall solve the constraints as done in Ref. [16]. Indeed, the constraints in Eq. (2.12) can be solved by introducing three unconstrained, antisymmetric tensor fields $c^{\nu\mu}$, $\bar{c}^{\nu\mu}$ and $b^{\nu\mu}$ and express c_T^μ, $\bar{c}_T^\mu$ and b_T^μ in terms of the former as follows

$$c_T^\mu = \hat{\nabla}_\nu c^{\nu\mu}, \quad \bar{c}_T^\mu = \hat{\nabla}_\nu \bar{c}^{\nu\mu}, \quad \text{and} \quad b_T^\mu = \hat{\nabla}_\nu b^{\nu\mu}. \quad (2.15)$$

Notice that $\hat{\nabla}_\nu c^{\nu\mu} = \partial_\nu c^{\nu\mu}$, and so on, since the metric $\hat{g}_{\mu\nu}$ is unimodular; however, we have decided to keep the covariant derivative for $\hat{g}_{\mu\nu}$ to render any comparison with the results in Ref. [16] straightforward.

We may now define the path integral of the theory by integrating over de unconstrained fields $c^{\nu\mu}$, $\overline{c}^{\nu\mu}$ and $b^{\nu\mu}$. But, unfortunately that path integral will not be well defined because the use of Eq. (2.15) will give rise to a new gauge invariance: the substitutions

$$c^{\nu\mu} \rightarrow c^{\nu\mu} + \hat{\nabla}_\rho d^{\rho\nu\mu}, \quad \overline{c}^{\nu\mu} \rightarrow \overline{d}^{\nu\mu} + \hat{\nabla}_\rho \overline{d}^{\rho\nu\mu},$$
$$b^{\nu\mu} \rightarrow b^{\nu\mu} + \hat{\nabla}_\rho b^{\rho\nu\mu} \qquad (2.16)$$

do not change the values of c_T^μ, $\overline{c}_T^\mu$ and b_T^μ, respectively, if $c^{\rho\nu\mu}$, $\overline{d}^{\rho\nu\mu}$ and $b^{\rho\nu\mu}$ are antisymmetric tensor fields. This gauge invariance comes from the mismatch between the 3 independent components of, say, c_T^μ and the 6 independent components of $c^{\rho\nu\mu}$. We shall fix this mismatch, and remove the corresponding gauge invariance, by gauge-fixing. This gauge-fixing will demand the introduction of new ghosts – called second generation ghosts, which will be antisymmetric tensor fields of rank 3. This second generation ghosts will give rise in turn to a new gauge symmetry, which will lead to the introduction of a third generation of new ghosts – rank 4 antisymmetric tensors. This process is called ghost for ghost process. As we shall see below the ghost for ghost process stops after introducing the rank 4 tensor ghosts if the dimension of space-time is 4, but this is not so if the dimension of space-time is greater than 4. We thus conclude that the field content of the quantum theory depends on the spacetime dimension.

2.1.1 BRST variations of the first generation ghosts

Let us work out the action of the BRST operator $\mathfrak{B}_D$ on the first generation ghosts $c^{\nu\mu}$, $\overline{c}^{\nu\mu}$ and $b^{\nu\mu}$, which are the rank 2 antisymmetric tensor fields introduced in Eq. (2.15).

Let $V^{\mu\nu_1\dots\nu_n}$ a rank $n+1$ antisymmetric tensor field. Then the following result

$$\hat{\nabla}_\mu V^{\mu\nu_1\dots\nu_n} = \partial_\mu \left(V^{\mu\nu_1\dots\nu_n} \right)$$

leads to

$$\mathfrak{B}_D \left(\hat{\nabla}_\mu V^{\mu\nu_1\dots\nu_n} \right) = \hat{\nabla}_\mu \left(\mathfrak{B}_D V^{\mu\nu_1\dots\nu_n} \right).$$

Taking into account the previous equation and Eqs. (2.9) and (2.15), one gets

$$\hat{\nabla}_\nu \left(\mathfrak{B}_D c^{\nu\mu} - c_T^\nu c_T^\mu \right) = 0.$$

The general solution to this equation reads

$$\mathfrak{B}_D c^{\nu\mu} = c_T^\nu c_T^\mu + i\hat{\nabla}_\rho d^{\rho\nu\mu}.$$

The symbol $d^{\rho\nu\mu}$ denotes a new ghost which is an antisymmetric tensor field of rank 3 and it is called a second generation ghost.

We are now ready to define the action of $\mathfrak{B}_D$ on the first generation ghost, antighost and auxiliary field:

$$\mathfrak{B}_D c^{\nu\mu} = c_T^\nu c_T^\mu + i\hat{\nabla}_\rho d^{\rho\nu\mu},$$
$$\mathfrak{B}_D \overline{c}^{\nu\mu} = ib^{\nu\mu},$$
$$\mathfrak{B}_D b^{\nu\mu} = 0. \qquad (2.17)$$

respectively. Notice that the two last equations above are consistent with Eqs. (2.10) and (2.11), respectively.

Let us express $\mathcal{L}_{\text{ghost}}^{(D)}$ in Eq. (2.14) in terms of the first generation ghosts:

$$\mathcal{L}_{\text{ghost},1}^{(D)} \equiv \mathcal{L}_{\text{ghost}}^{(D)} = -i\mathfrak{B}_D \left[\hat{g}_{\mu\nu} \hat{\nabla}_\rho \overline{c}^{\rho\nu} \cdot \partial_\lambda \hat{g}^{\lambda\mu} \right]$$
$$= \hat{g}_{\mu\nu} \hat{\nabla}_\rho b^{\rho\nu} \cdot \partial_\lambda \hat{g}^{\lambda\mu} + 2i\hat{\nabla}_{(\mu} \hat{\nabla}^\rho c_{\nu)\rho} \cdot \hat{\nabla}_\rho \overline{c}^{\rho\nu} \cdot \partial_\lambda \hat{g}^{\lambda\mu}$$
$$+ 2i\hat{g}_{\mu\nu} \hat{\nabla}_\rho \overline{c}^{\rho\mu} \cdot \partial_\lambda \hat{\nabla}^{(\lambda} \hat{\nabla}_\rho c^{\mu)\rho}. \qquad (2.18)$$

The symbol $(\mu_1, \mu_2\dots)$ stands for full symmetrization so that already symmetric objects are left invariant.

2.1.2 Second generation ghosts and BRST transformations

We have already discussed in the last paragraph of Sect. 2.1 that the transformations in Eq. (2.16) are gauge symmetries of the theory, the fields $d^{\rho\nu\mu}$, $\overline{d}^{\rho\nu\mu}$ and $b^{\rho\nu\mu}$ being referred to as second generation ghosts. It is apparent that the ghost Lagrangian in Eq. (2.18) is invariant under the changes in Eq. (2.16). Notice that in the previous subsection the field $d^{\rho\nu\mu}$ has shown up already in the BRST transformations of the first generation ghost $c^{\nu\mu}$ – see Eq. (2.17).

To remove the gauge invariance generated by the need of second generation ghosts one introduces gauge-fixing conditions. The gauge-fixing conditions we choose read

$$\nabla^{[\rho} c^{\nu\mu]} = 0 \quad \text{and} \quad \nabla^{[\rho} \overline{c}^{\nu\mu]} = 0. \qquad (2.19)$$

The Lagrange multiplier BRST doublets needed to implement them in the Lagrangian are

$$(\overline{d}_{\rho\nu\mu}, \overline{c}_{\rho\nu\mu}) \quad \text{and} \quad (b_{\rho\nu\mu}, c_{\rho\nu\mu}), \qquad (2.20)$$

respectively. The symbol $[\mu_1, \mu_2\dots]$ denotes antisymmetrization with regard to the indices so that antisymmetric indices are left invariant. Note that the gauge-fixing term for the gauge transformation of $b^{\nu\mu}$ is obtained using the BRST transformation of $\overline{c}^{\nu\mu}$.

Let us now introduce the BRST transformations of the second generation ghosts. The first equation in (2.17) and the requirement that $\mathfrak{B}_D^2 = 0$ leads to the following equation to be satisfied by $\mathfrak{B}_D d^{\rho\nu\mu}$:

$$\hat{\nabla}_\rho \left(i\mathfrak{B}_D d^{\rho\nu\mu} + c_T^\rho c_T^\nu c_T^\mu \right) = 0.$$

The general solution to this equation runs thus:

$$\mathfrak{B}_D d^{\rho\nu\mu} = i c_{\mathrm{T}}^\rho c_{\mathrm{T}}^\nu c_{\mathrm{T}}^\mu - \hat{\nabla}_\sigma t^{\sigma\rho\nu\mu}. \tag{2.21}$$

The symbol $t^{\sigma\rho\nu\mu}$ stands for a new ghost field which is a rank 4 antisymmetric field, i.e., a third generation ghost.

We are now ready to define the action of the BRST operator on the second generation ghosts:

$$\begin{aligned}
\mathfrak{B}_D d^{\rho\nu\mu} &= i c_{\mathrm{T}}^\rho c_{\mathrm{T}}^\nu c_{\mathrm{T}}^\mu - \hat{\nabla}_\sigma t^{\sigma\rho\nu\mu}, \\
\mathfrak{B}_D \overline{d}^{\rho\nu\mu} &= \overline{c}^{\rho\nu\mu}, \\
\mathfrak{B}_D b^{\rho\nu\mu} &= c^{\rho\nu\mu}, \\
\mathfrak{B}_D c^{\rho\nu\mu} &= \mathfrak{B}_D \overline{c}^{\rho\nu\mu} = 0.
\end{aligned} \tag{2.22}$$

The new fields $\overline{c}^{\rho\nu\mu}$ and $c^{\rho\nu\mu}$ constitute BRST doublets with $\overline{d}^{\rho\nu\mu}$ and $b^{\rho\nu\mu}$, respectively. Recall Eq. (2.20) and the key point that indices are raised and lowered with the unimodular metric $\hat{g}_{\mu\nu}$.

Furnished with the gauge-fixing conditions in Eq. (2.19) and the BRST transformations in Eq. (2.22), we are ready to introduce a ghost Lagrangian that removes the gauge symmetry arising from the introduction of second generation ghosts, i.e., the gauge invariance under the transformations in Eq. (2.16):

$$\begin{aligned}
\mathcal{L}_{\mathrm{ghost},2}^{(D)} &= \frac{i}{2} \mathfrak{B}_D \left[\overline{d}^{\rho\nu\mu} \hat{\nabla}_\rho c_{\nu\mu} + b^{\rho\nu\mu} \hat{\nabla}_\rho \overline{c}_{\nu\mu} \right] \\
&= \frac{i}{2} \Big[\overline{c}^{\rho\nu\mu} \hat{\nabla}_\rho c_{\nu\mu} - \hat{\nabla}_\rho \overline{d}^{\rho\nu\mu} \cdot \mathfrak{B}_D \left(\hat{g}_{\nu\sigma} \hat{g}_{\mu\kappa} \right) c^{\sigma\kappa} \\
&\quad - \hat{\nabla}^\rho \overline{d}_{\rho\nu\mu} \cdot \left(c_{\mathrm{T}}^\nu c_{\mathrm{T}}^\mu + i \hat{\nabla}_\sigma d^{\sigma\nu\mu} \right) + c^{\rho\nu\mu} \hat{\nabla}_\rho \overline{c}_{\nu\mu} \\
&\quad - \hat{\nabla}_\rho b^{\rho\nu\mu} \cdot \mathfrak{B}_D \left(\hat{g}_{\nu\sigma} \hat{g}_{\mu\kappa} \right) \overline{c}^{\sigma\kappa} - i \hat{\nabla}^\rho b_{\rho\nu\mu} \cdot b^{\nu\mu} \Big].
\end{aligned} \tag{2.23}$$

2.1.3 Third generation ghosts and their BRST transformations

At least one gauge symmetry still occurs in our theory – we have not discussed the Weyl symmetry yet. This gauge symmetry is generated by rank 4 antisymmetric fields and arises from the invariance of the ghost Lagrangian in Eq. (2.23) under the following transformations

$$\begin{aligned}
d^{\rho\nu\mu} &\to d^{\rho\nu\mu} - \hat{\nabla}_\sigma t^{\sigma\rho\nu\mu}, \\
\overline{d}^{\rho\nu\mu} &\to \overline{d}^{\rho\nu\mu} + \hat{\nabla}_\sigma \overline{t}^{\sigma\rho\nu\mu}, \quad b^{\rho\nu\mu} \to b^{\rho\nu\mu} + \hat{\nabla}_\sigma b^{\sigma\rho\nu\mu}, \\
c^{\rho\nu\mu} &\to c^{\rho\nu\mu} + \hat{\nabla}_\sigma c^{\sigma\rho\nu\mu}, \quad \overline{b}^{\rho\nu\mu} \to \overline{c}^{\rho\nu\mu} + \hat{\nabla}_\sigma \overline{c}^{\sigma\rho\nu\mu}.
\end{aligned} \tag{2.24}$$

The fields $t^{\sigma\rho\nu\mu}$, $\overline{t}^{\sigma\rho\nu\mu}$, $b^{\sigma\rho\nu\mu}$, $\overline{b}^{\sigma\rho\nu\mu}$, $c^{\sigma\rho\nu\mu}$ and $\overline{c}^{\sigma\rho\nu\mu}$, which generate the previous gauge transformations will be promoted to ghost fields becoming third generation ghosts.

To remove the gauge invariance of the Lagrangian under the gauge transformations in Eq. (2.24), we shall introduce – as done in Ref. [16] – the following gauge-fixing conditions:

$$\nabla^{[\sigma} d^{\rho\nu\mu]} = 0, \quad \nabla^{[\sigma} \overline{d}^{\rho\nu\mu]} = 0 \quad \text{and} \quad \nabla^{[\sigma} b^{\rho\nu\mu]} = 0. \tag{2.25}$$

The Lagrange multiplier BRST doublets used to implement the previous gauge-fixing conditions are the following

$$(\overline{t}_{\sigma\rho\nu\mu}, \overline{d}_{\sigma\rho\nu\mu}), \quad (c_{\sigma\rho\nu\mu}, d_{\sigma\rho\nu\mu}) \quad \text{and} \quad (\overline{c}_{\sigma\rho\nu\mu}, b_{\sigma\rho\nu\mu}),$$

respectively.

In the previous list, we have not included any gauge-fixing conditions for $c^{\rho\nu\mu}$ and $\overline{c}^{\rho\nu\mu}$ since they will be generated, via the action of the BRST operator, by the gauge-fixing terms in the Lagrangian that implement the gauge-fixing conditions for $b^{\rho\nu\mu}$ and $\overline{d}^{\rho\nu\mu}$, respectively – see Eq. (2.27) below.

Let us define next the BRST transformations for the third generation ghosts $t^{\sigma\rho\nu\mu}$, $\overline{t}^{\sigma\rho\nu\mu}$, $b^{\sigma\rho\nu\mu}$, $c^{\sigma\rho\nu\mu}$ and $\overline{c}^{\sigma\rho\nu\mu}$. Let us begin with the BRST transformation of $t^{\sigma\rho\nu\mu}$. Since we must have $\mathfrak{B}_D^2 = 0$, Eq. (2.21) leads to

$$\hat{\nabla}_\sigma \left(\mathfrak{B}_D t^{\sigma\rho\nu\mu} - i c_{\mathrm{T}}^\sigma c_{\mathrm{T}}^\rho c_{\mathrm{T}}^\nu c_{\mathrm{T}}^\mu \right) = 0.$$

The general solution to this equation reads

$$\mathfrak{B}_D t^{\sigma\rho\nu\mu} = i c_{\mathrm{T}}^\sigma c_{\mathrm{T}}^\rho c_{\mathrm{T}}^\nu c_{\mathrm{T}}^\mu.$$

No new ghosts occur this time since, in 4 dimensions, no rank 5 antisymmetric tensor exists. We are now in the right position to define the BRST transformations

$$\begin{aligned}
\mathfrak{B}_D t^{\sigma\rho\nu\mu} &= i c_{\mathrm{T}}^\sigma c_{\mathrm{T}}^\rho c_{\mathrm{T}}^\nu c_{\mathrm{T}}^\mu, \quad \mathfrak{B}_D \overline{t}^{\sigma\rho\nu\mu} = i \overline{d}^{\sigma\rho\nu\mu}, \\
\mathfrak{B}_D c^{\sigma\rho\nu\mu} &= i d^{\sigma\rho\nu\mu}, \quad \mathfrak{B}_D \overline{c}^{\sigma\rho\nu\mu} = i b^{\sigma\rho\nu\mu}, \\
\mathfrak{B}_D \overline{d}^{\sigma\rho\nu\mu} &= \mathfrak{B}_D d^{\sigma\rho\nu\mu} = \mathfrak{B}_D b^{\sigma\rho\nu\mu} = 0.
\end{aligned} \tag{2.26}$$

The gauge-fixing conditions in Eq. (2.25) and the BRST transformations in Eq. (2.26) lead to the following part of the BRST invariant Lagrangian which carries out the gauge-fixing of the symmetries in Eq. (2.24):

$$\begin{aligned}
\mathcal{L}_{\mathrm{ghost},3}^{(D)} &= \frac{-i}{6} \mathfrak{B}_D \left[-\overline{t}^{\sigma\rho\nu\mu} \left(\hat{\nabla}_\sigma d_{\rho\nu\mu} + \frac{\alpha}{4} d_{\sigma\rho\nu\mu} \right) \right. \\
&\quad \left. + c^{\sigma\rho\nu\mu} \hat{\nabla}_\sigma \overline{d}_{\rho\nu\mu} + \overline{c}^{\sigma\rho\nu\mu} \hat{\nabla}_\sigma b_{\rho\nu\mu} \right] \\
&= \frac{1}{6} \Big[-\overline{d}^{\sigma\rho\nu\mu} \hat{\nabla}_\sigma d_{\rho\nu\mu} - \frac{\alpha}{4} \overline{d}^{\sigma\rho\nu\mu} d_{\sigma\rho\nu\mu} \\
&\quad + i \hat{\nabla}^\sigma \overline{t}_{\sigma\rho\nu\mu} \cdot \left(i c_{\mathrm{T}}^\rho c_{\mathrm{T}}^\nu c_{\mathrm{T}}^\mu - \hat{\nabla}_\lambda t^{\lambda\rho\nu\mu} \right) \\
&\quad + d^{\sigma\rho\nu\mu} \hat{\nabla}_\sigma \overline{d}_{\rho\nu\mu} + i c^{\sigma\rho\nu\mu} \hat{\nabla}_\sigma \overline{c}_{\rho\nu\mu}
\end{aligned}$$

$$+ b^{\sigma\rho\nu\mu}\hat{\nabla}_\sigma b_{\rho\nu\mu} + i\overline{c}^{\sigma\rho\nu\mu}\hat{\nabla}_\sigma c_{\rho\nu\mu}$$
$$+ i\hat{\nabla}_\sigma\overline{t}^{\sigma\rho\nu\mu}\cdot\mathfrak{B}_D\left(\hat{g}_{\rho\kappa}\hat{g}_{\nu\tau}\hat{g}_{\mu\lambda}\right)d^{\kappa\tau\lambda}$$
$$- i\hat{\nabla}_\sigma c^{\sigma\rho\nu\mu}\cdot\mathfrak{B}_D\left(\hat{g}_{\rho\kappa}\hat{g}_{\nu\tau}\hat{g}_{\mu\lambda}\right)\overline{d}^{\kappa\tau\lambda}$$
$$- i\hat{\nabla}_\sigma\overline{c}^{\sigma\rho\nu\mu}\cdot\mathfrak{B}_D\left(\hat{g}_{\rho\kappa}\hat{g}_{\nu\tau}\hat{g}_{\mu\lambda}\right)b^{\kappa\tau\lambda}\Big]. \qquad (2.27)$$

2.1.4 The ghost gauge-fixing Lagrangian for the transverse diffeomorphisms

The gauge invariance of the theory under transverse diffeomorphisms is fully gauge-fixed in a BRST invariant way by considering a ghost Lagrangian, $\mathcal{L}^{(D)}_{\text{ghost}}$, which is the sum of $\mathcal{L}^{(D)}_{\text{ghost},1}$, $\mathcal{L}^{(D)}_{\text{ghost},2}$ and $\mathcal{L}^{(D)}_{\text{ghost},3}$ given in Eqs. (2.18), (2.23) and (2.27), respectively.

Next, following Ref. [16], we shall express $\mathcal{L}^{(D)}_{\text{ghost}}$ in terms of the Hodge duals of ghosts fields: Hodge duals with regard to the unimodular metric $g_{\mu\nu}$. Let us now defined these Hodge duals:

$$c^{\mu\nu} = (1/2)\varepsilon^{\mu\nu\rho\sigma}C_{\rho\sigma},$$
$$d^{\mu\rho} = \varepsilon^{\mu\nu\rho\sigma}D_\sigma,$$
$$t^{\mu\nu\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma}T.$$

and

$$\begin{pmatrix}\overline{c}^{\mu\nu}\\b^{\mu\nu}\end{pmatrix} = -(1/2)\varepsilon^{\mu\nu\rho\sigma}\begin{pmatrix}\overline{C}_{\rho\sigma}\\B_{\rho\sigma}\end{pmatrix},$$

$$\begin{pmatrix}\overline{t}^{\mu\nu\rho\sigma}\\\overline{d}^{\mu\nu\rho\sigma}\end{pmatrix} = -\varepsilon^{\mu\nu\rho\sigma}\begin{pmatrix}\overline{T}\\D\end{pmatrix},$$

$$\begin{pmatrix}\overline{d}^{\mu\nu\rho}\\\overline{c}^{\mu\nu\rho}\end{pmatrix} = -(1/2)\varepsilon^{\mu\nu\rho\sigma}\begin{pmatrix}\overline{D}_\sigma\\\overline{C}_\sigma\end{pmatrix},$$

$$\begin{pmatrix}c^{\mu\nu\rho\sigma}\\d^{\mu\nu\rho\sigma}\end{pmatrix} = -\varepsilon^{\mu\nu\rho\sigma}\begin{pmatrix}C\\D\end{pmatrix},$$

$$\begin{pmatrix}b^{\mu\nu\rho}\\c^{\mu\nu\rho}\end{pmatrix} = -(1/2)\varepsilon^{\mu\nu\rho\sigma}\begin{pmatrix}B_\sigma\\C_\sigma\end{pmatrix},$$

$$\begin{pmatrix}\overline{c}^{\mu\nu\rho\sigma}\\b^{\mu\nu\rho\sigma}\end{pmatrix} = -\varepsilon^{\mu\nu\rho\sigma}\begin{pmatrix}\overline{C}\\B\end{pmatrix}.$$

Hence, we get

$$\mathcal{L}^{(D)}_{\text{ghost}} = \mathcal{L}^{(D)}_{\text{ghost},1} + \mathcal{L}^{(D)}_{\text{ghost},2} + \mathcal{L}^{(D)}_{\text{ghost},3}, \qquad (2.28)$$

where

$$\mathcal{L}^{(D)}_{\text{ghost},1} = -\frac{1}{2}\hat{g}_{\mu\nu}\varepsilon^{\rho\nu\sigma\tau}\hat{\nabla}_\rho B^{\sigma\tau}\cdot\partial_\lambda\hat{g}^{\lambda\mu}$$
$$- \frac{i}{2}\varepsilon^{\chi\tau\rho\sigma}\varepsilon^{\nu\kappa\alpha\beta}\hat{g}_{\chi(\mu}\hat{\nabla}_{\nu)}\hat{\nabla}_\tau C_{\rho\sigma}\cdot\hat{\nabla}_\kappa\overline{C}_{\alpha\beta}\cdot\partial_\lambda\hat{g}^{\lambda\mu}$$
$$+ \frac{i}{2}\hat{g}_{\mu\nu}\varepsilon^{\nu\rho\sigma\tau}\hat{\nabla}_\rho\overline{C}^{\sigma\tau}\cdot\varepsilon^{\kappa\alpha\beta(\mu}\partial_\lambda\hat{\nabla}^{\lambda)}\hat{\nabla}_\kappa C_{\alpha\beta}, (2.29)$$

$$\mathcal{L}^{(D)}_{\text{ghost},2} = i\overline{C}^\sigma\hat{\nabla}^\rho C_{\rho\sigma} + \frac{3i}{4}\varepsilon^{\kappa\mu\nu\lambda}\hat{\nabla}^\rho\overline{D}^\sigma\cdot\hat{\nabla}_{[\rho}C_{\kappa\sigma]}\cdot\hat{\nabla}_\mu C_{\nu\lambda}$$
$$+ 2\hat{\nabla}^{[\lambda}\overline{D}^{\sigma]}\cdot\hat{\nabla}_\lambda D_\sigma$$

$$-iC^\sigma\hat{\nabla}^\rho\overline{C}_{\rho\sigma} + B^\sigma\hat{\nabla}^\rho B_{\rho\sigma} - 2i\hat{g}^{\mu\lambda}\hat{\nabla}^{[\rho}\overline{D}^{\sigma]}$$
$$\cdot C_{\mu\sigma}\mathfrak{B}_D\hat{g}_{\lambda\rho} + 2i\hat{g}^{\mu\lambda}\hat{\nabla}^{[\rho}B^{\sigma]}\cdot\overline{C}_{\mu\sigma}\mathfrak{B}_D\hat{g}_{\lambda\rho}, \qquad (2.30)$$

and

$$\mathcal{L}^{(D)}_{\text{ghost},3} = \Big[\overline{D}\hat{\nabla}_\mu D^\mu + \alpha\overline{D}D + D\hat{\nabla}_\mu\overline{D}^\mu + iC\hat{\nabla}_\mu\overline{C}^\mu$$
$$+ i\overline{C}\hat{\nabla}_\mu C^\mu + B\hat{\nabla}_\mu B^\mu$$
$$- \frac{1}{4}\hat{\nabla}^\sigma\overline{T}\cdot\hat{\nabla}_{[\sigma}C_{\nu\mu]}(\hat{\nabla}^\mu C^{\lambda\rho}\cdot\hat{\nabla}^\nu C_{\lambda\rho}$$
$$- 4\hat{\nabla}^\lambda C^{\mu\rho}\cdot\hat{\nabla}^\nu C_{\lambda\rho} + 2\hat{\nabla}^\lambda C^{\mu\rho}\cdot\hat{\nabla}_\lambda C^\nu{}_\rho$$
$$- 2\hat{\nabla}_\lambda C^{\mu\rho}\cdot\hat{\nabla}_\rho C^{\nu\lambda}) - i\hat{\nabla}^\mu\overline{T}\cdot\hat{\nabla}_\mu T + \Big(i\hat{\nabla}^\mu\overline{T}\cdot D^\nu$$
$$+ i\hat{\nabla}^\mu C\cdot\overline{D}^\nu + i\hat{\nabla}^\mu\overline{C}\cdot B^\nu\Big)\mathfrak{B}_D\hat{g}_{\mu\nu}\Big]. \qquad (2.31)$$

$\mathcal{L}^{(D)}_{\text{ghost}}$ above can be obtained from the corresponding expression in Ref. [16] by choosing in the latter the unimodular metric $\hat{g}_{\mu\nu}$ as the spacetime metric.

In Table 1, the reader may find a summary of the action of $\mathfrak{B}_D$ on the Hodge duals introduced above

2.2 Gauge-fixing the Weyl invariance and the complete BRST operator

Let us recall that in our formalism, in the path integral defining our theory, we must integrate over the unconstrained field $g_{\mu\nu}$, not over $\hat{g}_{\mu\nu}$. Also recall $\hat{g}_{\mu\nu}$ is defined in terms of $g_{\mu\nu}$ by Eq. (1.1). Now, it is the unimodular metric $\hat{g}_{\mu\nu}$, the metric that enters $\mathcal{L}^{(D)}_{\text{ghost}}$ and, hence, $\mathcal{L}^{(D)}_{\text{ghost}}$ depends on $g_{\mu\nu}$ only through $\hat{g}_{\mu\nu}$. It then follows that $\mathcal{L}^{(D)}_{\text{ghost}}$ has a gauge symmetry, the following Weyl symmetry:

$$g_{\mu\nu}(x) \rightarrow g_{\mu\nu}(x) + \delta_W g_{\mu\nu}(x), \quad \delta_W g_{\mu\nu}(x) = 2\theta(x)g_{\mu\nu}(x), \qquad (2.32)$$

where θ is an arbitrary scalar field.

The Weyl symmetry we have just mentioned has to be removed from the action in the path integral, lest the path integral be ill defined. We shall do so by using the BRST formalism.

Since one of the purposes of this paper is to make consistent the formalism of Ref. [16] with the classic linear splitting of the gravitational field, we shall introduce that splitting now:

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu}, \qquad (2.33)$$

and define the BRST transformations so that they are adapted to the splitting in question. In the previous equation $\overline{g}_{\mu\nu}$ is

Table 1 BRST transformations of the Hodge duals. N_{ghost} stands for the ghost number

	1st generation	2nd generation	3rd generation
$N_{ghost} = 0$	$\mathfrak{B}_D B_{\mu\nu} = 0$	$\mathfrak{B}_D B_\mu = C_\mu$	$\mathfrak{B}_D B = 0$
$N_{ghost} = 1$	$\mathfrak{B}_D C_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}c_T^\rho c_T^\sigma + i\left(\partial_\mu D_\nu - \partial_\nu D_\mu\right)$	$\mathfrak{B}_D C_\mu = 0$	$\mathfrak{B}_D C = iD$
$N_{ghost} = -1$	$\mathfrak{B}_D \overline{C}_{\mu\nu} = iB_{\mu\nu}$	$\mathfrak{B}_D \overline{C}_\mu = 0$	$\mathfrak{B}_D \overline{C} = iB$
$N_{ghost} = 2$	–	$\mathfrak{B}_D D_\mu = \frac{i}{3!}\varepsilon_{\mu\nu\rho\sigma}c_T^\nu c_T^\rho c_T^\sigma + \partial_\mu T$	$\mathfrak{B}_D D = 0$
$N_{ghost} = -2$	–	$\mathfrak{B}_D \overline{D}_\mu = \overline{C}_\mu$	$\mathfrak{B}_D \overline{D} = 0$
$N_{ghost} = 3$	–	–	$\mathfrak{B}_D T = -\frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}c_T^\mu c_T^\nu c_T^\rho c_T^\sigma$
$N_{ghost} = -3$	–	–	$\mathfrak{B}_D \overline{T} = i\overline{D}$

the background gravitational field and $h_{\mu\nu}$ is the quantum gravitational field.

Now, let R ($N_{\text{ghost}}(R) = 1$) denote the ghost associated to the Weyl transformation in Eq. (2.32), $\overline{R}$ ($N_{\text{ghost}}(\overline{R}) = -1$) the corresponding antighost field and L ($N_{\text{ghost}}(L) = 0$) the *Nakanishi-Lautrup* auxiliary field. Let $\mathfrak{B}_W$ denote the BRST operator for the Weyl symmetry in Eq. (2.32). Then, we define the action of $\mathfrak{B}_W$ on the fields as follows

$$\mathfrak{B}_W \overline{g}_{\mu\nu} = 0, \quad \mathfrak{B}_W h_{\mu\nu} = 2R\left(\overline{g}_{\mu\nu} + h_{\mu\nu}\right),$$
$$\mathfrak{B}_W R = 0, \mathfrak{B}_W \overline{R} = L, \quad \mathfrak{B}_W L = 0. \tag{2.34}$$

We will also need the action of $\mathfrak{B}_W$ on the first generation, the second generation, the third generation ghosts and the corresponding *Nakanishi-Lautrup* auxiliary fields introduced in the previous subsections. We define this action to be equal to zero. Hence, we have

$$\mathfrak{B}_W B_{\mu\nu} = 0, \quad \mathfrak{B}_W C_{\mu\nu} = 0, \quad \mathfrak{B}_W \overline{C}_{\mu\nu} = 0,$$
$$\mathfrak{B}_W D_\mu = 0,$$
$$\mathfrak{B}_W \overline{D}_\mu = 0, \mathfrak{B}_W T_{\mu\nu} = 0, \quad \mathfrak{B}_W \overline{T}_{\mu\nu} = 0, \quad \mathfrak{B}_W D = 0,$$
$$\mathfrak{B}_W \overline{D} = 0, \tag{2.35}$$

which are the fields entering the ghost Lagrangian in Eq. (2.28), in addition to the unimodular metric $\hat{g}_{\mu\nu}$.

Obviously, $\mathfrak{B}_W^2 = 0$ – recall that R is a Grassmann field. Equations (1.1) and (2.34) imply

$$\mathfrak{B}_W \hat{g}_{\mu\nu} = 0. \tag{2.36}$$

The fields R, $\overline{R}$ and L are scalar fields and, hence, the action on these fields of the BRST operator, $\mathfrak{B}_D$, associated to the transverse diffeomorphisms reads

$$\mathfrak{B}_D R = c_T^\rho \partial_\rho R, \quad \mathfrak{B}_D \overline{R} = c_T^\rho \partial_\rho \overline{R}, \quad \mathfrak{B}_D L = c_T^\rho \partial_\rho L,$$

where it is important to always bear in mind that $c_T^\mu = -\varepsilon^{\mu\nu\rho\sigma}\hat{\nabla}_\nu C_{\rho\sigma}$.

Let us show now that $\mathfrak{B}_D^2 = 0$ on R, $\overline{R}$ and L. Let F stand for any of those fields, then

$$\mathfrak{B}_D^2 F = \mathfrak{B}_D(c_T^\rho \partial_\rho F) = (\mathfrak{B}_D c_T^\rho)\partial_\rho F - c_T^\rho \partial_\rho \mathfrak{B}_D F$$
$$= c_T^\nu \partial_\nu c_T^\rho \partial_\rho F - c_T^\nu \partial_\nu(c_T^\rho \partial_\rho F) = 0.$$

We have taken into account that c_T^ρ is a Grassmann field and that Eqs. (2.9) and (2.13) hold.

Next, let us define the following linear operator

$$\mathfrak{B} = \mathfrak{B}_D + \mathfrak{B}_W \tag{2.37}$$

acting on the fields we have introduced so far.

Let us now prove that $\mathfrak{B}$ is a BRST operator, i.e., that $\mathfrak{B}^2 = 0$. Since we already know that $\mathfrak{B}_D^2 = 0$ and $\mathfrak{B}_W^2 = 0$, all that is left for us to do is show that

$$\mathfrak{B}_D \mathfrak{B}_W + \mathfrak{B}_W \mathfrak{B}_D = 0. \tag{2.38}$$

Taking into account Eqs. (2.8) and (2.34), one gets

$$\mathfrak{B}_D \mathfrak{B}_W h_{\mu\nu} = 2g_{\mu\nu}c_T^\rho \partial_\rho R - 2\mathfrak{B}_D h_{\mu\nu},$$
$$\mathfrak{B}_W \mathfrak{B}_D h_{\mu\nu} = -2g_{\mu\nu}c_T^\rho \partial_\rho R + 2\mathfrak{B}_D h_{\mu\nu}.$$
$$\mathfrak{B}_D \mathfrak{B}_W \overline{R} = \mathfrak{B}_D L = c_T^\rho \partial_\rho L,$$
$$\mathfrak{B}_W \mathfrak{B}_D \overline{R} = \mathfrak{B}_W \left(c_T^\rho \partial_\rho \overline{R}\right) = -c_T^\rho \partial_\rho L.$$

That $\mathfrak{B}_W c_T^\mu = 0$ leads immediately to the conclusion that Eq. (2.38) holds when acting on R and L. Now, recall that $\mathfrak{B}_W$ annihilates the first generation, the second generation, the third generation ghosts and the corresponding *Nakanishi-Lautrup* auxiliary fields, and that $\mathfrak{B}_D$ acting on those very fields *only* involves the fields in question and the unimodular metric $\hat{g}_{\mu\nu}$ – this metric being invariant under the Weyl transformations of $g_{\mu\nu}$. Hence, Eq. (2.38) also holds for the fields we have just mentioned.

We are now ready to carry out the BRST gauge-fixing of the Weyl gauge symmetry in Eq. (2.32) in a way which is consistent with the BRST gauge-fixing of the transverse diffeomorphisms. Indeed, we shall add to $\mathcal{L}_{\text{ghost}}^{(D)}$ in Eq. (2.28)

– which does the gauge-fixing of the transverse diffeomorphism invariance – the following term

$$
\begin{aligned}
\mathcal{L}_{\text{ghost}}^{(W)} &= \mathfrak{B}\big[\overline{\nabla}_\mu \overline{R} \cdot \overline{\nabla}^\mu (L - \alpha' h)\big] \\
&= \mathfrak{B}_D\big[\overline{\nabla}_\mu \overline{R} \cdot \overline{\nabla}^\mu (L - \alpha' h)\big] \\
&\quad + \mathfrak{B}_W\big[\overline{\nabla}_\mu \overline{R} \cdot \overline{\nabla}^\mu (L - \alpha' h)\big] \\
&= \overline{\nabla}_\mu i g\big[c_{\text{T}}^\rho \overline{\nabla}_\rho \overline{R}\big] \cdot \overline{\nabla}^\mu (L - \alpha' h) - \overline{\nabla}_\mu \overline{R} \cdot \overline{\nabla}^\mu \big[c_{\text{T}}^\rho \overline{\nabla}_\rho L\big] \\
&\quad + \alpha' \overline{\nabla}_\mu \overline{R} \cdot \overline{\nabla}^\mu \big[2\overline{\nabla}^\rho c_\rho^{\text{T}} + c_{\text{T}}^\rho \overline{\nabla}_\rho h \\
&\quad + 2\overline{\nabla}^\sigma c_{\text{T}}^\rho h_{\rho\sigma}\big] + \overline{\nabla}_\mu L \cdot \overline{\nabla}^\mu (L - \alpha' h) \\
&\quad + 2\alpha' (4 + h)\overline{\nabla}_\mu \overline{R} \cdot \overline{\nabla}^\mu R + 2\alpha' \overline{\nabla}_\mu \overline{R} \cdot R\overline{\nabla}^\mu h, \quad (2.39)
\end{aligned}
$$

where $h = \overline{g}^{\mu\nu} h_{\mu\nu}$ and $\overline{\nabla}_\mu$ is the covariant derivative with respect to $\overline{g}_{\mu\nu}$.

Notice that $\mathcal{L}_{\text{ghost}}^{(W)}$ is exact with regard to the complete BRST operator $\mathfrak{B}$ in Eq. (2.37), a must if the full power of the BRST formalism is to be taken advantage of. But what about $\mathcal{L}_{\text{ghost}}^{(D)}$ in Eq. (2.28)? The answer is in the affirmative. Indeed, $\mathcal{L}_{\text{ghost},1}^{(D)}$ in Eq. (2.18), $\mathcal{L}_{\text{ghost},2}^{(D)}$ in Eq. (2.23) and $\mathcal{L}_{\text{ghost},3}^{(D)}$ in Eq. (2.27) can be also expressed as follows

$$
\begin{aligned}
\mathcal{L}_{\text{ghost},1}^{(D)} &= -i\mathfrak{B}\left[\hat{g}_{\mu\nu}\hat{\nabla}_\rho \overline{c}^{\rho\nu} \cdot \partial_\lambda \hat{g}^{\lambda\mu}\right] \\
\mathcal{L}_{\text{ghost},2}^{(D)} &= \frac{i}{2}\mathfrak{B}\left[\overline{d}^{\rho\nu\mu}\hat{\nabla}_\rho c_{\nu\mu} + b^{\rho\nu\mu}\hat{\nabla}_\rho \overline{c}_{\nu\mu}\right] \\
\mathcal{L}_{\text{ghost},3}^{(D)} &= \frac{-i}{6}\mathfrak{B}\left[-\overline{t}^{\sigma\rho\nu\mu}\left(\hat{\nabla}_\sigma d_{\rho\nu\mu} + \frac{\alpha}{4}d_{\sigma\rho\nu\mu}\right)\right. \\
&\quad \left. + c^{\sigma\rho\nu\mu}\hat{\nabla}_\sigma \overline{d}_{\rho\nu\mu} + \overline{c}^{\sigma\rho\nu\mu}\hat{\nabla}_\sigma b_{\rho\nu\mu}\right],
\end{aligned}
$$

This is a consequence of the fact that, as we have seen above, the action of $\mathfrak{B}_W$ on any of the fields entering the definition of $\mathcal{L}_{\text{ghost},1}^{(D)}$, $\mathcal{L}_{\text{ghost},2}^{(D)}$ and $\mathcal{L}_{\text{ghost},3}^{(D)}$ yields zero. We thus conclude that fixing the gauge for the Weyl symmetry does not introduce any modification of $\mathcal{L}_{\text{ghost}}^{(D)}$ as given in Eqs. (2.28), (2.29), (2.30) and (2.31).

2.3 The complete BRST invariant action

In the previous subsections we have achieved, by using the BRST formalism, a complete gauge-fixing of the two gauge symmetries of our theory. Let us now quote the full action, $\mathcal{S}$, of our theory. This action reads

$$
\mathcal{S} = \mathcal{S}_{class} + 2\int d^4 x\, \mathcal{L}_{\text{ghost}}^{(D)} + 2\int d^4 x\, \mathcal{L}_{\text{ghost}}^{(W)}. \quad (2.40)
$$

$\mathcal{S}_{class}$, $\mathcal{L}_{\text{ghost}}^{(D)}$ and $\mathcal{L}_{\text{ghost}}^{(W)}$ are given in Eqs. (2.1), (2.28) and (2.39), respectively.

3 The free graviton propagator propagates gravitons into gravitons

In this section we shall put to work the action $\mathcal{S}$, in Eq. (2.40), and work out the free graviton propagator, first, and, then, show that the graviton propagator propagates free gravitons (i.e., helicity-two states) into free gravitons.

Let the background field $\overline{g}_{\mu\nu}$ in Eq. (2.33) be the Minkowski metric with signature $(-, +, +, +)$. Then, the free part of $\mathcal{S}$ needed for the computation of the free graviton propagator reads

$$
\begin{aligned}
\int d^4 x &\left\{\frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu} + \frac{1}{2}\left(\partial_\mu h^{\mu\nu}\right)^2 + \frac{1}{4}h_{\mu\nu}\partial^\mu \partial^\nu h \right. \\
&- \frac{1}{32}h\Box h + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\partial^\nu B^{\rho\sigma} \cdot \partial_\lambda h^{\lambda\mu} + B_\mu \partial_\nu B^{\nu\mu} \\
&+ B\partial^\mu B_\mu + \alpha' L\Box h - L\Box L - \overline{D}^\mu \left(\Box D_\mu - \partial_\mu \partial^\nu D_\nu\right) \\
&\left. + \overline{D}\partial^\mu D_\mu + \partial_\mu \overline{D}^\mu \cdot D + \alpha \overline{D}D\right\}. \quad (3.1)
\end{aligned}
$$

Hence, the free graviton propagator is obtained by inverting the following matrix

$$
\begin{array}{cccccc}
& h_{\rho\sigma} & B_{\rho\sigma} & B_\rho & B & L \\
\begin{array}{c} h_{\mu\nu} \\ B_{\mu\nu} \\ B_\mu \\ B \\ L \end{array} &
\left(\begin{array}{ccccc}
\mathcal{V}_{h,h} & 2\varepsilon^{\kappa\lambda\rho\sigma}p_\lambda \delta_\kappa^{(\nu}p^{\mu)} & 0 & 0 & -4\alpha' p^2 \eta^{\mu\nu} \\
-2\varepsilon^{\mu\nu\kappa(\rho}p^{\sigma)}p_\kappa & 0 & i4\eta^{\rho[\mu}p^{\nu]} & 0 & 0 \\
0 & -i4\eta^{\mu[\rho}p^{\sigma]} & 0 & -i4p^\mu & 0 \\
0 & 0 & i4p^\rho & 0 & 0 \\
-4\alpha' p^2 \eta^{\rho\sigma} & 0 & 0 & 0 & 4p^2
\end{array}\right),
\end{array}
$$

where $\mathcal{V}_{h,h} = -p^2 \eta^{\mu(\rho}\eta^{\sigma)\nu} + 2p^{(\mu}\eta^{\nu)(\sigma}p^{\rho)} - \frac{1}{2}(p^\mu p^\nu \eta^{\rho\sigma} + p^\rho p^\sigma \eta^{\mu\nu}) - \frac{1}{8}p^2 \eta^{\mu\nu}\eta^{\rho\sigma}$. For the free graviton propagator of our theory, we have obtained the following result:

$$
\begin{aligned}
&\langle h_{\mu_1\nu_1}(p)h_{\mu_2\nu_2}(-p)\rangle \\
&= \frac{i}{p^2 - i\varepsilon}\left\{A_1\left(\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} + \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2}\right) + A_2 \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\right. \\
&\quad + A_3 \frac{1}{p^2 - i\varepsilon}\left(\eta_{\mu_1\nu_1}p_{\mu_2}p_{\nu_2} + \eta_{\mu_2\nu_2}p_{\mu_1}p_{\nu_1}\right) \\
&\quad + A_4 \frac{1}{p^2 - i\varepsilon}\left(\eta_{\mu_1\mu_2}p_{\nu_1}p_{\nu_2} + \eta_{\mu_1\nu_2}p_{\nu_1}p_{\mu_2} + \eta_{\nu_1\mu_2}p_{\mu_1}p_{\nu_2}\right. \\
&\quad \left. + \eta_{\nu_1\nu_2}p_{\mu_1}p_{\mu_2}\right) + A_5 \frac{1}{(p^2 - i\varepsilon)^2}p_{\mu_1}p_{\nu_1}p_{\mu_2}p_{\nu_2}\Bigg\}, \quad (3.2)
\end{aligned}
$$

with

$$
A_1 = -\frac{1}{2}, \quad A_2 = \frac{3 + 32\alpha'^2}{8(1 + 8\alpha'^2)}, \quad A_3 = -1,
$$

$$
A_4 = \frac{1}{2}, \quad A_5 = 2.
$$

Now, the quantum state of a physical free graviton of momentum p^μ is described by the following of plane-wave

$$w^{(p)}_{\mu\nu}(x) = \epsilon_{\mu\nu}(p)\, e^{ipx}, \tag{3.3}$$

where $\epsilon_{\mu\nu}(p)$ is the polarization, which is symmetric, transverse(i.e., $p^\mu \epsilon_{\mu\nu}(p) = 0$) and traceless (i.e., $\eta^{\mu\nu}\epsilon_{\mu\nu}(p) = 0$. It can be easily shown that the previous physical graviton plane-wave is a solution to the equations of motion of the free action in (3.1). Then, the propagation of previous graviton wave is given by

$$\int d^4 y \left\{ -i \int \frac{d^4 q}{(2\pi)^4} \langle h_{\mu_1 \nu_1}(q) h_{\mu_2 \nu_2}(-q) \rangle e^{iq(x-y)} \right\} w^{(p)}_{\mu_2 \nu_2}(y),$$

which in turn yields the following physical free graviton plane-wave:

$$-\frac{i}{p^2 - i\varepsilon} \epsilon_{\mu\nu}(p)\, e^{ipx}. \tag{3.4}$$

We thus see that physical gravitons are propagated into physical gravitons. This result has a standard physical explanation. Indeed, the result in (3.4) comes in its enterity from the following part of the propagator in (3.2)

$$-\frac{\frac{1}{2}(\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} + \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2})}{k^2}.$$

And, it is well known [25] that the numerator of the previous expression is the sum of two summands. One is the projector onto the two dimensional space of physical polarizations, and reads

$$\sum_{\lambda=-,+} \epsilon^{(\lambda)}_{\mu_1\nu_1}(p)\epsilon^{(-\lambda)}_{\mu_2\nu_2}(p). \tag{3.5}$$

λ labels the two helicities of the physical graviton. The other summand yields a vanishing contribution when contracted with physical free graviton waves. Obviously, the expression in (3.4) comes completely from the action on $\epsilon_{\mu\nu}(p)\, e^{ipy}$ of the projector in (3.5), as demanded by unitarity [25].

Notice that the result in (3.4) and the corresponding analysis are the same as in General Relativity. Also notice that non-physical modes like

$$(p_\mu \epsilon_\nu(p) + p_\nu \epsilon_\mu(p))\, e^{ipy},$$

where $p^\mu \epsilon_\mu(p) = 0$, do not propagate, by means of the unimodular propagator above, into the physical graviton modes in (3.3).

We have exhibited that the unimodular theory defined by the action in (2.40) is a theory of gravitons, which only have

two physical polarizations. Further, the analysis carried out above indicates that the construction of the Fock space of the free theory by using the BRST charge – say, Q – obtained from the free action in (3.1) will lead to the conclusion that the free graviton states in (3.3) will be annihilated by Q, and, thus, they are physical sates, while all the other states will be either Q-exact (i.e., physically irrelevant) or non-physical (i.e., not annihilated by Q).

4 Obtaining Newton's law

In this section we shall show that Newton's law can be obtained when matter is coupled to our unimodular gravity theory, which is a check that we have a theory of gravitons interacting with matter.

In unimodular gravity, the graviton field $h_{\mu\nu}$ couples to the traceless part of the energy-momentum tensor $\hat{T}^{\mu\nu} \equiv T^{\mu\nu} - \eta^{\mu\nu} T^\rho_\rho/4$. This is equivalent to saying that it is the traceless part of graviton field, $\hat{h}_{\mu\nu} = h_{\mu\nu} - \eta_{\mu\nu} h/4$, the one which couples to the energy-momentum tensor $T^{\mu\nu}$. Hence, to retrieve the gravitational Newton's potential from unimodular gravity it is useful to compute the free two-point Green function of $\hat{h}_{\mu\nu}$. This Green function can be obtained from Eq. (3.2) and it reads

$$\langle \hat{h}_{\mu_1\nu_1}(p)\hat{h}_{\mu_2\nu_2}(-p)\rangle$$
$$= \frac{i}{p^2}\left\{ -\frac{1}{2}(\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2} + \eta_{\mu_1\nu_2}\eta_{\nu_1\mu_2} - \eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}) \right.$$
$$-\frac{1}{p^2}(\eta_{\mu_1\nu_1}p_{\mu_2}p_{\nu_2} + \eta_{\mu_2\nu_2}p_{\mu_1}p_{\nu_1}$$
$$+\frac{1}{2}\frac{1}{p^2}(\eta_{\mu_1\mu_2}p_{\nu_1}p_{\nu_2} + \eta_{\mu_1\nu_2}p_{\nu_1}p_{\mu_2} + \eta_{\nu_1\mu_2}p_{\mu_1}p_{\nu_2}$$
$$\left. +\eta_{\nu_1\nu_2}p_{\mu_1}p_{\mu_2}) + 2\frac{1}{p^4}p_{\mu_1}p_{\nu_1}p_{\mu_2}p_{\nu_2} \right\}. \tag{4.1}$$

The Gravitational Newton's potential can be obtained by computing, in the non-relativistic limit, the amplitude, $\mathcal{M}$, of the process in which one graviton is exchanged between two scalar particles with masses M_1 y M_2. The Feynman diagram describing this process is in Fig. 1. The amplitude in question is given by

$$\mathcal{M} = -\frac{i}{4} T^{(1)}_{\mu\nu}(p_1, p'_1)\langle \hat{h}^{\mu\nu}(k)\hat{h}^{\rho\sigma}(-k)\rangle T^{(2)}_{\rho\sigma}(p_2, p'_2), \tag{4.2}$$

where p_i y p'_i are the initial and final momenta, respectively, of the i particle, $i = 1, 2$. Hence, the transferred momentum k is given by $k = p_1 - p'_1 = p'_2 - p_2$ y $M_i^2 = -p_i^2 = -p'^2_i$. In Eq. (4.2) $T^{(i)}_{\mu\nu}(p_i, p'_i)$ denotes the on-shell lowest order
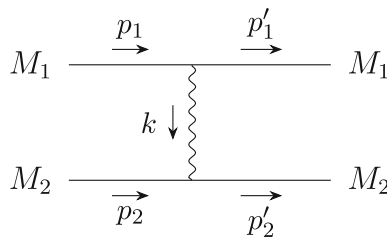
Fig. 1 The exchange of one graviton

contribution of the energy-momentum tensor between the states with momenta p_i y p_i'. $T_{\mu\nu}^{(i)}(p_i, p_i')$ is given by

$$T_{\mu\nu}^{(i)}(p_i, p_i') \equiv \langle p_i'|T_{\mu\nu}|p_i\rangle = p_{i\,\mu} p_{i\,\nu}' + p_{i\,\nu} p_{i\,\mu}' + \frac{1}{2}k^2 \eta_{\mu\nu},$$

which in the non-relativistic limit – hence $k^\mu = (0, \vec{k})$ – reads

$$T_{\mu\nu}^{(i)}(p_i, p_i') = -2M_i^2 \eta_{\mu 0} \eta_{\nu 0}.$$

We thus conclude that, in the non-relativistic, limit the amplitude $\mathcal{M}$ is equal to

$$\mathcal{M}^{nonrel} = -i M_1^2 M_2^2 \langle \hat{h}_{00}(k)\hat{h}_{00}(-k)\rangle,$$

where $\langle \hat{h}_{00}(k)\hat{h}_{00}(-k)\rangle$ is given in Eq. (4.1) and $k^\mu = (0, \vec{k})$.

Now, the Fourier transform of the Newton's potential is given by

$$\tilde{V}(\vec{k}) = \frac{\mathcal{M}^{nonrel}}{2M_1 2M_2} = -\frac{i}{4} M_1 M_2 \langle \hat{h}_{00}(k)\hat{h}_{00}(-k)\rangle.$$

From Eq. (4.1), we get $\langle \hat{h}_{00}(k)\hat{h}_{00}(-k)\rangle = -i/2k^2$ so that

$$\tilde{V}(\vec{k}) = -\frac{1}{8}\frac{M_1 M_2}{\vec{k}^2},$$

which precisely the Fourier transform of the Newton's potential one finds in elementary textbooks, in the unit system where $G = 1/(32\pi)$.

5 The Slavnov–Taylor equation and the linearized Slavnov–Taylor operator

As was discussed by the authors of Ref. [16] the construction of their BRST formalism involving antisymmetric tensor ghosts is such that the field content of the resulting BRST invariant theory depends on the dimension of spacetime. We have also analyzed this issue right below Eq. (2.16). It is then clear that dimensional regularization does not provide a BRST invariant regularization of the field theory in 4 dimensional spacetime – nor in any other spacetime dimension.

Since no regularization method that preserves general covariance, while keeping the spacetime dimension equal to 4, is known, one is lead to the conclusion that one has to take advantage of the algebraic renormalization techniques – see Ref. [26] – to keep the BRST symmetry of the renormalized theory under control. The implementation of algebraic renormalization techniques begins with the derivation of the formal Slavnov–Taylor equation, which is to be satisfied by the 1PI functional, and continues with the formulation, and solution, of a cohomology problem for the linearized Slavnov–Taylor operator. In this section, we will just obtain the Slavnov–Taylor equation for the 1PI functional and define the linearized Slavnov–Taylor and its cohomology problem. Since the theory is not renormalizable by power counting, the solution of this cohomology problem will demand the use of the highbrow algebraic cohomological and homological methods reviewed in Ref. [27] and this lies quite outside the scope of our paper.

Let us begin the derivation of the formal Slavnov–Taylor equation. Let $\mathcal{S}$ be the action in Eq. (2.40). Let Φ generically denote the fields of our theory. These fields are

$$h_{\mu\nu}, B_{\mu\nu}, B_\mu, B, C_{\mu\nu}, C_\mu, C, \overline{C}_{\mu\nu}, \overline{C}_\mu, \overline{C}, D_\mu, D, \overline{D}_\mu,$$
$$\overline{D}, T, \overline{T}, R, \overline{R} \text{ and } L.$$

Then, the generating functional of the complete Green functions of our theory reads

$$Z\left[J^{\mu\nu}, j^{\mu\nu}, ...\right] = \int \mathcal{D}\Phi \left[\mathcal{D}\text{ghost}\right] e^{i\mathcal{S}+i\int d^4x\, \mathcal{C}}, \quad (5.1)$$

where

$$\begin{aligned}
\mathcal{C} &= J^{\mu\nu}h_{\mu\nu} + j^{\mu\nu}B_{\mu\nu} + j^\mu B_\mu + jB + fL + \overline{\omega}^{\mu\nu}C_{\mu\nu} \\
&\quad + \overline{\omega}^\mu C_\mu + \overline{\omega}C + \overline{C}_{\mu\nu}\omega^{\mu\nu} \\
&\quad + \overline{C}_\mu \omega^\mu + \overline{C}\omega + \overline{q}R + \overline{R}q + \overline{k}^\mu D_\mu + \overline{k}D \\
&\quad + \overline{D}_\mu k^\mu + \overline{D}k + \overline{\chi}T + \overline{T}\chi + \xi^{\mu\nu}\mathfrak{B}h_{\mu\nu} + \phi\mathfrak{B}L \\
&\quad + \rho^{\mu\nu}\mathfrak{B}C_{\mu\nu} + \tau\mathfrak{B}R + \sigma\mathfrak{B}\overline{R} + \rho^\mu\mathfrak{B}D_\mu + \rho\mathfrak{B}T.
\end{aligned}$$

Recall that $\mathfrak{B}$ is the complete BRST operator in Eq. (2.37). Notice that we have included external fields coupled to the BRST variations

$$\mathfrak{B}h_{\mu\nu}, \mathfrak{B}L, \mathfrak{B}C_{\mu\nu}, \mathfrak{B}R, \mathfrak{B}\overline{R}, \mathfrak{B}D_\mu \text{ and } \mathfrak{B}T,$$

for those variations are composite operators.

Taking into account that, by definition, the external fields $J^{\mu\nu}, j^{\mu\nu},....$ are annihilated by $\mathfrak{B}$ and so it is $\mathcal{S}$, we conclude that

$$\left\langle \int d^4x \left(J^{\mu\nu}\mathfrak{B}h_{\mu\nu} + j^\mu C_\mu + f\mathfrak{B}L - \overline{\omega}^{\mu\nu}\mathfrak{B}C_{\mu\nu}\right.\right.$$
$$\left.- i\overline{\omega}D + iB_{\mu\nu}\omega^{\mu\nu} + iB\omega\right.$$

$$- \overline{q}\mathfrak{B}R + q\mathfrak{B}\overline{R} + \overline{k}^\mu \mathfrak{B}D_\mu + \overline{C}_\mu k^\mu$$

$$\left. - \overline{\chi}\mathfrak{B}T + i\overline{D}\chi \right) \right\rangle = 0. \tag{5.2}$$

where $\langle A \rangle \equiv \int \mathcal{D}\Phi \, A \cdot \exp[iS + iC]$. The previous equation is obtained by applying $\mathfrak{B}$ to both sides of Eq. (5.1).

Equation (5.2) can be recast into the following form

$$\left[\int d^4x \left(J^{\mu\nu}\frac{\delta}{\delta \xi^{\mu\nu}} + j^\mu \frac{\delta}{\delta \overline{\omega}^\mu} + f\frac{\delta}{\delta \phi} - \overline{\omega}^{\mu\nu}\frac{\delta}{\delta \rho^{\mu\nu}} \right. \right.$$
$$- i\overline{\omega}\frac{\delta}{\delta \overline{k}} + i\omega^{\mu\nu}\frac{\delta}{\delta j^{\mu\nu}} + i\omega\frac{\delta}{\delta j} - \overline{q}\frac{\delta}{\delta \tau} + q\frac{\delta}{\delta \sigma}$$
$$\left. \left. + \overline{k}^\mu \frac{\delta}{\delta \rho^\mu} - k^\mu \frac{\delta}{\delta \omega^\mu} - \overline{\chi}\frac{\delta}{\delta \rho} + i\chi\frac{\delta}{\delta k} \right) \right] Z = 0. \tag{5.3}$$

Let $\mathfrak{J}$ be given by

$$\mathfrak{J} = J^{\mu\nu}h_{\mu\nu} + j^{\mu\nu}B_{\mu\nu} + j^\mu B_\mu + jB + fL + \overline{\omega}^{\mu\nu}C_{\mu\nu}$$
$$+ \overline{\omega}^\mu C_\mu + \overline{\omega}C + \overline{C}_{\mu\nu}\omega^{\mu\nu} + \overline{C}_\mu \omega^\mu + \overline{C}\omega + \overline{q}R$$
$$+ \overline{R}q + \overline{k}^\mu D_\mu + \overline{k}D + \overline{D}_\mu k^\mu + \overline{D}k + \overline{\chi}T + \overline{T}\chi.$$

Then, the 1PI functional of our theory is defined as usual:

$$\Gamma[\Phi] = W[J^{\mu\nu}, j^{\mu\nu}, ...] - \int d^4x \, \mathfrak{J},$$

where $W[J^{\mu\nu}, j^{\mu\nu}, ...]$ is such that

$$Z[J^{\mu\nu}, j^{\mu\nu}, ...]/Z[0] = \exp\left(iW[J^{\mu\nu}, j^{\mu\nu}, ...]\right)$$

Using these definitions is not difficult to show that Eq. (5.3) is equivalent to the following equation

$$\int d^4x \left[\frac{\delta\Gamma}{\delta h_{\mu\nu}}\frac{\delta\Gamma}{\delta \xi^{\mu\nu}} + \frac{\delta\Gamma}{\delta B_\mu}C_\mu + \frac{\delta\Gamma}{\delta L}\frac{\delta\Gamma}{\delta \phi} + \frac{\delta\Gamma}{\delta C_{\mu\nu}}\frac{\delta\Gamma}{\delta \rho^{\mu\nu}} \right.$$
$$+ i\frac{\delta\Gamma}{\delta C}D + i\frac{\delta\Gamma}{\delta \overline{C}_{\mu\nu}}B_{\mu\nu} + i\frac{\delta\Gamma}{\delta \overline{C}}B + \frac{\delta\Gamma}{\delta R}\frac{\delta\Gamma}{\delta \tau} + \frac{\delta\Gamma}{\delta \overline{R}}\frac{\delta\Gamma}{\delta \sigma}$$
$$\left. + \frac{\delta\Gamma}{\delta D_\mu}\frac{\delta\Gamma}{\delta \rho^\mu} + \frac{\delta\Gamma}{\delta \overline{D}_\mu}\overline{C}_\mu + \frac{\delta\Gamma}{\delta T}\frac{\delta\Gamma}{\delta \rho} + i\frac{\delta\Gamma}{\delta \overline{T}}\overline{D} \right] = 0.$$

This is the Slavnov–Taylor equation. This equation governs the BRST equation at the quantum level.

By expanding $\Gamma[\Phi]$ in powers of $\hbar$, one reaches the conclusion that the Slavnov–Taylor equation implies that the one-loop contribution, $\Gamma_1[\Phi]$, to $\Gamma[\Phi]$ must satisfy the linearized Slavnov–Taylor equation, which reads

$$\Delta\Gamma_1[\Phi] = 0, \tag{5.4}$$

where Δ is the so-called linearized Slavnov–Taylor operator:

$$\Delta = \int d^4x \left[\frac{\delta\Gamma_0}{\delta h_{\mu\nu}}\frac{\delta}{\delta \xi^{\mu\nu}} + \frac{\delta\Gamma_0}{\delta \xi_{\mu\nu}}\frac{\delta}{\delta h^{\mu\nu}} + C_\mu \frac{\delta}{\delta B_\mu} + \frac{\delta\Gamma_0}{\delta L}\frac{\delta}{\delta \phi} \right.$$
$$+ \frac{\delta\Gamma_0}{\delta \phi}\frac{\delta}{\delta L} + \frac{\delta\Gamma_0}{\delta C_{\mu\nu}}\frac{\delta}{\delta \rho^{\mu\nu}} + \frac{\delta\Gamma_0}{\delta \rho_{\mu\nu}}\frac{\delta}{\delta C^{\mu\nu}}$$
$$+ iD\frac{\delta}{\delta C} + iB_{\mu\nu}\frac{\delta}{\delta \overline{C}_{\mu\nu}} + iB\frac{\delta}{\delta \overline{C}} + \frac{\delta\Gamma_0}{\delta R}\frac{\delta}{\delta \tau}$$
$$+ \frac{\delta\Gamma_0}{\delta \tau}\frac{\delta}{\delta R} + \frac{\delta\Gamma_0}{\delta \overline{R}}\frac{\delta}{\delta \sigma} + \frac{\delta\Gamma_0}{\delta D_\mu}\frac{\delta}{\delta \rho^\mu} + \frac{\delta\Gamma_0}{\delta \rho_\mu}\frac{\delta}{\delta D^\mu}$$
$$\left. + \overline{C}_\mu \frac{\delta}{\delta \overline{D}_\mu} + \frac{\delta\Gamma_0}{\delta T}\frac{\delta}{\delta \rho} + \frac{\delta\Gamma_0}{\delta \rho}\frac{\delta}{\delta T} + i\overline{D}\frac{\delta}{\delta \overline{T}} \right],$$

where

$$\Gamma_0 = S + \int d^4x \left[\xi^{\mu\nu}\mathfrak{B}h_{\mu\nu} + \phi\mathfrak{B}L + \rho^{\mu\nu}\mathfrak{B}C_{\mu\nu} + \tau\mathfrak{B}R \right.$$
$$\left. + \sigma\mathfrak{B}\overline{R} + \rho^\mu \mathfrak{B}D_\mu + \rho\mathfrak{B}T \right].$$

and S is given in Eq. (2.40).

It can be shown that the linearized Slavnov–Taylor operator is nilpotent

$$\Delta^2 = 0.$$

and it has ghost number equal to 1. Hence, any one-loop anomalous breaking, $\mathcal{A}$, of the linearized Slavnov–Taylor equation – thus breaking the BRST symmetry – must satisfy

$$\Delta\mathcal{A} = 0, \quad \mathcal{A} \neq \Delta\Xi,$$

$\mathcal{A}$ being a (infinite) linear combination of monomials in the fields and their derivatives which has ghost number equal to 1. Ξ is also a linear combination of monomial of the fields and their derivatives but it has got zero ghost number. The solution to this problem is a very involved one and lies well beyond the scope of paper. If the BRST symmetry can be restored at one-loop – i.e, $\mathcal{A} = 0$, one has to face the very same type of cohomology problem at two loops, and so on.

6 Conclusions

The main conclusion of the paper is that a modification of the BRST transformations of Ref. [16] can be combined with the BRST transformations arising from the extra Weyl symmetry of our unimodular theory into a set of transformations generated by a single BRST operator. In principle, that this combination be feasible is a non trivial fact, for it involves different types of BRST transformations. We obtained the full gauge-fixed action of the theory and use it to compute the Newton's potential and the Slavnov–Taylor equation, the latter governs full BRST symmetry of the theory at the quantum level.

Let us point out that the derivation of the Slavnov–Taylor equation is a must. Indeed, there is no known regularization method that preserves the symmetries of the theory and therefore non BRST invariant counterterms must be introduced to compensate for the breaking of BRST invariance due to the regularization and/or renormalization process one uses. Recall that the ghost content of the theory depends on the spacetime dimension: dimensional regularization does not furnish a BRST invariant regularized theory. Also recall that the construction of those non BRST invariant counterterms and the analysis of their consistency is guided by the cohomology of the linearized Slavnov–Taylor operator.

A final comment, in Ref. [28] another way of fixing the transverse gauge invariance of unimodular gravity is introduced. In [28] the unimodularity condition of the gravity field is introduced by using a Lagrange multiplier so that, again, the linear splitting of the gravity field leads to a meaningless quantum theory – the exponential parametrization would save the day. It is plain that the formalism developed in the previous sections can be combined easily with the BRST formalism in [28] without the need of a Lagrange multiplier. In other words the formalism of [28] can be adapted appropriately to quantize unimodular gravity with Weyl invariace.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This manuscript has no associated data in any data repository].

References

1. R. Carballo-Rubio, L.J. Garay, G. García-Moreno, Class. Quantum Gravity **39**(24), 243001 (2022). https://doi.org/10.1088/1361-6382/aca386. arXiv:2207.08499 [gr-qc]
2. E. Alvarez, E. Velasco-Aja. arXiv:2301.07641 [gr-qc]
3. J.J. van der Bij, H. van Dam, Y.J. Ng, Physica A **116**, 307–320 (1982). https://doi.org/10.1016/0378-4371(82)90247-3
4. A. Zee, Stud. Nat. Sci. **20**, 211–230 (1985). https://doi.org/10.1007/978-1-4684-8848-7_16
5. W. Buchmuller, N. Dragon, Phys. Lett. B **207**, 292–294 (1988). https://doi.org/10.1016/0370-2693(88)90577-1
6. M. Henneaux, C. Teitelboim, Phys. Lett. B **222**, 195–199 (1989). https://doi.org/10.1016/0370-2693(89)91251-3
7. W. Buchmuller, N. Dragon, JHEP **08**, 167 (2022). https://doi.org/10.1007/JHEP08(2022)167. arXiv:2203.15714 [hep-th]
8. A. Eichhorn, Class Quantum Gravity **30**, 115016 (2013). https://doi.org/10.1088/0264-9381/30/11/115016. arXiv:1301.0879 [gr-qc]
9. A. Padilla, I.D. Saltas, Eur. Phys. J. C **75**(11), 561 (2015). https://doi.org/10.1140/epjc/s10052-015-3767-0. arXiv:1409.3573 [gr-qc]
10. E. Álvarez, S. González-Martín, M. Herrero-Valea, C.P. Martín, JHEP **08**, 078 (2015). https://doi.org/10.1007/JHEP08(2015)078. arXiv:1505.01995 [hep-th]
11. R. Bufalo, M. Oksanen, A. Tureanu, Eur. Phys. J. C **75**(10), 477 (2015). https://doi.org/10.1140/epjc/s10052-015-3683-3. arXiv:1505.04978 [hep-th]
12. R. de León Ardón, N. Ohta, R. Percacci, Phys. Rev. D **97**(2), 026007 (2018). https://doi.org/10.1103/PhysRevD.97.026007. arXiv:1710.02457 [gr-qc]
13. G.P. De Brito, A. Eichhorn, A.D. Pereira, JHEP **09**, 100 (2019). https://doi.org/10.1007/JHEP09(2019)100. arXiv:1907.11173 [hep-th]
14. L. Baulieu, Phys. Lett. B **808**, 135591 (2020). https://doi.org/10.1016/j.physletb.2020.135591. arXiv:2004.05950 [hep-th]
15. G.P. de Brito, O. Melichev, R. Percacci, A.D. Pereira, JHEP **12**, 090 (2021). https://doi.org/10.1007/JHEP12(2021)090. arXiv:2105.13886 [gr-qc]
16. T. Kugo, R. Nakayama, N. Ohta, Phys. Rev. D **105**(8), 086006 (2022). https://doi.org/10.1103/PhysRevD.105.086006. arXiv:2202.03626 [hep-th]
17. E. Álvarez, J. Anero, C.P. Martin, E. Velasco-Aja, Phys. Rev. D **108**(2), 026013 (2023). https://doi.org/10.1103/PhysRevD.108.026013. arXiv:2304.05188 [hep-th]
18. N. Ohta, R. Percacci, A.D. Pereira, JHEP **06**, 115 (2016). https://doi.org/10.1007/JHEP06(2016)115. arXiv:1605.00454 [hep-th]
19. A. Nink. https://doi.org/10.25358/openscience-2790. arXiv:1701.08344 [hep-th]
20. E. Alvarez, D. Blas, J. Garriga, E. Verdaguer, Nucl. Phys. B **756**, 148–170 (2006). https://doi.org/10.1016/j.nuclphysb.2006.08.003. arXiv:hep-th/0606019
21. E. Alvarez, JHEP **03**, 002 (2005). https://doi.org/10.1088/1126-6708/2005/03/002. arXiv:hep-th/0501146
22. J. Anero, C.P. Martin, Phys. Rev. D **107**(4), 046001 (2023). https://doi.org/10.1103/PhysRevD.107.046001. arXiv:2211.01130 [hep-th]
23. L.J. Garay, G. García-Moreno, JHEP **03**, 027 (2023). https://doi.org/10.1007/JHEP03(2023)027. arXiv:2301.03503 [hep-th]
24. J. Anero, C.P. Martin, E. Velasco-Aja, JHEP **08**, 066 (2023). https://doi.org/10.1007/JHEP08(2023)066. arXiv:2306.16832 [hep-th]
25. M.J.G. Veltman, The quantum theory of gravitation. Conf. Proc. C **7507281**, 265–327 (1975)
26. O. Piguet, S.P. Sorella, Lect. Notes Phys. Monogr. **28**, 1–134 (1995). https://doi.org/10.1007/978-3-540-49192-7
27. G. Barnich, F. Brandt, M. Henneaux, Phys. Rep. **338**, 439–569 (2000). https://doi.org/10.1016/S0370-1573(00)00049-1. arXiv:hep-th/0002245
28. T. Kugo, R. Nakayama, N. Ohta, Phys. Rev. D **105**(10), 106006 (2022). https://doi.org/10.1103/PhysRevD.105.106006. arXiv:2202.10740 [hep-th]