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Conserved charges of the Kerr black hole revisited

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Abstract We revisit the Kerr black hole as cast in the Boyer–Lindquist, Kerr–Schild and Weyl canonical coordinates, and calculate its total mass/energy and total angular momentum using linearized gravity along with its background Killing isometries. We argue that the integration of the relevant gravitational flux does not depend on the geometry of the closed and simply connected spatial boundary provided it is also piecewise smooth.

Contents

1 Introduction

We are obviously living in a new era of gravitational physics ever since the first successful experimental detection of gravitational waves from the collision of two black holes [1] and the first image of a black hole distinguishing its horizon structure [2] in the way predicted by General Relativity. The gravitational wave observations so far are thought to consist of the products of collisions of compact astrophysical rotating objects. Since collisions of stationary black holes seem to form a rather large class in this catalogue, it is only natural that we rely heavily on the celebrated Kerr solution [3] in the analysis and understanding of the basic physical properties of these collision events.

The Kerr solution has, of course, been thoroughly studied over the years, and its basic physical and geometrical properties are well understood (see, e.g., [4,5] for a concise introduction and starting point). Specifically, the determination of its conserved gravitational charges, in particular its total mass/energy and total angular momentum, is standard textbook material [6]. Here we want to *revisit* the particular method based on linearized gravity and the use of background Killing isometries [7–9], which is in fact closely related to the celebrated Arnowitt–Deser–Misner (ADM) formulation [10] of General Relativity. We do this in three separate coordinate systems: the ubiquitous Boyer–Lindquist coordinates [11], the Kerr–Schild form [12,13] and the lesser-known Weyl canonical coordinates [5,14]. Our motivation in doing so is, first and foremost, to examine how the integration of the gravitational flux through "the boundary at infinity" depends on the geometry of the latter. For the three poses of the Kerr solution studied, "the cube at infinity" for the Kerr–Schild form and "the cylinder at infinity" for the Weyl canonical coordinates are certainly not smooth geometries as for "the sphere at infinity" for the Boyer–Lindquist case. As a by-product, we also explicitly verify that the whole procedure is indeed background gauge-invariant.

The paper is organized as follows. In Sect. 2, we start by giving a concise, self-contained description of how gravitational charges are defined using background Killing isometries. We explicitly calculate the total mass/energy and the total angular momentum of the Kerr black hole cast in the Boyer–Lindquist, Kerr–Schild and Weyl canonical coordinates in Sects. 3, 4, and 5, respectively. To our knowledge, the discussion presented in Sects. 4 and 5 is novel. We conclude with a brief summary and some remarks in Sect. 6. Finally,

Dedicated to the memory of Stanley Deser (March 19, 1931–April 21, 2023).

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we explicitly relegate some unwieldy formulas needed for the discussion in Sect. 5 to Appendix A.

2 Conserved charges in linearized gravity

Let us briefly recapitulate how gravitational charges that use background Killing isometries are defined. (Please refer to [7–9] for details.) Let h_{ab} denote the "deviation" of an asymptotically flat metric from its flat "background" $\bar{g}_{ab}$,[1] i.e., $h_{ab} \equiv g_{ab} - \bar{g}_{ab}$, where $\bar{g}_{ab}$ is the Ricci flat background with curvature tensors $\bar{R}_{ab} = 0 = \bar{R}$. It is also assumed that the deviation h_{ab} goes to zero "sufficiently fast" as one approaches the background $\bar{g}_{ab}$, which is typically located at "the boundary at infinity." It can be shown [7–9] that one can construct a conserved vector current $J^a := G_L^{ab}\bar{\xi}_b$, with $\bar{\nabla}_a J^a = 0$, out of the linearized Einstein tensor G_L^{ab} and the background Killing vector $\bar{\xi}^a$, which are well defined and smooth in the geometry described by $\bar{g}_{ab}$. Then the following gives a conserved and background gauge-invariant gravitational charge

$$Q(\bar{\xi}) = \frac{1}{8\pi}\int_\Sigma d^3x\,\sqrt{|\gamma|}\,n_a\,J^a = \frac{1}{8\pi}\oint_{\partial\Sigma} d^2x\,\sqrt{|q|}\,n_{[a}\,r_{b]}\,\ell^{ab}, \tag{2.1}$$

where ℓ_{ab} is the potential 2-form of the conserved vector current $J^a := \bar{\nabla}_b \ell^{ab}$.

Here we assume that the background admits a foliation

$$\bar{g}_{ab} := -n_a n_b + \gamma_{ab} = -n_a n_b + r_a r_b + q_{ab}, \tag{2.2}$$

where the induced non-degenerate metric γ_{ab} on the three-dimensional spacelike hypersurface Σ has a timelike normal n^a and $\gamma := \det \gamma_{ab}$ which are well defined everywhere. We also assume that the hypersurface Σ has a boundary $\partial\Sigma$ with a metric q_{ab}, with $q := \det q_{ab}$; n^a and the spacelike vector r^a are mutually orthogonal unit vectors to $\partial\Sigma$, with $n^a n_a = -1$ and $r^a r_a = 1$.

For the conventions adopted in this work, one has [7–9]

$$\ell^{ab}(\bar{\xi}) := \bar{\xi}_c \bar{\nabla}^{[a} h^{b]c} + \bar{\xi}^{[b}\bar{\nabla}_c h^{a]c} + h^{c[b}\bar{\nabla}_c \bar{\xi}^{a]} + \bar{\xi}^{[a}\bar{\nabla}^{b]} h + \frac{1}{2} h \bar{\nabla}^{[a}\bar{\xi}^{b]}, \tag{2.3}$$

where all raising and lowering of indices is done with respect to $\bar{g}_{ab}$, $\bar{\nabla}$ indicates the covariant derivative with respect to the background metric, and the antisymmetrization of indices is done with "weight 1," i.e. $A_{[ab]} \equiv \frac{1}{2}(A_{ab} - A_{ba})$.

All of these are of course well known. Our aim here is to carefully apply these to the celebrated Kerr metric written in three separate coordinates, and show how "delicate" the integration of the properly weighted 2-form potential ℓ on the two-dimensional boundary $\partial\Sigma$ (2.1) can be. For warming up, we start our discussion by first considering the Kerr black hole in the ubiquitous Boyer–Lindquist coordinates.

[1] Note that this discussion in fact holds for asymptotically maximally symmetric spacetimes, but we simply work with a vanishing cosmological constant in what follows.

3 Kerr black hole in Boyer–Lindquist coordinates

The Kerr black hole [3] in Boyer–Lindquist coordinates [4, 11] reads

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}\,dt\,d\phi + \frac{\Sigma}{\Delta}\,dr^2 + \Sigma\,d\theta^2 + \left(r^2 + a^2 + \frac{2a^2Mr\sin^2\theta}{\Sigma}\right)\sin^2\theta\,d\phi^2, \tag{3.1}$$

where the metric functions are given by

$$\Sigma(r,\theta) := r^2 + a^2\cos^2\theta \text{ and } \Delta(r) := r^2 + a^2 - 2Mr. \tag{3.2}$$

The relevant flat background is found by setting $M = 0$, $a = 0$ in (3.1):

$$\bar{g}_{ab} = \mathrm{diag}(-1, 1, r^2, r^2\sin^2\theta), \tag{3.3}$$

which admits the foliation (2.2) with

$$\gamma_{ab} = \mathrm{diag}(0, 1, r^2, r^2\sin^2\theta) \quad \text{and} \quad q_{ab} = \mathrm{diag}(0, 0, r^2, r^2\sin^2\theta), \quad n^a = (-\partial_t)^a \quad \text{and} \quad r^a = (\partial_r)^a. \tag{3.4}$$

Clearly, γ_{ab} is the metric in the three-dimensional Euclidean space Σ in spherical polar coordinates, and q_{ab} is the metric on a 2-sphere S_2 of radius r. The two-dimensional boundary $\partial\Sigma$ is found when one takes $r \to \infty$. The background clearly has two globally defined timelike $\bar{\xi}^a = (-\partial_t)^a$ and spacelike $\bar{\zeta}^a = (\partial_\phi)^a$ Killing vectors.

The relevant components of the 2-form potential (2.3) for each Killing vector are

$$\ell^{tr}(\bar{\xi}) = \frac{1}{2r^3}\left(\frac{2a^2Mr^2\sin^2\theta\,\Sigma'}{\Sigma^2} + \left(\frac{2r^2}{\Delta} + 1\right)\Sigma - r\Sigma' + a^2 - r^2\right), \tag{3.5}$$

$$\ell^{tr}(\bar{\zeta}) = \frac{aM\sin^2\theta\,(r\Sigma' + \Sigma)}{\Sigma^2}, \tag{3.6}$$

in terms of the metric functions (3.2), where a prime indicates the derivative with respect to the r coordinate. A careful integration gives

$$\oint_{S_2} r^2 \mathrm{d}\Omega\, n_{[a}\, r_{b]}\, \ell^{ab}(\bar{\xi}) = \frac{4\pi}{3}\left(\frac{r\left(a^2+3r^2\right)}{\Delta} + \frac{2a^3+3M\left(a^2-r^2\right)\arctan(a/r)+3ar(M-r)}{ar}\right), \tag{3.7}$$

where we have used $\mathrm{d}\Omega := \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$, which leads to

$$Q(\bar{\xi}) = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_2} r^2 \mathrm{d}\Omega\, n_{[a}\, r_{b]}\, \ell^{ab}(\bar{\xi}) = M, \tag{3.8}$$

the total mass of the Kerr black hole. Similarly, one finds

$$\oint_{S_2} r^2 \mathrm{d}\Omega\, n_{[a}\, r_{b]}\, \ell^{ab}(\bar{\zeta}) = 8\pi M r \arctan(a/r), \tag{3.9}$$

which yields

$$Q(\bar{\zeta}) = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_2} r^2 \mathrm{d}\Omega\, n_{[a}\, r_{b]}\, \ell^{ab}(\bar{\zeta}) = aM, \tag{3.10}$$

the total angular momentum of the Kerr black hole.

In hindsight, what we did was to first integrate the properly furbished 2-form potential ℓ over a constant radius 2-sphere S_2, and then take the limit as radius $r \to \infty$. It is perhaps worth considering the "shortcut" of first taking the $r \to \infty$ limit of the properly furbished 2-form potential ℓ first and then integrating the result on "the sphere at infinity" later. This interchange is in general clearly not up to par with mathematical rigor, but it should be possible for a "smooth" boundary $\partial\Sigma$ and a "smooth" potential ℓ. Thus, e.g.,

$$\frac{1}{8\pi}\oint \mathrm{d}\Omega \left(\lim_{r\to\infty} r^2\, n_{[a}\, r_{b]}\, \ell^{ab}(\bar{\xi})\right) = \frac{1}{8\pi}\oint \mathrm{d}\Omega\,(2M) = M\,,$$

$$\frac{1}{8\pi}\oint \mathrm{d}\Omega \left(\lim_{r\to\infty} r^2\, n_{[a}\, r_{b]}\, \ell^{ab}(\bar{\zeta})\right) = \frac{1}{8\pi}\oint \mathrm{d}\Omega\,(3aM\sin^2\theta) = aM\,, \tag{3.11}$$

which "validates" our hunch for the shortcut.

4 Kerr black hole in Kerr–Schild coordinates

Let us consider the Kerr black hole in its Kerr–Schild form [12,13,15,16]:

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \left(\eta_{ab} + \frac{2M}{U}\lambda_a\lambda_b\right)\mathrm{d}x^a\mathrm{d}x^b, \tag{4.1}$$

where λ^a is null and geodesic with respect to both the full metric g_{ab} and the flat Minkowski metric η_{ab}. Explicitly, in flat coordinates (t, x, y, z), one has

$$\lambda = \lambda_a \mathrm{d}x^a = \mathrm{d}t + \frac{r(x\mathrm{d}x+y\mathrm{d}y)+a(y\mathrm{d}x-x\mathrm{d}y)}{r^2+a^2} + \frac{z\mathrm{d}z}{r} \tag{4.2}$$

and

$$U = r + \frac{a^2 z^2}{r^3}, \tag{4.3}$$

where r is not a coordinate but a function of (x, y, z) and is defined by

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1. \tag{4.4}$$

Note that the Eddington-like coordinates [11] $(\bar{t}, r, \theta, \bar{\phi})$ can be obtained from (4.1) by the coordinate transformation

$$\begin{aligned} t &= \bar{t}. \\ x &= \sqrt{r^2+a^2}\,\sin\theta\,\cos\left(\bar{\phi}+\arctan(a/r)\right), \\ y &= \sqrt{r^2+a^2}\,\sin\theta\,\sin\left(\bar{\phi}+\arctan(a/r)\right), \\ z &= r\cos\theta. \end{aligned} \tag{4.5}$$

A final time-dependent coordinate transformation is required to arrive at the Kerr metric in Boyer–Lindquist coordinates (t, r, θ, ϕ) as in (3.1)

$$\mathrm{d}\bar{t} = \mathrm{d}t + \frac{2Mr}{\Delta}\mathrm{d}r,$$
$$\mathrm{d}\bar{\phi} = \mathrm{d}\phi + \frac{a}{\Delta}\mathrm{d}r.$$

One advantage of the Kerr–Schild form (4.1) is that the background metric is clearly the flat Minkowski metric η_{ab} in the usual Cartesian coordinates. This background admits the foliation (2.2) with

$$\gamma_{ab} = \mathrm{diag}(0, 1, 1, 1), \quad n^a = (-\partial_t)^a, \tag{4.6}$$

where γ_{ab} is the metric on the three-dimensional flat Euclidean space in the Cartesian coordinates. Now the question arises about the integration on the boundary of the space. The boundary relevant for the charge definition (2.1) can be thought of as "the cube at infinity" in Cartesian coordinates. This can be best described by first considering a cube centered at the origin and of side length $2L$, and then taking the $L \to \infty$ limit.

Any such cube of finite side length has six faces, and the integration on each face must be carried out with the following normal vectors and the corresponding two-dimensional metrics in mind:

$$\begin{aligned} x^a_\pm &= \left(\pm\partial_x\right)^a \quad \text{and} \quad \left(q_x\right)_{ab} = \mathrm{diag}(0, 0, 1, 1)\,, \\ y^a_\pm &= \left(\pm\partial_y\right)^a \quad \text{and} \quad \left(q_y\right)_{ab} = \mathrm{diag}(0, 1, 0, 1)\,, \\ z^a_\pm &= \left(\pm\partial_z\right)^a \quad \text{and} \quad \left(q_z\right)_{ab} = \mathrm{diag}(0, 1, 1, 0)\,. \end{aligned} \tag{4.7}$$

"The cube at infinity" presumably has its faces placed at the $L \to \infty$ limits of the coordinates along the direction of the normal vectors. Thus the integration on the boundary can be thought of as the sum of six separate integrals on each face of "the cube at infinity"

$$
\begin{aligned}
Q(\bar{\xi}) = \frac{1}{8\pi} \lim_{L\to\infty} \Bigg(& \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}y\,\mathrm{d}z\, n_{[a}\, x_{b]}\, \ell^{ab}(\bar{\xi})|_{x=L} \\
& - \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}y\,\mathrm{d}z\, n_{[a}\, x_{b]}\, \ell^{ab}(\bar{\xi})|_{x=-L} \\
& + \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}z\, n_{[a}\, y_{b]}\, \ell^{ab}(\bar{\xi})|_{y=L} \\
& - \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}z\, n_{[a}\, y_{b]}\, \ell^{ab}(\bar{\xi})|_{y=-L} \\
& + \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}y\, n_{[a}\, z_{b]}\, \ell^{ab}(\bar{\xi})|_{z=L} \\
& - \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}y\, n_{[a}\, z_{b]}\, \ell^{ab}(\bar{\xi})|_{z=-L} \Bigg) . \qquad (4.8)
\end{aligned}
$$

Using (4.4) and (4.7), this explicitly becomes

$$
\begin{aligned}
Q(\bar{\xi}) = \frac{1}{8\pi} \lim_{L\to\infty} \Bigg(& \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}y\,\mathrm{d}z\, \ell^{tx}(\bar{\xi})|_{x=L} \\
& - \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}y\,\mathrm{d}z\, \ell^{tx}(\bar{\xi})|_{x=-L} \\
& + \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}z\, \ell^{ty}(\bar{\xi})|_{y=L} \\
& - \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}z\, \ell^{ty}(\bar{\xi})|_{y=-L} \\
& + \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}y\, \ell^{tz}(\bar{\xi})|_{z=L} \\
& - \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}y\, \ell^{tz}(\bar{\xi})|_{z=-L} \Bigg) , \qquad (4.9)
\end{aligned}
$$

keeping a and M fixed, and keeping the definition of r (4.4) in mind. The relevant timelike Killing vector of the background is still $\bar{\xi}^a = \big(-\partial_t\big)^a$, which generates three nontrivial components for the 2-form potential ℓ (2.3)

$$
\begin{aligned}
\ell^{tx}(\bar{\xi}) = & \frac{Mr^2}{(a^2+r^2)^3(a^2z^2+r^4)^3} \\
& \times\Big(2r^{13}x + a^9yz^4 - 2a^8rxz^4 - 2a^7r^2yz^2(r^2-3z^2) \\
& - a^6r^3xz^2\big(4r^2+3(x^2+y^2)\big) \\
& + a^5r^4y\big(r^4+z^2(4x^2+4y^2+5z^2)\big) \\
& - a^4r^5x(r^2+z^2)(2r^2-x^2-y^2-2z^2) \\
& + 2a^3r^8y(3r^2-z^2) \\
& + a^2r^9x(5x^2+5y^2+4z^2) \\
& + ar^{10}y\big(5r^2-4(x^2+y^2+z^2)\big)\Big) , \qquad (4.10a)
\end{aligned}
$$

$$
\begin{aligned}
\ell^{ty}(\bar{\xi}) = & \frac{Mr^2}{(a^2+r^2)^3(a^2z^2+r^4)^3} \\
& \times\Big(2r^{13}y - a^9xz^4 - 2a^8ryz^4 + 2a^7r^2xz^2(r^2-3z^2) \\
& - a^6r^3yz^2\big(4r^2+3(x^2+y^2)\big) \\
& - a^5r^4x\big(r^4+z^2(4x^2+4y^2+5z^2)\big) \\
& - a^4r^5y(r^2+z^2)(2r^2-x^2-y^2-2z^2) \\
& + 2a^3r^8x(z^2-3r^2) \\
& + a^2r^9y(5x^2+5y^2+4z^2) \\
& + ar^{10}x\big(4(x^2+y^2+z^2)-5r^2\big)\Big) , \qquad (4.10b)
\end{aligned}
$$

$$
\begin{aligned}
\ell^{tz}(\bar{\xi}) = & \frac{Mr^3z}{(a^2+r^2)^2(a^2z^2+r^4)^3} \\
& \times\Big(2r^{10} - a^6z^2(x^2+y^2-2z^2) \\
& + a^4\big(r^4(3x^2+3y^2+4z^2)+r^2z^2(x^2+y^2+2z^2)\big) \\
& + a^2r^6(2r^2+5x^2+5y^2+4z^2)\Big) . \qquad (4.10c)
\end{aligned}
$$

To arrive at the expressions (4.10), we have utilized

$$
\frac{\partial r}{\partial x} = \frac{x}{U} , \qquad \frac{\partial r}{\partial y} = \frac{y}{U} , \qquad \frac{\partial r}{\partial z} = \frac{z}{U}\Big(1+\frac{a^2}{r^2}\Big) , \qquad (4.11)
$$

$$
\begin{aligned}
& \frac{\partial U}{\partial x} = \frac{x}{U}\Big(1-\frac{3a^2z^2}{r^4}\Big) , \qquad \frac{\partial U}{\partial y} = \frac{y}{U}\Big(1-\frac{3a^2z^2}{r^4}\Big) , \\
& \frac{\partial U}{\partial z} = \frac{z}{U}\Big(1-\frac{3a^2z^2}{r^4}\Big)\Big(1+\frac{a^2}{r^2}\Big) + \frac{2a^2z}{r^3} , \qquad (4.12)
\end{aligned}
$$

which follow from the definitions of r (4.4) and U (4.3), respectively.

Next we substitute (4.10) into (4.9). Note that the variable r (4.4) is symmetric in x and y by definition. Thus, we can interchange the latter in the integration for the terms that include, e.g., ℓ^{ty}. A careful consideration along the yz- and xz-surface leads to

$$
\begin{aligned}
Q(\bar{\xi}) = \frac{M}{4\pi} \lim_{L\to\infty} \Bigg(& \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}z\, \frac{2\,r(x,L,z)^3L}{(r^2+a^2)^3(r^4+a^2z^2)^3} \\
& \times\big(2r^{12} - 2a^8z^4 - a^6r^2z^2\big(4r^2+3(x^2+L^2)\big) \\
& - a^4r^4(r^2+z^2)(2r^2-x^2-L^2-2z^2) \\
& + a^2r^8(5x^2+5L^2+4z^2)\big) \\
& + \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}y\, \frac{r(x,y,L)^3L}{(r^2+a^2)^2(r^4+a^2L^2)^3} \\
& \times\big(2r^{10} - a^6L^2(x^2+y^2-2L^2) \\
& + a^4r^2\big(r^2(3x^2+3y^2+4L^2)+L^2(x^2+y^2+2L^2)\big) \\
& + a^2r^6(2r^2+5x^2+5y^2+4L^2)\big)\Bigg) , \qquad (4.13)
\end{aligned}
$$

where we have implicitly indicated the relevant $y = \pm L$ and $z = \pm L$ substitutions in r (4.4) by writing it with its pertinent arguments. Using (4.4) to eliminate x^2 and $x^2 + y^2$

combinations in the two integrands of (4.13), respectively, we arrive at

$$Q(\bar{\xi}) = \frac{M}{4\pi}\lim_{L\to\infty} \times \left(\int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}z\, \frac{2\,r(x,L,z)^3 L}{(r^2+a^2)(r^4+a^2z^2)^3} \times \left(2r^8 + a^4z^2(z^2-3r^2) + a^2r^4(r^2-z^2)\right) + \int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}y\, \frac{r(x,y,L)L}{\left(r^4+a^2L^2\right)^3} \times \left(2r^8 - a^4L^2(r^2-L^2) - a^2r^4(L^2-3r^2)\right)\right) \tag{4.14}$$

where the first integrand can be factorized to write

$$\begin{aligned} Q(\bar{\xi}) &= I_{xy} + I_{xz}\,, \\ I_{xy} &:= \frac{M}{4\pi}\lim_{L\to\infty}\left(\int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}y\, \frac{r(x,y,L)L}{\left(r^4+a^2L^2\right)^3} \times \left(2r^8 - a^4L^2(r^2-L^2) - a^2r^4(L^2-3r^2)\right)\right), \\ I_{xz} &:= \frac{M}{4\pi}\lim_{L\to\infty}\left(\int_{-L}^{L}\int_{-L}^{L} \mathrm{d}x\,\mathrm{d}z\, \frac{2\,r(x,L,z)^3 L}{(r^4+a^2z^2)} \times \left(\frac{r^2(r^4-3a^2z^2)}{(r^4+a^2z^2)^2} + \frac{1}{r^2+a^2}\right)\right). \end{aligned} \tag{4.15}$$

Since the solution of r (4.4) is not as simple as in the usual spherical coordinates, it is painful difficult to take these integrals in the Euclidean coordinates.

For dealing with I_{xy}, we first note that the relevant root of $r = r(x, y, L)$ (4.4) is

$$r = \frac{\sqrt{\sqrt{\left(-a^2+L^2+x^2+y^2\right)^2+4a^2L^2}-a^2+L^2+x^2+y^2}}{\sqrt{2}}. \tag{4.16}$$

This is to be substituted into I_{xy} in (4.15). Note, however, that the resultant expression is an analytic function of the rotation parameter a and that we are to first do the integration on the x and z variables, then set $x = \pm L$ and $z = \pm L$, and next take the $L \to \infty$ limit, keeping the parameter a finite and fixed the whole time. Thus the whole integral I_{xy} must be an analytic function of the parameter a, or better yet, $u := a/L$ about $u = 0$ for fixed a. Specifically, we find

$$\frac{2L}{\left(L^2+x^2+y^2\right)^{3/2}} + \frac{2u^2L^3\left(3\left(x^2+y^2\right)-2L^2\right)}{\left(L^2+x^2+y^2\right)^{7/2}} + \frac{3u^4L^5\left(8L^4-40L^2\left(x^2+y^2\right)+15\left(x^2+y^2\right)^2\right)}{4\left(L^2+x^2+y^2\right)^{11/2}}$$

for the integrand of I_{xy} to quintic order about $u = 0$. We next integrate this u-series term by term in both x and y over the square $x \in (-L, L)$ and $y \in (-L, L)$ to arrive at

$$I_{xy} = \frac{M}{108\pi}\lim_{u\to 0}\left(36\pi + \sqrt{3}\left(-32+u^2\right)u^2\right) = \frac{M}{3}. \tag{4.17}$$

As for I_{xz}, we similarly have

$$r = \frac{\sqrt{\sqrt{\left(-a^2+L^2+x^2+z^2\right)^2+4a^2z^2}-a^2+L^2+x^2+z^2}}{\sqrt{2}} \tag{4.18}$$

for the root of $r = r(x, L, z)$ (4.4), and the corresponding series expansion in u is

$$\frac{4L}{\left(L^2+x^2+z^2\right)^{3/2}} + \frac{4u^2L^3\left(x^2+L^2-4z^2\right)}{\left(L^2+x^2+z^2\right)^{7/2}} + \frac{9u^4L^5\left(8z^4-12z^2\left(x^2+L^2\right)+\left(x^2+L^2\right)^2\right)}{2\left(L^2+x^2+z^2\right)^{11/2}}$$

for the integrand of I_{xz} to quintic order about $u = 0$. We integrate term by term in both x and z over the square $x \in (-L, L)$ and $z \in (-L, L)$, and get

$$I_{xz} = \frac{M}{108\pi}\lim_{u\to 0}\left(72\pi + \sqrt{3}\left(32-u^2\right)u^2\right) = \frac{2M}{3}. \tag{4.19}$$

Hence, the total mass $Q(\bar{\xi})$ is

$$Q(\bar{\xi}) = M, \tag{4.20}$$

in accordance with (3.8).

The azimuthal (rotational) Killing vector reads

$$\bar{\zeta}^a = \left(x\,\partial_y - y\,\partial_x\right)^a. \tag{4.21}$$

in the Kerr–Schild coordinates. Using the same foliation (4.6) and the six boundary faces (4.7), we end up with $Q(\bar{\zeta})$, similar to (4.9). This time the three nontrivial components of the 2-form potential ℓ (2.3) are

$$\ell^{tx}(\bar{\zeta}) = \frac{Mr^2}{(a^2+r^2)^3(a^2z^2+r^4)^3} \times \left(a^8yz^4 - 2a^6r^2yz^2\left(r^2-2z^2\right) - 3a^5r^3xz^2\left(x^2+y^2\right)\right.$$

$$
\begin{aligned}
&+a^4r^4y\left(r^4-6r^2z^2+z^2\left(4x^2+4y^2+5z^2\right)\right)\\
&+a^3r^5x\left(r^2-z^2\right)\left(x^2+y^2\right)\\
&+2a^2r^6y\left(r^4-3r^2z^2+z^2\left(x^2+y^2+z^2\right)\right)\\
&+3ar^9x\left(x^2+y^2\right)\\
&+r^{10}y\left(r^2-2\left(x^2+y^2+z^2\right)\right)\Big), \qquad (4.22a)
\end{aligned}
$$

$$
\begin{aligned}
\ell^{ty}(\bar\zeta) = &-\frac{Mr^2}{(a^2+r^2)^3(a^2z^2+r^4)^3}\\
&\times\Big(a^8xz^4-2a^6r^2xz^2\left(r^2-2z^2\right)\\
&+3a^5r^3yz^2\left(x^2+y^2\right)\\
&+a^4r^4x\left(r^4-6r^2z^2+z^2\left(4x^2+4y^2+5z^2\right)\right)\\
&-a^3r^5y\left(r^2-z^2\right)\left(x^2+y^2\right)\\
&+2a^2r^6x\left(r^4-3r^2z^2+z^2\left(x^2+y^2+z^2\right)\right)\\
&-3ar^9y\left(x^2+y^2\right) \qquad (4.22b)\\
&+r^{10}x\left(r^2-2\left(x^2+y^2+z^2\right)\right)\Big),
\end{aligned}
$$

$$
\ell^{tz}(\bar\zeta) = -\frac{aMr^3(x^2+y^2)z(-3r^4+a^2z^2)}{(r^4+a^2z^2)^3}, \qquad (4.22c)
$$

where we have again used (4.11) and (4.12). We next substitute (4.22) into $Q(\bar\zeta)$. Once again using the symmetry $r(x,y)=r(y,x)$ (4.4), we can add the integrals on the xz- and yz-surface (in complete analogy to how (4.15) was found) to finally arrive at

$$
\begin{aligned}
&Q(\bar\zeta)=\frac{aM}{4\pi}\left(J_{xy}+J_{xz}\right),\\
&J_{xy}:=\\
&\lim_{L\to\infty}\left(\int_{-L}^{L}\int_{-L}^{L}\mathrm{d}x\,\mathrm{d}y\,\frac{r(x,y,L)L\left(r^2+a^2\right)\left(r^2-L^2\right)\left(3r^4-a^2L^2\right)}{\left(r^4+a^2L^2\right)^3}\right),\\
&J_{xz}:=\lim_{L\to\infty}\left(\int_{-L}^{L}\int_{-L}^{L}\mathrm{d}x\,\mathrm{d}z\right.\\
&\left.\frac{2r(x,L,z)^3L\left(r^2-z^2\right)\left(3r^6+a^2r^4-3a^4z^2-a^2r^2z^2\right)}{\left(r^2+a^2\right)\left(r^4+a^2z^2\right)^3}\right). \qquad (4.23)
\end{aligned}
$$

We calculate J_{xy} similarly to how we calculated I_{xy}. After the substitution of (4.16), the series expansion of the integrand of J_{xy} about $u=0$ is

$$
\begin{aligned}
&\frac{3L\left(x^2+y^2\right)}{\left(L^2+x^2+y^2\right)^{5/2}}-\frac{5\left(L^3\left(x^2+y^2\right)\left(4L^2-3\left(x^2+y^2\right)\right)\right)}{2\left(L^2+x^2+y^2\right)^{9/2}}u^2\\
&+\frac{21L^5\left(x^2+y^2\right)\left(8L^4-20L^2\left(x^2+y^2\right)+5\left(x^2+y^2\right)^2\right)}{8\left(L^2+x^2+y^2\right)^{13/2}}u^4\\
&+O(u^5).
\end{aligned}
$$

Similarly, the substitution of (4.18) gives the series expansion of the integrand of J_{xz} as

$$
\begin{aligned}
&\frac{6L\left(L^2+x^2\right)}{\left(L^2+x^2+z^2\right)^{5/2}}+\frac{5L^3\left(L^2+x^2\right)\left(L^2+x^2-6z^2\right)}{\left(L^2+x^2+z^2\right)^{9/2}}u^2\\
&+\frac{21L^5\left(L^2+x^2\right)\left(-16z^2\left(L^2+x^2\right)+\left(L^2+x^2\right)^2+16z^4\right)}{4\left(L^2+x^2+z^2\right)^{13/2}}u^4\\
&+O(u^5).
\end{aligned}
$$

The integration of these on the relevant squares then leads to

$$
\begin{aligned}
J_{xy}&=\frac{aM}{2430\pi}\lim_{u\to0}\left(810(\pi-\sqrt3)-495\sqrt3u^2+82\sqrt3u^4\right)\\
&=\frac{aM}{3}-\frac{aM\sqrt3}{3\pi}, \qquad (4.24)\\
J_{xz}&=\frac{aM}{2430\pi}\lim_{u\to0}\left(810(2\pi+\sqrt3)+495\sqrt3u^2-82\sqrt3u^4\right)\\
&=\frac{2aM}{3}+\frac{aM\sqrt3}{3\pi}, \qquad (4.25)
\end{aligned}
$$

giving the total angular momentum

$$
Q(\bar\zeta)=aM, \qquad (4.26)
$$

in accordance with (3.10).

5 Kerr black hole in Weyl canonical coordinates

The Kerr black hole can also be written in the Weyl canonical coordinates [5,14]

$$
\begin{aligned}
\mathrm{d}s^2=&-e^{2U}(\mathrm{d}t+A\,\mathrm{d}\phi)^2\\
&+e^{-2U}\left(e^{2k}\left(d\rho^2+\mathrm{d}z^2\right)+\rho^2\mathrm{d}\phi^2\right), \qquad (5.1)
\end{aligned}
$$

where

$$
e^{2U}=\frac{(R_++R_-)^2-4M^2+\frac{a^2}{M^2-a^2}(R_+-R_-)^2}{(R_++R_-+2M)^2+\frac{a^2}{M^2-a^2}(R_+-R_-)^2}, \qquad (5.2)
$$

$$
e^{2k}=\frac{(R_++R_-)^2-4M^2+\frac{a^2}{M^2-a^2}(R_+-R_-)^2}{4R_+R_-}, \qquad (5.3)
$$

$$
A=\frac{2aM\left(M+\frac{R_++R_-}{2}\right)\left(1-\frac{(R_+-R_-)^2}{4(M^2-a^2)}\right)}{\frac14(R_++R_-)^2-M^2+a^2\frac{(R_+-R_-)^2}{4(M^2-a^2)}}, \qquad (5.4)
$$

$$
\begin{aligned}
R_\pm&=\sqrt{\rho^2+\left(z\pm\sqrt{M^2-a^2}\right)^2}\\
&=r-M\pm\sqrt{M^2-a^2}\cos\theta. \qquad (5.5)
\end{aligned}
$$

The transformation back to Boyer–Lindquist coordinates (t,r,θ,ϕ) (3.1) is given by

$$
\rho=\sqrt{r^2-2Mr+a^2}\sin\theta, \qquad z=(r-M)\cos\theta. \qquad (5.6)
$$

The flat background can be obtained at the limit of $M = 0,\ a = 0$ in (5.1),

$$\bar{g}_{ab} = \mathrm{diag}(-1, 1, \rho^2, 1), \tag{5.7}$$

and it admits the foliation (2.2) with

$$\gamma_{ab} = \mathrm{diag}(0, 1, \rho^2, 1), \quad n^a = (-\partial_t)^a . \tag{5.8}$$

γ_{ab} is the metric in the three-dimensional Euclidean space in cylindrical polar coordinates. For this particular case, let us first consider a right-angled circular cylinder which has its axis of symmetry on the z-axis, with radius R and height $2L$. There are three faces in total, the two top and bottom caps/disks and the side surface, with the following normal vectors and the corresponding two-dimensional metrics, respectively:

$$z^a_\pm = (\pm\partial_z)^a \quad \text{and} \quad \left(q_z\right)_{ab} = \mathrm{diag}\left(0, 1, \rho^2, 0\right), \tag{5.9}$$

$$\rho^a = \left(\partial_\rho\right)^a \quad \text{and} \quad \left(q_\rho\right)_{ab} = \mathrm{diag}\left(0, 0, \rho^2, 1\right). \tag{5.10}$$

The boundary is then "the cylinder at infinity" found by taking the $R \to \infty$ and $L \to \infty$ limits simultaneously along the directions of the normal vectors. Finally, the background again has two globally defined timelike $\bar{\xi}^a = (-\partial_t)^a$ and spacelike $\bar{\zeta}^a = \left(\partial_\phi\right)^a$ Killing vectors, and the integration on the boundary amounts to three separate integrals on the three faces of "the cylinder at infinity"

$$\begin{aligned} Q(\bar{\xi}) &= \frac{1}{8\pi} \lim_{L\to\infty} \left(\int_0^{2\pi} \int_0^R d\rho\, d\phi\, \rho\, n_{[a}\, z_{+b]} \ell^{ab}(\bar{\xi})|_{z=L} \right. \\ &\quad \left. + \int_0^{2\pi} \int_0^R d\rho\, d\phi\, \rho\, n_{[a}\, z_{-b]} \ell^{ab}(\bar{\xi})|_{z=-L} \right) \\ &\quad + \frac{1}{8\pi} \lim_{R\to\infty} \int_0^{2\pi} \int_{-L}^L \mathrm{d}z\, d\phi\, \rho\, n_{[a}\, \rho_{b]} \ell^{ab}(\bar{\xi})|_{\rho=R} , \\ &= \frac{1}{4} \lim_{\substack{L\to\infty \\ R\to\infty}} \int_0^R d\rho\, \rho \left(\ell^{tz}(\bar{\xi})|_{z=L} - \ell^{tz}(\bar{\xi})|_{z=-L} \right) \\ &\quad + \frac{1}{4} \lim_{\substack{R\to\infty \\ L\to\infty}} \int_{-L}^L \mathrm{d}z\, R\, \ell^{t\rho}(\bar{\xi})|_{\rho=R}, \end{aligned} \tag{5.11}$$

where a and M are to be kept fixed, and we used the fact that all relevant parts are independent of ϕ. We relegate the explicit and unsavory forms of the two nontrivial components of the 2-form potential ℓ (2.3) for the two Killing vectors $\bar{\xi}$ and $\bar{\zeta}$ to Appendix A. We calculate the integrals separately as we did in Sect. 4 by defining

$$\begin{aligned} Q(\bar{\xi}) &= I_\rho + I_z , \\ I_\rho &:= \frac{1}{4} \lim_{\substack{L\to\infty \\ R\to\infty}} \int_0^R \mathrm{d}\rho\, \rho \left(\ell^{tz}(\bar{\xi})|_{z=L} - \ell^{tz}(\bar{\xi})|_{z=-L} \right), \end{aligned}$$

$$I_z := \frac{1}{4} \lim_{\substack{R\to\infty \\ L\to\infty}} \int_{-L}^L \mathrm{d}z\, R\, \ell^{t\rho}(\bar{\xi})|_{\rho=R} . \tag{5.12}$$

In Kerr–Schild coordinates, the integrands of $I_{xy},\ I_{xz}$ (4.15) and $J_{xy},\ J_{xz}$ (4.23) depended only on the rotation parameter a, and the mass parameter M came out as an overall factor. However, as can be seen from the relevant components of the 2-form potential ℓ in Appendix A, the integrands of I_ρ and I_z (5.12) explicitly depend on both a and M. Moreover, there is also the complication that arises from taking the two separate limits $R \to \infty$ and $L \to \infty$. In order to have better control over the geometry and these limits, we set $L = kR$, where $k > 0$ is a *finite chubbiness parameter* between the length L and the radius R of "the cylinder at infinity," just before the limiting step and take $R \to \infty$ later. In analogy to the discussion in Sect. 4, we define

$$u_L := \frac{a}{L},\ v_L := \frac{M}{L}, \qquad u_R := \frac{a}{R},\ v_R := \frac{M}{R} \tag{5.13}$$

as new (small) parameters which will allow us to expand the integrands of I_ρ and I_z (5.12) in a (double) Taylor series expansion[2] to evaluate I_ρ and I_z (5.12).

Thus, the expansion of the integrand of I_ρ is

$$\frac{4L^2\rho}{\left(L^2+\rho^2\right)^{3/2}} v_L + \frac{4L^3\rho}{\left(L^2+\rho^2\right)^2} \left(1 + \frac{L^2}{L^2+\rho^2}\right) v_L^2$$

to cubic order terms. Integrating this expression term by term on $\rho \in [0, R)$ and back-substituting v_L later, we obtain

$$I_\rho = \frac{1}{4} \lim_{\substack{L\to\infty \\ R\to\infty}} M \left(4 - \frac{4L}{\sqrt{L^2+R^2}} + \frac{3MR^2}{L\left(L^2+R^2\right)} + \frac{MR^2L}{\left(L^2+R^2\right)^2} \right). \tag{5.14}$$

Finally, letting $L = kR$, we can see the effect of the chubbiness parameter k on the result

$$\begin{aligned} I_\rho &= \frac{1}{4} \lim_{R\to\infty} M \\ &\quad \times \left(4 - \frac{4k}{\sqrt{k^2+1}} + \frac{3M}{R\,k\left(k^2+1\right)} + \frac{Mk}{R\left(k^2+1\right)^2} \right) \\ &= M - M\frac{k}{\sqrt{k^2+1}}. \end{aligned} \tag{5.15}$$

[2] For convenience, recall that a function $f(u, v)$ which is analytic about the point $(u, v) = (0, 0)$ can be expanded in a double Taylor series as

$$f(u, v) = \sum_{m\geq 0} \sum_{n\geq 0} \frac{1}{m!} \frac{1}{n!} \left(\frac{\partial^{m+n} f}{\partial u^m\, \partial v^n} \right)_{(0,0)} u^m\, v^n .$$

Similarly, the expansion of the integrand of I_z is

$$\frac{2R^3}{(R^2+z^2)^{3/2}}v_R + \frac{R^4}{(R^2+z^2)^2} \times\left(\frac{5}{2}+\frac{2z^2}{R^2+z^2}\right)v_R^2$$

to cubic order. Term-by-term integration of this expression on $z \in (-L, L)$ and back-substitution of v_R gives

$$I_z = \frac{1}{4}\lim_{\substack{R\to\infty\\L\to\infty}} M\left(\frac{4L}{\sqrt{L^2+R^2}} + \frac{ML^3}{(L^2+R^2)^2} + \frac{2ML}{L^2+R^2} + \frac{3M}{R}\arctan(L/R)\right). \tag{5.16}$$

Finally, with $L = kR$, we have

$$I_z = \frac{1}{4}\lim_{R\to\infty} M\left(\frac{4k}{\sqrt{k^2+1}} + \frac{Mk^3}{R\left(k^2+1\right)^2} + \frac{2Mk}{R\left(k^2+1\right)} + \frac{3M}{R}\arctan k\right) = M\frac{k}{\sqrt{k^2+1}}. \tag{5.17}$$

Substituting (5.15) and (5.17) in (5.12), the total mass $Q(\bar{\xi})$ is

$$Q(\bar{\xi}) = M, \tag{5.18}$$

as expected.

We proceed similarly for the Killing vector $\bar{\zeta}$ to calculate the angular momentum. We first write

$$\begin{aligned} Q(\bar{\zeta}) &= J_\rho + J_z\,, \\ J_\rho &:= \frac{1}{4}\lim_{\substack{L\to\infty\\R\to\infty}} \int_0^R d\rho\, \rho\left(\ell^{tz}(\bar{\zeta})|_{z=L} - \ell^{tz}(\bar{\zeta})|_{z=-L}\right), \\ J_z &:= \frac{1}{4}\lim_{\substack{R\to\infty\\L\to\infty}} \int_{-L}^L \mathrm{d}z\, R\,\ell^{t\rho}(\bar{\zeta})|_{\rho=R}\,. \end{aligned} \tag{5.19}$$

Analogously, a series expansion of the integrand of J_ρ gives

$$\frac{6L^3\rho^3}{(L^2+\rho^2)^{5/2}}u_L v_L - \frac{8L^4\rho^3}{(L^2+\rho^2)^3}u_L v_L^2,$$

whereas that of J_z is

$$\frac{3R^6}{(R^2+z^2)^{5/2}}u_R v_R - \frac{4R^7}{(R^2+z^2)^3}u_R v_R^2,$$

both to quartic order. Integrating on $\rho \in [0, R)$ and $z \in (-L, L)$, respectively, and back-substituting, we get

$$J_\rho = \frac{1}{4}\lim_{\substack{L\to\infty\\R\to\infty}} 2aM\left(2 - \frac{2L}{\sqrt{L^2+R^2}} - \frac{LR^2}{(L^2+R^2)^{3/2}} - \frac{MR^4}{L\left(L^2+R^2\right)^2}\right),$$

$$J_z = \frac{1}{4}\lim_{\substack{R\to\infty\\L\to\infty}} aM\left(\frac{4L}{\sqrt{L^2+R^2}} + \frac{2LR^2}{(L^2+R^2)^{3/2}} - \frac{2LMR^2}{(L^2+R^2)^2} - \frac{3LM}{L^2+R^2} - \frac{3M}{R}\arctan(L/R)\right).$$

Finally, letting $L = kR$, we have

$$\begin{aligned} J_\rho &= \frac{1}{4}\lim_{R\to\infty} 2aM\left(2 - 2\frac{k}{\sqrt{k^2+1}} - \frac{k}{(k^2+1)^{3/2}} - \frac{M}{Rk\left(k^2+1\right)^2}\right) \\ &= aM\left(1 - \frac{k}{\sqrt{k^2+1}} - \frac{1}{2}\frac{k}{(k^2+1)^{3/2}}\right), \\ J_z &= \frac{1}{4}\lim_{R\to\infty} aM\left(4\frac{k}{\sqrt{k^2+1}} + \frac{2k}{(k^2+1)^{3/2}} - \frac{2Mk}{R\left(k^2+1\right)^2} - \frac{3Mk}{R\left(k^2+1\right)} - \frac{3M}{R}\arctan k\right) \\ &= aM\left(\frac{k}{\sqrt{k^2+1}} + \frac{1}{2}\frac{k}{(k^2+1)^{3/2}}\right). \end{aligned} \tag{5.20}$$

Substituting (5.20) in (5.19), the total angular momentum $Q(\bar{\zeta})$ is

$$Q(\bar{\zeta}) = aM, \tag{5.21}$$

as expected.

6 Conclusions

In this work, we have concisely reviewed the construction of asymptotic conserved gravitational charges using background Killing vectors. One of the advantages of this description is that the spacetime whose charges are to be calculated does not need to have a Killing vector; rather it is enough if the spacetime to which it asymptotically tends admits one itself. Using this method, we have calculated the total mass/energy and the total angular momentum of the Kerr black hole with its timelike and rotational Killing vectors in the Boyer–Lindquist, Kerr–Schild and Weyl canonical coordinates in Sects. 3, 4, and 5, respectively, and shown that they all agree, explicitly verifying the background gauge independence of this prescription. Even though the calculations are harder for the Kerr–Schild and Weyl canonical coordinates, we have overcome the difficulties involved by introducing appropriate series expansions in suitably chosen “small parameters.” We would like to emphasize that, to our knowledge, the presentations given in Sects. 4 and 5 are original.

One of our main motivations was to examine how the gravitational flux integration through "the asymptotic boundary at infinity" depended on the geometry of the integration surface. Obviously, "the cube at infinity" for the Kerr–Schild form and "the cylinder at infinity" for the Weyl canonical coordinates are not smooth geometries as for "the sphere at infinity" for the Boyer–Lindquist coordinates. Hence, we were unable to determine the charges for the former via the shortcut of evaluating the limit of the relevant integrand at the spatial boundary and integrating the result on the boundary surface later, unlike the case for the Boyer–Lindquist coordinates. Moreover, for the calculation of the total angular momentum in the case of the Kerr–Schild form with its boundary as a "cube at infinity," note that the contribution of J_{xy} (4.24) is less than *half* that of J_{xz} (4.25), which lets us deduce that the relevant gravitational flux is more abundant along the rotation axis compared to the other axes. In Weyl canonical coordinates, we introduced a chubbiness parameter k when describing the spatial boundary. Although k has an effect on the contribution of the relevant gravitational fluxes along different directions (5.20), it is remarkable that the total charges remain independent of k.

For the case of the Kerr–Schild form, if we were to choose a spatial boundary as a "generic rectangular prism at infinity" with three unequal side lengths, say (L_x, L_y, L_z), instead of one (equal) side length L "cube at infinity," we would expect to recover the correct calculations again by using three series expansions along the respective pairwise side surfaces in the three "small parameters" $(a/L_x, a/L_y, a/L_z)$. In retrospect, we conjecture, at least heuristically, that provided the closed and simply connected spatial boundary is also piecewise smooth, i.e., a finite union of smooth surfaces, the integration of the relevant gravitational flux, apart from obvious complications resulting in the pertinent calculations, does not depend on the geometry of the boundary.

Finally, as advocates of the use of background Killing isometries for the determination of gravitational charges, the details of our calculations should have at least some pedagogical value for those who want to appreciate the delicate technicalities involved.

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Appendix A: The 2-form potential ℓ in Weyl coordinates

For the sake of completeness, here we list the relevant non-vanishing components of $\ell(\bar{\xi})$ and $\ell(\bar{\zeta})$ that are needed in determining the gravitational charges in the Weyl canonical coordinates of Sect. 5.

$$
\begin{aligned}
\ell^{t\rho}(\bar{\xi}) = \frac{1}{8\rho^3} \Bigg(& \frac{\rho^4\left(\frac{a^2(R_- - R_+)^2}{M^2-a^2}+(2M+R_-+R_+)^2\right)}{R_-^3 R_+} + \frac{\rho^4\left(\frac{a^2(R_- - R_+)^2}{M^2-a^2}+(2M+R_-+R_+)^2\right)}{R_- R_+^3} \\
& + \rho^2\left(\frac{\frac{a^2(R_- - R_+)^2}{M^2-a^2}+(2M+R_-+R_+)^2}{R_- R_+} - 4\right) - \frac{2\rho^4\left((a^2-M^2)(R_-+R_+)(2M+R_-+R_+)+a^2(R_- - R_+)^2\right)}{R_-^2 R_+^2 (a^2-M^2)} \\
& + \frac{8\left(\rho^2\left(a^2(R_- - R_+)^2-(a^2-M^2)(2M+R_-+R_+)^2\right)^2 - a^2 M^2 (2M+R_-+R_+)^2\left(4a^2-4M^2+(R_- - R_+)^2\right)^2\right)}{\left(a^2(R_- - R_+)^2-(a^2-M^2)(2M+R_-+R_+)^2\right)\left(4a^2(M^2-R_- R_+)-4M^4+M^2(R_-+R_+)^2\right)} \\
& + \frac{8a^2M^2\rho^2(2M+R_-+R_+)^2\left(4a^2-4M^2+(R_- - R_+)^2\right)^2\left(2a^2(R_-^2+R_+^2)-M^2(R_-+R_+)^2\right)}{R_- R_+\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)\left(4a^2(M^2-R_- R_+)-4M^4+M^2(R_-+R_+)^2\right)^2} \\
& - \frac{8\rho^4\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)\left(2a^2(R_-^2+R_+^2)-M^2(R_-+R_+)^2\right)}{R_- R_+\left(4a^2(M^2-R_- R_+)-4M^4+M^2(R_-+R_+)^2\right)^2} \\
& + \frac{8a^2M^2\rho^2(2M+R_-+R_+)^2\left(4a^2-4M^2+(R_- - R_+)^2\right)^2\left(2a^2\left(M(R_-+R_+)+R_-^2+R_+^2\right)-M^2(R_-+R_+)(2M+R_-+R_+)\right)}{R_- R_+\left(a^2(R_- - R_+)^2-(a^2-M^2)(2M+R_-+R_+)^2\right)^2\left(4a^2(M^2-R_- R_+)-4M^4+M^2(R_-+R_+)^2\right)} \\
& - \frac{8\rho^4\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)^2\left(2a^2\left(M(R_-+R_+)+R_-^2+R_+^2\right)-M^2(R_-+R_+)(2M+R_-+R_+)\right)}{R_- R_+\left(a^2(R_- - R_+)^2-(a^2-M^2)(2M+R_-+R_+)^2\right)^2\left(4a^2(M^2-R_- R_+)-4M^4+M^2(R_-+R_+)^2\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8a^2M^2\rho^2(2M+R_-+R_+)\left(4a^2-4M^2+(R_--R_+)^2\right)\left(4a^2(R_-+R_+)-4M^2(R_-+R_+)-4M(R_--R_+)^2-(R_--R_+)^2(R_-+R_+)\right)}{R_-R_+\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)\left(4a^2\left(M^2-R_-R_+\right)-4M^4+M^2(R_-+R_+)^2\right)}\\
&-\frac{8\rho^2\left(4a^2\left(-\left(\rho^2\left(M(R_-+R_+)+R_-^2+R_+^2\right)\right)-R_-R_+(M+R_-)(M+R_+)\right)+M^2(2M+R_-+R_+)\left(R_-R_+(2M+R_-+R_+)+2\rho^2(R_-+R_+)\right)\right)}{R_-R_+\left(4a^2\left(M^2-R_-R_+\right)-4M^4+M^2(R_-+R_+)^2\right)}\\
&+4\left(\rho^2-\frac{\rho^2\left(a^2(R_--R_+)^2-\left(a^2-M^2\right)(2M+R_-+R_+)^2\right)^2-a^2M^2(2M+R_-+R_+)^2\left(4a^2-4M^2+(R_--R_+)^2\right)^2}{\left(a^2(R_--R_+)^2-\left(a^2-M^2\right)(2M+R_-+R_+)^2\right)\left(4a^2\left(M^2-R_-R_+\right)-4M^4+M^2(R_-+R_+)^2\right)}\right)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\ell^{tz}(\bar{\xi}) =\ & \frac{1}{8}\left(\frac{\left(z-\sqrt{M^2-a^2}\right)\left(\frac{a^2(R_--R_+)^2}{M^2-a^2}+(2M+R_-+R_+)^2\right)}{R_-^3R_+}+\frac{\left(\sqrt{M^2-a^2}+z\right)\left(\frac{a^2(R_--R_+)^2}{M^2-a^2}+(2M+R_-+R_+)^2\right)}{R_-R_+^3}\right.\\
&-\frac{\frac{2a^2(R_--R_+)\left(\frac{z-\sqrt{M^2-a^2}}{R_-}-\frac{\sqrt{M^2-a^2}+z}{R_+}\right)}{M^2-a^2}+2(2M+R_-+R_+)\left(\frac{z-\sqrt{M^2-a^2}}{R_-}+\frac{\sqrt{M^2-a^2}+z}{R_+}\right)}{R_-R_+}\\
&+\frac{8\left(\frac{R_+\left(2a^2-M^2\right)\left(\sqrt{M^2-a^2}-z\right)}{R_-}-\frac{R_-\left(2a^2-M^2\right)\left(\sqrt{M^2-a^2}+z\right)}{R_+}+2M^2z\right)}{\rho^2\left(a^2(R_--R_+)^2-\left(a^2-M^2\right)(2M+R_-+R_+)^2\right)\left(4a^2\left(M^2-R_-R_+\right)-4M^4+M^2(R_-+R_+)^2\right)^2}\\
&\times\left(\rho^2\left(a^2(R_--R_+)^2-\left(a^2-M^2\right)(2M+R_-+R_+)^2\right)^2-a^2M^2(2M+R_-+R_+)^2\right.\\
&\left.\times\left(4a^2-4M^2+(R_--R_+)^2\right)^2\right)\\
&-\frac{8a^2(R_--R_+)\left(R_-\left(\sqrt{M^2-a^2}+z\right)+R_+\left(\sqrt{M^2-a^2}-z\right)\right)+\left(a^2-M^2\right)(2M+R_-+R_+)\left(R_-\left(\sqrt{M^2-a^2}+z\right)+R_+\left(z-\sqrt{M^2-a^2}\right)\right)}{\rho^2R_-R_+\left(a^2(R_--R_+)^2-\left(a^2-M^2\right)(2M+R_-+R_+)^2\right)^2\left(4a^2\left(M^2-R_-R_+\right)-4M^4+M^2(R_-+R_+)^2\right)}\\
&\times\left(\rho^2\left(a^2(R_--R_+)^2-\left(a^2-M^2\right)(2M+R_-+R_+)^2\right)^2-a^2M^2(2M+R_-+R_+)^2\right.\\
&\left.\times\left(4a^2-4M^2+(R_--R_+)^2\right)^2\right)\\
&+\frac{16}{\rho^2R_-R_+\left(a^2(R_--R_+)^2-\left(a^2-M^2\right)(2M+R_-+R_+)^2\right)\left(4a^2\left(M^2-R_-R_+\right)-4M^4+M^2(R_-+R_+)^2\right)}\\
&\times\Bigg(-a^2M^2(R_--R_+)(2M+R_-+R_+)^2\left(4a^2-4M^2+(R_--R_+)^2\right)\\
&\left(R_-\left(\sqrt{M^2-a^2}+z\right)+R_+\left(\sqrt{M^2-a^2}-z\right)\right)\Bigg).\\
&+\frac{1}{2}a^2M^2(2M+R_-+R_+)\left(4a^2-4M^2+(R_--R_+)^2\right)^2\left(R_-\left(\sqrt{M^2-a^2}+z\right)+R_+\left(z-\sqrt{M^2-a^2}\right)\right)\\
&+\rho^2\left(a^2(R_--R_+)^2-\left(a^2-M^2\right)(2M+R_-+R_+)^2\right)\\
&\times\left(a^2(R_--R_+)\left(R_-\left(\sqrt{M^2-a^2}+z\right)+R_+\left(\sqrt{M^2-a^2}-z\right)\right)\right.\\
&\left.\left.+\left(a^2-M^2\right)(2M+R_-+R_+)\left(R_-\left(\sqrt{M^2-a^2}+z\right)+R_+\left(z-\sqrt{M^2-a^2}\right)\right)\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\ell^{t\rho}(\bar{\zeta}) =\ & \frac{aM}{2\rho R_-R_+\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)^2}\\
&\times\Bigg(-2R_-R_+(2M+R_-+R_+)\left(4a^2-4M^2+(R_--R_+)^2\right)\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)\\
&+\rho^2(R_-+R_+)+\left(4a^2-4M^2+(R_--R_+)^2\right)\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)\\
&-2\rho^2(R_--R_+)^2(2M+R_-+R_+)\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)\\
&+2\rho^2(2M+R_-+R_+)\left(4a^2-4M^2+(R_--R_+)^2\right)\left(2a^2\left(M(R_-+R_+)+R_-^2+R_+^2\right)\right.\\
&\left.\left.-M^2(R_-+R_+)(2M+R_-+R_+)\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
\ell^{tz}(\bar{\zeta}) = & \frac{aM}{2R_-R_+\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right)^2}\times \\
& \times\Bigg(-2(R_- - R_+)(2M+R_-+R_+)\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right) \\
& \times\left(R_-\left(\sqrt{M^2-a^2}+z\right)+R_+\left(\sqrt{M^2-a^2}-z\right)\right) \\
& +\left(4a^2-4M^2+(R_- - R_+)^2\right)\left(M^2(2M+R_-+R_+)^2-4a^2(M+R_-)(M+R_+)\right) \\
& \times\left(R_-\left(\sqrt{M^2-a^2}+z\right)+R_+\left(z-\sqrt{M^2-a^2}\right)\right) \\
& -2(2M+R_-+R_+)\left(4a^2-4M^2+(R_- - R_+)^2\right)\times \\
& \times\left(R_+\left(\sqrt{M^2-a^2}-z\right)\left(2a^2(M+R_+)-M^2(2M+R_-+R_+)\right)-R_-\left(\sqrt{M^2-a^2}+z\right)\right. \\
& \left.\times\left(2a^2(M+R_-)-M^2(2M+R_-+R_+)\right)\right)\Bigg).
\end{aligned}
$$

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