



Consistent thermodynamics and topological classes for the four-dimensional Lorentzian charged Taub-NUT spacetimes

Di Wu^a

School of Physics and Astronomy, China West Normal University, Nanchong 637002, Sichuan, People's Republic of China

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Abstract In this paper, we derive the consistent thermodynamics of the four-dimensional Lorentzian Reissner-Nordström-NUT (RN-NUT), Kerr-Newman-NUT (KN-NUT), and RN-NUT-AdS spacetimes in the framework of the $(\psi - \mathcal{N})$ -pair formalism, and then investigate their topological numbers by using the uniformly modified form of the generalized off-shell Helmholtz free energy. We find that these solutions can be included into one of three categories of those well-known black hole solutions, which implies that these spacetimes should be viewed as generic black holes from the perspective of the topological thermodynamic defects. In addition, we demonstrate that although the existence of the NUT charge parameter seems to have no impact on the topological number of the charged asymptotically locally flat spacetimes, it has a remarkable effect on the topological number of the charged asymptotically locally AdS spacetime.

1 Introduction

In recent years, the study of the topology of black holes has attracted much interest and attention, including light rings [1–5], timelike circular orbits [6, 7], thermodynamics [8–17], phase transitions [18–23], as well as thermodynamic topological classifications [24–37], and so on. In particular, a novel approach has been recently suggested in Ref. [24] to study the thermodynamic topological properties of black holes by interpreting black hole solutions as topological thermodynamic defects, creating topological numbers, and then classifying all black holes into three separate categories based upon their different topological numbers, which sheds new light on the fundamental properties of black holes and gravity. Due to its adaptability and simplicity, the topological approach suggested in Ref. [24] quickly gained widespread acceptance.

As a result, it was successfully applied to explore the topological numbers of several well-known black hole solutions [25–36], for examples, the static Gauss-Bonnet-AdS black holes [25], the static black hole in nonlinear electrodynamics [26], the Kerr and Kerr-Newman black holes [27], the Kerr-AdS and Kerr-Newman-AdS as well as three-dimensional BTZ black holes [28], some static hairy black holes [29], the dyonic black hole in nonlinear electrodynamics [30], the black hole in de Sitter spacetimes [31], and the black hole in massive gravity [32, 33], the static dyonic AdS black holes in different ensembles [34], some Bardeen black holes [35], as well as the static Born-Infeld-AdS black holes [36]. Very recently, we have investigated the topological numbers for the cases of the four-dimensional Lorentzian Taub-NUT, Kerr-NUT, and Taub-NUT-AdS₄ spacetimes [37], and demonstrated that these spacetimes should be viewed as generic black holes from the viewpoint of the thermodynamic topological approach. It is then natural for us to extend that work to the more general charged cases with a pure electric charge to examine whether the four-dimensional Lorentzian charged Taub-NUT spacetimes are generic black holes, which serves as our motivation for the present work.

In this paper, within the framework of the $(\psi - \mathcal{N})$ -pair formalism, we will first utilize the generalized Komar superpotential [38] to derive the consistent thermodynamics of the four-dimensional Lorentzian Reissner-Nordström-NUT (RN-NUT), Kerr-Newman-NUT (KN-NUT), and RN-NUT-AdS spacetimes, and to investigate their topological number via the uniformly modified form of the generalized off-shell Helmholtz free energy. We find that these spacetimes should also be viewed as generic black holes from the thermodynamic topological perspective.

The remaining part of this paper is organized as follows. In Sect. 2, we give a brief review of the novel thermodynamic topological approach for Taub-NUT-type spacetimes. In Sect. 3, we first derive the consistent formulation of ther-

^ae-mail: wdcwnu@163.com (corresponding author)

modynamic properties of the four-dimensional Lorentzian RN-NUT spacetime and then investigate its topological number. In Sect. 4, we turn to discuss the case of the Lorentzian KN-NUT spacetime. In Sect. 5, we then extend to discuss the more general Lorentzian RN-NUT-AdS₄ spacetime. Finally, we present our conclusions in Sect. 6.

2 Thermodynamic topological approach for Taub-NUT-type spacetimes

In accordance with the thermodynamic topological approach suggested in Ref. [37], it is possible to introduce the modified form of the generalized off-shell Helmholtz free energy

$$\mathcal{F} = M - \frac{S}{\tau} - \psi \mathcal{N}, \tag{1}$$

for a Taub-NUT-type black hole thermodynamical system with the mass M , the entropy S , and the Misner potential ψ , as well as the gravitational Misner charge \mathcal{N} , where τ is an extra variable that can be treated as the inverse temperature of the cavity surrounding the Taub-NUT-type black hole. Only when $\tau = T^{-1}$, the modified form of the generalized Helmholtz free energy (1) is on-shell and reduces to the common Helmholtz free energy: $F = M - TS - \psi \mathcal{N}$ of the Taub-NUT-type black holes [39–41].

According to Ref. [24], a core vector ϕ is defined as

$$\phi = \left(\frac{\partial \mathcal{F}}{\partial r_h}, -\cot \Theta \csc \Theta \right), \tag{2}$$

where the two parameters satisfy the ranges: $0 < r_h < +\infty$, $0 \leq \Theta \leq \pi$, respectively. The component ϕ^Θ is divergent at $\Theta = 0$ and $\Theta = \pi$, indicating that the direction of the vector is outward there.

One can define the topological current by using Duan’s ϕ -mapping topological current theory [42–44] as follows:

$$j^\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\rho} \epsilon_{ab} \partial_\nu n^a \partial_\rho n^b, \quad \mu, \nu, \rho = 0, 1, 2, \tag{3}$$

where $\partial_\nu = \partial/\partial x^\nu$ and $x^\nu = (\tau, r_h, \Theta)$. The unit vector n is $n = (n^r, n^\Theta)$, where $n^r = \phi^{r_h}/|\phi|$ and $n^\Theta = \phi^\Theta/|\phi|$. Since it is easy to prove that the above current (3) is conserved, and one can quickly obtain $\partial_\mu j^\mu = 0$ and then indicate that the topological current is a δ -function of the field configuration [43, 44]

$$j^\mu = \delta^2(\phi) J^\mu \left(\frac{\phi}{x} \right), \tag{4}$$

where the three dimensional Jacobian $J^\mu(\phi/x)$ obeys: $\epsilon^{ab} J^\mu(\phi/x) = \epsilon^{\mu\nu\rho} \partial_\nu \phi^a \partial_\rho \phi^b$. It is easy to demonstrate that j^μ equals to zero only when $\phi^a(x_i) = 0$, and one can easily

obtain the topological number W as follows:

$$W = \int_\Sigma j^0 d^2x = \sum_{i=1}^N \beta_i \eta_i = \sum_{i=1}^N w_i, \tag{5}$$

where β_i is the positive Hopf index that counts the number of the loops of the vector ϕ^a in the ϕ -space when x^μ are around the zero point z_i , while $\eta_i = \text{sign}(J^0(\phi/x)_{z_i}) = \pm 1$ is the Brouwer degree, and w_i is the winding number for the i -th zero point of ϕ that is contained in the domain Σ . In addition, if two different closed curves Σ_1 and Σ_2 enclose the same zero point of ϕ , the corresponding winding number must equal. On the other hand, if there is no zero point of ϕ in the enclosed region, one must have $W = 0$.

Note that the local winding number w_i can be used to characterize the local thermodynamic stability, with positive and negative values corresponding to thermodynamically stable and unstable black holes, respectively, and the global topological number W represents the difference between the number of thermodynamically stable black holes and the number of thermodynamically unstable black holes of a classical black hole solution at a fixed temperature [24]. Therefore, not only can one distinguish between different black hole phases (thermodynamically stable or unstable) of the same black hole solution at a specific temperature based upon the local winding number, but also can classify the black hole solutions based upon the global topological number. Furthermore, according to this classification, black holes with the same global topological number (even if they are of different geometric types) have similar thermodynamical topology properties.

3 RN-NUT spacetime

As the simplest charged case, we will investigate the four-dimensional Lorentzian RN-NUT solution [45–47], and adopt the following line element in which the Misner strings are symmetrically distributed along the polar axis:

$$ds^2 = -\frac{f(r)}{r^2 + n^2} (dt + 2n \cos \theta d\varphi)^2 + \frac{r^2 + n^2}{f(r)} dr^2 + (r^2 + n^2) (d\theta^2 + \sin^2 \theta d\varphi^2), \tag{6}$$

where

$$f(r) = r^2 - 2mr - n^2 + q^2,$$

in which m , n and q are the mass, the NUT charge, and the electric charge parameters, respectively. The event horizon radius: $r_h = m + \sqrt{m^2 + n^2 - q^2}$ is the largest root of the equation: $f(r_h) = 0$. In addition, the electromagnetic gauge potential one-form is given by

$$\mathbb{A} = \frac{qr}{r^2 + n^2} (dt + 2n \cos \theta d\varphi), \tag{7}$$

with which a convenient gauge choice is made so that its temporal component vanishes at infinity.

The metric (6) with the Abelian gauge potential (7) is an exact solution to the field equations derived from the Lagrangian density: $\mathcal{L} = \sqrt{-g}(R - F^2)/(16\pi)$. For later convenience, we introduce a generalized Komar superpotential (see Eq. (5.20) of Ref. [38]) as follows:

$$\Xi^{ab}[\xi] = \nabla^a \xi^b - \nabla^b \xi^a + (4\xi^c F^{ab} + 2\xi^a F^{bc} + 2\xi^b F^{ca})\mathbb{A}_c, \tag{8}$$

associated with a Killing vector ξ and the Faraday-Maxwell field strength tensor: $F_{ab} = \nabla_a \mathbb{A}_b - \nabla_b \mathbb{A}_a$ defined by $F = d\mathbb{A}$. It can be shown that $\nabla_b \Xi^{ab} = 0$.

3.1 Consistent thermodynamics

As we are only focused on the purely electrically charged RN-NUT solutions, so we will first rederive the consistent thermodynamics of the four-dimensional Lorentzian RN-NUT spacetime within the framework of the $(\psi - \mathcal{N})$ -pair formalism.

The Bekenstein–Hawking entropy is one quarter of the area of the event horizon

$$S = \frac{A}{4} = \pi(r_h^2 + n^2), \tag{9}$$

the Gibbons-Hawking temperature is proportional to the surface gravity κ on the event horizon

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi(r_h^2 + n^2)} = \frac{1}{4\pi r_h} \left(1 - \frac{q^2}{r_h^2 + n^2}\right), \tag{10}$$

in which a prime represents the partial derivative with respect to its variable.

Secondly, the total electric charge distribution over a two-dimensional sphere with a finite radius r is given by the Gauss' integral

$$Q(r) = \frac{-1}{4\pi} \int_{S^2} \star F = q \frac{r^2 - n^2}{r^2 + n^2},$$

which clearly shows it is radial-dependently distributed and should be on the Misner string singularities [48]. So the electric charge on the event horizon is

$$Q_h = q \frac{r_h^2 - n^2}{r_h^2 + n^2}. \tag{11}$$

The corresponding electrostatic potential at the event horizon simply reads

$$\Phi = (\mathbb{A}_\mu \chi^\mu)|_{r=r_h} = \frac{qr_h}{r_h^2 + n^2}, \tag{12}$$

where $\chi = \partial_t$ is the timelike Killing vector normal to the event horizon.

In the Lorentzian RN-NUT spacetime, there are additional Killing horizons (north/south pole axes) connected to the Misner strings with the associated Misner potential being

$$\psi = \frac{1}{8\pi n}. \tag{13}$$

In the language of exterior differential forms, the Hodge dual two-form corresponding to the generalized Komar superpotential (8) can be obtained as

$$\begin{aligned} \star \Xi[\chi] &= \frac{2n}{(r^2 + n^2)^2} \left(-f + \frac{4q^2 r^2}{r^2 + n^2}\right) dr \wedge \\ &\quad (dt + 2n \cos \theta d\phi) \\ &\quad - \left[f' - \frac{2rf}{r^2 + n^2} + \frac{2q^2 r(r^2 - n^2)}{(r^2 + n^2)^2} \right] \sin \theta d\theta \wedge d\phi, \end{aligned} \tag{14}$$

for the timelike Killing vector $\chi = \partial_t$.

Using the above generalized Komar superpotential two-form (14) to replace the ordinary Komar one and following the same pattern of the $(\psi - \mathcal{N})$ -pair formalism as did in Refs. [40,49] (namely, one deliberately separates the integral into three parts: the spatial infinity, the horizon, and two Misner string tubes), one can derive the integral Bekenstein–Smarr-like mass formula

$$M = 2TS + \Phi Q_h + 2\psi \mathcal{N}, \tag{15}$$

and then verify that the differential first law can also be satisfied:

$$dM = TdS + \Phi dQ_h + \psi d\mathcal{N}. \tag{16}$$

In the derivation of the Smarr-like formula, one can define the conserved mass as

$$\begin{aligned} M &= \frac{-1}{8\pi} \int_{S_\infty^2} \star \Xi[\chi] = \frac{-1}{4} \int_0^\pi d\theta (\sqrt{-g} \Xi^t r) |_{r \rightarrow \infty} \\ &= m, \end{aligned} \tag{17}$$

which exactly coincides with the Komar mass calculated via the usual Komar integral at infinity, since the well fall-off asymptotic behavior of the Maxwell field. In the computation, we has used the determinant $\sqrt{-g} = (r^2 + n^2) \sin \theta$. On the hand hand, we have instead

$$\frac{-1}{8\pi} \int_{S_h^2} \star \Xi[\chi] = 2TS + \Phi Q_h. \tag{18}$$

The last thermodynamic quantity of the Misner charge can be easily determined via the above Bekenstein–Smarr-like mass formula as

$$\mathcal{N} = \frac{4\pi n^3}{r_h} \left[-1 + q^2 \frac{3r_h^2 + n^2}{(r_h^2 + n^2)^2} \right], \tag{19}$$

which is non-globally conserved. Alternately, it can also be evaluated via the Misner tubes integral

$$\begin{aligned} \mathcal{N} &= \frac{n}{2} \int_{(T_+ - T_-)} \star \Xi[\chi] = \pi n \int_{r_h}^{\infty} dr (\sqrt{-g} \Xi^{t\theta}) \Big|_{\theta=0}^{\theta=\pi} \\ &= -8\pi n^3 \frac{(r^2 + n^2)(r_h - m) - q^2 r_h}{(r_h^2 + n^2)^2}, \end{aligned}$$

which reproduces the above expression (19) after using the identity $m = (r_h^2 - n^2 + q^2)/(2r_h)$.

It can be further identified that all the above thermodynamic quantities are related to the Gibbs free energy of the four-dimensional Lorentzian RN-NUT spacetime [50]

$$G = M - TS - \psi \mathcal{N} - \Phi Q_h, \tag{20}$$

whose expression can be obtained via a Wick-rotated back procedure from the Euclidean action of the Euclidean RN-NUT spacetime:

$$\begin{aligned} I_E &= \frac{1}{16\pi} \int_M d^4x \sqrt{g} (R - F^2) \\ &+ \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h} (K - K_0), \end{aligned} \tag{21}$$

where h is the determinant of the induced metric h_{ij} , K is the trace of the extrinsic curvature tensor defined on the boundary with this induced metric, and K_0 is the subtracted one of the massless uncharged Taub-NUT solution as the reference background. The computation of the Euclidean action integral yields the following expression for the Gibbs free energy

$$G = \frac{I_E}{\beta} = \frac{m}{2} - q^2 \frac{r_h(r_h^2 - n^2)}{2(r_h^2 + n^2)^2} = \frac{1}{2}(M - \Phi Q_h), \tag{22}$$

where $\beta = 1/T$ is the interval of the time coordinate.

By the way, it should be noted that the above results are completely consistent with those given in Ref. [51] without any “derivation”.

3.2 Topological number

Next, we will investigate the topological number of the four-dimensional Lorentzian RN-NUT spacetime. We note that the Helmholtz free energy simply reads

$$F = G + \Phi Q_h = M - TS - \psi \mathcal{N}. \tag{23}$$

Replacing T with $1/\tau$ in Eq. (23) and using $m = (r_h^2 - n^2 + q^2)/(2r_h)$, then the generalized off-shell Helmholtz free energy is

$$\mathcal{F} = \frac{r_h}{2} - \frac{\pi(r_h^2 + n^2)}{\tau} + q^2 \frac{r_h(r_h^2 - n^2)}{2(r_h^2 + n^2)^2}. \tag{24}$$

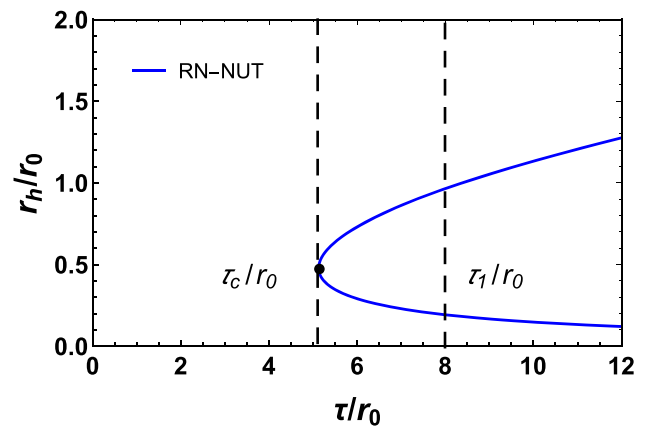


Fig. 1 Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $n/r_0 = 1$ and $q/r_0 = 1$. The generation point for the RN-NUT spacetime is represented by the black dot with τ_c . At $\tau = \tau_1$, there are two RN-NUT spacetimes

Adopting the definition of Eq. (2), the components of the vector ϕ can be easily calculated as follows:

$$\begin{aligned} \phi^{r_h} &= \frac{1}{2} - \frac{2\pi r_h}{\tau} - q^2 \frac{r_h^4 - 6n^2 r_h^2 + n^4}{2(r_h^2 + n^2)^3}, \\ \phi^\Theta &= -\cot \Theta \csc \Theta. \end{aligned} \tag{25}$$

By solving the equation: $\phi^{r_h} = 0$, one can arrive at a curve on the $r_h - \tau$ plane. For the four-dimensional RN-NUT spacetime, one can obtain

$$\tau = \frac{4\pi r_h (r_h^2 + n^2)^3}{(r_h^2 + n^2)^3 - q^2 (r_h^4 - 6n^2 r_h^2 + n^4)}. \tag{26}$$

We point out that Eq. (26) consistently reduces to the one obtained in the case of the four-dimensional RN black hole [24] when the NUT charge parameter n is turned off. Note that the generation point satisfies the constraint conditions:

$$\frac{\partial \tau}{\partial r_h} = 0, \quad \frac{\partial^2 \tau}{\partial r_h^2} > 0. \tag{27}$$

Taking $q/r_0 = 1$ and $n/r_0 = 1$ for the four-dimensional Lorentzian RN-NUT spacetime, we plot in Figs. 1 and 2, respectively, for the zero points of the component ϕ^{r_h} , and for the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau = 7r_0$ in which r_0 is an arbitrary length scale set by the size of a cavity enclosing the RN-NUT spacetime. From Fig. 1, one generation point can be found at $\tau/r_0 = \tau_c/r_0 = 5.13$. It is clear that the RN-NUT spacetime behaves like the RN black hole, showing that the NUT charge parameter appears to have no impact on the thermodynamic topological classification for the static charged asymptotically locally flat spacetime. Consequently, it would be fascinating to learn more about the relationship between geometric topology and thermodynamic topology: for example, it would be very inter-

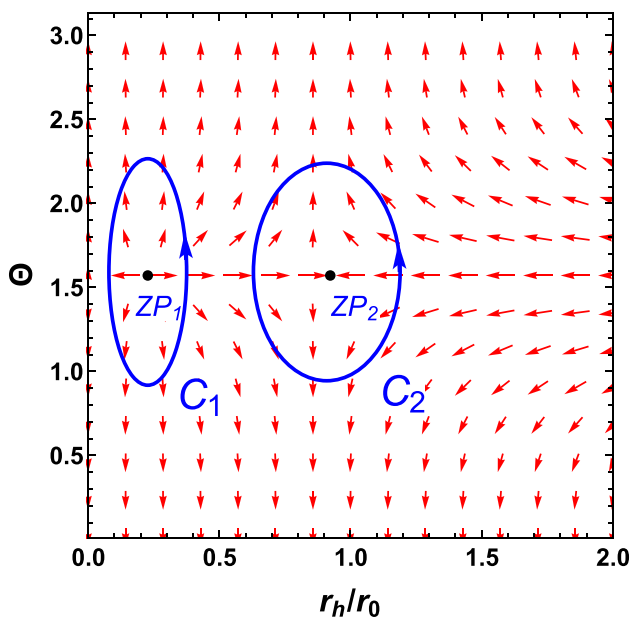


Fig. 2 The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the RN-NUT spacetime with $n/r_0 = 1$, $q/r_0 = 1$ and $\tau/r_0 = 7$. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.23, \pi/2)$, and $(0.86, \pi/2)$, respectively. The blue contours C_i are closed loops enclosing the zero points

esting to investigate the topological number of ultraspinning black holes [52–59] and their usual counterparts.

In Fig. 2, the zero points are located at $(r_h/r_0, \Theta) = (0.23, \pi/2)$, and $(0.86, \pi/2)$, respectively. Thus, one can read the winding numbers w_i for the blue contours C_i : $w_1 = 1$, $w_2 = -1$, which are similar to those of the RN black hole [24]. In terms of the topological global properties, one can easily obtain the topological number $W = 0$ for the RN-NUT spacetime from Fig. 2, which is also the same one as that of the RN black hole. As a result, based upon the viewpoint of the thermodynamic topological numbers, the Lorentzian RN-NUT spacetime should be welcomed into the black hole family. Furthermore, it can be indicated that, while the RN-NUT spacetime and RN black hole are evidently distinguished in geometric topology, they belong to the same class in terms of thermodynamic topology.

4 KN-NUT spacetime

In this section, we will focus on the case of a rotating charged Taub-NUT spacetime by considering the four-dimensional KN-NUT solution [60–63], whose line element with the Misner strings symmetrically distributed along the rotation axis

is written in the Boyer-Lindquist coordinates as:

$$ds^2 = -\frac{\Delta(r)}{\Sigma} [dt + (2n \cos \theta - a \sin^2 \theta) d\varphi]^2 + \frac{\Sigma}{\Delta(r)} dr^2 + \Sigma d\theta^2 + \frac{\sin^2 \theta}{\Sigma} [adt - (r^2 + a^2 + n^2) d\varphi]^2, \quad (28)$$

where

$$\Sigma = r^2 + (n + a \cos \theta)^2, \quad \Delta(r) = r^2 - 2mr - n^2 + a^2 + q^2,$$

in which m , n , a and q are the mass, the NUT charge, the rotation and the electric parameters, respectively. The event horizon radius is $r_h = m + \sqrt{m^2 + n^2 - a^2 - q^2}$. In addition, the electromagnetic gauge potential one-form is given by

$$\mathbb{A} = \frac{qr}{\Sigma} [dt + (2n \cos \theta - a \sin^2 \theta) d\varphi], \quad (29)$$

in a gauge that its temporal component vanishes at infinity.

4.1 Consistent thermodynamics

Now, we investigate the consistent thermodynamics of the four-dimensional Lorentzian KN-NUT spacetime within the framework of the $(\psi - \mathcal{N})$ -pair formalism. The Bekenstein–Hawking entropy is taken as one quarter of the event horizon area:

$$S = \frac{A}{4} = \pi(r_h^2 + a^2 + n^2). \quad (30)$$

The Hawking temperature is proportional to the surface gravity κ on the event horizon

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi(r_h^2 + a^2 + n^2)} = \frac{r_h - m}{2\pi(r_h^2 + a^2 + n^2)}. \quad (31)$$

The angular velocity at the event horizon and the Misner potential are, respectively,

$$\Omega = \frac{a}{r_h^2 + a^2 + n^2}, \quad \psi = \frac{1}{8\pi n}. \quad (32)$$

The electric charge Q on the event horizon can be computed as

$$Q_h = \frac{-1}{4\pi} \int_{S_h^2} {}^*F = q \frac{(r_h^2 + a^2)^2 - n^4}{(r_h^2 + a^2 + n^2)^2 - 4a^2n^2}, \quad (33)$$

and its corresponding electrostatic potential at the event horizon is

$$\Phi = (\mathbb{A}_\mu \chi^\mu)|_{r=r_h} = \frac{qr_h}{r_h^2 + a^2 + n^2}, \quad (34)$$

where $\chi = \partial_t + \Omega \partial_\varphi$ is the Killing vector normal to the event horizon.

As for the conserved mass, one can compute it just like the non-rotating case and get

$$M = \frac{-1}{8\pi} \int_{S^2_\infty} \star \Xi[\partial_t] = m, \tag{35}$$

for the timelike Killing vector ∂_t . One can note that it exactly coincides with the Komar mass evaluated via the usual Komar integral at infinity.

One can anticipate that both the first law and the Bekenstein–Smarr mass formula for the Lorentzian KN-NUT spacetime should read

$$dM = TdS + \Omega dJ_h + \Phi dQ_h + \psi d\mathcal{N}, \tag{36}$$

$$M = 2TS + 2\Omega J_h + \Phi Q_h + 2\psi \mathcal{N}, \tag{37}$$

from which one can first solve J_h in terms of \mathcal{N} from the integral Smarr-like formula (37), and then solve $\mathcal{N} = \mathcal{N}(r_h, q, n, a)$ from the differential first law. After abandoning an integration constant, one can finally get the expressions for the gravitational Misner charge and the angular momentum as follows:

$$\mathcal{N} = \frac{4\pi n^3}{r_h} \left\{ -1 + q^2 \left[3r_h^6 + (7n^2 - 3a^2)r_h^4 + (5n^4 + 2a^2n^2 - 7a^4)r_h^2 - (a^2 - n^2)^3 \right] \times [(r_h^2 + a^2 + n^2)^2 - 2a^2n^2]^{-2} \right\}, \tag{38}$$

$$J_h = a \left\{ m + \frac{n^2}{r_h} + \frac{q^2n^2}{r_h} \left[3r_h^6 + (7a^2 - 3n^2)r_h^4 + (5a^4 + 2a^2n^2 - 7n^4)r_h^2 + (a^2 - n^2)^3 \right] \times [(r_h^2 + a^2 + n^2)^2 - 4a^2n^2]^{-2} \right\}. \tag{39}$$

We would like to point out that the above two expressions are identical to those of \tilde{N} and \tilde{J} given by Eqs. (3.15) and (3.16) in Ref. [41] in the case when the magnetic charge parameter is turned off, namely, the asymptotic magnetic charge vanishes. The first law and the Bekenstein–Smarr mass formula precisely correspond to the magnetic version of the full cohomogeneity first law [41] when the magnetic charge parameter is set to zero.

On the other hand, one can utilize the generalized Komar superpotential (8) rather than the usual Komar one with respect to the Killing vector $\chi = \partial_t + \Omega \partial_\varphi$ and follow the same paradigm of the $(\psi - \mathcal{N})$ -pair formalism as did in Ref. [64] to derive the integral Bekenstein–Smarr-like mass formula (37) and then check that the above thermodynamic quantities simultaneously satisfy the differential first law as well. In this way, it is facilitated to use the GRTensor II package to perform the algebraic manipulation to get the above involved expressions of J_h and \mathcal{N} . Here, we will not repeat the “derivation” but just provide a simple and equivalent way

to evaluate the horizon angular momentum J_h by using our generalized Komar superpotential (8):

$$J_h = \frac{1}{16\pi} \int_{S^2_h} \star \Xi[\partial_\varphi] = \frac{1}{8} \int_0^\pi d\theta \sqrt{-g} \Xi^{tr}[\partial_\varphi] \Big|_{r=r_h}, \tag{40}$$

which reproduces the above expression after using $m = (r_h^2 - n^2 + a^2 + q^2)/(2r_h)$.

Adopting the same procedure as did in Sect. 3.1, the calculation of the Euclidean action integral (21) of the KN-NUT spacetime yields the Gibbs free energy

$$G = \frac{m}{2} - \frac{q^2r_h(r^2 + a^2 - n^2)}{2(r_h^2 + a^2 + n^2)^2 - 8a^2n^2} = \frac{M - \Phi Q_h}{2}, \tag{41}$$

which coincides with the result of Eq. (3.1) given in Ref. [41] in the case when the magnetic charge parameter $g = 0$ is turned off. Furthermore, it can be also identified as

$$G = M - TS - \psi \mathcal{N} - \Omega J_h - \Phi Q_h. \tag{42}$$

4.2 Topological number

In order to obtain the thermodynamic topological number of the KN-NUT spacetime, we need to get the expression of the generalized off-shell Helmholtz free energy in advance. The Helmholtz free energy is given by

$$F = G + \Omega J_h + \Phi Q_h = M - TS - \psi \mathcal{N}. \tag{43}$$

It is a simple matter to obtain the generalized off-shell Helmholtz free energy as

$$\begin{aligned} \mathcal{F} = M - \frac{S}{\tau} - \psi \mathcal{N} &= \frac{r_h^2 + a^2}{2r_h} - \frac{\pi(r_h^2 + a^2 + n^2)}{\tau} \\ &+ \frac{q^2(r_h^2 + a^2 - n^2)}{2r_h[(r_h^2 + a^2 + n^2)^2 - 4a^2n^2]^2} \left[(r_h^2 + a^2)^3 \right. \\ &\left. + 2(r_h^4 + 4a^2r_h^2 - a^4)n^2 + (r_h^2 + a^2)n^4 \right]. \end{aligned} \tag{44}$$

Then, the components of the vector ϕ are given by

$$\begin{aligned} \phi^{r_h} &= \frac{r_h^2 - a^2}{2r_h^2} - \frac{q^2 X}{2r_h^2[(r_h^2 + a^2 + n^2)^2 - 4a^2n^2]^3} \\ &\quad - \frac{2\pi r_h}{\tau} \\ \phi^\Theta &= -\cot \Theta \csc \Theta, \end{aligned} \tag{45}$$

where

$$\begin{aligned} X &= r_h^{12} + 3(2a^2 - n^2)r_h^{10} + (15a^4 + 39a^2n^2 - 14n^4)r_h^8 \\ &\quad + 2(10a^6 + 41a^4n^2 - 20a^2n^4 - 7n^6)r_h^6 \\ &\quad + 3(a^2 - n^2)(5a^6 + 15a^4n^2 + 27a^2n^4 + n^6)r_h^4 \\ &\quad + (a^2 - n^2)^3(6a^4 + 3a^2n^2 - n^4)r_h^2 + a^2(a^2 - n^2)^5. \end{aligned}$$

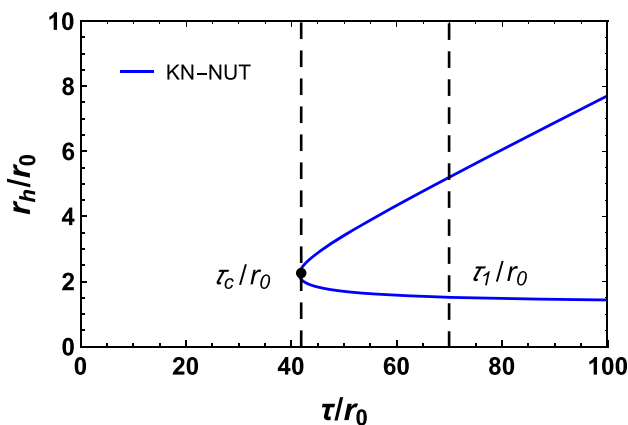


Fig. 3 Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane with $n/r_0 = 1, q/r_0 = 1$ and $a/r_0 = 1$. The generation point for the KN-NUT spacetime is represented by the black dot with τ_c . At $\tau = \tau_1$, there are two KN-NUT spacetimes. Obviously, the topological number is: $W = 1 - 1 = 0$

Therefore, by solving the equation: $\phi^{r_h} = 0$, one can obtain

$$\tau = \frac{4\pi r_h^3 [(r_h^2 + a^2 + n^2)^2 - 4a^2 n^2]^3}{(r_h^2 - a^2) [(r_h^2 + a^2 + n^2)^2 - 4a^2 n^2]^3 - q^2 X} \tag{46}$$

as the zero point of the vector field ϕ^{r_h} . We also point out that Eq. (46) consistently reduces to the one obtained in the case of the four-dimensional Kerr–Newman black hole [27] when the NUT charge parameter n vanishes.

Taking $n/r_0 = 1$ and $q/r_0 = 1$ as well as $n/r_0 = 1$ for the KN-NUT spacetime, we plot the zero points of the component ϕ^{r_h} in Fig. 3, and the unit vector field n on a portion of the $\Theta - r_h$ plane in Fig. 4 with $\tau/r_0 = 50$, respectively. In Fig. 3, one generation point can be found at $\tau/r_0 = \tau_c/r_0 = 41.49$. At $\tau = \tau_1$, there are one thermodynamically unstable KN-NUT spacetime and one thermodynamically stable KN-NUT spacetime, just like the Kerr–Newman black hole [27]. In Fig. 4, one can observe that the zero points are located at $(r_h/r_0, \Theta) = (1.70, \pi/2)$, and $(3.41, \pi/2)$, respectively. Based upon the local property of the zero points, we can obtain the topological number: $W = 1 - 1 = 0$ for the KN-NUT spacetime, which is the same one as that of the Kerr–Newman black hole [27]. Therefore, the four-dimensional KN-NUT spacetime should be present in the large family of black holes. Additionally, it can be concluded that even though the KN-NUT spacetime and Kerr–Newman black hole have undoubtedly distinct geometric topologies, they are the same type from the viewpoint of the thermodynamic topology, just like the RN-NUT spacetime and RN black hole, which have been addressed in Sect. 3 and Ref. [24], respectively.

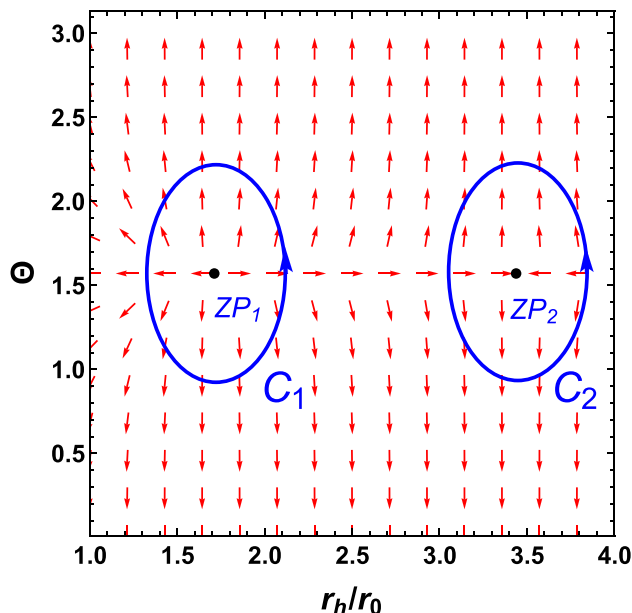


Fig. 4 The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the KN-NUT spacetime with $n/r_0 = 1, q/r_0 = 1, a/r_0 = 1$ and $\tau/r_0 = 50$. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (1.70, \pi/2)$, and $(3.41, \pi/2)$, respectively. The blue contour C_i are closed loops enclosing the zero points

5 RN-NUT-AdS4 spacetime

In this section, we turn to explore the Lorentzian charged Taub-NUT spacetime with an negative cosmological constant, namely, the Lorentzian RN-NUT-AdS4 spacetime, whose metric and Abelian gauge potential are still given by Eqs. (6)–(7), but now $f(r) = r^2 - 2mr - n^2 + q^2 + (r^4 + 6n^2r^2 - 3n^4)/l^2$, in which the AdS radius l is related to the thermodynamic pressure $P = 3/(8\pi l^2)$ of the four-dimensional AdS black hole [65–67]. One can show that the generalized Komar superpotential now obey an identity: $\nabla_b \Xi^{ab} = -6\xi^a/l^2$ in the present case.

5.1 Consistent thermodynamics

We now investigate the thermodynamical properties within the $(\psi - \mathcal{N})$ -pair formalism of the four-dimensional Lorentzian RN-NUT-AdS spacetime. Since we are extending the results already appeared in the Sect. 3.1, so we will mainly collect the needed expressions and just outline the different aspects. For the event horizon r_h , which is the location of the largest root of the radial function: $f(r_h) = 0$, the Bekenstein–Hawking entropy is

$$S = \pi(r_h^2 + n^2), \tag{47}$$

while the Hawking temperature has a different expression

$$T = \frac{f'(r_h)}{4\pi(r_h^2 + n^2)} = \frac{1}{4\pi r_h} \left(1 - \frac{q^2}{r_h^2 + n^2} + 3\frac{r_h^2 + n^2}{l^2} \right). \tag{48}$$

On the event horizon, the electric charge and its corresponding electrostatic potential

$$Q_h = q \frac{r_h^2 - n^2}{r_h^2 + n^2}, \quad \Phi = \frac{q r_h}{r_h^2 + n^2}. \tag{49}$$

The conformal mass can be evaluated as

$$M = m, \tag{50}$$

which is associated with the timelike Killing vector: $\chi = \partial_t$. The Misner potential is

$$\psi = \frac{1}{8\pi n}. \tag{51}$$

Using the metric determinant: $\sqrt{-g} = (r^2 + n^2) \sin \theta$, one can define the thermodynamic volume

$$V = 2\pi \int_0^\theta d\theta \int_0^{r_h} \sqrt{-g} dr = \frac{4}{3} \pi r_h (r_h^2 + 3n^2), \tag{52}$$

which is conjugate to the pressure: $P = 3/(8\pi l^2)$.

Within the framework of the extended phase space, one can substitute the above thermodynamical quantities into the Bekenstein–Smarr mass formula

$$M = 2TS + \Phi Q_h + 2\psi \mathcal{N} + 2VP, \tag{53}$$

and use the identity: $m = (r_h^2 - n^2 + q^2)/(2r_h) + (r_h^4 + 6n^2 r_h^2 - 3n^4)/(2l^2 r_h)$ to acquire the expression of the gravitational Misner charge:

$$\mathcal{N} = \frac{4\pi n^3}{r_h} \left[-1 + 3\frac{r_h^2 - n^2}{l^2} + q^2 \frac{3r_h^2 + n^2}{(r_h^2 + n^2)^2} \right]. \tag{54}$$

Then one can verify that they also completely satisfy the first law:

$$dM = TdS + \Phi dQ_h + \psi d\mathcal{N} + VdP. \tag{55}$$

We would like to point out that the mass formulae presented here exactly correspond to the magnetic-type first law and Smarr-like mass formula of the unconstrained $\psi - \mathcal{N}$ pair formalism of the consistent thermodynamics of the dyonic RN-NUT-AdS₄ spacetimes [49, 50] when the asymptotic magnetic charge is turned off. In particular, the above expression for the Misner charge (54) coincides with that of $N^{(2)}$ explicitly given by Eq. (57) in Ref. [49] after setting the magnetic charge parameter to zero.

One can also follow the same steps as did in Refs. [40, 49] to derive the above Smarr-like formula. To do so, in addition to use the generalized Komar superpotential two-form (14), one must also introduce a dual Killing co-potential ${}^*\omega$ to

cancel the divergence at infinity. We shall not repeat this algebraic exercise here. Instead, there is another simple way to regulate the divergence by making a subtraction from the massless pure NUT-charged background.

One can obtain the Gibbs free energy [50]

$$G = \frac{m}{2} - \frac{q^2 r_h (r_h^2 - n^2)}{2(r_h^2 + n^2)^2} - \frac{r_h (r_h^2 + 3n^2)}{2l^2} = \frac{M - \Phi Q_h}{2} - VP, \tag{56}$$

which coincides with those computed via the Euclidean action integral, namely $G = I/\beta$. In order to obtain this result, one can calculate the Euclidean action [50, 68] for the Euclidean spacetime

$$I_E = \frac{1}{16\pi} \int_M d^4x \sqrt{g} \left(R + \frac{6}{l^2} - F^2 \right) + \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{h} \left[K - \frac{2}{l} - \frac{l}{2} \mathcal{R}(h) \right], \tag{57}$$

where K and $\mathcal{R}(h)$ are the extrinsic curvature and Ricci scalar of the boundary metric $h_{\mu\nu}$, respectively. In order to remove the divergence, the action includes, in addition to the ordinary Einstein–Hilbert term, the Gibbons–Hawking boundary term and the corresponding AdS boundary counterterms [69–73]. Note that the Gibbs free energy (56) should also be identified with

$$G = M - TS - \psi \mathcal{N} - \Phi Q_h. \tag{58}$$

5.2 Topological number

In the following, we will investigate the topological number of the four-dimensional Lorentzian RN-NUT-AdS spacetime. The Helmholtz free energy simply reads

$$F = G + \Phi Q_h = M - TS - \psi \mathcal{N}. \tag{59}$$

Replacing T with $1/\tau$ and substituting $l^2 = 3/(8\pi P)$, then the generalized off-shell Helmholtz free energy is given by

$$\mathcal{F} = \frac{r_h}{2} - \frac{\pi(r_h^2 + n^2)}{\tau} + \frac{q^2 r_h (r_h^2 - n^2)}{2(r_h^2 + n^2)^2} + \frac{4\pi P}{3} r_h (r_h^2 + 3n^2). \tag{60}$$

Thus, the components of the vector ϕ are obtained as follows:

$$\begin{aligned} \phi^{r_h} &= \frac{1}{2} - \frac{2\pi r_h}{\tau} - q^2 \frac{r_h^4 - 6n^2 r_h^2 + n^4}{2(r_h^2 + n^2)^3} \\ &\quad + 4\pi P (r_h^2 + n^2), \\ \phi^\Theta &= -\cot \Theta \csc \Theta, \end{aligned} \tag{61}$$

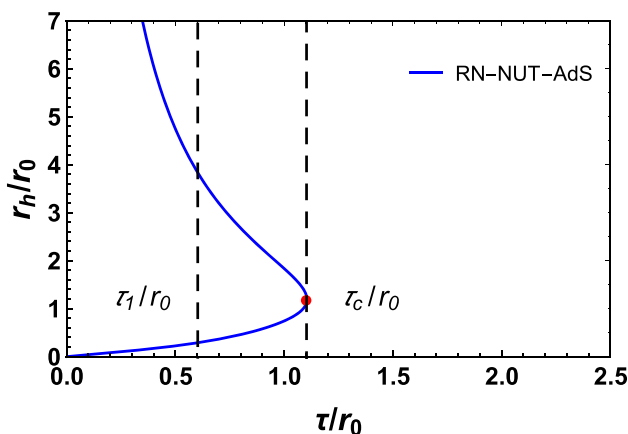


Fig. 5 Zero points of the vector ϕ^{rh} shown on the $r_h - \tau$ plane with $q/r_0 = 1, n/r_0 = 1,$ and $Pr_0^2 = 0.2$ for the RN-NUT-AdS₄ spacetime. The annihilation point for this spacetime is represented by the red dot with τ_c . There are two RN-NUT-AdS₄ spacetimes when $\tau = \tau_1$. Clearly, the topological number is: $W = -1 + 1 = 0$

from which one can obtain the zero point of the vector field ϕ^{rh} as

$$\tau = \frac{4\pi r_h (r_h^2 + n^2)^3}{8\pi P (r_h^2 + n^2)^4 + (r_h^2 + n^2)^3 - q^2 (r_h^4 - 6n^2 r_h^2 + n^4)}, \tag{62}$$

which consistently reduces to the one obtained in the four-dimensional RN-AdS₄ black hole case [24] when the NUT charge parameter n is turned off. We also point out that the annihilation point satisfies the constraint conditions:

$$\frac{\partial \tau}{\partial r_h} = 0, \quad \frac{\partial^2 \tau}{\partial r_h^2} < 0. \tag{63}$$

Taking the pressure $Pr_0^2 = 0.2$ and the NUT charge parameter $n/r_0 = 1$ as well as the electric charge parameter $q/r_0 = 1$ for the RN-NUT-AdS₄ spacetime, we plot the zero points of ϕ^{rh} in the $r_h - \tau$ plane in Fig. 5 and the unit vector field n on a portion of the $\Theta - r_h$ plane with $\tau = r_0$ in Fig. 6, respectively. In Fig. 5, one annihilation point can be found at $\tau/r_0 = \tau_c/r_0 = 1.10$. From Fig. 6, one can find that the zero points are located at $(r_h/r_0, \Theta) = (0.74, \pi/2)$, and $(1.84, \pi/2)$, respectively. According to the conclusions in Ref. [15], it can be inferred that the second-order phase transition occurs in the RN-NUT-AdS₄ spacetime system. Thus, it is very interesting to explore the phase transitions of the RN-NUT-AdS₄ spacetime, such as the Hawking-Page phase transitions [74] and the P-V criticality [75] to check the correctness of the above conjecture. In addition, for the RN-NUT-AdS₄ spacetime, we observe that the topological number is: $W = 0$, and is different from that of the RN-AdS₄ black hole, which has: $W = 1$ [24], because the third zero point of the RN-AdS₄ black hole solution vanishes once

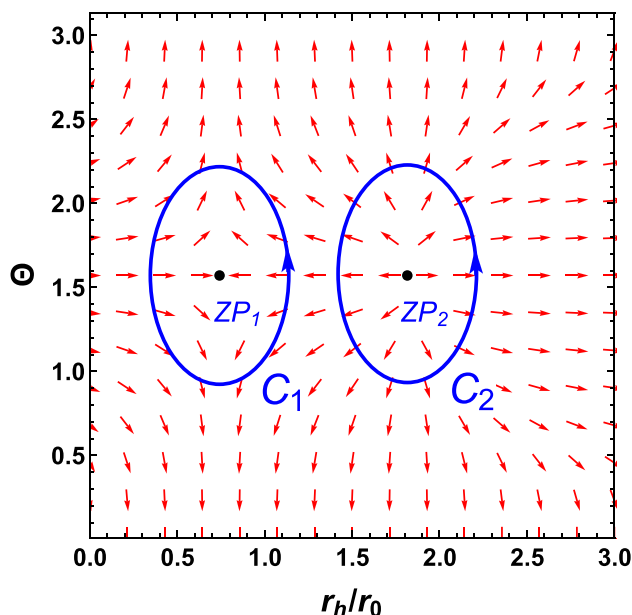


Fig. 6 The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $q/r_0 = 1, n/r_0 = 1, Pr_0^2 = 0.2$ and $\tau/r_0 = 1$ for the RN-NUT-AdS₄ spacetime. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (0.74, \pi/2), (1.84, \pi/2)$ for ZP₁ and ZP₂, respectively. The blue contours C_i are closed loops surrounding the zero points

Table 1 The topological number W , numbers of generation and annihilation points for the four-dimensional Lorentzian charged Taub-NUT spacetimes

Solutions	W	Generation point	Annihilation point
RN-NUT	0	1	0
KN-NUT	0	1	0
RN-NUT-AdS	0	0	1

the NUT charge parameter is introduced. Therefore, it indicates that the NUT charge parameter has a remarkable effect on the topological number for the static charged asymptotically local AdS spacetime. As a result, at least according to the viewpoint of the thermodynamic topological approach, the Lorentzian RN-NUT-AdS₄ spacetime should be included into a member of the black hole family.

6 Conclusions

Our results found in the present paper are now summarized in the following Table 1.

In this paper, we first derive the consistent thermodynamics of the four-dimensional Lorentzian charged RN-NUT, KN-NUT, and RN-NUT-AdS spacetimes within the framework of the $(\psi - \mathcal{N})$ -pair formalism, which exactly cor-

respond to the magnetic version of the full cohomogeneity (unconstrained) first law [41, 49, 50] of the dyonic NUT-charged spacetimes when the magnetic charge parameter vanishes. Then we investigate their topological numbers by using the uniformly modified form of the generalized off-shell Helmholtz free energy. We found that the RN-NUT spacetime has: $W = 0$, which is the same one as that of the RN black hole [24]. We showed that the KN-NUT spacetime has: $W = 0$, which is identical to that of the Kerr–Newman black hole [27]. In addition, we also indicated that the RN-NUT-AdS₄ spacetime has: $W = 0$, which is different from that of the RN-AdS₄ black hole ($W = 1$) [24]. Therefore, one can conclude that although the existence of the NUT charge parameter seems to have no impact on the topological number of the charged asymptotically locally flat spacetimes, it has an important effect on the topological number of the charged asymptotically locally AdS spacetime. Furthermore, it can be demonstrated that the four-dimensional RN-NUT, KN-NUT and RN-NUT-AdS spacetimes should be treated as generic black holes from the standpoint of the thermodynamic topological approach.

There are two promising further topics that can be pursued in the future. As mentioned above, one intriguing topic is to explore the phase transitions of the RN-NUT-AdS₄ spacetime. Another one is to extend the present work to the more general dyonic cases [76] and higher-even dimensional cases [77, 78].

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Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: There is no external data associated with the manuscript.]

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