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Classifying topology of consistent thermodynamics of the four-dimensional neutral Lorentzian NUT-charged spacetimes

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Abstract In this paper, via employing the uniformly modified form of the generalized off-shell Helmholtz free energy, we investigate the topological numbers for the four-dimensional neutral Lorentzian Taub–NUT, Taub–NUT–AdS and Kerr–NUT spacetimes, and find that these solutions can also be classified into one of three types of those well-known black hole solutions, which implies that these spacetimes should be viewed as generic black holes from the viewpoint of the thermodynamic topological approach.

Contents

1 Introduction

In the great family of four-dimensional exact solutions in General Relativity, the Lorentzian NUT-charged spacetimes belong to a very important class of asymptotically locally flat solutions to the Einstein field equations [1,2]. Since its birth, the prevailing view has commonly excluded the Lorentzian NUT-charged spacetimes from the big family of black holes, due to the Misner's identification of the NUT charge parameter as unphysical [3]. However, in recent years, there has been a surge in the studies that exploit the Lorentzian NUT-charged spacetimes as black holes to explore their physical properties, including thermodynamics [4–25], geodesic motion [26], Kerr/CFT correspondence [27,28], black hole shadow [29], weak cosmic censorship conjecture [30,31], holographic complexity [32], and so on [33–39]. Thus, a question arises naturally as to whether the Lorentzian NUT-charged spacetimes are generic black holes.

On the other hand, the Lorentzian NUT-charged spacetimes have many peculiar properties that are mainly due to the presence of the wire/line singularities at the polar axes ($\theta = 0$ and $\theta = \pi$), which are often called as the Misner strings. For instance: (I) The NUT charge has many different meanings and interpretations [40–44], and there are many different explanations of the physical origin of the NUT-charged spacetimes [45,46]. Up to date, there is no unified explanation for these facts; (II) The Taub–NUT de Sitter spacetimes not only provide counterexamples to the maximal mass conjecture but also violate the entropic N-bound [47–49]; (III) The thermodynamic volume of the Euclidean Taub–NUT–AdS spacetime can be negative [50], thus violating the reverse isoperimetric inequality [51]. Therefore, it is very interesting to explore the physics behind these peculiar properties of the Lorentzian NUT-charged spacetimes.

Very recently, by treating the black hole solutions as topological defects and using the generalized off-shell free energy, Wei et al. [52] constructed the topological numbers which are independent of the intrinsic parameters of black holes and divided some black hole solutions into three categories according to their different topological numbers.

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Because of its simplicity and easy maneuverability of the procedure, the thermodynamic topological approach proposed in Ref. [52] soon attracted a great deal of attention and was then successfully applied to calculate the topological numbers of other black hole solutions [53–61]. It is natural to investigate the topological numbers of the NUT-charged spacetimes so as to decree whether they are black holes or not. In this paper, we shall investigate the topological numbers for the four-dimensional neutral Lorentzian Taub–NUT, Taub–NUT–AdS and Kerr–NUT spacetimes by first employing the uniformly modified form of the generalized off-shell Helmholtz free energy, and find that from the thermodynamic topological standpoint, these spacetimes should be viewed as generic black holes also.

The organization of this paper is outlined as follows. In Sect. 2, we give a brief review of the thermodynamic topological approach proposed in Ref. [52]. In Sect. 3, we first recall the consistent formulation of thermodynamic properties of the four-dimensional static Lorentzian Taub–NUT spacetime and then investigate its topological number. In Sect. 4, we turn to discuss the case of the rotating Lorentzian Kerr–NUT spacetime. In Sect. 5, we then extend to discuss the more general static Lorentzian Taub–NUT–AdS$_4$ spacetime. Finally, our conclusion and outlook are given in Sect. 6.

2 A brief review of thermodynamic topological approach

Following the thermodynamic topological approach proposed in Ref. [52], one can first introduce the generalized off-shell Helmholtz free energy

$$\mathscr{F} = M - \frac{S}{\tau} \tag{2.1}$$

for a black hole thermodynamical system with the mass M and the entropy S, where τ is an extra variable that can be regarded as the inverse temperature of the cavity surrounding the black hole. Only when $\tau = T^{-1}$, the generalized Helmholtz free energy (2.1) is on-shell and reduces to the black hole's ordinary Helmholtz free energy $F = M - TS$ [62–64].

In Ref. [52], a key vector ϕ is defined as

$$\phi = \left(\frac{\partial \mathscr{F}}{\partial r_h},\ -\cot\Theta\csc\Theta\right), \tag{2.2}$$

where the two parameters obey $0 < r_h < +\infty$, $0 \leq \Theta \leq \pi$, respectively. The component ϕ^Θ is divergent at $\Theta = 0$ and $\Theta = \pi$, implying that the direction of the vector is outward here.

A topological current can be defined using the Duan's ϕ-mapping topological current theory [65–68] as follows:

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\epsilon_{ab}\partial_\nu n^a \partial_\rho n^b, \quad \mu, \nu, \rho = 0, 1, 2, \tag{2.3}$$

where $\partial_\nu = \partial/\partial x^\nu$ and $x^\nu = (\tau,\ r_h,\ \Theta)$. The unit vector n reads as $n = (n^r, n^\Theta)$, where $n^r = \phi^{r_h}/\|\phi\|$ and $n^\Theta = \phi^\Theta/\|\phi\|$. Since it is easy to demonstrate that the aforementioned current (2.3) is conserved, and one can quickly arrive at $\partial_\mu j^\mu = 0$ and then demonstrate that the topological current is a δ-function of the field configuration [67–69]

$$j^\mu = \delta^2(\phi) J^\mu\left(\frac{\phi}{x}\right), \tag{2.4}$$

where the three dimensional Jacobian $J^\mu(\phi/x)$ satisfies: $\epsilon^{ab}J^\mu(\phi/x) = \epsilon^{\mu\nu\rho}\partial_\nu\phi^a\partial_\rho\phi^b$. It is simple to show that j^μ equals to zero only when $\phi^a(x_i) = 0$, and one can deduce the topological number W as follows:

$$W = \int_\Sigma j^0 d^2x = \sum_{i=1}^{N} \beta_i \eta_i = \sum_{i=1}^{N} w_i, \tag{2.5}$$

where β_i is the positive Hopf index counting the number of the loops of the vector ϕ^a in the ϕ-space when x^μ are around the zero point z_i, while $\eta_i = \mathrm{sign}(J^0(\phi/x)_{z_i}) = \pm 1$ is the Brouwer degree, and w_i is the winding number for the i-th zero point of ϕ that is contained in the domain Σ.

3 Four-dimensional Lorentzian Taub–NUT spacetime

As the simplest case, we will investigate the four-dimensional Lorentzian Taub–NUT spacetime solution [2], and adopt the following line element in which the Misner strings [3] are symmetrically distributed along the polar axis:

$$\begin{aligned} ds^2 = &-\frac{f(r)}{r^2+n^2}(dt + 2n\cos\theta\, d\varphi)^2 + \frac{r^2+n^2}{f(r)}dr^2 \\ &+(r^2+n^2)(d\theta^2 + \sin^2\theta\, d\varphi^2), \end{aligned} \tag{3.1}$$

where $f(r) = r^2 - 2mr - n^2$, in which m and n are the mass and NUT charge parameters, respectively. The event horizon radius r_h is the largest root of the equation $f(r_h) = 0$, namely, $r_h = m + \sqrt{m^2+n^2}$.

3.1 Consistent thermodynamics

First, let's briefly recapitulate the $(\psi - \mathcal{N})$-pair formalism [4, 5] of the consistent thermodynamic of the four-dimensional Lorentzian Taub–NUT spacetime.

The Bekenstein–Hawking entropy is taken as one quarter of the area of the event horizon

$$S = \frac{\mathscr{A}}{4} = \pi(r_h^2 + n^2), \tag{3.2}$$

and the Hawking temperature is thought of as being proportional to the surface gravity κ on the event horizon

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi(r_h^2+n^2)} = \frac{r_h - m}{2\pi(r_h^2+n^2)} = \frac{1}{4\pi r_h}, \tag{3.3}$$

in which a prime denotes the partial derivative with respective to its variable.

In the $(\psi - \mathscr{N})$-pair formalism, one intentionally divides the Komar integral into three patches: the spatial infinity, the horizon, and two Misner string tubes, and defines a non-globally conserved Misner charge. For the global conserved charge, the Komar mass related to the timelike Killing vector ∂_t is: $M = m$, which is computed via the Komar integral at infinity.

There are also other Killing horizons associated with the Misner strings in the Lorentzian Taub–NUT spacetime, namely, the north/south pole axis is a Killing horizon related to the Killing vector [5]: $k = \partial_t + \partial_\varphi/(2n)$, whose corresponding surface gravity can be calculated as

$$\hat{\kappa} = \frac{1}{2n}, \tag{3.4}$$

and the associated Misner potential is

$$\psi = \frac{\hat{\kappa}}{4\pi} = \frac{1}{8\pi n}. \tag{3.5}$$

It is a simple matter to check that the above thermodynamic quantities simultaneously fulfil both the differential and integral mass formulae:

$$dM = TdS + \psi d\mathscr{N}, \tag{3.6}$$
$$M = 2TS + 2\psi\mathscr{N}, \tag{3.7}$$

with the gravitational Misner charge

$$\mathscr{N} = -\frac{4\pi n^3}{r_h}, \tag{3.8}$$

being conjugate to the Misner potential ψ.

The expression of the Helmholtz free energy is then given as [4,5,15]

$$F = M - TS - \psi\mathscr{N} = \frac{m}{2}. \tag{3.9}$$

3.2 Topological number

Next, we will obtain the topological number of the four-dimensional Lorentzian Taub–NUT spacetime. The expression for the Helmholtz free energy of the Lorentzian Taub–NUT spacetime can be recovered via a Wick-rotated back procedure from the Euclidean action of the Euclidean Taub–NUT spacetime:

$$I_E = \frac{1}{16\pi}\int_M d^4x\sqrt{g}R + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}(K - K_0), \tag{3.10}$$

where h is the determinant of the induced metric h_{ij}, K is the trace of the extrinsic curvature tensor defined on the boundary with this metric, and K_0 is the subtracted one of the massless Taub–NUT solution as the reference background.

The free energy calculated by the action integral is: $I/\beta = m/2 = F$, where $\beta = 4\pi r_h$ is the interval of the time coordinate. Replacing T with $1/\tau$ in Eq. (3.9) and using $m = (r_h^2 - n^2)/(2r_h)$, the generalized off-shell Helmholtz free energy is modified as

$$\mathscr{F} = M - \frac{S}{\tau} - \psi\mathscr{N} = \frac{r_h}{2} - \frac{\pi(r_h^2+n^2)}{\tau}. \tag{3.11}$$

Utilizing the definition of Eq. (2.2), the components of the vector ϕ can be easily computed as follows:

$$\phi^{r_h} = \frac{1}{2} - \frac{2\pi r_h}{\tau}, \quad \phi^\Theta = -\cot\Theta\csc\Theta. \tag{3.12}$$

By solving the equation: $\phi^{r_h} = 0$, one can obtain a curve on the $r_h - \tau$ plane, which is just a straight line for the four-dimensional Taub–NUT spacetime:

$$\tau = 4\pi r_h. \tag{3.13}$$

For the Lorentzian Taub–NUT spacetime, we plot, respectively, the zero points of the component ϕ^{r_h} in Fig. 1, and the unit vector field n on a portion of the $\Theta - r_h$ plane in Fig. 2 with $\tau = 4\pi r_0$ in which r_0 is an arbitrary length scale set by the size of a cavity enclosing the Taub–NUT spacetime. From Fig. 1, one can observe that the behavior of the Taub–NUT spacetime resembles that of the Schwarzschild black hole, and this indicates that the NUT charge parameter seems to have no effect on the thermodynamic topological classification for neutral static asymptotically locally flat spacetime. Therefore, it would be interesting to explore the connection between the geometric topology and the thermodynamic topology, for instance, to investigate the topological number of the ultraspinning black holes [70–77] and their usual counterparts.

In Fig. 2, the zero point is located at $r_h = r_0$ and $\Theta = \pi/2$. one can determine the winding number w for an arbitrary

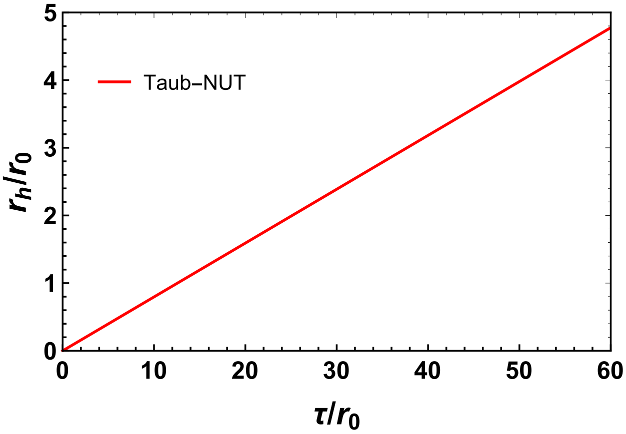

Fig. 1 Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane. There is only one Taub–NUT spacetime for any value of τ

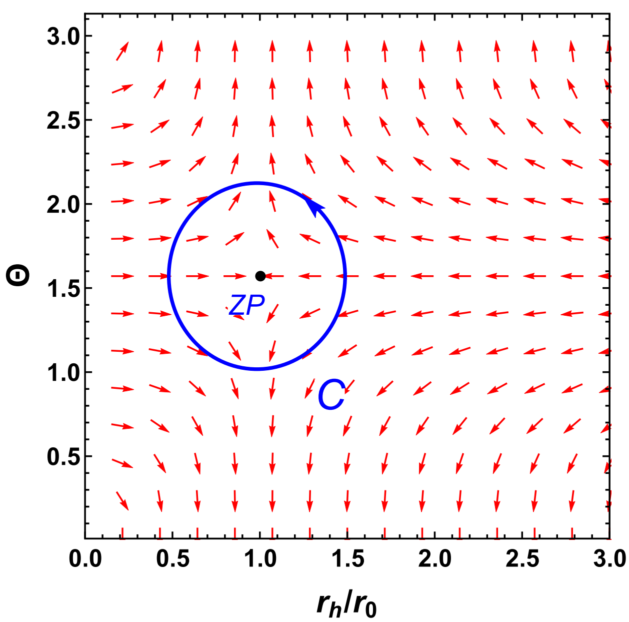

Fig. 2 The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane for the Taub–NUT spacetime with $\tau/r_0 = 4\pi$. The zero point (ZP) marked with black dot is at $(r_h/r_0, \Theta) = (1, \pi/2)$. The blue contour C is a closed loop enclosing that zero point

loop since it is independent of the loops that surround the zero point; for instance, through a look at the blue contour C in Fig. 2. If the calculation is made for the forward rotation in the anticlockwise direction, then the winding number is $w = -1$, which coincides with those of the Schwarzschild black hole [52] and the $d \geq 6$ singly rotating Kerr black hole [58]. In this sense, the Lorentzian Taub–NUT spacetime behaves like a genuine black hole. Since the winding number is related to local thermodynamic stability, with positive and negative values corresponding to stable and unstable black hole solutions [52] respectively, one can naturally conclude that the Taub–NUT spacetime is an unstable black hole in an arbitrary given temperature just like the Schwarzschild black hole. Turning to the topological global properties, we have the topological number $W = -1$ for the Taub–NUT spacetime from Fig. 2, which is also the same as the results of the Schwarzschild black hole and the $d \geq 6$ singly rotating Kerr black hole. Therefore, from the point of view of topological numbers, the Lorentzian Taub–NUT spacetime should be accepted into the black hole family. In addition, it can be concluded that although the Taub–NUT spacetime and Schwarzschild black hole are obviously different in geometric topology aspect, they are the same class from the perspective of the thermodynamic topology.

4 Lorentzian Kerr–NUT spacetime

In this section, we will extend the above discussion to the case of a rotating NUT-charged spacetime by considering the four-dimensional Kerr–NUT spacetime [78–81], whose line element with the Misner strings symmetrically distributed along the rotation axis is

$$ds^2 = -\frac{\Delta(r)}{\Sigma}\big[dt + (2n\cos\theta - a\sin^2\theta)d\varphi\big]^2 + \frac{\Sigma}{\Delta(r)}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\big[adt - (r^2 + a^2 + n^2)d\varphi\big]^2, \quad (4.1)$$

where

$$\Sigma = r^2 + (n + a\cos\theta)^2, \quad \Delta(r) = r^2 - 2mr - n^2 + a^2,$$

in which m and a are the mass and rotation parameters, respectively. The horizon is determined by: $\Delta(r_h) = 0$, which gives $r_h = m \pm \sqrt{m^2 + n^2 - a^2}$.

4.1 Consistent thermodynamics

Now we briefly recall the $(\psi - \mathcal{N})$-pair formalism [7] of the consistent thermodynamics of the four-dimensional Lorentzian Kerr–NUT spacetime. The Bekenstein–Hawking entropy is taken as one quarter of the event horizon area:

$$S = \frac{\mathscr{A}}{4} = \pi(r_h^2 + n^2 + a^2), \quad (4.2)$$

while the Hawking temperature is proportional to the surface gravity κ on the event horizon

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi(r_h^2 + n^2 + a^2)} = \frac{r_h - m}{2\pi(r_h^2 + n^2 + a^2)}. \quad (4.3)$$

The angular velocity at the event horizon and the Misner potential are given by

$$\Omega = \frac{a}{r_h^2 + n^2 + a^2}, \quad \psi = \frac{1}{8\pi n}. \quad (4.4)$$

As for the global conserved charge, the Komar mass $M = m$ at infinity is related to the timelike Killing vector ∂_t.

It is easy to check that the above thermodynamic quantities satisfy the differential first law and integral Bekenstein–Smarr mass formula simultaneously,

$$dM = TdS + \Omega dJ + \psi d\mathcal{N}, \tag{4.5}$$

$$M = 2TS + 2\Omega J + 2\psi\mathcal{N}, \tag{4.6}$$

with the gravitational Misner charge and the angular momentum

$$\mathcal{N} = -\frac{4\pi n^3}{r_h}, \quad J = \Big(m + \frac{n^2}{r_h}\Big)a, \tag{4.7}$$

being conjugate to the Misner potential ψ and the angular velocity, respectively. Note that both of them do not have a global character, due to having a location-dependent factor $1/r_h$.

With the help of the above expressions and using $m = (r_h^2 - n^2 + a^2)/(2r_h)$, the Gibbs free energy of the Kerr–NUT spacetime reads

$$G \equiv M - TS - \psi\mathcal{N} - \Omega J = \frac{m}{2}, \tag{4.8}$$

which is identical to the one calculated via the action integral [7], just like the non-rotating case.

4.2 Topological number

In order to calculate the thermodynamical topological number of the Kerr–NUT spacetime, we then consider the Helmholtz free energy which is given by

$$F = G + \Omega J = M - TS - \psi\mathcal{N}. \tag{4.9}$$

It coincides with the result of Eq. (4.19) given in Ref. [8] in the case of the parameters $e = g = 0$ are turned off.

It is straightforward to modify the generalized Helmholtz free energy as

$$\mathscr{F} = M - \frac{S}{\tau} - \psi\mathcal{N} = \frac{r_h^2 + a^2}{2r_h} - \frac{\pi(r_h^2 + n^2 + a^2)}{\tau}. \tag{4.10}$$

Then, the components of the vector ϕ are given by

$$\phi^{r_h} = \frac{r_h^2 - a^2}{2r_h^2} - \frac{2\pi r_h}{\tau}, \quad \phi^{\Theta} = -\cot\Theta\csc\Theta. \tag{4.11}$$

Therefore, using the thermodynamic topological approach and solving the equation: $\phi^{r_h} = 0$, we get

$$\tau = \frac{4\pi r_h^3}{r_h^2 - a^2} \tag{4.12}$$

as the zero point of the vector field ϕ^{r_h}.

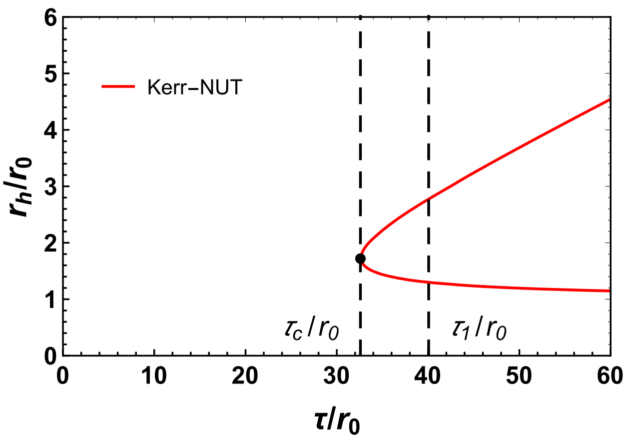

Fig. 3 Zero points of the vector ϕ^{r_h} shown in the $r_h - \tau$ plane. The generation point for the Kerr–NUT spacetime is represented by the black dot with τ_c. At $\tau = \tau_1$, there are two Kerr–NUT spacetimes. Obviously, the topological number is: $W = -1 + 1 = 0$

Taking $a = r_0$ for the Kerr–NUT spacetime, we plot the zero points of the component ϕ^{r_h} in Fig. 3, and the unit vector field n on a portion of the $\Theta - r_h$ plane in Fig. 4 with $\tau/r_0 = 40$, respectively. In Fig. 3, one generation point can be found at $\tau/r_0 = \tau_c/r_0 = 32.65$. In Fig. 4, the zero points are located at $(r_h/r_0, \Theta) = (1.30, \pi/2)$, and $(2.77, \pi/2)$, respectively. Based upon the local property of the zero points, we can obtain the topological number: $W = -1 + 1 = 0$ for the Kerr–NUT spacetime, which is same as that of the Kerr black hole [58]. Thus, the big family of the black holes should contain the four-dimensional Kerr–NUT spacetime. In addition, it also can be concluded that although the Kerr–NUT spacetime and Kerr black hole are obviously different in geometric topology aspect, they seem to be the same kind from the perspective of the thermodynamic topology, just like the Taub–NUT spacetime and Schwarzschild black hole as discussed in Sect. 3 and Ref. [52], respectively.

5 Lorentzian Taub–NUT–AdS$_4$ spacetime

In this section, we turn to explore the Lorentzian Taub–NUT spacetime with an negative cosmological constant, namely, the Lorentzian Taub–NUT–AdS$_4$ spacetime, whose metric is still given by Eq. (3.1), but now $f(r) = r^2 - 2mr - n^2 + (r^4 + 6n^2r^2 - 3n^4)/l^2$, in which the AdS radius l appears in the thermodynamic pressure $P = 3/(8\pi l^2)$ of the 4-dimensional AdS black hole [51,82].

5.1 Consistent thermodynamics

We now recollect the thermodynamical properties of the four-dimensional Taub–NUT–AdS spacetime. The Bekenstein–

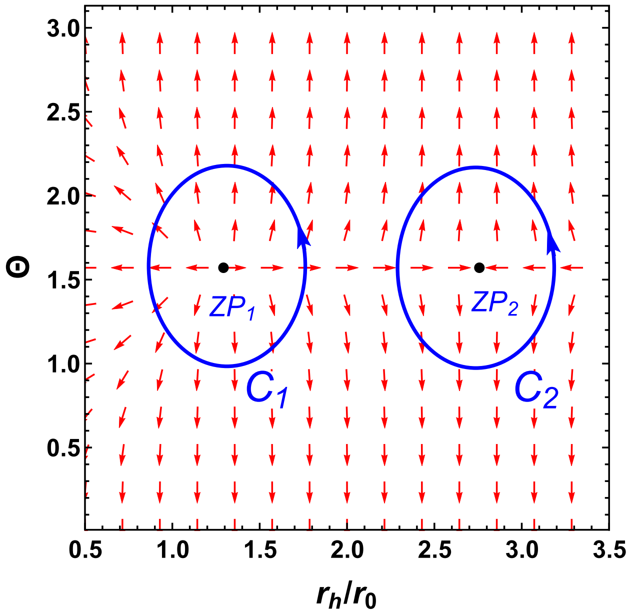

Fig. 4 The red arrows represent the unit vector field n on a portion of the $r_h-\Theta$ plane for the Kerr–NUT spacetime with $a/r_0 = 1$ and $\tau/r_0 = 40$. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (1.30, \pi/2)$, and $(2.77, \pi/2)$ for ZP_1 and ZP_2, respectively. The blue contours C_i are closed loops surrounding the zero points

Hawking entropy is taken as one quarter of the event horizon area:

$$S = \frac{\mathscr{A}}{4} = \pi(r_h^2 + n^2), \tag{5.1}$$

and the Hawking temperature is assumed to be proportional to the surface gravity κ on the event horizon

$$T = \frac{\kappa}{2\pi} = \frac{f'(r_h)}{4\pi(r_h^2+n^2)} = \frac{r_h - m}{2\pi(r_h^2+n^2)} + \frac{r_h^3 + 3n^2 r_h}{\pi l^2(r_h^2+n^2)}, \tag{5.2}$$

where r_h is the location of the event horizon.

The conformal mass is: $M = m$, and the Misner potential is

$$\psi = \frac{1}{8\pi n}. \tag{5.3}$$

It is easy to verify that the above thermodynamic quantities satisfy the differential first law and integral Bekenstein–Smarr mass formula simultaneously [4,5],

$$dM = TdS + \psi d\mathcal{N} + VdP, \tag{5.4}$$
$$M = 2TS + 2\psi\mathcal{N} - 2VP, \tag{5.5}$$

with the gravitational Misner charge and the thermodynamic volume

$$\mathcal{N} = -\frac{4\pi n^3}{r_h}\Big[1 + \frac{3(n^2 - r_h^2)}{l^2}\Big],$$
$$V = \frac{4}{3}\pi r_h(r_h^2 + 3n^2), \tag{5.6}$$

being conjugate to the Misner potential ψ and the pressure $P = 3/(8\pi l^2)$, respectively.

5.2 Topological number

One can find that the Helmholtz free energy reads [4]

$$F \equiv M - TS - \psi\mathcal{N} = \frac{m}{2} - \frac{r_h(r_h^2+3n^2)}{2l^2}, \tag{5.7}$$

which coincides with those computed via the action integral, namely $F = I/\beta$. In order to get this result, one can calculate the Euclidean action [4] for the Euclidean spacetime

$$I_E = \frac{1}{16\pi}\int_M d^4x\sqrt{g}\Big(R + \frac{6}{l^2}\Big) + \frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}K$$
$$-\frac{1}{8\pi}\int_{\partial M} d^3x\sqrt{h}\Big[\frac{2}{l} + \frac{l}{2}\mathscr{R}(h)\Big], \tag{5.8}$$

where K and $\mathscr{R}(h)$ are the extrinsic curvature and Ricci scalar of the boundary metric, respectively. In order to cancel the divergence, the above action also contains the Gibbons–Hawking boundary term and the necessary AdS boundary counterterms [83–87], apart from the standard Einstein–Hilbert term.

It is now a position to investigate the topological number of the four-dimensional Lorentz Taub–NUT–AdS spacetime. Replacing T with $1/\tau$ and substituting $l^2 = 3/(8\pi P)$, thus the modified form of the generalized off-shell Helmholtz free energy is

$$\mathscr{F} = M - \frac{S}{\tau} - \psi\mathcal{N} = \frac{r_h}{2} + \frac{4\pi P}{3}r_h(r_h^2+3n^2) - \frac{\pi(r_h^2+n^2)}{\tau}. \tag{5.9}$$

Then, the components of the vector ϕ are obtained as follows:

$$\phi^{r_h} = \frac{1}{2} + 4\pi P(r_h^2+n^2) - \frac{2\pi r_h}{\tau}, \quad \phi^{\Theta} = -\cot\Theta\csc\Theta. \tag{5.10}$$

from which one can get the zero point of the vector field ϕ^{r_h} as

$$\tau = \frac{4\pi r_h}{8\pi P(r_h^2+n^2)+1}. \tag{5.11}$$

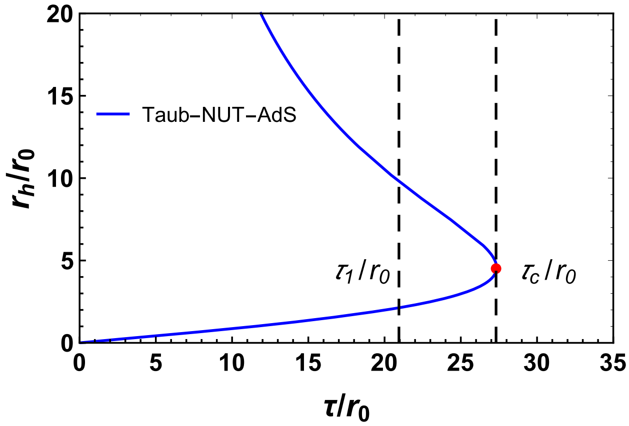

Fig. 5 Zero points of the vector ϕ^{r_h} shown on the $r_h - \tau$ plane with $Pr_0^2 = 0.002$ for the Taub–NUT–AdS$_4$ spacetime. The annihilation point for this spacetime is represented by the red dot with τ_c. There are two Taub–NUT–AdS$_4$ spacetimes when $\tau = \tau_1$. Obviously, the topological number is: $W = 1 - 1 = 0$

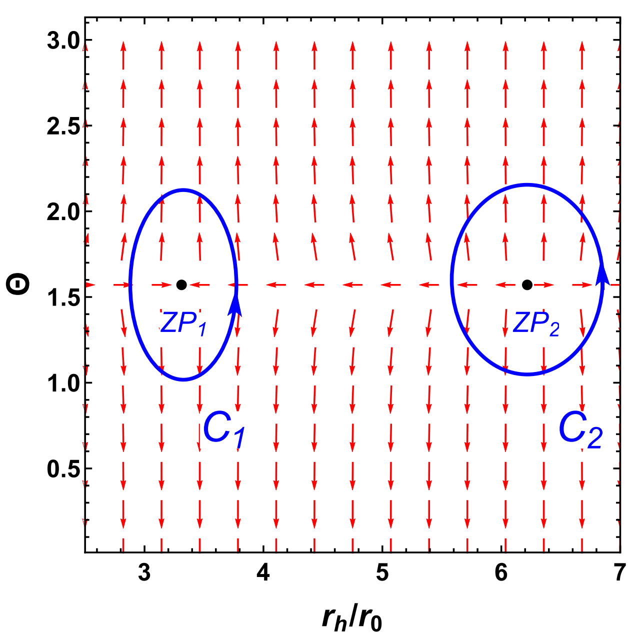

Fig. 6 The red arrows represent the unit vector field n on a portion of the $r_h - \Theta$ plane with $Pr_0^2 = 0.002$ and $\tau/r_0 = 26$ for the Taub–NUT–AdS$_4$ spacetime. The zero points (ZPs) marked with black dots are at $(r_h/r_0, \Theta) = (3.32, \pi/2), (6.30, \pi/2)$ for ZP$_1$ and ZP$_2$, respectively. The blue contours C_i are closed loops surrounding the zero points

Note that Eq. (5.11) consistently reduces to the one obtained in the four-dimensional Schwarzschild–AdS black hole case [61] when the NUT charge parameter n vanishes.

Taking the pressure $Pr_0^2 = 0.002$ and the NUT charge parameter $n/r_0 = 1$, in Figs. 5 and 6, we plot the zero points of ϕ^{r_h} in the $r_h - \tau$ plane and the unit vector field n on a portion of the $\Theta - r_h$ plane for the Taub–NUT–AdS$_4$ spacetime. For the Taub–NUT–AdS$_4$ spacetime, we observe that the topological number is $W = 0$, which is the same as that of the Schwarzschild–AdS$_4$ black hole [55,61], and this implies that the NUT charge parameter also seems to have no impact on the thermodynamic topological classification for the neutral static asymptotically local AdS spacetime. Furthermore, it indicates that the Taub–NUT–AdS$_4$ solution can still be categorized as one of the three types of known black hole solutions [52]. As a result, at least according to the thermodynamic topological approach, the Lorentzian Taub–NUT–AdS$_4$ spacetime should be included into a member of the black hole family.

Table 1 The topological number W, numbers of generation and annihilation points for the four-dimensional neutral Lorentzian NUT-charged spacetimes

Solutions	W	Generation point	Annihilation point
Taub–NUT	−1	0	0
Kerr–NUT	0	1	0
Taub–NUT–AdS	0	0	1

6 Conclusion and outlook

Our results obtained in this paper are summarized in Table 1. In this work, we have employed the uniformly modified form of the generalized off-shell Helmholtz free energy and investigated the topological numbers for the four-dimensional uncharged Lorentzian Taub–NUT, Taub–NUT–AdS and Kerr–NUT spacetimes. We showed that the Taub–NUT spacetime has: $W = -1$, which is the same as that of the Schwarzschild black hole [52]. We found that the Kerr–NUT spacetime has: $W = 0$, which is identical to that of the Kerr black hole [58]. We also demonstrated that the Taub–NUT–AdS$_4$ spacetime has: $W = 0$, which coincides with that of the Schwarzschild–AdS$_4$ black hole [61]. Therefore, one can conclude that the existence of the NUT charge parameter seems has no impact on the topological number of the neutral asymptotically locally flat/AdS spacetimes. It can be inferred that the four-dimensional Taub–NUT, Taub–NUT–AdS and Kerr–NUT spacetimes should be viewed as generic black holes from the viewpoint of the thermodynamic topological approach.

There are two promising topics to be pursued in the future. The first intriguing topic is to explore the phase transitions of the NUT-charged AdS spacetimes, such as the Hawking–Page phase transitions [88] and the P–V criticality [89], and we expect that this will help to unravel the mystery of the NUT-charged spacetimes. The second interesting topic is to extend the present work to the more general charged RN–NUT(–AdS) and rotating charged Kerr–Newman–NUT cases.

Perhaps, one has just touched the tip of an iceberg, much more needs to be explored.

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