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Asymptotically flat vacuum solution for a rotating black hole in a modified gravity theory

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Abstract The theory of $f(R)$-gravity is one of the theories of modified Einstein gravity. The vacuum solution, on the other hand, of the field equation is the solution for black hole geometry. We establish here an asymptotically flat rotating black hole solution in an $f(R)$-gravity. This essentially leads to the modified solution to the Kerr black hole. This solution exhibits the change in fundamental properties of the black hole and its geometry. It particularly shows that radii of marginally stable and bound orbits and black hole event horizon increase compared to those in Einstein gravity, depending on the modified gravity parameter. It further argues for faster spinning black holes with spin (Kerr) parameter greater than unity, without any naked singularity. This supports the weak cosmic censorship hypothesis.

1 Introduction

General relativistic gravity of Einstein turns out to be a remarkable discovery to explain a range of astrophysical sources, apart from its theoretical integrity, even after more than 100 years of its original discovery. Eventually, all the predictions of Einstein's gravity proved to be correct, particularly after the direct detection of gravitational wave in 2015 [1]. In fact, the said discovery could be considered 'three in one': direct confirmation of gravitational wave, spinning black hole and binary black hole.

Although to understand coalescence of, e.g., black holes and to probe the underlying gravitational radiation, strong field general relativity (GR) or numerical relativity is indispensable, most of the direct tests of GR are done based on weak field approximation. Therefore, the global validity of GR in the strong field regime, i.e. the true nature of gravity close to the source of gravity, remains questionable. Hence,

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no one can rule out possible modification to GR in natural systems, particularly when the theory is asymptotically flat. Asymptotic flatness assures reduction of modified GR to GR and to Minkowskian with distance from the source. Therefore, even if close to the source, i.e. a compact object like black hole, neutron star, actual gravitational theory is modified GR, the same theory will be able to explain any solar-system based or Earth-based experiment.

One such example of modified GR is the theory of f(R)-gravity [2,3], which was explored to explain sub- and super-Chandrasekhar limiting mass white dwarfs, in a unified theory, what GR as such could not. They are possibly leading to under- and over-luminous type Ia supernovae under the same model framework. Recently, we also established an asymptotically flat vacuum solution, unlike that for a white dwarf, of $f(R)$-gravity in spherical symmetry [4]. This is essentially a modified solution for the Schwarzschild, hence nonrotating, black hole. We showed that depending on the modified gravity parameter, various basic characteristics of the black hole, e.g. marginally stable and bound circular orbits, event horizon etc., change. We also showed that for a very hot accretion flow, critical/sonic point location changes in modified GR. There are other explorations of black hole in modified GR as well [5–7].

However, most of the cosmic objects are rotating, hence more realistic, at least in general, black holes are expected to be rotating. The same goes with other compact objects described by non-vacuum solutions. What if, a black hole is rotating in modified GR, more precisely in $f(R)$-gravity? In other words, how the Kerr solution changes in the $f(R)$-gravity?

In this work, we establish an asymptotically flat solution for a rotating black hole in modified GR. In place of obtaining a solution from the appropriate Einstein action for a modified GR, we rely on the Newman–Janis algorithm (NJA) [8]. We know that based on NJA the Kerr black hole solution

can be derived from the Schwarzschild solution by making an elementary transformation involved with complex numbers. The basic idea is, as if due to the choice of coordinates combining realistic coordinates and metric parameters, the Kerr metric appeared to be diagonal and also spherical symmetric, like the Schwarzschild black hole. However, once it is expanded in realistic coordinates it turns out to have off-diagonal terms with axially symmetric nature of the metric. We plan to implement NJA in the modified Schwarzschild metric under $f(R)$-gravity [4] to obtain the corresponding modified Kerr solution. To the best of our knowledge, there is no venture towards this solution before this work. Once we obtain the modified Kerr solution, we explore various basic characteristics of the metric, e.g. radius of event horizon, marginally stable and bound circular orbits, various components of epicyclic oscillation frequency, orbital angular frequency, etc., with the change of black hole spin and modified gravity parameter.

The paper is organized as follows. In the next two sections, we recapitulate the basic formalism of obtaining modified GR based field equation in $f(R)$-gravity and its solution for an asymptotically flat non-rotating black hole, respectively, in Sects. 2 and 3. Thereafter, we establish a rotating black hole solution in Sect. 4 based on NJA. Further, we discuss the nature of singularity of the metric and horizons in, respectively, Sects. 5 and 6. For the latter, first we present the numerical solution and then approximate analytical solution. Subsequently, we explore various fundamental orbits, as in GR, in this modified gravity framework for a test particle motion in Sect. 7 and corresponding fundamental oscillation frequencies in Sect. 8. We conclude our work in Sect. 9.

2 Basic formalism of field equation

In GR, the Einstein–Hilbert action produces the field equation. With the metric signature $(+ - - -)$ in 4-dimension it is given by [9]

$$S = \int \left[\frac{c^4}{16\pi G} R + \mathcal{L}_M \right], \tag{1}$$

where c is the speed of light, R is the scalar curvature such that $R = R_{\mu\nu} g^{\mu\nu}$, often called Ricci scalar, with $R_{\mu\nu}$ being Ricci tensor, G is Newton's gravitation constant, $\mathcal{L}_M$ is the Lagrangian of the matter field and $g = det(g_{\mu\nu})$ is the determinant of the metric tensor $g_{\mu\nu}$. Varying this action w.r.t. $g_{\mu\nu}$ and equating it to zero with appropriate boundary condition produces the Einstein's field equation for GR, given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{2}$$

where $T_{\mu\nu}$ is the energy–momentum tensor of the matter field. This equation relates the matter to the curvature of the spacetime. In case of modified GR, here $f(R)$ gravity, the Ricci scalar in Einstein–Hilbert action is replaced by $f(R)$ (being a function of the Ricci scalar). The action is then represented as

$$S = \int \left[\frac{c^4}{16\pi G} f(R) + \mathcal{L}_M \right]. \tag{3}$$

Now varying this modified action w.r.t $g_{\mu\nu}$ with appropriate boundary condition gives a modified version of the field equation, which is given by [10–12]

$$F(R)G_{\mu\nu} + \frac{1}{2} g_{\mu\nu}[RF(R) - f(R)]$$
$$- (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box)F(R) = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{4}$$

where $F(R) = \frac{d}{dR} f(R)$, $\Box$ is the d'Alembertian operator given by $\Box = \nabla^\mu \nabla_\mu$ and ∇_μ is the covariant derivative. For $f(R) = R$, this equation reduces to the well-known Einstein field equation in GR.

Now for the vacuum solution the energy–momentum tensor vanishes, i.e. $T_{\mu\nu} = 0$, and the equation reduces to

$$F(R)G_{\mu\nu} + \frac{1}{2} g_{\mu\nu} [RF(R) - f(R)]$$
$$- (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) F(R) = 0. \tag{5}$$

The trace of this equation is given by

$$RF(R) - 2f(R) + 3\Box F(R) = 0. \tag{6}$$

Substituting $f(R)$ from Eq. (6) into Eq. (5), we have

$$F R_{\mu\nu} - \nabla_\mu \nabla_\nu F = \frac{1}{4} g_{\mu\nu} (RF - \Box F). \tag{7}$$

3 Solution for a nonrotating black hole

Here we briefly recapitulate a solution for a non-rotating black hole in $f(R)$-gravity obtained earlier [4]. The vacuum solution of a spherically symmetric and static system can be written in the form of $g_{\mu\nu} = \text{diag} \left(s(r), -p(r), -r^2, -r^2 \sin^2\theta \right)$. Now we assume that $F(R)$ has a form such that, $F(r) = 1 + B/r$. Hence, as $r \to \infty$, $F(r) \to 1$, which generates the usual theory of GR. Note that $B \le 0$ to guarantee the attractive nature of gravity [4]. Now from Eq. (6) we have [13]

$$2 \frac{X'}{X} + r \frac{F'}{F} \frac{X'}{X} - 2r \frac{F''}{F} = 0 \tag{8}$$

and

$$-4s + 4X - 4rs\frac{F'}{F} + 2r^2s'\frac{F'}{F}$$
$$+2rs\frac{X'}{X} - r^2s'\frac{X'}{X} + 2r^2\,s'' = 0, \tag{9}$$

where $X(r) = p(r)\,s(r)$.

Solving Eqs. (8) and (9), and applying the boundary condition $X(r) \to 1$ as $r \to \infty$, $X(r)$ can be found as [4]

$$X(r) = \frac{16r^4}{(B + 2r)^4}. \tag{10}$$

Putting Eq. (10) in Eq. (9) we obtain the series solution for $s(r)$ (for $B \neq 0$) as

$$s(r) = \frac{-16 + 2BC_1 + 32\log 2 + (BC_2 + 8)\,i\pi}{2B^2}r^2$$
$$+1 + \frac{B(-24 + BC_2)}{24r} + \frac{B^2 - \frac{1}{16}B^3C_2}{r^2}$$
$$+\frac{-B^3 + \frac{11}{160}B^4C_2}{r^3} + \frac{188B^4 - 13B^5C_2}{192r^4} + \cdots, \tag{11}$$

where C_1 and C_2 are constants of integrations which can be obtained by arguing that the metric needs to behave as Schwarzschild metric at a large distance, which requires the coefficient of r^2 to vanish and coefficient of $1/r$ to be -2, which gives

$$C_2 = \frac{24(B - 2)}{B^2}, \tag{12}$$

$$C_1 = -8\frac{B(-1 + \log 4) + (-3 + 2B)\,i\pi}{B^2}. \tag{13}$$

Thus, the temporal component of the metric turns out to be

$$g_{tt} = s(r) = 1 - \frac{2}{r} - \frac{(-6 + B)\,B}{2r^2}$$
$$+\frac{B^2(-66 + 13B)}{20r^3} - \frac{B^3(-156 + 31B)}{48r^4}$$
$$+\frac{3B^4(-57 + 11B)}{56r^5} + O\left[r^{-6}\right]. \tag{14}$$

Thus the radial component of the metric can be found as $g_{rr} = -p(r)$, where $p(r) = X(r)/s(r)$, and thus the power series solution takes the form as

$$p(r) = 1 + \frac{2 - 2B}{r} + \frac{(-1 + B)(-4 + 3B)}{r^2}$$
$$-\frac{(-2 + B)(80 + B(-160 + 83B))}{20r^3}$$
$$+\frac{16 - 52B + \frac{1}{60}B^2(3732 + B(-1917 + 338B))}{r^4}$$
$$+\frac{32 - 128B + \frac{1008B^2}{5} - 155B^3 + \frac{6002B^4}{105} - \frac{6431B^5}{840}}{r^5}$$
$$+O\left[r^{-6}\right]. \tag{15}$$

4 Rotating black hole

4.1 Revisiting basics of Newman–Janis algorithm

After the original discovery of the Kerr metric, Newman and Janis showed that the solution could be derived from the Schwarzschild solution by making an elementary transformation involved with complex numbers, assuming the black hole to be spinning. The spin (angular momentum per unit mass) of black hole comes into the solution as an arbitrary parameter. The static spherically symmetric metric and the line element could be written in the general form in $(+---)$ convention as [14]

$$ds^2 = s(r)\,dt^2 - p(r)\,dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right). \tag{16}$$

In the null coordinates, this line element can be written, by advancing the time coordinate as $dt = du + \hat{f}\,dr$ and setting $\hat{f} = [s(r)/p(r)]^{-\frac{1}{2}}$, as

$$ds^2 = s(r)\,du^2 + 2\left[s(r)\,p(r)\right]^{\frac{1}{2}}\,du\,dr$$
$$-r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right). \tag{17}$$

Thus, the contravariant form of the metric can be written as

$$g^{\mu\nu} = \begin{pmatrix} 0 & [s(r)\,p(r)]^{-\frac{1}{2}} & 0 & 0 \\ \cdot & -\frac{1}{p(r)} & 0 & 0 \\ \cdot & \cdot & -\frac{1}{r^2} & 0 \\ \cdot & \cdot & \cdot & -\frac{1}{r^2\sin^2\theta} \end{pmatrix}. \tag{18}$$

Here "."s in Eq. (18) indicate that the metric is symmetric and will have the same elements as in the upper triangle. The contravariant form of the metric can be written so that it can be expressed in terms of its null tetrads [8,15,16] as

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu\overline{m}^\nu - m^\nu\overline{m}^\mu, \tag{19}$$

where the null tetrads satisfy the conditions

$$l_\mu l^\mu = m_\mu m^\mu = n_\mu n^\mu = 0,$$
$$l_\mu n^\mu = -m_\mu\overline{m}^\mu = 1,$$
$$l_\mu m^\mu = n_\mu m^\mu = 0, \tag{20}$$

with the bar indicating the complex conjugate.

Putting the elements of the metric from Eqs. (18) to (19), along with Eq. (20), the null tetrads are found to be

$$l^\mu = \delta_1^\mu, \tag{21}$$

$$n^\mu = -\frac{1}{2}\frac{1}{p(r)}\delta_1^\mu + [s(r)\,p(r)]^{-\frac{1}{2}}\delta_0^\mu, \tag{22}$$

$$m^\mu = \frac{1}{\sqrt{2}\,r}\left(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right). \tag{23}$$

Then following NJA, we proceed by making a complex transformation as

$$
\begin{aligned}
u &\to u' = u - ia\cos\theta,\\
r &\to r' = r + ia\cos\theta,\\
\theta &\to \theta' = \theta,\\
\phi &\to \phi' = \phi.
\end{aligned}
\tag{24}
$$

By considering this as a complex rotation of the $\theta - \phi$ plane, the tetrads can be obtained as

$$
l^{\mu} = \delta_{1}^{\mu},
\tag{25}
$$

$$
n^{\mu} = -\frac{1}{2}\frac{1}{p(r,\theta)}\delta_{1}^{\mu} + [s(r,\theta)\ p(r,\theta)]^{-\frac{1}{2}}\delta_{0}^{\mu},
\tag{26}
$$

$$
g_{\mu\nu} =
\begin{pmatrix}
s(r,\theta) & 0 & 0 & a\sin^{2}\theta\left((X(r,\theta))^{\frac{1}{2}} - s(r,\theta)\right)\\
. & -\frac{\Sigma}{\frac{\Sigma\,s(r,\theta)}{X(r,\theta)} + a^{2}\sin^{2}\theta} & 0 & 0\\
. & . & -\Sigma & 0\\
. & . & . & -\sin^{2}\theta\left(\Sigma + a^{2}\sin^{2}\theta\left(2(X(r,\theta))^{\frac{1}{2}} - s(r)\right)\right)
\end{pmatrix},
\tag{32}
$$

$$
m^{\mu} = \frac{1}{\sqrt{2}\,(r + ia\cos\theta)}\left(ia\sin\theta\,(\delta_{0}^{\mu} - \delta_{1}^{\mu}) + \delta_{2}^{\mu}\right.
$$
$$
\left. + \frac{i}{\sin\theta}\delta_{3}^{\mu}\right).
\tag{27}
$$

Note that $s(r,\theta)$ and $p(r,\theta)$ in Eq. (26) are completely different from $s(r)$ and $p(r)$ in Eq. (22) (and in Eqs. (11) and (15); also see [17–19]). In fact, the new functions are functions of both r and θ, while the old ones are functions of only r.

From Eq. (19), the contravariant form of the metric is obtained as

$$
g^{\mu\nu} =
\begin{pmatrix}
-\frac{a^{2}\sin^{2}\theta}{\Sigma}\,[s(r,\theta)\ p(r,\theta)]^{-\frac{1}{2}} + \frac{a^{2}\sin^{2}\theta}{\Sigma} & 0 & -\frac{a}{\Sigma}\\
. & -\frac{1}{p(r,\theta)} - \frac{a^{2}\sin^{2}\theta}{\Sigma} & 0 & \frac{a}{\Sigma}\\
. & . & -\frac{1}{\Sigma} & 0\\
. & . & . & -\frac{1}{\Sigma\sin^{2}\theta}
\end{pmatrix},
\tag{28}
$$

where $\Sigma = r^{2} + a^{2}\cos^{2}\theta$. The inverse of this metric, i.e. its covariant form, is

$$
g_{\mu\nu} =
\begin{pmatrix}
s(r,\theta)\,[s(r,\theta)\ p(r,\theta)]^{\frac{1}{2}} & 0 & & a\sin^{2}\theta\left([s(r,\theta)\ p(r,\theta)]^{\frac{1}{2}} - s(r,\theta)\right)\\
. & 0 & 0 & -a\,[s(r,\theta)\ p(r,\theta)]^{\frac{1}{2}}\sin^{2}\theta\\
. & . & -\Sigma & 0\\
. & . & . & -\sin^{2}\theta\left(\Sigma + a^{2}\sin^{2}\theta\left(2[s(r,\theta)\ p(r,\theta)]^{\frac{1}{2}} - s(r)\right)\right)
\end{pmatrix}.
\tag{29}
$$

Now we redefine the coordinates u and ϕ such that, $du = dt + g(r)\ dr$ and $d\phi = d\varphi + h(r)\ dr$, with g and h as

$$
g(r) = -\frac{(p(r,\theta))^{\frac{1}{2}}\left(\Sigma + a^{2}\sin^{2}\theta[s(r,\theta)\ p(r,\theta)]^{\frac{1}{2}}\right)}{(s(r))^{\frac{1}{2}}\left(\Sigma + a^{2}\sin^{2}\theta\ e^{2\lambda(r,\theta)}\right)},
\tag{30}
$$

$$
h(r) = -\frac{a\ p(r)}{\Sigma + a^{2}\sin^{2}\theta\ p(r)},
\tag{31}
$$

in a new coordinate system. This leads to all the non-diagonal elements, except $g_{\phi t}$, go to zero. This transforms the metric to Boyer–Lindquist coordinate system. Now putting $X(r,\theta) = p(r,\theta)\,s(r,\theta)$, the metric in this coordinate system takes the form

which essentially leads to the counter part of rotating black hole of the metric in Eq. (16).

4.2 Transformation of specific functions under NJA and modified Kerr metric

Equipped with the knowledge of NJA, the angular momentum parameter can be easily incorporated in the non-rotating vacuum solution. For this we first proceed by noting that while we make the complex transformation, the coordinates r and u are complexified and a new parameter a is introduced. However, since in the end one needs a real spacetime, a function Q must remain real and so its changes are given as [16,20]

$$
Q(r) \to Q(r,\bar{r}),
\tag{33}
$$

so that the functions $1/r^{2n}$ and $1/r^{2n+1}$ must be written as

$$
\frac{1}{r^{2n}} \to \frac{1}{(r\bar{r})^{n}},
\tag{34}
$$

$$\frac{1}{r^{2n+1}} = \frac{1}{r^{2n}}\frac{1}{r} \rightarrow \frac{1}{(r\bar{r})^n}\frac{1}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right). \qquad (35)$$

Now suppose the function $Q(r, \bar{r})$ has some terms of $\frac{1}{(r\bar{r})^n}$ and $\frac{1}{(r\bar{r})^n}\frac{1}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)$ with at least one of them having a non-zero coefficient, then after the complex transformation of $u \rightarrow u' = u - ia\cos\theta,\ r \rightarrow r' = r + ia\cos\theta,\ \theta \rightarrow \theta' = \theta,\ \phi \rightarrow \phi' = \phi$, the components of $Q(r, \bar{r})$ will transform as

$$\frac{1}{r^{2n}} \equiv \frac{1}{(r\bar{r})^n} \rightarrow \frac{1}{[(r+ia\cos\theta)(r-ia\cos\theta)]^n} = \frac{1}{\Sigma^n} \qquad (36)$$

and similarly

$$\frac{1}{r^{2n+1}} \equiv \frac{1}{(r\bar{r})^n}\frac{1}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)$$
$$\rightarrow \frac{1}{[(r+ia\cos\theta)(r-ia\cos\theta)]^n}\frac{1}{2}\left(\frac{1}{r+ia\cos\theta}\right.$$
$$\left.+\frac{1}{r-ia\cos\theta}\right) = \frac{r}{\Sigma^{n+1}}. \qquad (37)$$

Thus, after the complex transformation, the function $Q(r)$ transforms to $Q(r, \theta)$.[1] Applying Eqs. (36) and (37) to the functions $X(r)$, $s(r)$ and $p(r)$, we have

$$X(r, \theta) = \frac{\left(r^2 + a^2\cos^2\theta\right)^2}{\left(\left(\frac{B}{2}+r\right)^2 + a^2\cos^2\theta\right)^2}, \qquad (38)$$

$$s(r, \theta) = 1 - \frac{2r}{r^2 + a^2\cos^2\theta} - \frac{(-6+B)B}{2\left(r^2 + a^2\cos^2\theta\right)}$$
$$+ \frac{B^2(-66+13B)r}{20\left(r^2 + a^2\cos^2\theta\right)^2} - \frac{B^3(-156+31B)}{48\left(r^2 + a^2\cos^2\theta\right)^2}$$
$$+ \frac{3B^4(-57+11B)r}{56\left(r^2 + a^2\cos^2\theta\right)^3} + \cdots, \qquad (39)$$

$$p(r, \theta) = 1 + \frac{(2-2B)r}{r^2 + a^2\cos^2\theta} + \frac{(-1+B)(-4+3B)}{r^2 + a^2\cos^2\theta}$$
$$- \frac{(-2+B)(80+B(-160+83B))r}{20\left(r^2 + a^2\cos^2\theta\right)^2}$$
$$+ \frac{16 - 52B + \frac{1}{60}B^2(3732+B(-1917+338B))}{\left(r^2 + \cos^2\theta\right)^2}$$
$$+ \frac{\left(32 - 128B + \frac{1008B^2}{5} - 155B^3 + \frac{6002B^4}{105} - \frac{6431B^5}{840}\right)r}{\left(r^2 + \cos^2\theta\right)^3}$$
$$+ \cdots. \qquad (40)$$

Thus Eqs. (32), (38)–(40) essentially complete our development of the metric which is the asymptotically flat vacuum solution for a rotating black hole in a modified gravity. It can

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[1] $Q(r)$ and $Q(r, \theta)$ are not necessarily equal.

be easily seen that by setting $B = 0$, we obtain the usual Kerr-metric.

5 Source and singularity

From Eq. (32) we see that the metric becomes singular, when $s(r)$ or $p(r)$ becomes singular and that happens when $\Sigma = 0$, since Σ is present at the denominator in both. This shows that the metric becomes singular for [20]

$$r = 0, \quad \theta = \frac{\pi}{2}.$$

This can be seen to be a geometric singularity by computing the curvature contraction $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$. Further, it is an extended singularity, rather than 'point-like' singularity (as in Schwarzschild metric).

Now defining local rectangular coordinate system

$$x = r\sin\theta\cos\phi + \alpha\sin\theta\sin\phi,$$
$$y = r\sin\theta\sin\phi - \alpha\sin\theta\cos\phi,$$
$$z = r\cos\theta,$$

we immediately see that $r = 0,\ \theta = \pi/2$ corresponds to $x^2 + y^2 = \alpha^2$ and $z = 0$. Consequently, the physical singularity of the Kerr metric is a ring singularity. With the small B approximation as made in Sect. 6.2 below, the term involved with spin angular momentum transforms as $\alpha \approx a - 1.5B$ (as will be clearer in Sect. 6.2 below), thus the radius and angular position as, respectively,

$$\alpha = a - 1.5B, \quad \theta = \frac{\pi}{2}. \qquad (41)$$

Therefore, the singularity can be seen to be on a circle of radius α around the origin in the $z = 0$ plane. The solution can be considered to lie uniformly distributed on this circle, bounding an interior disc $\sqrt{x^2 + y^2} \leq \alpha$. This singularity signifies the presence of a rotating black hole and is termed as ring singularity.

6 Horizons

In addition to the ring-like curvature singularity, there are also additional coordinate singularities. Such coordinate singularities can be removed by suitable choice of coordinates, but they often underlie important physical phenomenon and have geometric description. Considering the Boyer–Lindquist coordinates for the metric given by (32), we define Δ as

$$\Delta = \Sigma\frac{s(r, \theta)}{X(r, \theta)} + a^2\sin^2\theta, \qquad (42)$$

Table 1 Numerical values of r_H for varying B and a

r_H	a										
		0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
B	0	2.00	1.99	1.98	1.95	1.92	1.87	1.80	1.71	1.60	1.44	1.00
	-0.1	2.15	2.14	2.13	2.11	2.07	2.02	1.96	1.88	1.78	1.64	1.42
	-0.2	2.30	2.29	2.28	2.26	2.22	2.18	2.12	2.05	1.95	1.83	1.66
	-0.3	2.45	2.44	2.42	2.41	2.37	2.33	2.28	2.21	2.12	2.01	1.86
	-0.4	2.60	2.59	2.58	2.55	2.52	2.48	2.43	2.37	2.29	2.19	2.06
	-0.5	2.73	2.74	2.72	2.70	2.67	2.63	2.59	2.52	2.45	2.36	2.24
	-1	3.46	3.45	3.45	3.43	3.41	3.37	3.33	3.28	3.23	3.16	3.07
	-1.5	4.17	4.16	4.15	4.14	4.12	4.09	4.06	4.02	3.97	3.91	3.84
	-2	4.86	4.86	4.85	4.84	4.82	4.79	4.76	4.73	4.68	4.63	4.58
	-2.5	5.55	5.45	5.54	5.53	5.51	5.49	5.46	5.43	5.39	5.34	5.29
	-3	6.23	6.25	6.22	6.21	6.19	6.17	6.15	6.12	6.08	6.04	5.99

then $g_{rr} = -\Sigma/\Delta$, which becomes singular when $\Delta = 0$. The solution of r for $\Delta = 0$ gives two real values $r_\pm$ of which $r_- \leq r_+$. These radii are referred to as outer (r_+) and inner (r_-) horizons; the former is called the event horizon and the later one Cauchy horizon, and the region $r < r_+$ is referred to as the 'interior' of the black hole. It can be shown that the event horizon marks the point of no return. Now since r_- lies inside the event horizon and no actual observer can have access to the interior of the event horizon, we avoid any discussion about the inner horizon r_-.

6.1 Numerical solution

From Eqs. (32), (38)–(40) we obtain the metric components as a series solution and substituting them in equation (42) effectively gives Δ. Now $\Delta = 0$ has been numerically solved in order to obtain event horizon r_H which is r_+. We will obtain an analytic approximation of the result in the next section. Tables 1 and 2 show r_H for different a and B in the equatorial plane.

Tables 1 and 2 show that r_H monotonically increases with the increase of $|B|$ and monotonically decreases with the increase of a. From Table 2 and Fig. 1 it can be seen that unlike in Kerr metric, $|a_{max}| > 1$ is allowed due to $B < 0$. The variation of maximum a, i.e. a_{max}, for varying B is shown in Fig. 2. It can be seen from the Fig. 2 that $|a_{max}|$ varies almost linearly with B. Exploring and interpreting these results with the exact solutions is beyond the scope of this work. We will look at an analytic approximation of the above feature and report the result in the next section, where we will calculate $|a_{max}|$. We will confirm that indeed $|a_{max}|$ is allowed to be greater than unity in modified gravity and also varies approximately linearly with B.

6.2 Analytical approximation

In order to assure the possibility of analytical solutions, we consider very small modifications to GR and hence we take $B/r \ll 1$. Thus we take only terms up to r^{-2}, the functions $s(r, \theta)$ and $p(r, \theta)$ can then be written as

$$s(r, \theta) = 1 - \frac{2r}{r^2 + a^2\cos^2\theta} - \frac{(-6 + B)B}{2(r^2 + a^2\cos^2\theta)} + O[r^{-3}], \tag{43}$$

$$p(r, \theta) = 1 + \frac{(2 - 2B)r}{r^2 + a^2\cos^2\theta} + \frac{(-1 + B)(-4 + 3B)}{r^2 + a^2\cos^2\theta} + O\left[r^{-3}\right]. \tag{44}$$

Taking terms upto r^{-2}, in Boyer–Lindquist coordinate system, the metric can be recast from Eqs. (32), (43), (44) and taking further $B \ll 1$ and having $X \approx 1$, the nonzero component of the metric comes out to be

$$g_{\mu\nu} = \begin{pmatrix} 1 - \frac{2r}{\Sigma} - \frac{\beta}{\Sigma} & 0 & 0 & \frac{2a\sin^2\theta}{\Sigma}(2r + \beta) \\ . & -\frac{\Sigma}{\Delta} & 0 & 0 \\ . & . & -\Sigma & 0 \\ . & . & . & -\sin^2\theta\left(r^2 + a^2 + \frac{a^2\sin\theta(2r+\beta)}{\Sigma}\right) \end{pmatrix}, \tag{45}$$
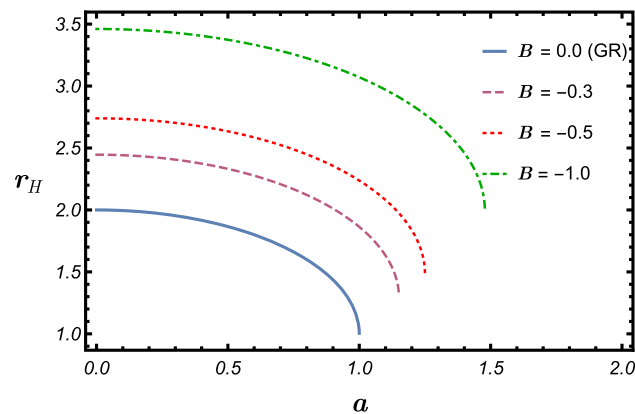
where $\Delta \approx r^2 + a^2 - 2r - \beta$, and $\beta = (-6 + B)B/2 \approx -3B$. [2] Thus the line element is of the form

$$ds^2 = \left(1 - \frac{2r}{\Sigma} - \frac{\beta}{\Sigma}\right)dt^2 + \left(\frac{4a\sin^2\theta}{\Sigma}(2r + \beta)\right)dtd\varphi$$
$$- \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \sin^2\theta$$
$$\times \left(r^2 + a^2 + \frac{a^2\sin\theta(2r + \beta)}{\Sigma}\right)d\varphi^2. \tag{46}$$

---

[2] Note, $B \leq 0 \rightarrow \beta \geq 0$ as $B = -\beta/3$.

Table 2 Numerical values of r_H for varying B and a, with $a > 1$

| r_H | a | | | |
	1.1	1.2	1.3	1.4
B 0	–	–	–	–
− 0.1	–	–	–	–
− 0.2	1.24	–	–	–
− 0.3	1.63	–	–	–
− 0.4	1.87	1.44	–	–
− 0.5	2.08	1.84	–	–
− 0.6	2.27	2.08	–	–
− 0.7	2.46	2.29	2.03	–
− 0.8	2.63	2.48	2.28	–
− 0.9	2.80	2.67	2.49	2.20
− 1	2.97	2.85	2.69	2.46

Fig. 2 Variation of maximum possible spin of black hole a_{max} as a function of modified gravity parameter B

Fig. 1 Variation of event horizon r_H as a function of spin of black hole a for different modified gravity parameter B

Fig. 3 Approximate analytic solution for r_H with the change of a for different B

This line element matches exactly with the results of black hole theories with higher-dimensional branes [21,22]. This shows that the work presented here gives a more general metric and includes the results from higher-dimensional branes. The effects of higher-dimensional branes come from a specialized case where the modification to gravity has been taken to be very small.
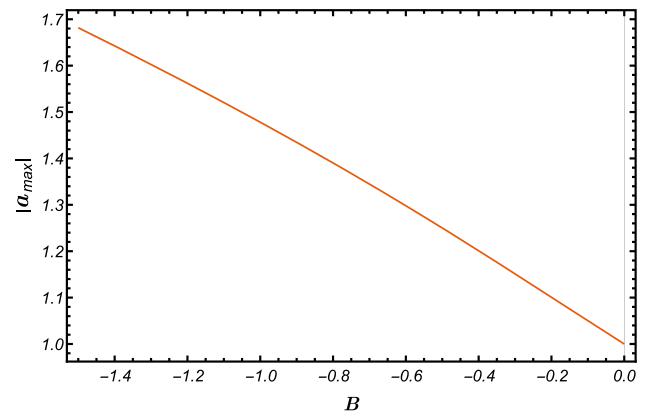
Now to find the horizons in this case, the equation $\Delta = 0$ has to be solved which approximately becomes, from Eq. (42),

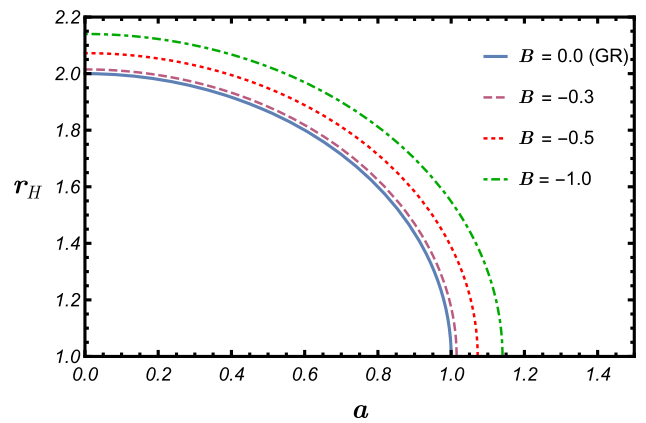$$\Delta = \Sigma \left(1 - \frac{2r}{\Sigma} - \frac{\beta}{\Sigma}\right) + a^2\sin^2\theta \approx 0, \qquad (47)$$

which gives

$$\Delta = r^2 - 2r + \left(a^2 - \beta\right) = 0. \qquad (48)$$

Thus, to the first order in B, we obtain $\Delta = r^2 + a^2 - 2r - \beta$. Now solving the quadratic equation (48) gives two three-surfaces of constant r as

$$r_\pm = 1 \pm \sqrt{1 - \left(a^2 - \beta\right)}.$$

These surfaces give the outer (r_+) and inner (r_-) horizons. Thus, the event horizon takes the form as

$$r_H = r_+ = 1 + \sqrt{1 - a^2 + \beta}. \qquad (49)$$

It can be easily seen that by setting $B = 0$, we recover the well-known results of the event horizon in Kerr metric, $r_{H_0} = 1 + \sqrt{1 - a^2}$, which confirms the validity of analytical solutions.

Figure 3 shows how r_H varies with a based on analytical approximate solution. It can be seen from Table 1 that for $a = 0$ the results match quite well with the analytical results presented here. However, as $|B|$ increases, the value deviates a lot from the actual solution, which is because we have taken only terms up to r^{-2} in $s(r)$ and $p(r)$ in analytical calculation. Quantitatively, when $B \approx -0.1$, very small compared to r, the numerical solution matches with the approximate analytical solution; thus, the analytical approximation is valid

for the $B \geq -0.1$ realm, so that

$$\left(r_{H_{analytical}}\right)_{B \geq -0.1} \approx \left(r_{H_{numerical}}\right)_{B \geq -0.1}.$$

From Eq. (49), for r_H to be real we must have

$$1 - \left(a^2 - \beta\right) \geq 0,$$
$$\rightarrow |a| \leq \sqrt{1+\beta}.$$

Thus,

$$|a_{max}| = \sqrt{1+\beta}, \tag{50}$$
$$|a_{max}| \approx 1 + \beta/2 \approx 1 - 1.5B \geq |a_{max}|_{kerr}. \tag{51}$$

From Eq. (50) the maximum value of $|a_{max}|$ obtained to be different from that obtained from Kerr metric and because $\beta \geq 0$, black holes can have spin parameter of value more than unity, i.e. $|a| \geq 1$. The linear dependence of spin on modified gravity parameter can also be seen from Eq. (51) which nearly matches with Fig. 2.

Interestingly, this approximate analytical solution matches exactly with the Kerr–Newman metric if we replace β with $-Q^2$, where Q is the charge of the black hole. However, we know that the Kerr-Newman solution is a vacuum solution of the Einstein's field equation when the integrand of action is a scalar curvature (Ricci scalar) dependent on the parameters M, a and Q. Hence, this approximate solution due to the perturbative correction to GR can be treated as the solution of Einstein's field equation itself with appropriate redefinition of the action and parameter(s). However, in general the solution $(g_{\mu\nu})$ obtained in Sect. 4 can be understood as the one corresponding to an appropriate choice of $f(R)$ and then $F(R)$ satisfying Eq. (5).

7 Orbits in equatorial plane

Due to the source having an angular momentum, the system's geometry is no longer spherical and is only axisymmetric. Only the components of the angular momentum along the symmetry axis are conserved. There are orbits confined to the equatorial plane ($\theta = \pi/2$), but the general orbit is not necessarily on the plane. However, to present a manageable solution, we consider the equatorial plane in this section. Thus, from Eqs. (32), (38)–(40) we can construct two Killing vectors corresponding to energy and angular momentum. The energy arises from the timelike Killing vector $K_\mu = \partial_t$, and the Killing vector whose conserved quantity is the magnitude of the angular momentum is given by $L = \partial_\varphi$. Thus, we can construct the conserved quantities as E and L as the conserved energy per unit mass and angular momentum per unit mass along the symmetry axes, which can be expressed as [23]

$$E = -K_\mu u^\mu \tag{52}$$

and

$$L = L_\mu u^\mu. \tag{53}$$

Now by inspecting the metric we have

$$E = -g_{tt}u^t - g_{t\varphi}u^\varphi, \tag{54}$$
$$L = g_{t\varphi}u^t + g_{\varphi\varphi}u^\varphi. \tag{55}$$

These Eqs. (52)–(55) can be solved for u^t and u^φ to find

$$u^t = \frac{1}{\Delta}\left(g_{\varphi\varphi}E + g_{t\varphi}L\right), \tag{56}$$
$$u^\varphi = -\frac{1}{\Delta}\left(g_{tt}L + g_{t\varphi}E\right), \tag{57}$$

where $\Delta = \left(g_{t\varphi}\right)^2 - g_{\varphi\varphi}g_{tt}$.

7.1 Marginally bound circular orbit

From normalization condition of four-velocity $\boldsymbol{u} \cdot \boldsymbol{u} = 1$, together with $u^\theta = 0$, we obtain a radial equation for $u^r = dr/d\tau$ as

$$g_{tt}\left(u^t\right)^2 + g_{rr}\left(u^r\right)^2 + 2g_{t\varphi}u^t u^\varphi + g_{\varphi\varphi}\left(u^\varphi\right)^2 = 1. \tag{58}$$

Thus Eqs. (56)–(58) essentially calculate u^r as a function of E, L, r, a and B. The effective potential can now be defined as [23,24]

$$V_{eff}(E, L, r, a, B) := r^3\left(u^r\right)^2. \tag{59}$$

Now for circular orbits we must have the radial velocity to vanish and hence the effective potential must vanish. Thus for equilibrium condition, we must have an extremum in V_{eff}. Therefore, we obtain the relations

$$V_{eff} = 0, \quad \frac{\partial V_{eff}}{\partial r} = 0. \tag{60}$$

It can be shown that unbound circular orbits have $E > 1$. Given an infinitesimal outward perturbation, a particle in such an orbit will escape infinity. Bound orbits exist for $r > r_{mb}$, where r_{mb} is the radius of the marginally bound circular orbit with $E = 1$. Thus, solving Eq. (60) with condition $E = 1$, we obtain the value of $r = r_{mb}$. From Fig. 4 the effect of B on r_{mb} can be seen, and that r_{mb} increases with increasing $|B|$ for a fixed a, and r_{mb} decreases with the increase of a for a fixed B. It also can be seen that setting $B = 0$ gives the same results as in GR.

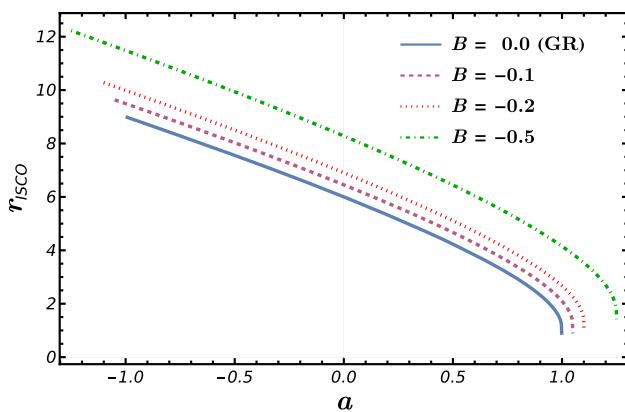## 7.2 Innermost stable circular orbit

To find the innermost stable circular orbit, we opt for the same V_{eff} as defined in Sect. 7.1. Since we are considering circular orbits, Eq. (60) is still valid. All the bound circular orbits are not stable. For stability condition, we must have
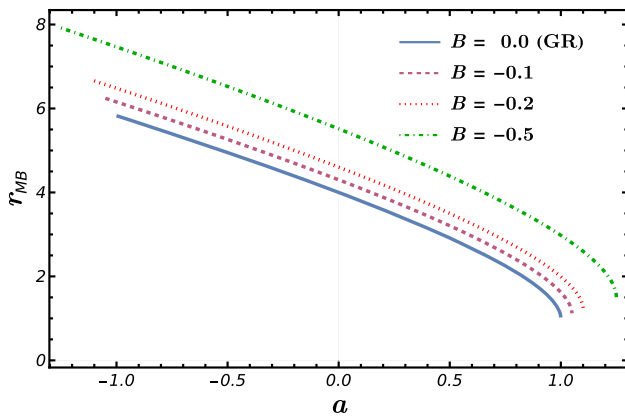
Fig. 4 Variation of marginally bound orbit as a function of spin of black hole for different B

Fig. 5 Variation of marginally stable circular orbit as a function of spin of black hole for different B

the condition

$$\frac{\partial^2 V_{eff}}{\partial r^2} \leq 0. \qquad (61)$$

Now, the minimum radius (innermost orbit) that satisfies Eqs. (60) and (61) is termed as Innermost Stable Circular Orbit (ISCO) and the radius named as r_{ISCO}. Numerically solving these three equations simultaneously we obtain the variation of r_{ISCO} shown in Fig. 5. Similar to the case of r_{mb}, here we see r_{ISCO} increases with increasing $|B|$ for a fixed a, and r_{ISCO} decreases with the increase of a for a fixed B. Also, it can be easily verified that as $B = 0$, the results of GR are preserved.

8 Epicyclic frequency in modified gravity

In this section we will briefly describe the derivation of epicyclic oscillation frequencies for the stationary, axisymmetric metric from the effective potential for circular geodesics, depicting the spacetime around a rotating black

hole. From Eqs. (32)–(40) the line element can essentially be expressed as

$$ds^2 = g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2, \qquad (62)$$

with $g_{\mu\nu}$ as a function of r and θ and a symmetry along ϕ and t. It is most straightforward to obtain the epicyclic frequencies for a metric that can be expressed in this form. Epicyclic frequencies originate from the the relaxation of the circular orbits under external perturbation and it must be that this frequencies solely depend on the structure of the spacetime.

Now the similar normalization condition as in Eq. (58) along with Eqs. (56) and (57) but without a fixed θ, hence with u^θ, can be rewritten as

$$g_{rr}\left(u^r\right)^2 + g_{\theta\theta}\left(u^\theta\right)^2 = \mathcal{V}_{eff}, \qquad (63)$$

where the effective potential can be defined as

$$\mathcal{V}_{eff} = \frac{\left(E^2 - g_{tt}\right)g_{\varphi\varphi} + \left(2LE + g_{t\varphi}\right)g_{t\varphi} + L^2 g_{tt}}{\left(g_{t\varphi}^2 - g_{tt}g_{\varphi\varphi}\right)\Delta}. \qquad (64)$$

For circular orbits in the equatorial plane we have $u^r = u^\theta = 0$, which implies $\mathcal{V}_{eff} = 0$, and $\dot{u}^r = \dot{u}^\theta = 0$ give $\partial_r \mathcal{V}_{eff} = \partial_\theta \mathcal{V}_{eff} = 0$. From these three conditions E and L can be obtained as [25]

$$E = -\frac{g_{tt} + \Omega g_{t\varphi}}{\sqrt{-g_{tt} - 2g_{t\varphi}\Omega - g_{\varphi\varphi}\Omega^2}}, \qquad (65)$$

$$L = -\frac{g_{t\varphi} + \Omega g_{\varphi\varphi}}{\sqrt{-g_{tt} - 2g_{t\varphi}\Omega - g_{\varphi\varphi}\Omega^2}} \qquad (66)$$

and the orbital angular frequency is given by [25]

$$\Omega \equiv 2\pi \nu_\varphi = \frac{-\partial_r g_{t\phi} \pm \sqrt{\left(\partial_r g_{t\varphi}\right)^2 - \partial_r g_{\varphi\varphi}\partial_r g_{tt}}}{\partial_r g_{\varphi\varphi}}, \qquad (67)$$
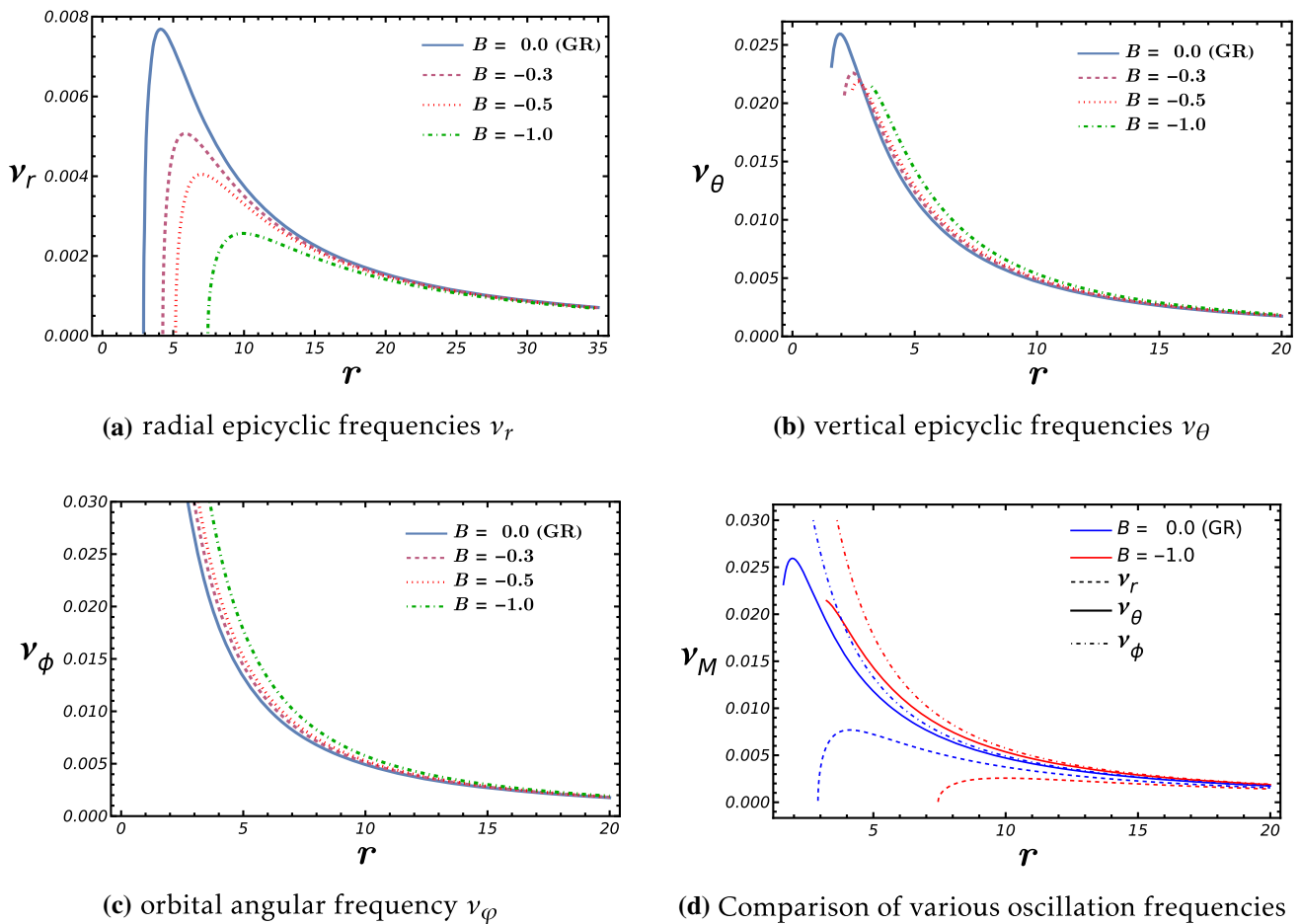
where the positive (negative) sign in Eq. (67) refers to the co-rotating (counter-rotating) orbits with respect to the black hole spin. Equation (67) also defines the quantity ν_φ which is the frequency in which the particles move around the black hole in circular orbits. Now the proper angular momentum (ℓ) can be derived to be

$$\ell = -\frac{g_{t\varphi} + \Omega g_{\varphi\varphi}}{g_{tt} + \Omega g_{t\varphi}}. \qquad (68)$$

For finding the epicyclic frequencies, we first consider the perturbation to the radial (r) and vertical (θ) coordinates so that

$$r(t) \approx r_0 + \delta r(t), \quad \theta(t) \approx \theta_0 + \delta\theta(t), \qquad (69)$$

where the perturbations are considered to be $\delta r(t) \sim e^{i\Omega_r t}$ and $\delta\theta(t) \sim e^{i\Omega_\theta t}$, so as to have equations for harmonic

(a) radial epicyclic frequencies ν_r

(b) vertical epicyclic frequencies ν_θ

(c) orbital angular frequency ν_φ

(d) Comparison of various oscillation frequencies

Fig. 6 Profiles of the Keplerian frequency ν_φ and the epicyclic frequencies ν_r (radial) and ν_θ (vertical) in the modified theory with spin parameter set to $a = 0.8$. The modified gravity parameter B has been set to $0.0, -0.3, -0.5, -1.0$ in **a**–**c**. In **d** $B = 0$ and -1.0 for bottom and top curves respectively for ν_φ and ν_θ, while for top and bottom curves for ν_r

oscillator of the form

$$\frac{d^2\delta r}{dt^2} + \Omega_r^2 \delta r = 0, \tag{70}$$

$$\frac{d^2\delta\theta}{dt^2} + \Omega_\theta^2 \delta\theta = 0. \tag{71}$$

Here r_0 is the radius of the circular orbit and $\theta_0 = \pi/2$, is the angle at which the equatorial plane exists. Now expanding the R.H.S. of Eq. (63) into second-order Taylor series along with the radial (r) and vertical (θ) components, replacing r and θ from Eq. (69), using Eqs. (70) and (71), and after some simple algebra we obtain [25,26]

$$\Omega_r^2 = (2\pi \nu_r)^2 = -\frac{1}{2g_{rr}(u^t)^2}\frac{\partial^2 \mathcal{V}_{eff}}{\partial r^2}, \tag{72}$$

$$\Omega_\theta^2 = (2\pi \nu_\theta)^2 = -\frac{1}{2g_{\theta\theta}(u^t)^2}\frac{\partial^2 \mathcal{V}_{eff}}{\partial \theta^2}. \tag{73}$$

The dependence of the frequencies on B arises from various metric components. The explicit forms of the frequencies are huge and hence are not included in this work. Rather, we

shall provide a numerical estimations of these frequencies. It should also be noted that these frequencies are observables and will be the key in estimating the most favored value of B from observational data.

The behaviors of ν_r and ν_θ are shown in Fig. 6a, b with a fixed spin parameter $a = 0.8$. From Fig. 6 it can be seen that ν_r decreases, while ν_θ and ν_ϕ increase, with the increase of $|B|$, at a given r (particularly away from the black hole). However, the peak of ν_θ decreases with increasing $|B|$. Also ν_r vanishes at a larger radius with a smaller peak with increasing $|B|$. It can be easily seen from Eq. (67) that the GR result, i.e. $\Omega \sim (r^{3/2} \pm a)^{-1}$, can be found by setting $B = 0$.

9 Conclusion

The idea of modified GR is in the literature for sometime, but its indispensable usefulness was not very clear. Although Starobinsky argued for R^2-gravity (a kind of $f(R)$-gravity) to explain inflation [27], it was not clear if all the gravity theo-

ries are the same. In last one decade or so, the authors however showed that R^2-gravity could be useful to sort out problems lying with neutron stars and white dwarfs [2,3,28,29] as well. Nevertheless, none of these solutions is black hole (vacuum) solution. In this work, we establish an asymptotically flat vacuum solution of the axially symmetric field equation in a modified GR, more precisely $f(R)$-gravity. The solution particularly describes the spacetime geometry around a rotating black hole, i.e. the modified Kerr black hole solution, for the first time of this kind to the best of our knowledge.

It shows that depending on the modified gravity parameter, all the fundamental properties of the black hole change, e.g. the radii of black hole, marginally stable and bound circular orbits increase. Therefore, based on the observed size, e.g. by Event Horizon Telescope (EHT) image, the inference or estimate of spin of black hole would be incorrect unless proper theory is used. If indeed the gravity theory is based on an $f(R)$-gravity, the GR based inference of spin of the black hole would actually underestimate it. This has many far reaching astrophysical implications.

The solution also implies that the naked singularity, as formed at the Kerr parameter $a > 1$, need not necessarily produce in modified GR. This naturally has important implications to the cosmic censorship hypothesis [30,31]. Therefore, black holes, according to this gravity theory, can spin faster without forming naked singularity depending on the modified gravity parameter.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is completely a theoretical work, hence no observed data directly involved in the project.]

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