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# Multi-Hamiltonian formulations and stability of higher-derivative extensions of 3*d* Chern–Simons

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**Abstract** Most general third-order 3*d* linear gauge vector field theory is considered. The field equations involve, besides the mass, two dimensionless constant parameters. The theory admits two-parameter series of conserved tensors with the canonical energy-momentum being a particular representative of the series. For a certain range of the model parameters, the series of conserved tensors include bounded quantities. This makes the dynamics classically stable, though the canonical energy is unbounded in all the instances. The free third-order equations are shown to admit constrained multi-Hamiltonian form with the 00-components of conserved tensors playing the roles of corresponding Hamiltonians. The series of Hamiltonians includes the canonical Ostrogradski's one, which is unbounded. The Hamiltonian formulations with different Hamiltonians are not connected by canonical transformations. This means, the theory admits inequivalent quantizations at the free level. Covariant interactions are included with spinor fields such that the higher-derivative dynamics remains stable at interacting level if the bounded conserved quantity exists in the free theory. In the first-order formalism, the interacting theory remains Hamiltonian and therefore it admits quantization, though the vertices are not necessarily Lagrangian in the third-order field equations.

## 1 Introduction

Classical dynamics and quantization of various higher-derivative models are discussed once and again for decades. Among most frequently studied specific models we can mention Pais-Uhlenbeck (PU) oscillator [1], Podolsky and Lee–Wick electrodynamics [2–4], higher-derivative extensions of the Chern–Simons model [5], higher-derivative Yang–Mills models [6], conformal gravity [7], various higher-derivative higher-spin fields theories [8–10], modified theories of gravity [11], including critical gravity [12]. The higher-derivative models often reveal remarkable various properties comparing to the counterparts without higher derivatives. In particular, the inclusion of higher derivatives improves the convergency in the field theory both at classical and quantum level in many models. Also the conformal symmetry often requires inclusion of higher derivatives in the field equations.

The higher-derivative dynamics are also notorious for the classical and quantum instability. The key point, where the problem can be immediately seen is that the canonical energy is unbounded for general higher-derivative Lagrangian systems. Several exceptions are known [13–18] of the higher-derivative models such that have bounded canonical energy. In all these cases, the energy is on-shell bounded because of strong constraints among the field equations. At the quantum level, the instability reveals itself by the ghost poles in the propagator and it is related to the problem of unbounded spectrum of energy. In its turn, the unbounded energy spectrum results from the fact that the canonical Hamiltonian, being the phase space equivalent of canonical Noether's energy, is unbounded due to the higher derivatives.

In the work [19], it was noticed that the broad class of higher-derivative models are stable at classical level because they admit conserved tensors with bounded 00-component. The bounded conserved quantity turns out different from the canonical energy which can be unbounded for the same dynamics. Furthermore, these models admit non-canonical Lagrange anchors.[1] The class of higher-derivative systems considered in Ref. [19] covers a variety of well-known

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[1] The concept of Lagrange anchor was introduced in Ref. [20] to covariantly quantize not necessarily Lagrangian field theories. Later, it was established that the Lagrange anchor maps global symmetries to conserved currents [21]. The canonical Lagrange anchor, being an identity map, is always admitted by the Lagrangian equations, and it identifies a symmetry with the characteristics of conserved quantity. This can be understood as the Noether theorem in a different wording.

higher-derivative models, including PU oscillator and Podolsky electrodynamics. In Ref. [22], this class is further generalized, and it covers also extended Maxwell–Chern–Simons models. In all the examples of higher-derivative models considered in [19,22], the conserved tensor with bounded 00-component turns out connected with the space-time translations by a non-canonical Lagrange anchor. In this sense, the bounded conserved tensor can be interpreted as a non-canonical energy-momentum. The conservation law makes the theory stable at classical level irrespectively to interpretation of the conserved quantity. Also notice that all the considered examples [19,22] of stable higher-derivative models admit the interactions such that do not spoil classical stability. Further examples of stable interactions can be found in [23–25] for various higher-derivative models. In all these models, the canonical energy is unbounded at free level, while the stability is due to another bounded conserved quantity.

The Hamiltonian formalism for non-singular higher-derivative theories was introduced by Ostrogradski [26]. Its generalization for singular Lagrangians was first worked out in the paper [27]. The general constrained Hamiltonian formalism of higher-derivative systems was further developed since that in various directions. In particular it was adapted for higher-derivative gravity in a series of works starting from [28], for recent developments and further references see [29,30]. Notice that all these reformulations are connected by canonical transformations, so they cannot replace the unbounded Hamiltonian with any bounded quantity. The canonical Hamiltonian, being a canonical energy expressed in terms of phase space variables, is always unbounded for non-degenerate higher-derivative systems. Once the higher-derivative Lagrangian is degenerate, the phase space variables are subject to constraints. On the constraint surface, the canonical Hamiltonian can be bounded if the constraints are strong enough. The examples of this phenomenon are the same as previously mentioned cases of on-shell bounded canonical energy. The paper [31] demonstrates that once a Lagrange anchor is admitted by the equations of motion, the first-order formalism of the theory admits a constrained Hamiltonian formulation. If the model admits multiple Lagrange anchors, the first-order formalism will be multi-Hamiltonian. Furthermore, it is the conserved quantity connected to the time-shift symmetry by the Lagrange anchor which serves as Hamilton function. With this regard, the higher-derivative field theories of this class are expected to admit multi-Hamiltonian formalism where some of Hamiltonians are bounded. Once the classical Hamiltonian is bounded, the theory, being canonically quantized with respect to corresponding Poisson bracket, has a good chance to remain stable at quantum level.

The paper [19] provides a list of examples of higher-derivative systems admitting multiple Lagrange anchors, including the PU oscillator. By the above mentioned reasons, every model on this list has to be a multi-Hamiltonian system. It has been earlier noticed that the free PU oscillator admits alternative Hamiltonian formulations [32,33]. It has been observed that the series of canonically inequivalent Hamiltonians includes the bounded ones, while the canonical Ostrogradski Hamiltonian is unbounded. Later, the multi-Hamiltonian formulations of PU oscillator have been re-derived and re-interpreted from various viewpoints in [23,34–38]. All these observations can be summarized in the statement that the PU oscillator of order $2n$ admits the $n$-parameter series of alternative Hamiltonians and associated Poisson brackets. Once the equations of motion admit a Hamiltonian formulation with bounded Hamilton functions, the dynamics is stable classically and quantum-mechanically. It is also worth to notice that the PU oscillator equation of motion admits the interaction vertices such that do not spoil the classical stability [19,39]. These vertices are non-Lagrangian, while the interacting higher-derivative equations, being brought to the first-order formalism, still remain Hamiltonian with positive Hamilton function [23,24]. In this way, the PU oscillator equation admits inclusion of interactions such that leave the dynamics stable beyond the free level and admit Hamiltonian formulation. Notice that the stability of PU oscillator with the Lagrangian interaction vertices is studied once and again for decades. In some cases, the model admits isles of stability, see e.g. [40–44] for the most recent results and review, while it is unstable in general, unlike the case of above mentioned non-Lagrangian interactions.

If the equations of motion admit a Lagrange anchor, the dynamics have to admit a constrained Hamiltonian formulation [31]. With multiple Lagrange anchors, the dynamics should be multi-Hamiltonian. In general, the construction of Hamiltonian formulation for a given Lagrange anchor is implicit [31]. A direct relation between the Lagrange anchor and corresponding Hamiltonian formalism has been established for the PU oscillator in [23]. In [19,39], the interactions are introduced, being compatible with the Lagrange anchor. The stable interactions are found by means of the factorization method [19] and proper deformation method [39]. These two methods are equivalent [25] in principle, though they apply different techniques. Recently, more examples has become known of stable interaction vertices in various higher-derivative models with unbounded canonical energy at free level. The examples include PU theory [19,23,37,38], Podolsky electrodynamics [19], and higher-derivative exten-

Footnote 1 continued
The non-canonical Lagrange anchor also connects symmetries with characteristics of conserved currents, though it is not an identity map. Every Lagrange anchor connects any conserved current to a certain symmetry. Once the dynamical system, be it Lagrangian or not, admits different Lagrange anchors, the same symmetry can be connected with different conserved quantities.

sions of the Chern–Simons theory [22]. The stable interaction vertices are explicitly covariant in all the field theoretical examples, though they do not follow from the least action principle. The existence of a Lagrange anchor, however, implies that these models have to admit the Hamiltonian description at interacting level.

To the best of our knowledge, no explicit example has been known yet of higher-derivative field theory admitting multi-Hamiltonian formulation. In this work, we construct the multi-Hamiltonian formulation for higher-derivative extensions of Chern–Simons theory. The canonical unbounded Hamiltonian is included into the two-parametric series of admissible Hamilton functions. The series can also include bounded Hamiltonians in some cases. The existence of a bounded Hamiltonian depends on the parameters in the third-order equations. We also demonstrate that the covariant interactions exist such that the higher-derivative theory still admits bounded Hamiltonian, and therefore it remains stable at interacting level if the free model was stable.

We consider the class of theories of the vector field $A = A_\mu dx^\mu$ in $3d$ Minkowski space with the free action functional

$$S[A] = \frac{1}{2}\int *A \wedge (\alpha_0 m^2 A + \alpha_1 m * dA + a_2 * d * dA + \alpha_3 m^{-1} * d * d * dA + \cdots). \quad (1)$$

Here, $d$ is the de-Rham differential, $*$ is the Hodge star operator, $m$ is a dimensional constant, $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$ are the dimensionless constant real parameters. Depending on the values of the parameters, the action (1) can reproduce various $3d$ field theories, including the Chern–Simons–Proca theory [45,46], topologically massive gauge theory [47,48], Maxwell–Chern–Simons–Proca model [49,50], Lee–Wick electrodynamics [3,4] and extended Chern–Simons [5]. The classical stability of the model (1) is considered in the works [22,51]. In Ref. [51], it has been found that the model admits multiple conserved tensors being connected with the time translation by the Lagrange anchors. The anchors are polynomials in the Chern–Simons operator $*d$. The set of conserved quantities can include bounded ones. This depends on the roots of the characteristic equation

$$\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3 + \cdots = 0, \quad (2)$$

Here, $z$ is considered as a formal complex-valued variable, and $\alpha_k$ are the parameters of the model (1). As is established in [51], the model (1) admits a bounded conserved tensor and, hence, it is stable iff all the non-zero simple roots of Eq. (2) are real, while zero root may have the maximal multiplicity 2, and no roots occur with a higher multiplicity.

In this paper, we focus at the model (1) with at maximum third-order derivatives, i.e. the action reads

$$S[A] = \frac{1}{2}\int *A \wedge (\alpha_1 m * dA + \alpha_2 * d * dA + m^{-1} * d * d * dA), \quad (3)$$

with $\alpha_1, \alpha_2$ being two independent dimensionless parameters. This model has been proposed in [5] as the third-order extension of Chern–Simons theory. The model is obviously gauge-invariant. We construct the constrained multi-Hamiltonian formalism for this model. For similar reasons, the more general case (1) has to be a multi-Hamiltonian system, with a broader class of admissible Hamiltonians depending on the structure of roots in (2). As the construction of multi-Hamiltonian formalism becomes more cumbersome with growth of the order of derivatives, we do not go beyond the third-order models in this paper.[2]

Let us explain what do we understand by constrained multi-Hamiltonian formalism. At first, notice the obvious fact that the higher-derivative field equations can be always reduced to the first-order derivatives in time by introducing extra fields absorbing the higher time derivatives. We denote the original and extra fields by $\varphi^a(\vec{x})$. The first-order equations are said to be multi-Hamiltonian if there exists $k$-parametric series of Hamiltonians $H(\beta, \varphi, \nabla\varphi, \nabla^2\varphi, \nabla^3\varphi, \ldots)$ and Poisson brackets $\{\varphi^a(\vec{x}), \varphi^b(\vec{y})\}_\beta$, with $\beta_1, \ldots, \beta_k$ being constant parameters and $\nabla$ denoting derivatives by space $\vec{x}$, such that the equations constitute constrained Hamiltonian system with any $\beta$, i.e.

$$\dot{\varphi}^a = \{\varphi^a, H_T(\beta)\}_\beta\,, \quad (4)$$

$$H_T(\beta) = H(\beta, \varphi, \nabla\varphi, \nabla^2\varphi, \nabla^3\varphi, \ldots) + \lambda^A T_A(\varphi, \nabla\varphi, \nabla^2\varphi, \nabla^3\varphi, \ldots)\,;$$
$$T_A(\varphi, \nabla\varphi, \nabla^2\varphi, \nabla^3\varphi, \ldots) = 0. \quad (5)$$

The rhs of Eq. (4) does not depend on the parameters $\beta$, while both the total Hamiltonian $H_T(\beta)$ and the Poisson bracket do. In the other wording, changing values of parameters $\beta$, we simultaneously change Hamiltonian $H_T(\beta)$ and Poisson brackets $\{\cdot,\cdot\}_\beta$ in such a way that the equations of motion (4) remain intact.

Any higher-derivative Lagrangian field theory always admits at least one Hamiltonian formulation which can be constructed by the Ostrogradski method in the unconstrained case, and by various generalizations [27–30] developed for the constrained systems. In this paper, we develop the Hamiltonian formalism of higher-derivative field theory in several respects by the example of the model (3). At first, the third-order extension of the Chern–Simons model (3) is shown to admit a two-parameter series of constrained Hamiltonian formulations. The Hamiltonians from this series can be bounded from below in some cases, depending upon param-

[2] The multi-Hamiltonian formulation for the gauge-invariant extension of the Chern–Simons model of the fourth order have been constructed in [52].

eters $\alpha_1, \alpha_2$, even though Ostrogradski's Hamiltonian of the model is unbounded in all the instances. The second is that the free higher-derivative equations of this model admit inclusion of covariant interactions which do not break the stability if the theory have bounded conserved quantity at free level. Furthermore, the stable theory admits constrained Hamiltonian formulation at interacting level with a bounded Hamilton function.

Let us also remark that the multi-Hamiltonian formulation helps to resolve the discrepancy between classical stability of higher-derivative dynamics and quantum instability which is connected to the unboundedness of canonical Hamiltonian. As it is noticed in [22], the stable higher-derivative extensions of the Chern–Simons model realize the reducible representations which are decomposed into the unitary irreps in some cases. In the other cases, the representations are non-unitary or non-decomposable. If the model admits only unbounded conserved tensors, it corresponds to a non-unitary representation, while the models with unitary representations admit non-Ostrogradski's bounded Hamiltonians. If the theory is quantized with the bounded Hamiltonian, and the commutation relations are imposed in accordance with the corresponding Poisson brackets, the theory will be quantum-mechanically stable, as it is at the classical level.

Let us make some comments on the interactions which do not break the stability of the higher-derivative theory. An example of stable couplings in the model (1) has been noticed in [22] in the case involving massive Proca term, so it is the theory without gauge symmetry. In the present paper, we consider the gauge model (3) and introduce gauge-invariant interaction with spinors. This class of interactions can be viewed as a generalization to the non-minimal stable couplings of $d = 4$ Podolsky electrodynamics to the spinor matter proposed in Ref. [19].

The article is organized as follows. In Sect. 2, we describe conserved tensors of the third-order model (3). We also relate the existence of bounded conserved tensors with the structure of the corresponding Poincaré group representation. In doing that, we mostly follow the general prescriptions of [22] and [51]. The section is self-contained, however. In Sect. 3, the multi-Hamiltonian formulation is constructed with the Hamiltonians defined by the conserved tensors of Sect. 2. In Sect. 4, we introduce the interactions with spin $1/2$ such that do not break the stability of higher-derivative theory if the theory is stable at free level. After that, we demonstrate that the higher-derivative interacting theory still admits Hamiltonian formulation in all the instances, even if the vertices are not Lagrangian.

## 2 Conserved tensors

For the action (3), the Lagrange equations read

$$\frac{\delta S}{\delta A} \equiv \left(\alpha_1 m * d + \alpha_2 * d * d + \frac{1}{m} * d * d * d\right) A = 0. \tag{6}$$

The third-order time derivatives are involved in these equations. That is why, the conserved quantities can involve the second-order time derivatives.

The equations (6) correspond to a reducible representation of the Poincaré group. Specifics of the representation depends on the constants $\alpha_1, \alpha_2$. Different cases are distinguished by the structure of roots in the characteristic equation

$$z^3 + \alpha_2 z^2 + \alpha_1 z = 0 \tag{7}$$

associated to the field Eq. (6). Here, $z$ is a formal unknown variable, and $\alpha_1, \alpha_2$ are the parameters of the model. There are the following different cases distinguished by the structure of roots for the variable $z$:

$$\begin{array}{lllll}
\text{(A)} & \alpha_1 \neq 0, & {\alpha_2}^2 - 4\alpha_1 > 0, & \text{two simple real nonzero roots, and one simple zero root;} \\
\text{(B)} & \alpha_1 = 0, & \alpha_2 \neq 0, & \text{one simple real nonzero root, and one zero root of multiplicity two;} \\
\text{(C)} & \alpha_1 \neq 0, & {\alpha_2}^2 - 4\alpha_1 = 0, & \text{one real nonzero root of multiplicity two, and one simple zero root;} \\
\text{(D)} & \alpha_1 = 0, & \alpha_2 = 0, & \text{one zero root of multiplicity three;} \\
\text{(E)} & \alpha_1 \neq 0, & {\alpha_2}^2 - 4\alpha_1 < 0, & \text{two simple complex conjugate roots, and one simple zero root.}
\end{array} \tag{8}$$

In cases A and B, the representation is unitary and reducible. In case A, the representation is decomposed into two irreducible sub-representations. Each one corresponds to a self-dual massive spin 1, while the masses can be different. In case B, the set of sub-representations includes a massless spin 1 and a massive spin 1 subject to a self-duality condition. Cases C and D correspond to reducible indecomposable non-unitary representations. These two options are distinguished by different multiplicity of the multiple real root in Eq. (7). In case E, the representation is irreducible and non-unitary. So, one can see that the field Eq. (6) can describe either unitary or non-unitary representations of the $3d$ Poincaré group depending on the relations between the parameters $\alpha_1, \alpha_2$.

The third-order field Eq. (6) admits two-parameter series of on-shell conserved second-rank tensors

$$T_{\mu\nu}(\beta_1, \beta_2) = \beta_1 (T_1)_{\mu\nu} + \beta_2 (T_2)_{\mu\nu}, \tag{9}$$

where $\beta_1, \beta_2$ are the real constant parameters, and $(T_a)_{\mu\nu}$, $a = 1, 2$ read

$$(T_1)_{\mu\nu} = \frac{1}{m}\Big\{(G_\mu F_\nu + G_\nu F_\mu - \eta_{\mu\nu} G_\rho F^\rho) + \alpha_2 m(F_\mu F_\nu - \frac{1}{2}\eta_{\mu\nu} F_\rho F^\rho)\Big\},$$
$$(T_2)_{\mu\nu} = \frac{1}{m^2}\Big\{(G_\mu G_\nu - \frac{1}{2}\eta_{\mu\nu} G_\rho G^\rho) - \alpha_1 m^2(F_\mu F_\nu - \frac{1}{2}\eta_{\mu\nu} F_\rho F^\rho)\Big\}. \quad (10)$$

Here we use the notation[3]

$$F_\mu \equiv \varepsilon_{\mu\rho\nu}\partial^\rho A^\nu = (*dA)_\mu, \quad G_\mu \equiv \partial_\mu \partial^\nu A_\nu - \Box A_\mu = (*d*dA)_\mu, \quad \varepsilon_{012} = 1. \quad (11)$$

Tensor $T_1$ is the canonical energy-momentum for the action (3), while $T_2$ is another independent conserved quantity. As $F$ and $G$ are gauge invariant quantities, the tensor (9) is gauge invariant with any $\beta$. Also notice that $F_i, G_i, i = 1, 2$ define independent unconstrained Cauchy data for the field Eq. (6). Once $T_1$ is linear in $G$, it is unbounded anyway. The general entry of the series (9) is bilinear in both $G$ and $F$. So, $T(\beta)$ can be bounded, in principle, if $\beta_2 \neq 0$.

The conserved tensors of the series (9) are connected to the invariance of the model with respect to the space-time translations if the parameters meet the condition

$$\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2 \neq 0. \quad (12)$$

This connection can be traced by the Lagrange anchor method along the same lines as in the paper [51]. From this perspective, any representative of the series (9) satisfying condition (12) can be viewed as energy-momentum.

The 00-component of the conserved tensor $T(\beta_1, \beta_2)$ from the series (9) can be bounded or unbounded from below depending on the parameters $\alpha$ involved in the Eq. (6) and on specific values of $\beta$. Once the representation is unitary [that corresponds to the cases A,B in classification (8)], the bounded representatives exist with certain $\beta$'s, as we shall see in the next section. For non-unitary representations (the cases C,D,E), the 00-component of the conserved tensor $T(\beta)$ is unbounded in all the instances. As the existence of bounded conservation law provides the classical stability of the model, the theory is stable if the parameters of the model meet the conditions (8.A) or (8.B), and it is unstable in all the other cases. The canonical energy $(T_1)_{00}$ is always unbounded.

The conserved tensors are defined modulo on-shell vanishing terms. So, we have equivalence classes of conserved quantities which coincide on-shell, being off-shell different. The choice of specific representative of the equivalence class is a natural ambiguity in the definition of conserved quantity.

[3] The Minkowski metric is taken with mostly negative signature.

We mention this ambiguity because it has a natural counterpart in the Hamiltonian formalism considered in the next section. As far as the linear Eq. (3) admit bilinear gauge invariant conserved tensors (9), it is natural to consider the series up to quadratic on-shell vanishing terms. The most general gauge-invariant bilinear and symmetric representative in the equivalence class of $T_{\mu\nu}(\beta_1, \beta_2)$ (9) reads

$$T_{\mu\nu}(\beta_1, \beta_2, \beta_3, \beta_4) = T_{\mu\nu}(\beta_1, \beta_2) + \frac{\beta_3}{2m}\Big(F_\mu \frac{\delta S}{\delta A^\nu} + F_\nu \frac{\delta S}{\delta A^\mu}\Big) + \frac{\beta_4}{2m^2}\Big(G_\mu \frac{\delta S}{\delta A^\nu} + G_\nu \frac{\delta S}{\delta A^\mu}\Big). \quad (13)$$

Two real parameters $\beta_3, \beta_4$ label different representatives of the same equivalence class of conserved tensors, while $\beta_1, \beta_2$ determine the equivalence class of conserved tensor as such. Only one of two constants $\beta_3, \beta_4$ is independent. The other one can be absorbed by the multiplication of the equations of motion by the constant overall factor.

In the next section, we construct a multi-Hamiltonian formulation where 00-components of the conserved tensors $T_{\mu\nu}(\beta_1, \beta_2, \beta_3, \beta_4)$ (13) serve as Hamiltonians, and all the values of the parameters $\beta_1$ and $\beta_2$, being subject to condition (12), are admissible. For reasons of convenience, we consider all the cases in a uniform way, be the Hamiltonian bounded or not.

## 3 Multi-Hamiltonian formulation

The multi-Hamiltonian formalism is constructed for the Eq. (6) in three steps. First, the higher-derivative equations are reduced to the first-order in time by introducing extra variables to absorb the time derivatives of the original field $A$. The first-order equations are split in two subsets. The first one includes the evolutionary type equations, while the other equations are the constraints. The latter ones do not involve the time derivatives of the fields. Second, the 00-component of the most general conserved tensor of the series (9) is taken as the Hamiltonian of the model. As far as the considered model is constrained, the Hamiltonian involves a linear combination of constraints. Third, the series of Poisson bracket is found for the series of Hamiltonians such that the evolutionary-type equations of motion take the constrained multi-Hamiltonian form (4).

Let us reduce the third-order field Eq. (6) to the first order in time $x^0$. Introduce new fields absorbing the first- and second-order time derivatives of original field $A_i, i = 1, 2$, while the time derivatives of $A_0$ eventually drop out from the equations. We chose the gauge-invariant quantities $F_i$, $G_i$, $i = 1, 2$ (11) as new variables absorbing the time derivatives of $A$,

$$F_i = \varepsilon_{ij}(\dot{A}_j - \partial_j A_0),$$
$$G_i = -\ddot{A}_i + \partial_i \dot{A}_0 + \partial_j(\partial_j A_i - \partial_i A_j), \quad i, j = 1, 2, \quad (14)$$

with $\varepsilon_{ij} = \varepsilon_{0ij}$ being the $2d$ Levi-Civita symbol. Substituting these variables into (6), we arrive at the following first-order equations in terms of the fields $A_\mu$, $F_i$, $G_i$:

$$\begin{aligned}
&\dot{A}_i = \partial_i A_0 - \varepsilon_{ij} F_j,\\
&\dot{F}_i = \varepsilon_{ij}\big[\partial_k(\partial_k A_j - \partial_j A_k) - G_j\big],\\
&\dot{G}_i = \varepsilon_{ij}\big[\partial_k(\partial_k F_j - \partial_j F_k) + m(\alpha_2 G_j + \alpha_1 m F_j)\big], \qquad (15)\\
&\Theta \equiv \varepsilon_{ij}\partial_i\left(\frac{1}{m}G_j + \alpha_2 F_j + \alpha_1 m A_j\right) = 0. \qquad (16)
\end{aligned}$$

In terms of fields $A$, $F$, $G$, the evolutionary equations (15) represent the first-order form of the space components of the Lagrange Eq. (6). The zero component of the original field Eq. (6) is a constraint (16), which does not involve the time derivatives. Since the constraint $\Theta$ conserves with account for the evolutionary equations, no secondary constraints are imposed on the fields. The first-order equations (15), (16) are obviously equivalent to the original third-order ones (6).

In the first-order formalism, the equations are invariant under the gauge transformation

$$\delta_\xi A_0 = \partial_0 \xi(x), \qquad \delta_\xi A_i = \partial_i \xi(x), \qquad \delta_\xi F_i = \delta_\xi G_i = 0\,, \qquad (17)$$

where $\xi$ is the gauge transformation parameter, being arbitrary function of $x$. In what follows, it is natural to consider the field $A_0$ as the Lagrange multiplier associated to the constraint (16). This interpretation is consistent with the gauge transformation (17) which includes the time derivative of the gauge parameter, as it should be for Lagrange multiplier in the constrained Hamiltonian formalism.

In the first-order formalism, the 00-component of the conserved tensor (9) reads

$$\begin{aligned}
T_{00}(\beta_1,\beta_2) = \frac{1}{2m^2}\Big\{&\beta_2\big[\partial_i F_j(\partial_i F_j - \partial_j F_i) + (G_i)^2\big]\\
&+ 2m\beta_1\big[\partial_i F_j(\partial_i A_j - \partial_j A_i) + G_i F_i)\big]\\
&+ m^2(\beta_1\alpha_2 - \beta_2\alpha_1)\big[\partial_i A_j(\partial_i A_j - \partial_j A_i)\\
&+ (F_i)^2\big]\Big\}. \qquad (18)
\end{aligned}$$

We treat this quantity as the series of on-shell Hamiltonians parameterized by constants $\alpha$, $\beta$. Off-shell, the Hamiltonian can be a sum of (18) and constraints. We chose the following ansatz for the total Hamiltonian:

$$\begin{aligned}
H_T(\beta_1,\beta_2,\gamma) \equiv T_{00}(\beta_1,\beta_2) + \Bigg[&\frac{\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma}A_0\\
&+ \frac{1}{m}\frac{\beta_1\beta_2 + \alpha_1\beta_2\gamma - \alpha_2\beta_1\gamma}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma}\varepsilon_{ij}\partial_i A_j\\
&+ \frac{1}{m^2}\frac{\beta_2^2 + \beta_1\gamma}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma}\varepsilon_{ij}\partial_i F_j\Bigg]\Theta, \qquad (19)
\end{aligned}$$

where $\beta_1$, $\beta_2$, $\gamma$ are constant parameters. On account of the constraint (16), the quantities (18) and (19) coincide on shell. The parameter $\gamma$ is introduced to control the inclusion of the constraint term into the Hamiltonian. The admissible values of the parameters $\beta$ and $\gamma$ subject to conditions

$$\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2 \neq 0, \quad \beta_1 - \alpha_2\beta_2 - \alpha_1\gamma \neq 0. \qquad (20)$$

Here, the first condition implies that the conserved quantity (18) is connected to the invariance of the model (6) with respect to the time translations, see Eq. (12). Both the conditions (20) ensure that the numerical factor at the Lagrange multiplier $A_0$ in the Hamiltonian (19) is nonzero and nonsingular. Once these two requirements are met, any conserved quantity (18) can serve as the Hamiltonian with appropriate Poisson bracket.

Now, let us seek for the Poisson brackets among the fields $A_i$, $F_i$, $G_i$, $i = 1, 2$ such that the Eqs. (15), (16) take the constrained multi-Hamiltonian form (4), (5) with the Hamiltonian defined by relations (18), (19) and the constraint (16). Given the series of Hamiltonians (18), (19) and the r.h.s. of the equations (15), we arrive at the system of linear algebraic equations defining the series of Poisson brackets $\{\cdot,\cdot\}_{\beta,\gamma}$:

$$\begin{aligned}
\{A_i, H_T(\beta,\gamma)\}_{\beta,\gamma} &= \partial_i A_0 - \varepsilon_{ij} F_j,\\
\{F_i, H_T(\beta,\gamma)\}_{\beta,\gamma} &= \varepsilon_{ij}\big[\partial_k(\partial_k A_j - \partial_j A_k) - G_j\big],\\
\{G_i, H_T(\beta,\gamma)\}_{\beta,\gamma} &= \varepsilon_{ij}\big[\partial_k(\partial_k F_j - \partial_j F_k)\\
&\quad + m(\alpha_2 G_j + \alpha_1 m F_j)\big]. \qquad (21)
\end{aligned}$$

The Poisson bracket, being defined by these equations, involves five independent parameters $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $\gamma$. The bracket eventually reads

$$\begin{aligned}
\{G_i(\vec{x}), G_j(\vec{y})\}_{\beta,\gamma} &= m^3\frac{(\alpha_1 - \alpha_2^2)\beta_1 + \alpha_1\alpha_2\beta_2}{\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2}\varepsilon_{ij}\delta(\vec{x} - \vec{y}),\\
\{F_i(\vec{x}), G_j(\vec{y})\}_{\beta,\gamma} &= m^2\frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2}\varepsilon_{ij}\delta(\vec{x} - \vec{y}),\\
\{F_i(\vec{x}), F_j(\vec{y})\}_{\beta,\gamma} &= \{A_i(\vec{x}), G_j(\vec{y})\}_{\beta,\gamma}\\
&= m\frac{-\beta_1}{\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2}\varepsilon_{ij}\delta(\vec{x} - \vec{y}),\\
\{A_i(\vec{x}), F_j(\vec{y})\}_{\beta,\gamma} &= \frac{\beta_2}{\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2}\varepsilon_{ij}\delta(\vec{x} - \vec{y}),\\
\{A_i(\vec{x}), A_j(\vec{y})\}_{\beta,\gamma} &= \frac{1}{m}\frac{\gamma}{\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2}\varepsilon_{ij}\delta(\vec{x} - \vec{y}). \qquad (22)
\end{aligned}$$

The accessory parameter $\gamma$ controls the constraint terms in the total Hamiltonian (19). As is seen, the same parameter defines the Poisson bracket between the components $A_i$ of gauge potential. This parameter does not contribute to the Poisson brackets between the physical observables, being the functions of the gauge-invariant quantities $F_i$, $G_i$,

and the strength $\varepsilon_{ij}\partial_i A_j$. That is why, $\gamma$ can be considered as an accessory parameter. Inclusion of $\gamma$-terms into total Hamiltonian and brackets allows us to literally reproduce in Hamiltonian form the first-order dynamical Eq. (15) for all the quantities, be they gauge-invariant or not.

Let us make one more comment on the meaning of the accessory parameter $\gamma$ which defines the bracket between $A_i$ and does not affect on the brackets of gauge-invariant quantities. Notice that the Poisson brackets in gauge theory have the inherent ambiguities. The general study of these ambiguities can be found in Ref. [53]. In context of the bracket (22), one of these ambiguities turns out relevant. It is related to the option of redefining the Poisson bracket by adding the bi-vector, being the wedge product of gauge symmetry generator to another vector. This redefinition does not affect the brackets between gauge-invariant observables, while it can alter the brackets of non-gauge-invariant quantities. The bracket (22) involves the ambiguous terms of this type, and it is the ambiguity which is controlled by the accessory parameter $\gamma$.

The problem of identification of ambiguous terms in the Poisson bracket is a subtle issue. The Poisson bracket (22) is ultralocal between components of $A_i$ with no derivatives involved, while the generator of the gauge symmetry for $A_i$ (17) involves a derivative. Thus, the ambiguous terms in the Poisson bracket cannot be absorbed by adding the wedge product of the gauge symmetry generator, being a derivative, to another vector, being a polynomial in the partial derivatives $\partial_i$. The problem is solved by including the inverse Laplace operator $\Delta^{-1} = (\partial_i\partial_i)^{-1}$ into the coefficient at the gauge generator. The space non-locality of this type is usually considered as admissible for the constrained Hamiltonian formalism in the field theory.[4] To represent the bracket (22) between the components of $A_i$ in terms of gauge generators, we use the following identical representation for the $2d$ Levi-Civita tensor $\varepsilon_{ij}$:

$$\varepsilon_{ij} = \frac{1}{2\Delta}(\varepsilon_{im}\partial_m\partial_j - \varepsilon_{jm}\partial_m\partial_i). \tag{23}$$

Substituting $\varepsilon_{ij}$ from this relation into rhs of the Poisson bracket for the potential components, we rewrite the bracket in the form

$$\{A_i(\vec{x}),\, A_j(\vec{y})\}_{\beta,\gamma} = \frac{1}{2}(V_i(\gamma)\partial_j - V_j(\gamma)\partial_i)\delta(\vec{x}-\vec{y}),$$
$$V_i(\gamma) = \frac{\gamma}{m(\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2)}\frac{\varepsilon_{im}\partial_m}{\Delta}. \tag{24}$$

Here, all the partial derivatives act on argument $\vec{x}$ in the delta-function. Once the operator $\partial_i$ is a gauge generator for the field $A_i$, the vector $V_j(\gamma)$ parametrizes the ambiguity in the Poisson bracket. Thus, we treat the parameter $\gamma$ as inherent ambiguity of the Poisson bracket in the gauge theory outlined in Ref. [53].

Let us summarize all the aspects related to the ambiguity in parametrization of the multi-Hamiltonian formulation of Eq. (6). The Hamiltonian and brackets (19), (22) involve five parameters. Two of them, $\alpha_1$ and $\alpha_2$, define the original Eq. (6). The constants $\beta_1, \beta_2$ parameterize the series of conserved tensors tensors (9). These tensors admit gauge-invariant re-definitions by on-shell vanishing terms (13), with one more parameter in control of corresponding ambiguity. The 00-components of the conserved tensors are chosen as Hamiltonians for the first-order formulation (15), (16) of the original third-order Eq. (6). In this way, the ambiguity in the off-shell definition of the conserved tensors is converted into the ambiguity in the constraint terms of the Hamiltonian. The later ambiguity does not contribute to the equations of motion for the gauge-invariant quantities $F_i, G_i, \varepsilon_{ij}\partial_i A_j$, while the equations of motion for the potential components $A_i$ can alter. We seek for a series of the Hamiltonians and Poisson brackets such that the Hamiltonian equations literally reproduce the first-order form (15) of the original third-order Eq. (3) for all the variables, including the original vector field. In this case, one and the same parameter has to control the ambiguity in the Hamiltonian and Poisson bracket. It is the parameter $\gamma$. In the free theory, $\gamma$ can be set to an arbitrary value. This corresponds to the choice of the representative in the equivalence class in the series of Hamiltonian formulations with the Hamiltonian (18), (19) and Poisson bracket (22). Thus, $\gamma$ is an accessory parameter in the series of Hamiltonian formulations unless the interaction is introduced. We keep $\gamma$ in the Hamiltonian formulation throughout this section to have the contact with Sect. 4, where the couplings are introduced with spinors. As we will see, this parameter becomes essential for inclusion of consistent interactions in the non-linear model.

The Hamiltonians in the series (19) can be bounded from below provided for the parameters are subject to certain conditions. Let us elaborate on the issue of the boundedness. As is seen from Eq. (19), the Hamiltonian is the sum of the 00-component of the conserved tensor (18) and a constrained term. We ignore the constrained term as the boundedness matters only on-shell. The 00-component of the conserved tensor is defined by relation (18). In the notation (11), we rewrite the rhs of (18) in the form

$$T_{00}(\beta_1,\beta_2) = \frac{1}{2m^2}\Big\{\beta_2 G_\mu G_\mu + 2m\beta_1 G_\mu F_\mu + m^2(\beta_1\alpha_2 - \beta_2\alpha_1)F_\mu F_\mu\Big\}, \tag{25}$$

[4] For example, the inverse Laplacian contributes to the Dirac brackets of vector potential to electric strength in the Maxwell electrodynamics in the Coulomb gauge.

where summation over repeated at one level index $\mu = 0, 1, 2$ is implied.[5] Once $\beta_2 = 0$, this expression determines the 00-component of the canonical energy-momentum tensor $(T_1)_{00}$, and it is unbounded for all the values of parameters $\alpha_1, \alpha_2$. This happens just because the 00-component of canonical energy-momentum is linear in $G_i$, while $G_i$ are independent Cauchy data for Eq. (15). Once $\beta_2 \neq 0$, we rewrite (25) as the linear combination of two Euclidean squares

$$T_{00}(\beta_1, \beta_2) = \frac{1}{2m^2}\left\{\beta_2 X_\mu X_\mu + \frac{-\beta_1^2 + \alpha_2\beta_1\beta_2 - \alpha_1\beta_2^2}{\beta_2} m^2 F_\mu F_\mu\right\},$$
$$X_\mu = G_\mu + m\frac{\beta_1}{\beta_2} F_\mu. \tag{26}$$

This expression is the quadratic form in the variables $G_\mu$, $F_\mu$. The variables $X_\mu$, $F_\mu$ diagonalize the quadratic form (26). Once the quadratic form is brought to the diagonal form, it is positive if all the coefficients are nonnegative at the squares of the variables. The last fact implies the following conditions on the parameters $\alpha_1, \alpha_2, \beta_2, \beta_2$:

$$\beta_2 > 0, \qquad -\beta_1^2 + \alpha_2\beta_1\beta_2 - \alpha_1\beta_2^2 \geq 0. \tag{27}$$

The equality sign in the second inequality should be excluded, because the conservation law is unrelated to the time translations in this case, and it does not lead to any Hamiltonian (19) [see conditions (20)]. Finally, we conclude that the Hamiltonian (19) is bounded from below if the parameters of the model $\alpha_1, \alpha_2, \beta_1, \beta_2$ are subject to the conditions

$$\beta_2 > 0, \quad -\beta_1^2 + \alpha_2\beta_1\beta_2 - \alpha_1\beta_2^2 > 0. \tag{28}$$

Relations (28) imply certain restrictions on possible values of parameters of model $\alpha_1, \alpha_2$. In cases A,B in classification (8), these conditions can be satisfied by appropriate values of the parameters $\beta_1$, $\beta_2$. In cases C, D, E of classification (8), conditions (28) are inconsistent. As we see, the bounded Hamiltonians are included in the series (19) once the Eq. (6) transform under unitary representations of the Poincaré group. For non-unitary representations, every Hamiltonian is unbounded in the series. We finally notice that condition (28) is more restrictive than (12). Thus, any bounded conserved quantity serves as a Hamiltonian. The Ostrogradski Hamiltonian, being included in the series (19) with $\beta_1 = 1, \beta_2 = 0$, is always unbounded.

For every $\beta, \gamma$, the Poisson bracket (22) is a non-degenerate tensor, so it has an inverse, being a symplectic two-form. The latter defines the series of Hamiltonian action functionals

$$S(\beta, \gamma) = \int \left\{\frac{\beta_1^2 - \alpha_2\beta_1\beta_2 + \alpha_2\beta_2^2}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma}\left(\alpha_1 m A_i + 2\alpha_2 F_i + \frac{2}{m} G_i\right)\right.$$
$$\varepsilon_{ij}\dot{A}_j + \frac{1}{m}\frac{\beta_1^2 + ((\alpha_2^2 - \alpha_1)\beta_1 - \alpha_1\alpha_2\beta_2)\gamma}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma}$$
$$\varepsilon_{ij} F_i \dot{F}_j + \frac{2}{m^2}\frac{\beta_1\beta_2 + (\alpha_2\beta_1 - \alpha_1\beta_2)\gamma}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma}\varepsilon_{ij} G_i \dot{F}_j$$
$$\left. + \frac{1}{m^3}\frac{\beta_2^2 + \beta_1\gamma}{\beta_1 - \alpha_2\beta_2 - \alpha_1\gamma}\varepsilon_{ij} G_i \dot{G}_j - H_T(\beta, \gamma)\right\} d^3x, \tag{29}$$

where $H_T(\beta, \gamma)$ denotes the total Hamiltonian (19).

For $\beta_1 = 1, \beta_2 = \gamma = 0$, we get Ostrogradski's action for the variational model (3):

$$S_{\text{Canonical}} = \int \left\{(\alpha_1 m A_i + 2\alpha_2 F_i + \frac{2}{m} G_i)\varepsilon_{ij}\dot{A}_j\right.$$
$$\left. - \frac{1}{m}\varepsilon_{ij} F_i \dot{F}_j - A_0\Theta - (T_1)_{00}\right\} d^3x, \tag{30}$$

where $(T_1)_{00}$ is the 00-component of the canonical energy-momentum tensor. The formula (30) follows from (29) for all the values of parameters $\alpha_1, \alpha_2$ of the model (6).

For $\beta_2 \neq 0$, we get the non-canonical Hamiltonian actions that still result to the same original Eq. (6). Different actions in the series (29) are not connected by a canonical transformation. This is obvious because the Hamiltonian in the series (19) can be bounded from below, while the canonical Hamiltonian (30) is always unbounded.

The Poincaré invariance can be questioned of the non-canonical Hamiltonian actions (29), and hence the covariance of the corresponding quantum theory may seem in question. We do not elaborate on this issue here, while we claim that the quantum theory associated to any model in the series (29) is Poincaré-invariant. The argument is that the original higher-derivative theory admits the series of covariant Lagrange anchors [22]. It is the series of anchors which underlies the multi-Hamiltonian formulation (29). One more reason is provided by the fact that every Hamiltonian in the series (18) is 00-component of the second rank tensor (13). All the entries of the series transform in the same way, including Ostrogradski's Hamiltonian.

## 4 Stable interactions with spinor field

As we have seen above, the higher-derivative extensions of the Chern–Simons theory admit multi-Hamiltonian formulations. In some cases, the Hamiltonians are bounded. In this section, we provide an example of couplings to spinors such that the theory still has bounded Hamiltonian and therefore it remains stable at interacting level.

In [19], the stable interaction is included for the higher-derivative Podolsky's electrodynamics in the dimension $d = 4$. The stable interaction is non-Lagrangian in $d = 4$, while the Hamiltonian formalism is not considered there. So, the

[5] No Minkowski metric is involved in this summation, it is just Euclidean product.

possibility could be questioned of the canonical quantization of the interacting model even without gauge invariance. The three-dimensional model admits more options than its four-dimensional counterpart, because (due to the presence of the Chern–Simons term) it can describe a variety of reducible representations of the $3d$ Poincaré group. Below we introduce the interaction mostly following the lines of [24] with regard to the $d = 3$ specifics, and then we construct the Hamiltonian formalism for the interacting theory.

Let us introduce coupling of the vector field $A$ and 2-component spinor field $\psi_a$, $a = 1, 2$ ($\overline{\psi}_a$ stands for conjugate spinor) by imposing the following non-linear field equations

$$\frac{\delta S}{\delta A^\mu} - J_\mu(\overline{\psi}, \psi) \equiv \varepsilon_{\mu\rho\nu}\partial^\rho\Big(\frac{1}{m}G^\nu + \alpha_2 F^\nu + m\alpha_1 A^\nu\Big) - e\overline{\psi}\gamma_\mu\psi = 0,$$
$$(i\gamma^\mu D_\mu - m)\psi = 0, \quad \overline{\psi}(i\gamma^\mu \overleftarrow{D}_\mu + m) = 0. \tag{31}$$

Here, $J_\mu = e\overline{\psi}\gamma_\mu\psi$ is the current of the spinor field, $\gamma$'s are the $3d$ gamma matrices, and $D$ is the covariant derivative,

$$D_\mu = \partial_\mu - ie\mathcal{A}_\mu, \qquad \overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu + ie\mathcal{A}_\mu,$$
$$\mathcal{A}_\mu = g_3\frac{1}{m^2}G_\mu + g_2\frac{1}{m}F_\mu + g_1 A_\mu. \tag{32}$$

The spinors $\psi$, $\overline{\psi}$ are Grassmann odd fields. The spinor field $\psi$ and its conjugate $\overline{\psi}$ are considered as independent variables. The real constants $g_1$, $g_2$, $g_3$ are dimensionless parameters of interaction. The parameter $e$ is a coupling constant.

In general, the interaction vertices are non-Lagrangian in the Eq. (31). The Lagrangian case corresponds to $g_1 \neq 0$, $g_2 = g_3 = 0$ in (32). The Lagrangian model is unstable, while the stability can be retained by admitting non-Lagrangian higher-derivative contributions to the interaction, i.e. by $g_2 \neq 0$, $g_3 \neq 0$. As we shall demonstrate in this section, with non-Lagrangian stable interactions, the Eqs. (31), (32) still admit constrained Hamiltonian formulation with on-shell bounded Hamiltonian.

The consistency of interaction implies that the gauge transformation (17) is complimented by the standard $U(1)$-transformation for the spinor field

$$\delta_\xi\psi = -ieg_1\psi\xi(x), \quad \delta_\xi\overline{\psi} = ieg_1\overline{\psi}\xi(x). \tag{33}$$

The non-linear theory describes propagation of the gauge field $A$ coupled to the spinor $\psi$ in the gauge-invariant way.

The Eq. (31) admits the second-rank conserved tensor

$$T_{\mu\nu}(g) = T_{\mu\nu}(\beta_1, \beta_2) + \frac{i}{4}\overline{\psi}[\gamma_\mu D_\nu + \gamma_\nu D_\mu - \gamma_\mu\overleftarrow{D}_\nu - \gamma_\nu\overleftarrow{D}_\mu]\psi - \frac{1}{2}\eta_{\mu\nu}\overline{\psi}[(i\gamma^\rho D_\rho - m) - (i\gamma^\rho\overleftarrow{D}_\rho + m)]\psi, \tag{34}$$

where $T_{\mu\nu}(\beta_1, \beta_2)$ stands for the conserved tensor (9) of the free theory with the parameters $\beta$ fixed by the interaction constants in the following way

$$\beta_1 = g_1 - \alpha_1 g_3, \quad \beta_2 = g_2 - \alpha_2 g_3. \tag{35}$$

We chose the conserved tensor in the form (34) because its 00-component does not involve time derivatives of the spinor field. Once the time derivatives of the spinor filed are not involved in $T_{00}(g)$, the conserved tensor still admits by redefinition of on-shell vanishing terms that involve the derivatives of the vector field. The structure of this term is analogous to (13), so we do not write these contributions explicitly.

Upon inclusion of interaction, the deformation is still conserved of a single representative from the series of conserved tensors (9) admitted at free level. The parameters $\beta_1, \beta_2$ in this conserved tensor are fixed by the interaction constants by the formula (35).

The procedure of construction of the conserved tensor (34) is analogous to that from [19], Sect. 4.2, where the couplings of Podolsky's electrodynamics with the spinor matter are considered. This procedure preserves the relationship between the conserved tensor and space-time translations. In particular, (34) is related to the invariance of model w.r.t. the space-time translations if this is true in the linear approximation. The necessary and sufficient condition to connect the conserved tensor (34) to the invariance of model w.r.t. the space-time translations follows form (12) and (35). Substituting (35) into (12), we get

$$g_1^2 + \alpha_1 g_2^2 + \alpha_1^2 g_3^2 - \alpha_1 g_1 g_2 + (\alpha_2^2 - 2\alpha_1)g_1 g_3 - \alpha_1\alpha_2 g_2 g_3 \neq 0. \tag{36}$$

In what follows, we consider the interactions (31), whose parameters satisfy this condition. By this reason, we consider (34) as the energy-momentum tensor of the non-linear theory (31).

The 00-component of the tensor (34) reads

$$T_{00}(g) = T_{00}(g_1 - \alpha_1 g_3, g_2 - \alpha_2 g_3) + \frac{1}{2}\overline{\psi}[i(\gamma_i\partial_i - \gamma_i\overleftarrow{\partial}_i) + 2e\gamma_i\mathcal{A}_i - 2m]\psi. \tag{37}$$

Depending on the values of the parameters $g$, this quantity can be bounded or unbounded from below.[6] The necessary and sufficient condition for that follows from (28). It reads

$$g_2 - \alpha_2 g_3 > 0, \quad g_1^2 + \alpha_1 g_2^2 + \alpha_1^2 g_3^2 - \alpha_1 g_1 g_2 + (\alpha_2^2 - 2\alpha_1)g_1 g_3 - \alpha_1\alpha_2 g_2 g_3 > 0. \tag{38}$$

[6] With the cubic interaction contribution, the conserved tensor (37) is no longer bounded in the strict sense. By saying 'bounded' we mean that the quadratic contribution in the conserved quantity is bounded. The latter property is interpreted as stability of the theory with respect to small fluctuations of initial data, and it is not considered as obstruction to the stability of the model. For example, the energy-momentum of spinor electrodynamics includes cubic term.

In case of minimal interaction $g_1 = 1$, $g_2 = g_3 = 0$, the equations of motion (31) are Lagrangian. However, the Lagrangian non-linear theory is unstable because the canonical energy of the model is unbounded. Once the condition (38) is satisfied, the model is stable, while the field Eqs. (31) and (32) are non-Lagrangian.

Let us bring the theory (31) to the form of constrained Hamiltonian dynamics. The first-order formulation for the model (31) is constructed in the same way as in the linear case. The variables $A_i$, $F_i$, $G_i$ are introduced by the recipe (14) to absorb the time derivatives of $A$. For these fields, we get three equations of evolutionary type

$$\begin{aligned}
&\dot{A}_i = \partial_i A_0 - \varepsilon_{ij} F_j,\\
&\dot{F}_i = \varepsilon_{ij}\big[\partial_k(\partial_k A_j - \partial_j A_k) - G_j\big],\\
&\dot{G}_i = \varepsilon_{ij}\big[\partial_k(\partial_k F_j - \partial_j F_k) + m(\alpha_2 G_j + \alpha_1 m F_j) - J_j\big].
\end{aligned} \tag{39}$$

Obviously, the first pair of equations in this system have the same form as in (15) because they just define the new fields introduced to absorb the time derivatives of the original field $A$. The third equation represents the first-order form of the original field equations, so it involves the interaction. With account of the interaction, the constraint reads

$$\Theta \equiv \varepsilon_{ij}\partial_i\left(\frac{1}{m}G_j + \alpha_2 F_j + \alpha_1 m A_j\right) - J_0 = 0. \tag{40}$$

Equations (39), (40) are complimented by the equations for the spinors from (31):

$$\begin{aligned}
&\dot{\psi} = \gamma_0\{\gamma_j\partial_j + ie\gamma_i\mathcal{A}_i - ie\gamma_0\mathcal{A}_0 - im\}\psi,\\
&\dot{\overline{\psi}} = \overline{\psi}\{\gamma_j \overleftarrow{\partial}_j - ie\gamma_i\mathcal{A}_i + ie\gamma_0\mathcal{A}_0 + im\}\gamma_0.
\end{aligned} \tag{41}$$

These equations are of the first order from the outset. In this way, we have the first-order formulation for the model (31) which includes Eqs. (39), (40) and (41).

We chose the following ansatz for the total Hamiltonian:

$$H_T(g) = T_{00}(g) + \mathcal{A}_0\Theta, \tag{42}$$

where $g_1, g_2, g_3$ are the parameters. On account of (37), the Hamiltonian describes the same conserved quantity as the 00-component of the tensor (34). Substituting (42) into (4), (5), we arrive at the system of linear algebraic equations defining the series of Poisson brackets:

$$\begin{aligned}
&\{A_i, H_T(g)\}_g = \partial_i A_0 - \varepsilon_{ij} F_j,\\
&\{F_i, H_T(g)\}_g = \varepsilon_{ij}\big[\partial_k(\partial_k A_j - \partial_j A_k) - G_j\big],\\
&\{G_i, H_T(g)\}_g = \varepsilon_{ij}\big[\partial_k(\partial_k F_j - \partial_j F_k)\\
&\qquad\qquad + m(\alpha_2 G_j + \alpha_1 m F_j) - J_j\big],\\
&\{\psi, H_T(g)\}_g = \gamma_0\{\gamma_j\partial_j + ie\gamma_i\mathcal{A}_i - ie\gamma_0\mathcal{A}_0 - im\}\psi,\\
&\{\overline{\psi}, H_T(g)\}_g = \overline{\psi}\{\gamma_j \overleftarrow{\partial}_j - ie\gamma_i\mathcal{A}_i + ie\gamma_0\mathcal{A}_0 + im\}\gamma_0.
\end{aligned} \tag{43}$$

These relations should take into account the Grassmann parity of the fields, so it is an even $Z_2$-graded Poisson bracket. In particular, the brackets are symmetric of the spinor fields $\psi, \overline{\psi}$.

Equation (43) are consistent if the interaction parameters satisfy condition (36). The structure of the Poisson bracket, however, depends on the relations between the interaction parameters $g_1, g_2, g_3$. Below, we focus on the case $g_1 \neq 0$, while the other cases can be treated in a similar way. The Poisson bracket, being defined by Eq. (43), reads

$$\begin{aligned}
\{G_i(\vec{x}), G_j(\vec{y})\}_g &= m^3\frac{(\alpha_1-\alpha_2^2)g_1 + \alpha_1\alpha_2 g_2 - \alpha_1^2 g_3}{g_1^2 + \alpha_1 g_2^2 + \alpha_1^2 g_3^2 - \alpha_1 g_1 g_2 + (\alpha_2^2 - 2\alpha_1)g_1 g_3 - \alpha_1\alpha_2 g_2 g_3}\varepsilon_{ij}\delta(\vec{x}-\vec{y}),\\
\{F_i(\vec{x}), G_j(\vec{y})\}_g &= m^2\frac{\alpha_2 g_1 - \alpha_1 g_2}{g_1^2 + \alpha_1 g_2^2 + \alpha_1^2 g_3^2 - \alpha_1 g_1 g_2 + (\alpha_2^2 - 2\alpha_1)g_1 g_3 - \alpha_1\alpha_2 g_2 g_3}\varepsilon_{ij}\delta(\vec{x}-\vec{y}),\\
\{F_i(\vec{x}), F_j(\vec{y})\}_g &= \{A_i(\vec{x}), G_j(\vec{y})\}_g = m\frac{(\alpha_1 g_3 - g_1)}{g_1^2 + \alpha_1 g_2^2 + \alpha_1^2 g_3^2 - \alpha_1 g_1 g_2 + (\alpha_2^2 - 2\alpha_1)g_1 g_3 - \alpha_1\alpha_2 g_2 g_3}\varepsilon_{ij}\delta(\vec{x}-\vec{y}),\\
\{A_i(\vec{x}), F_j(\vec{y})\}_g &= \frac{g_2 - \alpha_2 g_3}{g_1^2 + \alpha_1 g_2^2 + \alpha_1^2 g_3^2 - \alpha_1 g_1 g_2 + (\alpha_2^2 - 2\alpha_1)g_1 g_3 - \alpha_1\alpha_2 g_2 g_3}\varepsilon_{ij}\delta(\vec{x}-\vec{y}),\\
\{A_i(\vec{x}), A_j(\vec{y})\}_g &= \frac{1}{m}\frac{-\alpha_1 g_3^2 + \alpha_2 g_2 g_3 + g_1 g_3 - g_2^2}{g_1(g_1^2 + \alpha_1 g_2^2 + \alpha_1^2 g_3^2 - \alpha_1 g_1 g_2 + (\alpha_2^2 - 2\alpha_1)g_1 g_3 - \alpha_1\alpha_2 g_2 g_3)}\varepsilon_{ij}\delta(\vec{x}-\vec{y}).
\end{aligned} \tag{44}$$

The spinor field $\psi$ and its Dirac conjugate $\psi^\dagger = \overline{\psi}\gamma_0$ are conjugate w.r.t. to the graded canonical bracket,

$$\begin{aligned}
&\{\psi_a^\dagger(\vec{x}), \psi_b(\vec{y})\}_g = i\eta_{ab}\delta(\vec{x}-\vec{y}),\\
&\{\psi_a(\vec{x}), \psi_b(\vec{y})\}_g = \{\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})\}_g = 0.
\end{aligned} \tag{45}$$

As is seen from these relations, the Poisson bracket is unique in the non-linear theory (31). No free parameters are involved in the Poisson bracket (44), (45) besides the coupling constants $g$.

With no arbitrary parameters involved in the Hamiltonian formulation, the non-linear theory (31) is not multi-Hamiltonian anymore, while the free limit admits the two-parameter series of Hamiltonian formulations (19), (22). This means, the interaction preserves one of possible Hamilto-

nian formulations admitted by the free theory. This fact can be explained in various ways. The most simple explanation is that upon inclusion of interaction, the deformation of the unique entry still conserves of the series of tensors (9). The parameters of series (34) are fixed by the interaction constants in the non-linear theory. It is the sole conserved tensor which defines the unique Hamiltonian at interacting level, while the corresponding Poisson bracket is fixed by the Hamiltonian.

For every $g$ the Poisson bracket (44) is a non-degenerate tensor, so it has an inverse, being a symplectic two-form. The latter defines the Hamiltonian action functional

$$S(g) = \int \Big\{ g_1 \left(\alpha_1 m A_i + \alpha_2 F_i + \frac{1}{m} G_i\right) \varepsilon_{ij} \dot{A}_j + \frac{1}{m}(g_1 - \alpha_2 g_2 - \alpha_1 g_3)\varepsilon_{ij} F_i \dot{F}_j + \frac{g_2}{m^2}\varepsilon_{ij} G_i \dot{F}_j + \frac{g_3}{m^3}\varepsilon_{ij} G_i \dot{G}_j + \psi^\dagger \dot{\psi} - H_T(g) \Big\} d^3x, \tag{46}$$

where $H_T(g)$ denotes the total Hamiltonian (42). For the minimal interaction $g_1 = 1, g_2 = g_3 = 0$, we get the standard Ostrogradski action

$$S(g) = \int \Big\{ (\alpha_1 m A_i + \alpha_2 F_i + \frac{1}{m} G_i)\varepsilon_{ij} \dot{A}_j + \frac{1}{m}\varepsilon_{ij} F_i \dot{F}_j + \psi^\dagger \dot{\psi} - A_0 \Theta - T_{00}(g)\Big|_{g_1=1, g_2=g_3=0} \Big\} d^3x. \tag{47}$$

For non-minimal interactions, we have the Hamiltonian action functional (46). This action is not canonically equivalent to (47). The non-minimal interactions are consistent with the bounded Hamiltonian (42), while the Ostrogradski Hamiltonian, which is associated with the minimal interaction, is unbounded in all the instances.

In this way, we see that the higher-derivative field equations (6) are compatible with inclusion of non-minimal explicitly covariant interactions (31) such that the theory still admits the Hamiltonian formalism with bounded Hamiltonian if the model has a bounded conserved quantity at the free level.

## 5 Concluding remarks

Let us summarize and discuss the results. First, we have seen that the third-order extension of the Chern–Simons admits a two-parameter series of conserved tensors. If the Eq. (6) describes unitary representations [cases (A),(B) in classification (8)], the bounded conserved quantities are included in the series. If the representations are non-unitary and/or indecomposable [cases (C), (D), (E) in classification (8)], all the conserved quantities are unbounded in the series. The series includes the canonical energy-momentum which is unbounded in all the cases. Second, we construct the constrained multi-Hamiltonian formalism for the higher-derivative Eq. (6). The 00-components of conserved tensors serve as Hamiltonians in this formalism. The formulations with different Hamiltonians and Poisson brackets result in the same equations, while the formulations are not connected by canonical transformations. For the cases with unitary representations, there are bounded Hamiltonians in the series. The Ostrogradski Hamiltonian, being included in the series, is unbounded. Third, we introduce explicitly the Poincaré-covariant and gauge-invariant stable interactions in higher-derivative dynamics. If the free theory has a bounded conserved quantity, it is still conserved at interacting level. After that, we demonstrate that the covariant and stable higher-derivative interacting theory admits the Hamiltonian formulation with the bounded Hamiltonian.

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