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On flux integrals for generalized Melvin solution related to simple finite-dimensional Lie algebra

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Abstract A generalized Melvin solution for an arbitrary simple finite-dimensional Lie algebra $\mathcal{G}$ is considered. The solution contains a metric, n Abelian 2-forms and n scalar fields, where n is the rank of $\mathcal{G}$. It is governed by a set of n moduli functions $H_s(z)$ obeying n ordinary differential equations with certain boundary conditions imposed. It was conjectured earlier that these functions should be polynomials—the so-called fluxbrane polynomials. These polynomials depend upon integration constants q_s, $s = 1, \ldots, n$. In the case when the conjecture on the polynomial structure for the Lie algebra $\mathcal{G}$ is satisfied, it is proved that 2-form flux integrals Φ^s over a proper $2d$ submanifold are finite and obey the relations $q_s \Phi^s = 4\pi n_s h_s$, where the $h_s > 0$ are certain constants (related to dilatonic coupling vectors) and the n_s are powers of the polynomials, which are components of a twice dual Weyl vector in the basis of simple (co-)roots, $s = 1, \ldots, n$. The main relations of the paper are valid for a solution corresponding to a finite-dimensional semi-simple Lie algebra $\mathcal{G}$. Examples of polynomials and fluxes for the Lie algebras A_1, A_2, A_3, C_2, G_2 and $A_1 + A_1$ are presented.

1 Introduction

In this paper we start with a generalization of a Melvin solution [1], which was presented earlier in Ref. [2]. It appears in the model which contains a metric, n Abelian 2-forms and $l \geq n$ scalar fields. This solution is governed by a certain non-degenerate (quasi-Cartan) matrix $(A_{ss'})$, $s, s' = 1, \ldots, n$. It is a special case of the so-called generalized fluxbrane solutions from Ref. [3]. For fluxbrane solutions see Refs. [4–28] and the references therein. The appearance of fluxbrane solutions was motivated by superstring/M theory.

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The generalized fluxbrane solutions from Ref. [3] are governed by moduli functions, $H_s(z) > 0$, defined on the interval $(0, +\infty)$, where $z = \rho^2$ and ρ is a radial variable. These functions obey a set of n non-linear differential master equations governed by the matrix $(A_{ss'})$, equivalent to Toda-like equations, with the following boundary conditions imposed: $H_s(+0) = 1$, $s = 1, \ldots, n$.

In this paper we assume that $(A_{ss'})$ is a Cartan matrix for some simple finite-dimensional Lie algebra $\mathcal{G}$ of rank n ($A_{ss} = 2$ for all s). According to a conjecture suggested in Ref. [3], the solutions to the master equations with the boundary conditions imposed are polynomials:

$$H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \tag{1.1}$$

where the $P_s^{(k)}$ are constants. Here $P_s^{(n_s)} \neq 0$ and

$$n_s = 2 \sum_{s'=1}^{n} A^{ss'}, \tag{1.2}$$

where we denote $(A^{ss'}) = (A_{ss'})^{-1}$. The integers n_s are components of a twice dual Weyl vector in the basis of simple (co-)roots [29].

The set of fluxbrane polynomials H_s defines a special solution to open Toda chain equations [30,31] corresponding to a simple finite-dimensional Lie algebra $\mathcal{G}$ [32]. In Refs. [2,33] a program (in Maple) for the calculation of these polynomials for the classical series of Lie algebras (A-, B-, C- and D-series) was suggested. It was pointed out in Ref. [3] that the conjecture on the polynomial structure of $H_s(z)$ is valid for Lie algebras of the A- and C-series. In Ref. [34] the conjecture from Ref. [3] was verified for the Lie algebra E_6 and certain duality relations for six E_6-polynomials were proved. In Sect. 2 we present the generalized Melvin solution from

Ref. [2]. In Sect. 3 we deal with the generalized Melvin solution for an arbitrary simple finite-dimensional Lie algebra $\mathcal{G}$. Here we calculate 2-form flux integrals $\Phi^s = \int_{M_*} F^s$, where F^s are 2-forms and M_* is a certain $2d$ submanifold. These integrals (fluxes) are finite when moduli functions are polynomials. In Sect. 3 we consider examples of fluxbrane polynomials and fluxes for the Lie algebras: A_1, A_2, A_3, C_2, G_2 and $A_1 + A_1$.

2 The solutions

We consider a model governed by the action

$$S = \int d^D x \sqrt{|g|} \Big\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \frac{1}{2} \sum_{s=1}^{n} \exp[2\lambda_s(\varphi)](F^s)^2 \Big\} \tag{2.1}$$

where $g = g_{MN}(x) dx^M \otimes dx^N$ is a metric, $\varphi = (\varphi^\alpha) \in \mathbb{R}^l$ is a set of scalar fields, $(h_{\alpha\beta})$ is a constant symmetric non-degenerate $l \times l$ matrix ($l \in \mathbb{N}$), $F^s = dA^s = \frac{1}{2} F^s_{MN} dx^M \wedge dx^N$ is a 2-form, λ_s is a 1-form on $\mathbb{R}^l$: $\lambda_s(\varphi) = \lambda_{s\alpha}\varphi^\alpha$, $s = 1, \ldots, n$; $\alpha = 1, \ldots, l$. Here $(\lambda_{s\alpha})$, $s = 1, \ldots, n$, are dilatonic coupling vectors. In (2.1) we denote $|g| = |\det(g_{MN})|$, $(F^s)^2 = F^s_{M_1 M_2} F^s_{N_1 N_2} g^{M_1 N_1} g^{M_2 N_2}$, $s = 1, \ldots, n$.

Here we start with a family of exact solutions to field equations corresponding to the action (2.1) and depending on one variable ρ. The solutions are defined on the manifold

$$M = (0, +\infty) \times M_1 \times M_2, \tag{2.2}$$

where M_1 is a one-dimensional manifold (say S^1 or $\mathbb{R}$) and M_2 is a (D-2)-dimensional Ricci-flat manifold. The solution reads [2]

$$g = \Big(\prod_{s=1}^{n} H_s^{2h_s/(D-2)} \Big) \Big\{ w d\rho \otimes d\rho + \Big(\prod_{s=1}^{n} H_s^{-2h_s} \Big) \rho^2 d\phi \otimes d\phi + g^2 \Big\}, \tag{2.3}$$

$$\exp(\varphi^\alpha) = \prod_{s=1}^{n} H_s^{h_s \lambda_s^\alpha}, \tag{2.4}$$

$$F^s = q_s \left(\prod_{s'=1}^{n} H_{s'}^{-A_{ss'}} \right) \rho d\rho \wedge d\phi, \tag{2.5}$$

$s = 1, \ldots, n$; $\alpha = 1, \ldots, l$, where $w = \pm 1$, $g^1 = d\phi \otimes d\phi$ is a metric on M_1 and g^2 is a Ricci-flat metric on M_2. Here $q_s \neq 0$ are integration constants, $q_s = -Q_s$ in the notations of Ref. [2], $s = 1, \ldots, n$.

The functions $H_s(z) > 0$, $z = \rho^2$, obey the master equations

$$\frac{d}{dz} \left(\frac{z}{H_s} \frac{d}{dz} H_s \right) = P_s \prod_{s'=1}^{n} H_{s'}^{-A_{ss'}}, \tag{2.6}$$

with the following boundary conditions:

$$H_s(+0) = 1, \tag{2.7}$$

where

$$P_s = \frac{1}{4} K_s q_s^2, \tag{2.8}$$

$s = 1, \ldots, n$. The boundary condition (2.7) guarantees the absence of a conic singularity [in the metric (2.3)] for $\rho = +0$.

The parameters h_s satisfy the relations

$$h_s = K_s^{-1}, \quad K_s = B_{ss} > 0, \tag{2.9}$$

where

$$B_{ss'} \equiv 1 + \frac{1}{2 - D} + \lambda_{s\alpha} \lambda_{s'\beta} h^{\alpha\beta}, \tag{2.10}$$

$s, s' = 1, \ldots, n$, with $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$. In the relations above we denote $\lambda_s^\alpha = h^{\alpha\beta}\lambda_{s\beta}$ and

$$(A_{ss'}) = (2B_{ss'}/B_{s's'}). \tag{2.11}$$

The latter is the so-called quasi-Cartan matrix.

We note that the constants $B_{ss'}$ and $K_s = B_{ss}$ have a certain mathematical sense. They are related to scalar products of certain vectors U^s (brane vectors, or U-vectors), which belong to a certain linear space ("truncated target space", for our problem it has dimension $l + 2$), i.e. $B_{ss'} = (U^s, U^{s'})$ and $K_s = (U^s, U^s)$ [35–37]. The scalar products of such a type are of physical significance, since they appear for various solutions with branes, e.g. black branes, S-branes, fluxbranes etc. Several physical parameters in multidimensional models with branes, e.g. the Hawking-like temperatures and the entropies of black holes and branes, PPN parameters, Hubble-like parameters, fluxes etc., contain such scalar products; see [36,37] and Sect. 3 of this paper. The relation (2.11) defines generalized intersection rules for branes which were suggested in [35]. The constants K_s are invariants of dimensional reduction. It is well known, see [37] and the references therein, that $K_s = 2$ for branes in numerous supergravity models, e.g. in dimensions $D = 10, 11$.

It may be shown that if the matrix $(h_{\alpha\beta})$ has an Euclidean signature and $l \geq n$, and $(A_{ss'})$ is a Cartan matrix for a simple Lie algebra $\mathcal{G}$ of rank n, there exists a set of co-vectors $\lambda_1, \ldots, \lambda_n$ obeying (2.11) (for $l = n$ see Remark 1 in the next section). Thus the solution is valid at least when $l \geq n$ and the matrix $(h_{\alpha\beta})$ is positive-definite.

The solution under consideration is a special case of the fluxbrane (for $w = +1$, $M_1 = S^1$) and S-brane ($w = -1$) solutions from [3] and [25], respectively.

If $w = +1$ and the (Ricci-flat) metric g^2 has a pseudo-Euclidean signature, we get a multidimensional generalization of Melvin's solution [1].

In our notations Melvin's solution (without scalar field) corresponds to $D = 4$, $n = 1$, $l = 0$, $M_1 = S^1$ ($0 < \phi < 2\pi$), $M_2 = \mathbb{R}^2$, $g^2 = -dt \otimes dt + dx \otimes dx$ and $\mathcal{G} = A_1$.

For $w = -1$ and g^2 of Euclidean signature we obtain a cosmological solution with a horizon (as $\rho = +0$) if $M_1 = \mathbb{R}$ ($-\infty < \phi < +\infty$).

3 Flux integrals for a simple finite-dimensional Lie algebra

Here we deal with the solution which corresponds to a simple finite-dimensional Lie algebra $\mathcal{G}$, i.e. the matrix $A = (A_{ss'})$ is coinciding with the Cartan matrix of this Lie algebra. We put also $n = l$, $w = +1$ and $M_1 = S^1$, $h_{\alpha\beta} = \delta_{\alpha\beta}$ and denote $(\lambda_{sa}) = (\lambda_s^a) = \boldsymbol{\lambda}_s$, $s = 1, \ldots, n$.

Due to (2.9)–(2.11) we get

$$K_s = \frac{D-3}{D-2} + \boldsymbol{\lambda}_s^2, \tag{3.1}$$

$h_s = K_s^{-1}$, and

$$\boldsymbol{\lambda}_s \boldsymbol{\lambda}_l = \frac{1}{2} K_l A_{sl} - \frac{D-3}{D-2} \equiv \Gamma_{sl}, \tag{3.2}$$

$s, l = 1, \ldots, n$. [Equation (3.1) is a special case of (3.2)].

It follows from (2.9)–(2.11) that

$$\frac{h_i}{h_j} = \frac{K_j}{K_i} = \frac{B_{jj}}{B_{ii}} = \frac{B_{ji}}{B_{ii}} \frac{B_{jj}}{B_{ij}} = \frac{A_{ji}}{A_{ij}} \tag{3.3}$$

for any $i \neq j$ obeying $A_{ij}, A_{ji} \neq 0$; $i, j = 1, \ldots, n$. It may be readily shown from (3.3) that the ratios $\frac{h_i}{h_j} = \frac{K_j}{K_i}$ are fixed numbers for any given Cartan matrix (A_{ij}) of a simple (finite-dimensional) Lie algebra $\mathcal{G}$. (This follows from (3.3) and the connectedness of the Dynkin diagram of a simple Lie algebra.) The ratios (3.3) may be written as follows:

$$\frac{h_i}{h_j} = \frac{K_j}{K_i} = \frac{r_j}{r_i} \tag{3.4}$$

$i \neq j$, where $r_i = (\alpha_i, \alpha_i)$ is the length squared of a simple root α_i corresponding to the Lie algebra $\mathcal{G}$. Here we use the notations $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$; $i, j = 1, \ldots, n$. Equation (3.4) implies

$$K_i = \frac{1}{2} K r_i, \tag{3.5}$$

$i = 1, \ldots, n$, where $K > 0$. (For simply laced (A, D, E) Lie algebras all r_i are equal.)

Remark 1 For large enough K in (3.5) there exist vectors $\boldsymbol{\lambda}_s$ obeying (3.2) [and hence (3.1)]. Indeed, the matrix (Γ_{sl}) is positive-definite if $K > K_*$, where K_* is some positive number. Hence there exists a matrix Λ, such that $\Lambda^T \Lambda = \Gamma$. We put $(\Lambda_{as}) = (\lambda_s^a)$ and get the set of vectors obeying (3.2).

Now let us consider the oriented 2-dimensional manifold $M_* = (0, +\infty) \times S^1$. The flux integrals

$$\Phi^s = \int_{M_*} F^s = \int_0^{+\infty} d\rho \int_0^{2\pi} d\phi \; \rho \mathcal{B}^s(\rho^2) = 2\pi \int_0^{+\infty} d\rho \; \rho \mathcal{B}^s(\rho^2), \tag{3.6}$$

where

$$\mathcal{B}^s(\rho^2) = q_s \prod_{l=1}^{n} (H_l(\rho^2))^{-A_{sl}}, \tag{3.7}$$

are convergent for all s, if the conjecture for the Lie algebra $\mathcal{G}$ (on polynomial structure of moduli functions H_s) is obeyed for the Lie algebra $\mathcal{G}$ under consideration.

Indeed, due to the polynomial assumption (1.1) we have

$$H_s(\rho^2) \sim C_s \rho^{2n_s}, \quad C_s = P_s^{(n_s)}, \tag{3.8}$$

as $\rho \to +\infty$; $s = 1, \ldots, n$. From (3.7), (3.8) and the equality $\sum_1^n A_{sl} n_l = 2$, following from (1.2), we get

$$\mathcal{B}^s(\rho^2) \sim q_s C^s \rho^{-4}, \quad C^s = \prod_{l=1}^{n} C_l^{-A_{sl}}, \tag{3.9}$$

and hence the integral (3.6) is convergent for any $s = 1, \ldots, n$.

By using the master equations (2.6) we obtain

$$\int_0^{+\infty} d\rho \rho \mathcal{B}^s(\rho^2) = q_s P_s^{-1} \frac{1}{2} \int_0^{+\infty} dz \frac{d}{dz} \left(\frac{z}{H_s} \frac{d}{dz} H_s \right) = \frac{1}{2} q_s P_s^{-1} \lim_{z \to +\infty} \left(\frac{z}{H_s} \frac{d}{dz} H_s \right) = \frac{1}{2} n_s q_s P_s^{-1}, \tag{3.10}$$

which implies [see (2.8)]

$$\Phi^s = 4\pi n_s q_s^{-1} h_s, \tag{3.11}$$

$s = 1, \ldots, n$.

Thus, any flux Φ^s depends upon one integration constant $q_s \neq 0$, while the integrand form F^s depends upon all constants: $q_1, \ldots, q_n$.

We note that for $D = 4$ and $g^2 = -dt \otimes dt + dx \otimes dx$, q_s is coinciding with the value of the x-component of the sth magnetic field on the axis of symmetry.

In the case of the Gibbons–Maeda dilatonic generalization of the Melvin solution, corresponding to $D = 4$, $n = l = 1$ and $\mathcal{G} = A_1$ [5], the flux from (3.11) ($s = 1$) is in agreement with that considered in Ref. [26]. For Melvin's case and some higher dimensional extensions (with $\mathcal{G} = A_1$) see also Ref. [14].

Due to (3.4) the ratios

$$\frac{q_i \Phi^i}{q_j \Phi^j} = \frac{n_i h_i}{n_j h_j} = \frac{n_i r_j}{n_j r_i} \tag{3.12}$$

are fixed numbers depending upon the Cartan matrix (A_{ij}) of a simple finite-dimensional Lie algebra $\mathcal{G}$.

Remark 2 The relation for flux integrals (3.11) is also valid when the matrix $(A_{ss'})$ is a Cartan matrix of a finite-dimensional semi-simple Lie algebra $\mathcal{G} = \mathcal{G}_1 \oplus \cdots \oplus \mathcal{G}_k$, where $\mathcal{G}_1, \ldots, \mathcal{G}_k$ are simple Lie (sub)algebras. In this case the Cartan matrix (A_{ij}) has a block-diagonal form, i.e. $(A_{ij}) = \mathrm{diag}\left(\left(A^{(1)}_{i_1 j_1}\right), \ldots, \left(A^{(k)}_{i_k j_k}\right)\right)$, where $\left(A^{(a)}_{i_a j_a}\right)$ is the Cartan matrix of the Lie algebra $\mathcal{G}_a$, $a = 1, \ldots, k$. The set of polynomials in this case splits in a direct union of sets of polynomials corresponding to the Lie algebras $\mathcal{G}_1, \ldots, \mathcal{G}_k$. Equations (3.4) and (3.12) are valid, when the indices i, j correspond to one ath block, $a = 1, \ldots, k$. The quantities $q_i \Phi^i$ and $q_j \Phi^j$ corresponding to different blocks are independent. Equation (3.5) should be replaced by

$$K_{i_a} = \frac{1}{2} K^{(a)} r_{i_a}, \quad K^{(a)} > 0, \tag{3.13}$$

for any index i_a corresponding to the ath block; $a = 1, \ldots, k$. The existence of dilatonic coupling vectors $\boldsymbol{\lambda}_s$ obeying (3.2) [(and (3.1)] just follows from the arguments of Remark 1, if we put all $K^{(a)} = K > 0$.

The manifold $M_* = (0, +\infty) \times S^1$ is isomorphic to the manifold $\mathbb{R}^2_* = \mathbb{R}^2 \setminus \{0\}$. The solution (2.3)–(2.5) may be understood (or rewritten by pull-backs) as defined on the manifold $\mathbb{R}^2_* \times M_2$, where the coordinates ρ, ϕ are understood as coordinates on $\mathbb{R}^2_*$. They are not globally defined. One should consider two charts with coordinates ρ, $\phi = \phi_1$ and ρ, $\phi = \phi_2$, where $\rho > 0$, $0 < \phi_1 < 2\pi$ and $-\pi < \phi_2 < \pi$. Here $\exp(i\phi_1) = \exp(i\phi_2)$. In both cases we have $x = \rho \cos\phi$ and $y = \rho \sin\phi$, where x, y are standard coordinates of $\mathbb{R}^2$. Using the identity $\rho d\rho \wedge d\phi = dx \wedge dy$ we get

$$F^s = q_s \prod_{s'=1}^{n} (H_{s'}(x^2 + y^2))^{-A_{ss'}} dx \wedge dy, \tag{3.14}$$

$s = 1, \ldots, n$. The 2-forms (3.14) are well defined on $\mathbb{R}^2$. Indeed, due to the conjecture from Ref. [3] any polynomial $H_s(z)$ is a smooth function on $\mathbb{R} = (-\infty, +\infty)$ which obeys $H_s(z) > 0$ for $z \in (-\varepsilon_s, +\infty)$, where $\varepsilon_s > 0$. This is valid due to the conjecture from Ref. [3] $H_s(z) > 0$ for $z > 0$ and $H_s(+0) = 1$. Thus, $\left(\prod_{s'=1}^{n} \left(H_{s'}\left(x^2 + y^2\right)\right)^{-A_{ss'}}\right)$ is a smooth function since it is a composition of two well-defined smooth functions $\left(\prod_{s'=1}^{n} (H_{s'}(z))^{-A_{ss'}}\right)$ and $z = x^2 + y^2$.

Now we show that there exist 1-forms A^s obeying $F^s = dA^s$ which are globally defined on $\mathbb{R}^2$. We start with the open submanifold $\mathbb{R}^2_*$. The 1-forms

$$A^s = \left(\int_0^{\rho} d\bar{\rho} \bar{\rho} \mathcal{B}^s(\bar{\rho}^2)\right) d\phi = \frac{1}{2} \left(\int_0^{\rho^2} d\bar{z} \mathcal{B}^s(\bar{z})\right) d\phi \tag{3.15}$$

are well defined on $\mathbb{R}^2_*$ (here $d\phi = (x^2 + y^2)^{-1}(-y dx + x dy)$) and obey $F^s = dA^s$, $s = 1, \ldots, n$. Using the master equation (2.6) we obtain

$$\begin{aligned} A^s &= \frac{q_s}{2P_s} \left(\int_0^{\rho^2} d\bar{z} \frac{d}{d\bar{z}} \left(\frac{\bar{z}}{H_s(\bar{z})} \frac{d}{d\bar{z}} H_s(\bar{z})\right)\right) d\phi \\ &= \frac{2h_s}{q_s} \frac{H_s'(\rho^2)}{H_s(\rho^2)} \rho^2 d\phi, \end{aligned} \tag{3.16}$$

$s = 1, \ldots, n$. Here $H'_s = \frac{d}{dz} H_s$. Due to the relation $\rho^2 d\phi = -y dx + x dy$, we obtain

$$A^s = \frac{2h_s}{q_s} \frac{H_s'(x^2 + y^2)}{H_s(x^2 + y^2)} (-y dx + x dy), \tag{3.17}$$

$s = 1, \ldots, n$. The 1-forms (3.17) are well-defined smooth 1-forms on $\mathbb{R}^2$.

We note that in the case of the Gibbons–Maeda solution [5] corresponding to $D = 4$, $n = l = 1$ and $\mathcal{G} = A_1$ the gauge potential from (3.16) coincides (up to notations) with that considered in Ref. [7].

Now we verify our result (3.11) for flux integrals by using the relations for the 1-forms A^s. Let us consider a $2d$ oriented manifold (disk) $D_R = \{(x, y) : x^2 + y^2 \leq R^2\}$ with the boundary $\partial D_R = C_R = \{(x, y) : x^2 + y^2 = R^2\}$. C_R is a circle of radius R. It is an $1d$ oriented manifold with the orientation (inherited from that of D_R) obeying the relation $\int_{C_R} d\phi = 2\pi$. Using the Stokes–Cartan theorem we get

$$\int_{D_R} F^s = \int_{D_R} dA^s = \int_{C_R} A^s = \frac{4\pi h_s}{q_s} \frac{H_s'(R^2)}{H_s(R^2)} R^2, \tag{3.18}$$

$s = 1, \ldots, n$. By using the asymptotic relation (3.8) we find

$$\lim_{R \to +\infty} \int_{D_R} F^s = \frac{4\pi h_s n_s}{q_s}, \tag{3.19}$$

$s = 1, \ldots, n$, in agreement with (3.11).

Remark 3 We note (for completeness) that the metric and scalar fields for our solution with $w = +1$ and $l = n$ can be extended to the manifold $\mathbb{R}^2 \times M_2$. Indeed, in the coordinates x, y the metric (2.3) and scalar fields (2.4) read as follows:

$$g = \left(\prod_{s=1}^{n} H_s^{2h_s/(D-2)}\right)\Big\{dx \otimes dx + dy \otimes dy + f(-y dx + x dy)^2 + g^2\Big\}, \tag{3.20}$$

$$\varphi^a = \sum_{s=1}^{n} h_s \lambda_s^a \ln H_s, \tag{3.21}$$

$a = 1, \ldots, l$. Here $H_s = H_s(x^2 + y^2)$, $s = 1, \ldots, n$, and $f = f(x^2 + y^2)$, where

$$f(z) = \left(\left(\prod_{s=1}^{n} (H_s(z))^{-2h_s}\right) - 1\right) z^{-1}, \tag{3.22}$$

for $z \neq 0$ and $f(0) = \lim_{z\to 0} f(z)$ (the limit does exist). The function $f(z)$ is smooth in the interval $(-\varepsilon, +\infty)$ for some $\varepsilon > 0$. Indeed, it is smooth in the interval $(0, +\infty)$ and holomorphic in the domain $\{z | 0 < |z| < \varepsilon\}$ for a small enough $\varepsilon > 0$. Since the limit $\lim_{z\to 0} f(z)$ does exist the function $f(z)$ is holomorphic in the disc $\{z | |z| < \varepsilon\}$ and hence it is smooth in the interval $(-\varepsilon, +\infty)$. This implies that the metric is smooth on the manifold $\mathbb{R}^2 \times M_2$. (See the text after Eq. (3.14).) The scalar fields are also smooth on $\mathbb{R}^2 \times M_2$.

4 Examples

Here we present fluxbrane polynomials corresponding to the Lie algebras $A_1, A_2, A_3, C_2, G_2, A_1 + A_1$ and related fluxes. Here as in [32] we use other parameters p_s instead of P_s:

$$p_s = P_s / n_s, \tag{4.1}$$

$s = 1, \ldots, n$.

A_1**-case.** The simplest example occurs in the case of the Lie algebra $A_1 = sl(2)$. Here $n_1 = 1$. We get [3]

$$H_1 = 1 + p_1 z \tag{4.2}$$

and

$$\Phi^1 = 4\pi q_1^{-1} h_1, \tag{4.3}$$

which is also valid for Melvin's solution with $D = 4$ and $h_1 = 2$.

A_2**-case.** For the Lie algebra $A_2 = sl(3)$ with the Cartan matrix

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{4.4}$$

we have [3,25,32] $n_1 = n_2 = 2$ and

$$H_1 = 1 + 2p_1 z + p_1 p_2 z^2, \tag{4.5}$$

$$H_2 = 1 + 2p_2 z + p_1 p_2 z^2. \tag{4.6}$$

We get in this case

$$(\Phi^1, \Phi^2) = 8\pi h (q_1^{-1}, q_2^{-1}), \tag{4.7}$$

where $h_1 = h_2 = h$.

A_3**-case.** The polynomials for the A_3-case read as follows [32,33]:

$$H_1 = 1 + 3p_1 z + 3p_1 p_2 z^2 + p_1 p_2 p_3 z^3, \tag{4.8}$$

$$H_2 = 1 + 4p_2 z + 3\big(p_1 p_2 + p_2 p_3\big) z^2 + 4p_1 p_2 p_3 z^3 + p_1 p_2^2 p_3 z^4, \tag{4.9}$$

$$H_3 = 1 + 3p_3 z + 3p_2 p_3 z^2 + p_1 p_2 p_3 z^3. \tag{4.10}$$

Here we have $(n_1, n_2, n_3) = (3, 4, 3)$ and

$$(\Phi^1, \Phi^2, \Phi^3) = 4\pi h (3q_1^{-1}, 4q_2^{-1}, 3q_3^{-1}) \tag{4.11}$$

with $h_1 = h_2 = h_3 = h$.

C_2**-case.** For the Lie algebra $C_2 = so(5)$ with the Cartan matrix

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \tag{4.12}$$

we get $n_1 = 3$ and $n_2 = 4$. For C_2-polynomials we obtain [25,32]

$$H_1 = 1 + 3p_1 z + 3p_1 p_2 z^2 + p_1^2 p_2 z^3, \tag{4.13}$$

$$H_2 = 1 + 4p_2 z + 6p_1 p_2 z^2 + 4p_1^2 p_2 z^3 + p_1^2 p_2^2 z^4. \tag{4.14}$$

In this case we find

$$(\Phi^1, \Phi^2) = 4\pi (3h_1 q_1^{-1}, 4h_2 q_2^{-1}) \tag{4.15}$$

where $h_1 = 2h_2$.

G_2**-case.** For the Lie algebra G_2 with the Cartan matrix

$$(A_{ss'}) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \tag{4.16}$$

we get $n_1 = 6$ and $n_2 = 10$. In this case the fluxbrane polynomials read [25,32]

$$H_1 = 1 + 6p_1 z + 15p_1 p_2 z^2 + 20p_1^2 p_2 z^3 + 15p_1^3 p_2 z^4 + 6p_1^3 p_2^2 z^5 + p_1^4 p_2^2 z^6, \tag{4.17}$$

$$H_2 = 1 + 10p_2 z + 45p_1 p_2 z^2 + 120p_1^2 p_2 z^3 + p_1^2 p_2(135p_1 + 75p_2)z^4 + 252p_1^3 p_2^2 z^5 + p_1^3 p_2^2\left(75p_1 + 135p_2\right)z^6 + 120p_1^4 p_2^3 z^7 + 45p_1^5 p_2^3 z^8 + 10p_1^6 p_2^3 z^9 + p_1^6 p_2^4 z^{10}. \tag{4.18}$$

We are led to the relations

$$(\Phi^1, \Phi^2) = 4\pi(6h_1 q_1^{-1}, 10h_2 q_2^{-1}) \tag{4.19}$$

where $h_1 = 3h_2$.

$(A_1 + A_1)$**-case.** For the semi-simple Lie algebra $A_1 + A_1$ we obtain $n_1 = n_2 = 1$,

$$H_1 = 1 + p_1 z, \quad H_2 = 1 + p_2 z, \tag{4.20}$$

and

$$(\Phi^1, \Phi^2) = 4\pi(q_1^{-1} h_1, q_2^{-1} h_2), \tag{4.21}$$

where h_1 and h_2 are independent, as well as the quantities $q_1\Phi^1$ and $q_2\Phi^2$.

5 Conclusions

Here we have considered a multidimensional generalization of Melvin's solution corresponding to a simple finite-dimensional Lie algebra $\mathcal{G}$. We have assumed that the solution is governed by a set of n fluxbrane polynomials $H_s(z)$, $s = 1, \ldots, n$. These polynomials define special solutions to open Toda chain equations corresponding to the Lie algebra $\mathcal{G}$.

The polynomials $H_s(z)$ depend also upon parameters q_s, which are coinciding for $D = 4$ (up to a sign) with the values of colored magnetic fields on the axis of symmetry.

We have calculated $2d$ flux integrals $\Phi^s = \int F^s$, $s = 1, \ldots, n$. Any flux Φ^s depends only upon one parameter q_s, while the integrand F^s depends upon all parameters $q_1, \ldots, q_n$. The relation for flux integrals (3.11) is also valid when the matrix $(A_{ss'})$ is a Cartan matrix of a finite-dimensional semi-simple Lie algebra $\mathcal{G}$.

Here we have considered examples of polynomials and fluxes for the Lie algebras A_1, A_2, A_3, C_2, G_2 and $A_1 + A_1$. The approach of this paper will be used for a calculation of certain flux integrals for forms F^s of arbitrary ranks corresponding to certain fluxbrane solutions (of electric type by p-brane notation or magnetic type by fluxbrane classification[1]) governed by fluxbrane polynomials [38].

An open problem is to find the fluxes for the solutions which are related to infinite-dimensional Lorentzian Kac–Moody algebras, e.g. hyperbolic ones [39,40]. In this case one should deal with phantom scalar fields in the model (2.1) and non-polynomial solutions to Eqs. (2.6). Another possibility is to study the convergence of flux integrals for non-polynomial solutions for moduli functions corresponding to non-Cartan matrices $(A_{ss'})$ (e.g. for the model with two 2-forms from Ref. [41]).

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[1] We remind the reader that an electric (magnetic) p-brane corresponds to a magnetic (electric) $F(D-3-p)$ fluxbrane; see [3] and the references therein.

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