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Open string T-duality in double space

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Abstract The role of double space is essential in the new interpretation of T-duality and consequently in an attempt to construct M-theory. The case of the open string is missing in such an approach because until now there has been no appropriate formulation of open string T-duality. In the previous paper (Sazdović, From geometry to non-geometry via T-duality, arXiv:1606.01938, 2017), we showed how to introduce vector gauge fields A_a^N and A_i^D at the end-points of an open string in order to enable open string invariance under local gauge transformations of the Kalb–Ramond field and its T-dual "restricted general coordinate transformations". We demonstrated that gauge fields A_a^N and A_i^D are T-dual to each other. In the present article we prove that all above results can be interpreted as coordinate permutations in double space.

1 Introduction

It is well known that M-theory unifies all five consistent superstring theories by a web of T and S dualities. In order to formulate M-theory we should construct one theory which contains the initial theory (any of the five consistent ones) and all corresponding dual ones.

The $2D$ dimensional double space with the coordinates $Z^M = (x^\mu, y_\mu)$ (which are the coordinates of initial space x^μ and its T-dual y_μ) offers many benefits in the interpretation of T-duality. In fact, in such a space, the T-duality transformations can be realized simply by exchanging places of some coordinates x^a, along which we performed T-duality and the corresponding dual coordinates y_a [2,3]. It contains the initial and all corresponding T-dual theories. Realization of such a program for T-duality in the bosonic case has been done: for a flat background in Ref. [2] and for a weakly curved background, with linear dependence on coordinates, in Ref. [3]. We hope that S-duality, which can be understood as a transformation of a dilaton background field, can be successfully incorporated in such a procedure.

T-duality for superstrings is a non-trivial extension of the bosonic case. In Ref. [4] we extended such an approach to type II theories. In fact, doubling all bosonic coordinates we have unified types IIA and IIB theories. The formulation of M-theory should include T-dualization along fermionic variables, also. T-dualization along all fermionic coordinates in fermionic double space (where we doubled all fermionic variables) has been considered in Ref. [5].

The remaining step is to extend interpretation of T-duality in double space (which we earlier proposed for the case of the closed string) to the case of the open string, also. This will be done in the present article.

The difference between open and closed string appears at the open string end-points. Until recently, background fields along Neumann and Dirichlet directions A_a^N and A_i^D (N and D denote components with Neumann and Dirichlet boundary conditions) are treated in a different way [6,7]. The Neumann vector field has been introduced in the Lagrangian through the coupling with $\dot{x}^a$. On the other hand, the Dirichlet vector field has been introduced as a consistency condition without contributions to the Lagrangian. In order to realize a double space formulation in the open string case we should treat Neumann and Dirichlet vector fields in the same way. This has recently been done in Ref. [1].

In Ref. [8] it has been shown how to introduce vector gauge fields A_a^N in order that open string retain the symmetries of the closed string. Note that according to Ref. [1], beside the well-known local gauge invariance of the Kalb–Ramond field we used its T-dual "restricted general coordinate transformations", which include transformations of the background fields but do not include transformations of the coordinates. So, the above interpretation of the T-duality in double space will confirm the expressions for T-dual closed string background fields $G_{\mu\nu}$ and $B_{\mu\nu}$ (as in Refs. [2,3]) and

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gives the same expressions for T-dual vector fields ${}^\star A^a_D$ and ${}^\star A^i_N$ as those obtained in Ref. [1] with Buscher's procedure.

2 T-duality of the open string

In this section we will introduce some well-known features of the bosonic string and briefly repeat the main results of Ref. [1]. We will adapt T-duality to be in compliance with boundary conditions on the open string end-points.

We will consider vector gauge fields: A^N_a with Neumann boundary conditions which compensate for the not implemented gauge symmetry of the Kalb–Ramond field at the open string end-points and A^D_i with Dirichlet boundary conditions, which compensate for the not implemented restricted general coordinate transformations at the open string end-points. We will show that field A^D_i is T-dual to the A^N_a one, and that the general coordinate transformations are T-dual to gauge symmetry.

2.1 Closed and open bosonic string

Let us start with the action for the closed bosonic string [8–10]

$$S[x]=\frac{1}{2\pi\alpha'}\int_\Sigma d^2\xi\sqrt{-g}\times\left[\frac{1}{2}g^{\alpha\beta}G_{\mu\nu}[x]+\frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}B_{\mu\nu}[x]\right]\partial_\alpha x^\mu\partial_\beta x^\nu,\quad(\varepsilon^{01}=-1). \tag{2.1}$$

It propagates in D-dimensional space-time with a background defined by the space metric $G_{\mu\nu}$ and the Kalb–Ramond field $B_{\mu\nu}$. We denote the string coordinates by $x^\mu(\xi)$, $\mu=0,1,...,D-1$ and the intrinsic world-sheet metric by $g_{\alpha\beta}$. The integration goes over the two-dimensional world-sheet Σ with coordinates ξ^α ($\xi^0=\tau$, $\xi^1=\sigma$) and α' is the Regge slope parameter. Since the constant $\frac{1}{2\pi\alpha'}$ appears in many expressions, from now on we will denote it $\kappa=\frac{1}{2\pi\alpha'}$.

In the conformal gauge $g_{\alpha\beta}=e^{2F}\eta_{\alpha\beta}$ this action can be rewritten in terms of light-cone coordinates $\xi^\pm=\frac{1}{2}(\tau\pm\sigma)$, $\partial_\pm=\partial_\tau\pm\partial_\sigma$ as

$$S=\kappa\int_\Sigma d^2\xi\ \partial_+x^\mu\Pi_{+\mu\nu}\partial_-x^\nu, \tag{2.2}$$

with the following combination of the background fields:

$$\Pi_{\pm\mu\nu}=B_{\mu\nu}\pm\frac{1}{2}G_{\mu\nu}. \tag{2.3}$$

In the string theory, variation of the action (2.2) with respect to x^μ produces not only the equation of motion

$$\partial_+\partial_-x^\mu+(\Gamma^\mu_{\nu\rho}-B^\mu_{\ \nu\rho})\partial_+x^\nu\partial_-x^\rho=0, \tag{2.4}$$

but also the boundary conditions

$$\gamma^{(0)}_\mu(x)\delta x^\mu/_{\sigma=\pi}-\gamma^{(0)}_\mu(x)\delta x^\mu/_{\sigma=0}=0, \tag{2.5}$$

where $\Gamma^\mu_{\nu\rho}$ is Christoffel symbol and we introduce the useful variable

$$\gamma^{(0)}_\mu(x)\equiv\frac{\delta S}{\delta x'^\mu}=\kappa[2B_{\mu\nu}\dot{x}^\nu-G_{\mu\nu}x'^\nu]. \tag{2.6}$$

From now on, we will denote the boundary of the open string $\partial\Sigma$, so that we can rewrite Eq. (2.5) as follows:

$$\gamma^{(0)}_\mu(x)\delta x^\mu/_{\partial\Sigma}=0. \tag{2.7}$$

As a consequence of periodicity, the boundary conditions are trivially satisfied in the closed string case. In the open string case there are two different ways to satisfy the boundary conditions. For some coordinates x^a ($a=0,1,\ldots,p$) we will choose the Neumann boundary conditions, when variations of the string end-points $\delta x^a/_{\partial\Sigma}$ are arbitrary and for the other ones, x^i ($i=p+1,\ldots,D-1$), we will choose the Dirichlet boundary conditions, when the edges of the string are fixed, $\dot{x}^i/_{\partial\Sigma}=0$. In order to satisfy the Neumann boundary conditions according to (2.5) we should take $\gamma^{(0)}_a(x)/_{\partial\Sigma}=0$.

It s well known that closed string theory is invariant under the following infinitesimal transformations: local gauge transformations of the Kalb–Ramond field,

$$\delta_\Lambda G_{\mu\nu}=0,\qquad\delta_\Lambda B_{\mu\nu}=\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu, \tag{2.8}$$

and general coordinate transformations,

$$\delta_\xi G_{\mu\nu}=-2\left(D_\mu\xi_\nu+D_\nu\xi_\mu\right),\quad \delta_\xi B_{\mu\nu}=-2\,\xi^\rho B_{\rho\mu\nu}+2\partial_\mu(B_{\nu\rho}\xi^\rho)-2\partial_\nu(B_{\mu\rho}\xi^\rho). \tag{2.9}$$

These transformations are connected by T-duality [1,11]. Let us stress that according to Ref. [1] we are not going to add transformations of the coordinates to (2.9). So, we will call these "restricted general coordinate transformations".

Both of the above symmetries fail at the open string end-points. In order to restore these symmetries the gauge fields have to be introduced. To restore local gauge symmetry we introduce the vector fields A^N_a with Neumann boundary conditions (see Ref. [8]), while to restore restricted general coordinate transformations we introduce the vector fields A^D_i with the Dirichlet boundary conditions (see Ref. [1]). Note that as a consequence of the boundary conditions only parts of these gauge fields survive.

So, the action for the open bosonic string with the above boundary conditions is [1]

$$S_{\rm open}[x]=\kappa\int_\Sigma d^2\xi\,\partial_+x^\mu\pi_{+\mu\nu}\partial_-x^\nu+2\kappa\int_{\partial\Sigma}d\tau\times\left(A^N_a[x]\dot{x}^a-\frac{1}{\kappa}A^D_i[x]G^{-1ij}\gamma^{(0)}_j(x)\right)$$

$$= \kappa \int_\Sigma d^2\xi\, \partial_+ x^\mu \pi_{+\mu\nu} \partial_- x^\nu + 2\kappa \eta^{\alpha\beta} \int_{\partial\Sigma} d\tau\, \mathcal{A}_{\alpha\mu}[x]\, \partial_\beta x^\mu, \tag{2.10}$$

where following Ref. [1] we introduced the effective variables $\mathcal{A}_{\pm\mu}(V) = \{\mathcal{A}_{\pm a}(V), \mathcal{A}_{\pm i}(V)\}$ defined as

$$\mathcal{A}_{\pm a}(V) \equiv A^N_a(V), \qquad \mathcal{A}_{\pm i}(V) = 2\Pi_{\mp ij} G^{-1jk} A^D_k(V), \tag{2.11}$$

and for simplicity we assumed that the metric tensor has the form

$$G_{\mu\nu} = \begin{pmatrix} G_{ab} & 0 \\ 0 & G_{ij} \end{pmatrix}. \tag{2.12}$$

We introduced a pair of effective vector fields $\mathcal{A}_{\alpha\mu} = \{\mathcal{A}_{0\mu}, \mathcal{A}_{1\mu}\}$ instead of the initial one $A_\mu = \{A^N_a, A^D_i\}$. So, we doubled the number of vector fields, but there are two constraints on the effective vector fields,

$$\begin{aligned} &\mathcal{A}_{1a}(V) = 0, \\ &\mathcal{A}_{0i}(V) = -(BG^{-1})_i \\ &\mathcal{A}_{1j}(V) - \mathcal{A}_{1i}(G^{-1}BV). \end{aligned} \tag{2.13}$$

In the literature $A^N_a[x]$ is known as a massless vector field on the Dp-brane, while the $A^D_i[x]$ are known as massless scalar oscillations orthogonal to the Dp-brane.

Let us briefly discuss the appearance of two types of vector fields and see the advantage of each of them. The $p+1$-dimensional Neumann gauge field A^N_a is the standard one and it couples with $\dot{x}^a$, as usual. The $D-p-1$-dimensional Dirichlet gauge field A^D_i is a nonstandard one and it couples with the term $G^{-1ij}\gamma^{(0)}_j$, which contains both $\dot{x}^i$ and x'^i. The fields coupled with x'^μ behave unusually and it is useful to treat them separately. So, we denote the fields coupled with $\dot{x}^\mu$ by $\mathcal{A}_{0\mu}$ and that coupled with x'^μ by $\mathcal{A}_{1\mu}$. But in such an approach, instead of one D-dimensional vector $A_\mu = \{A^N_a, A^D_i\}$ we have two effective D-dimensional vectors $\mathcal{A}_{\alpha\mu} = \{\mathcal{A}_{0\mu}, \mathcal{A}_{1\mu}\}$. This is the source of the constraints (2.13).

2.2 Choice of background

The space-time equations of motion are a consequence of the absence of the conformal anomaly. For the closed string case in the lowest order in the slope parameter α', it produces [12]

$$\begin{aligned} &\beta^G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} B_{\mu\rho\sigma} B_\nu{}^{\rho\sigma} + 2D_\mu \partial_\nu \Phi = 0, \\ &\beta^B_{\mu\nu} \equiv D_\rho B^\rho{}_{\mu\nu} - 2\partial_\rho \Phi B^\rho{}_{\mu\nu} = 0, \\ &\beta^\Phi \equiv 4(\partial\Phi)^2 - 4D_\mu \partial^\mu \Phi + \frac{1}{12} B_{\mu\nu\rho} B^{\mu\nu\rho} \\ &\qquad + 4\pi\kappa(D-26)/3 - R = 0. \end{aligned} \tag{2.14}$$

Here $B_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$ is the field strength of the Kalb–Ramond field $B_{\mu\nu}$, and $R_{\mu\nu}$ and D_μ are the Ricci tensor and covariant derivative with respect to the space-time metric.

For the same reason, for the open string case there are additional space-time equations of motion [13]. In our notation they take the following form:

$$\begin{aligned} &\beta_a \equiv -\frac{1}{2} \mathcal{B}_a{}^b \partial_b \Phi + \mathcal{G}^{-1bc}_E \partial_c \mathcal{B}_{ba} + \mathcal{G}^{-1bc}_E \\ &\left(\frac{1}{2} \mathcal{B}_a{}^d B_{dbe} \mathcal{B}^e{}_c + K^\mu_{ac} B_{\mu\nu} \partial_b f^\nu \right) = 0, \\ &\beta_\mu \equiv \frac{1}{2} \partial_\mu \Phi + \mathcal{G}^{-1ab}_E \left(\frac{1}{2} \mathcal{B}_b{}^c B_{\mu ac} - K_{\mu ab} \right) = 0, \end{aligned} \tag{2.15}$$

where

$$\mathcal{B}_{ab} = B_{ab} + \partial_a A^N_b - \partial_b A^N_a, \qquad \mathcal{G}^E_{ab} = G_{ab} - 4\mathcal{B}_{ac} G^{-1cd} \mathcal{B}_{db} \tag{2.16}$$

and K^μ_{ab} is the extrinsic curvature.

We will consider the simplest solutions of the closed string part,

$$G_{\mu\nu} = \text{const}, \quad B_{\mu\nu} = \text{const}, \quad \Phi = \text{const}, \quad D = 26, \tag{2.17}$$

which satisfy Eq. (2.14). For the open string part (2.15), we will assume that the vector fields are linear in the coordinates [1]

$$A^N_a(x) = A^0_a - \frac{1}{2} F^{(a)}_{ab} x^b, \qquad A^D_i(x) = A^0_i - \frac{1}{4} F^{(s)}_{ij} x^j, \tag{2.18}$$

so that the corresponding field strengths are constant. The infinitesimal coefficients $F^{(a)}_{ab}$ and $F^{(s)}_{ij}$ are defined as

$$F^{(a)}_{ab} = \partial_a A^N_b - \partial_b A^N_a, \qquad F^{(s)}_{ij} = -2(\partial_i A^D_j + \partial_j A^D_i). \tag{2.19}$$

Note that the $F^{(a)}_{ab}$ is antisymmetric in a, b indices while the $F^{(s)}_{ij}$ is symmetric in i, j indices. Since we are working with a plane Dp-brane the extrinsic curvature is zero and because Φ, $\mathcal{B}_{ab}$ and $\mathcal{G}^E_{ab}$ are constant, both β_a and β_μ vanish.

So, our choices of the background fields (2.17) and (2.18) satisfy all space-time equations of motion.

2.3 Sigma-model T-duality of the open string

The T-dualization procedure of the theory described by the action (2.10) with the background fields (2.17) and (2.18) has been performed in Ref. [1]. The T-dualization of the vector background fields A^N_a and A^D_i is non-trivial because these fields are coordinate dependent and it is not possible to apply the standard Buscher procedure [16,17]. Instead, the T-dualization procedure of Ref. [15], which works in the absence of a global symmetry, has been applied.

So, applying the T-dualization procedure along all coordinates, the T-dual action has been obtained:

$$
\begin{aligned}
{}^\star S[y] &= \frac{\kappa^2}{2}\int_\Sigma d^2\xi\, \partial_+ y_\mu \theta_-^{\mu\nu}\partial_- y_\nu + 2\kappa\int_{\partial\Sigma} d\tau \\
&\quad\times\left(A_i^D(V)G^{-1ij}\dot{y}_j - \frac{1}{\kappa}A_a^N(V)\,{}^\star\gamma_{(0)}^a\right)\\
&= \frac{\kappa^2}{2}\int_\Sigma d^2\xi\, \partial_+ y_\mu \theta_-^{\mu\nu}\partial_- y_\nu + 2\kappa\eta^{\alpha\beta}\int_{\partial\Sigma} d\tau\, {}^\star\mathcal{A}_\alpha^\mu(V)\partial_\beta y_\mu,
\end{aligned}
\tag{2.20}
$$

where

$$
\theta_\pm^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}\Pi_\pm G^{-1})^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa}(G_E^{-1})^{\mu\nu} \tag{2.21}
$$

and

$$
G^E_{\mu\nu} \equiv G_{\mu\nu} - 4(BG^{-1}B)_{\mu\nu}, \qquad \theta^{\mu\nu} \equiv -\frac{2}{\kappa}(G_E^{-1}BG^{-1})^{\mu\nu} \tag{2.22}
$$

are the symmetric and antisymmetric parts of $\theta_\pm^{\mu\nu}$. In the literature, $G^E_{\mu\nu}$ is known as the open string metric and $\theta^{\mu\nu}$ as a non-commutative parameter.

The T-dual action (2.20) should have the same form as the initial one (2.10) but in terms of T-dual fields. So, we can express the T-dual background fields in terms of the initial ones,

$$
\begin{aligned}
&{}^\star\Pi_+^{\mu\nu} = \frac{\kappa}{2}\theta_-^{\mu\nu}, \quad {}^\star A_D^a(V) = G_E^{-1ab}A_b^N(V),\\
&{}^\star A_N^i(V) = G^{-1ij}A_j^D(V).
\end{aligned}
\tag{2.23}
$$

The first relation can be rewritten as

$$
{}^\star G^{\mu\nu} = (G_E^{-1})^{\mu\nu}, \quad {}^\star B^{\mu\nu} = \frac{\kappa}{2}\theta^{\mu\nu}. \tag{2.24}
$$

Note that the T-dual vector background fields do not depend on y_μ but on

$$
V^\mu = -\kappa\,\theta^{\mu\nu}y_\nu + G_E^{-1\mu\nu}\,\tilde{y}_\nu, \tag{2.25}
$$

which is a function of both y_μ and its double

$$
\tilde{y}_\mu = \int(d\tau\, y'_\mu + d\sigma\,\dot{y}_\mu). \tag{2.26}
$$

With the help of (2.23) we can find the effective T-dual vector fields in analogy with Eq. (2.11),

$$
\begin{aligned}
&{}^\star\mathcal{A}_\pm^a(V) = 2\,{}^\star\Pi_\mp^{ab}\,{}^\star G_{bc}^{-1}\,{}^\star A_D^c(V) = \kappa\,\theta_\pm^{ab}A_b^N(V),\\
&{}^\star\mathcal{A}_\pm^i(V) = {}^\star A_N^i(V) = G^{-1ij}A_i^D(V).
\end{aligned}
\tag{2.27}
$$

We introduced two effective T-dual vector fields ${}^\star\mathcal{A}_\alpha^\mu = \{{}^\star\mathcal{A}_0^\mu, {}^\star\mathcal{A}_1^\mu\}$ instead of the initial one ${}^\star A^\mu = \{{}^\star A_D^a, {}^\star A_N^i\}$, but we have two constraints:

$$
\begin{aligned}
&{}^\star\mathcal{A}_0^a(V) = -2({}^\star B\,{}^\star G^{-1})^a{}_b\,{}^\star\mathcal{A}_1^b(V) = 2(G^{-1}B)^a{}_b\,{}^\star\mathcal{A}_1^b(V),\\
&{}^\star\mathcal{A}_1^i(V) = 0.
\end{aligned}
\tag{2.28}
$$

The explanation for the two types of vector fields is the same as that at the end of Sect. 2.1 for the original vector fields. We will see that effective fields naturally appear in T-duality transformation laws for the open string (2.34). In terms of their combinations $\mathcal{A}_{\pm\mu} = \mathcal{A}_{0\mu} \pm \mathcal{A}_{1\mu}$, the T-duality relations for the vector fields obtain the simple form ${}^\star\mathcal{A}_\pm^\mu \cong \kappa\,\theta_\pm^{\mu\nu}\mathcal{A}_{\pm\nu}$ (see (3.12)).

We can define the field strengths of the vector fields rewriting the interaction term as follows:

$$
S_A = 2\kappa\eta^{\alpha\beta}\int_{\partial\Sigma} d\tau\,\mathcal{A}_{\alpha\mu}[x]\,\partial_\beta x^\mu = \kappa\int_\Sigma d^2\xi\,\partial_+ x^\mu\mathcal{F}_{\mu\nu}\partial_- x^\nu, \tag{2.29}
$$

and similarly for the T-dual case. The field strength is simply defined in terms of effective fields and in the most general T-dual case that leads to [1]

$$
\begin{aligned}
{}^\star\mathcal{F}^{\mu\nu} &= {}^\star\mathcal{F}_{(a)}^{\mu\nu} + \frac{1}{2}{}^\star\mathcal{F}_{(s)}^{\mu\nu} = \eta^{\alpha\beta}[\partial_\alpha^\mu\,{}^\star\mathcal{A}_\beta^\nu(V) - \partial_\alpha^\nu\,{}^\star\mathcal{A}_\beta^\mu(V)]\\
&\quad -\varepsilon^{\alpha\beta}[\partial_\alpha^\mu\,{}^\star\mathcal{A}_\beta^\nu(V) + \partial_\alpha^\nu\,{}^\star\mathcal{A}_\beta^\mu(V)],
\end{aligned}
\tag{2.30}
$$

where $\partial_\alpha^\mu \equiv \frac{\partial}{\partial y_\mu^\alpha} = \{\frac{\partial}{\partial y_\mu}, -\frac{\partial}{\partial\tilde{y}_\mu}\}$ are derivatives with respect to the variables $y_\mu^\alpha = \{y_\mu^0 = y_\mu,\ y_\mu^1 = -\tilde{y}_\mu\}$.

Note that beside the standard antisymmetric part ${}^\star\mathcal{F}_{(a)}^{\mu\nu}$ it contains the unusual symmetric part ${}^\star\mathcal{F}_{(s)}^{\mu\nu}$ also. The source of the last one is the vector field ${}^\star\mathcal{A}_1^\mu$ (which originally has been introduced as a field coupled with x'^μ) and derivatives with respect to $\tilde{y}_\mu$.

So, the advantage of introducing effective fields is higher than the price we paid, the constraints (2.13) and (2.28). Technically, we can consider them as gauge fixing of some additional gauge symmetry for effective vector fields.

For the initial and T-dual open strings, the boundary conditions at the string end-points take the form

$$
\gamma_\mu^{(0)}\delta x^\mu/_{\partial\Sigma} = 0, \qquad {}^\star\gamma_{(0)}^\mu\delta y_\mu/_{\partial\Sigma} = 0. \tag{2.31}
$$

Here $\gamma_\mu^{(0)}$ (defined in (2.6) for a closed string) now receives a new infinitesimal term,

$$
\begin{aligned}
\gamma_\mu^{(0)}(x) &\equiv \frac{\delta S}{\delta x'^\mu} = \kappa[2B_{\mu\nu}\dot{x}^\nu - G_{\mu\nu}x'^\nu - 2\mathcal{A}_{1\nu}\Delta(\sigma)]\\
&= \kappa[2B_{\mu\nu}\dot{x}^\nu - G_{\mu\nu}x'^\nu + 2A_i^D\Delta(\sigma)],
\end{aligned}
\tag{2.32}
$$

while for the T-dual theory we have

$$
\begin{aligned}
{}^\star\gamma_{(0)}^\mu(y) &\equiv \frac{\delta\,{}^\star S}{\delta y'_\mu} = \kappa[2\,{}^\star B^{\mu\nu}\dot{y}_\nu - {}^\star G^{\mu\nu}y'_\nu - 2\,{}^\star\mathcal{A}_1^\nu\Delta(\sigma)]\\
&= \kappa[\kappa\theta^{\mu\nu}\dot{y}_\nu - (G_E^{-1})^{\mu\nu}y'_\nu + 2\,G_E^{-1ab}A_b^N\Delta(\sigma)],
\end{aligned}
\tag{2.33}
$$

where $\Delta(\sigma) \equiv \delta(\sigma - \pi) - \delta(\sigma)$.

The terms with the vector field A_i^D in (2.32) and A_b^N (2.33) are irrelevant in the expressions for the actions (2.10) and

(2.20), because they appear as an infinitesimal of the second order terms.

2.4 T-duality transformations of the open string

The T-dual transformation laws for the open string, connecting the initial and corresponding T-dual variables, take the form [1,18]

$$\partial_\pm x^\mu \cong -\kappa\theta_\pm^{\mu\nu}\partial_\pm y_\nu \pm 4\kappa\theta_\pm^{\mu\nu}\mathcal{A}_{\pm\nu}(V)\Delta(\sigma),\\ \partial_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}\partial_\pm x^\nu \pm 4\mathcal{A}_{\pm\mu}(x)\Delta(\sigma), \tag{2.34}$$

where the symbol $\cong$ denotes the T-duality relation.

In fact the second transformation (2.34) can be obtained after T-dualization of the action (2.20). Equations (2.34) are the inverse. Both transformations differ from the closed string ones by the infinitesimal term which contains the vector background fields $\mathcal{A}_{\pm\mu}$.

In terms of the covariant derivatives,

$$D_\pm x^\mu = \partial_\pm x^\mu + 2(G^{-1})^{\mu\nu}\mathcal{A}_{\pm\nu}\Delta(\sigma),\\ D_\pm y_\mu = \partial_\pm y_\mu + 2({}^\star G^{-1})_{\mu\nu}{}^\star\mathcal{A}_\pm^\nu\Delta(\sigma), \tag{2.35}$$

we can rewrite the transformations (2.34) in the simple form

$$D_\pm x^\mu \cong -\kappa\theta_\pm^{\mu\nu}D_\pm y_\nu,\quad D_\pm y_\mu \cong -2\Pi_{\mp\mu\nu}D_\pm x^\nu. \tag{2.36}$$

From the first equation (2.34) we can find the T-dual transformation laws for $\dot{x}^\mu$ and x'^μ,

$$\dot{x}^\mu \cong -\kappa\theta^{\mu\nu}[\dot{y}_\nu - 4\mathcal{A}_{1\nu}\Delta(\sigma)] + G_E^{-1\mu\nu}[y'_\nu - 4\mathcal{A}_{0\nu}\Delta(\sigma)]\\ = -\kappa\theta^{\mu\nu}\dot{y}_\nu + G_E^{-1\mu\nu}y'_\nu + 4{}^\star\mathcal{A}_1^\mu\Delta(\sigma) \tag{2.37}$$

$$x'^\mu \cong -\kappa\theta^{\mu\nu}[y'_\nu - 4\mathcal{A}_{0\nu}\Delta(\sigma)] + G_E^{-1\mu\nu}[\dot{y}_\nu - 4\mathcal{A}_{1\nu}\Delta(\sigma)]\\ = -\kappa\theta^{\mu\nu}y'_\nu + G_E^{-1\mu\nu}\dot{y}_\nu + 4{}^\star\mathcal{A}_0^\mu\Delta(\sigma), \tag{2.38}$$

and from the second one the inverse transformations become

$$\dot{y}_\mu \cong -2B_{\mu\nu}\dot{x}^\nu + G_{\mu\nu}x'^\nu + 4\mathcal{A}_{1\mu}\Delta(\sigma), \tag{2.39}$$

$$y'_\mu \cong G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu}x'^\nu + 4\mathcal{A}_{0\mu}\Delta(\sigma). \tag{2.40}$$

Using the expression for the canonical momentum of the original and of the T-dual theory

$$\pi_\mu \equiv \frac{\delta S}{\delta\dot{x}^\mu} = \kappa[G_{\mu\nu}\dot{x}^\nu - 2B_{\mu\nu}x'^\nu + 2\mathcal{A}_{0\mu}\Delta(\sigma)],\\ {}^\star\pi^\mu \equiv \frac{\delta\,{}^\star S}{\delta\dot{y}_\mu} = \kappa[(G_E^{-1})^{\mu\nu}\dot{y}_\nu - \kappa\theta^{\mu\nu}y'_\nu + 2{}^\star\mathcal{A}_0^\mu\Delta(\sigma)], \tag{2.41}$$

we can rewrite the transformations (2.38) and (2.40) in the canonical form

$$\kappa\, x'^\mu \cong {}^\star\pi^\mu + 2\kappa\,{}^\star\mathcal{A}_0^\mu\Delta(\sigma),\\ \pi_\mu + 2\kappa\mathcal{A}_{0\mu}\Delta(\sigma) \cong \kappa y'_\mu. \tag{2.42}$$

This relation connect momenta and winding numbers.

We can rewrite the transformations (2.37) and (2.39) in the form

$$-\kappa\,\dot{x}^\mu \cong {}^\star\gamma_{(0)}^\mu - 2\kappa\,{}^\star\mathcal{A}_1^\mu\Delta(\sigma),\\ \gamma_\mu^{(0)} - 2\kappa\mathcal{A}_{1\mu}\Delta(\sigma) \cong -\kappa\,\dot{y}_\mu, \tag{2.43}$$

where $\gamma_\mu^{(0)}$ is defined in (2.32) and ${}^\star\gamma_{(0)}^\mu$ in (2.33).

It was shown in Ref. [11] that π_μ is a generator of general coordinate transformations, while x'^μ is generator of gauge symmetry. In Ref. [1] the T-duality relation between $\dot{x}^\mu$ and ${}^\star\gamma_{(0)}^\mu$ (as well as between $\dot{y}_\mu$ and $\gamma_\mu^{(0)}$) has been established. Equations (2.42) and (2.43) are an extension of T-duality to the open string case. The additional $\mathcal{A}_\mu$-dependent terms stem from variations of the arguments of vector background fields.

Note that the momentum π_μ and variable $\gamma_\mu^{(0)}(x)$, as well as $\partial_\alpha x^\mu = \{\dot{x}^\mu, x'^\mu\}$, are components of the same world-sheet vector:

$$\pi_\mu^\alpha \equiv \frac{\delta S}{\delta\partial_\alpha x^\mu} = \{\pi_\mu, \gamma_\mu^{(0)}(x)\}. \tag{2.44}$$

From now on we will call $\gamma_\mu^{(0)}(x)$ the σ-momentum. We can rewrite Eqs. (2.42) and (2.43) in the form

$$\pi_\mu^\alpha \cong -\kappa\varepsilon^{\alpha\beta}\partial_\beta y_\mu + 2\kappa\eta^{\alpha\beta}\mathcal{A}_{\beta\mu}\Delta(\sigma),\\ {}^\star\pi^{\alpha\mu} \cong -\kappa\varepsilon^{\alpha\beta}\partial_\beta x^\mu + 2\kappa\eta^{\alpha\beta}{}^\star\mathcal{A}_\beta^\mu\Delta(\sigma). \tag{2.45}$$

Therefore, T-duality interchanges Neumann with Dirichlet gauge fields. It also interchanges $\dot{x}^\mu$ and x'^μ with ${}^\star\gamma_{(0)}^\mu$ and ${}^\star\pi^\mu$ as well as $\dot{y}_\mu$ and y'_μ with $\gamma_\mu^{(0)}$ and π_μ.

3 T-dual background fields of open string in double space

Following Refs. [2–5,14] we are going to introduce a double space in order to offer a simple interpretation of T-dualization as a coordinate permutation in double space. Let us start with the T-dual transformation laws (2.34). We can express them in a useful form, where on the left hand side we put the terms with the world-sheet antisymmetric tensor $\varepsilon_\alpha{}^\beta$ (note that $\varepsilon_\pm{}^\pm = \pm 1$)

$$\pm\partial_\pm y_\mu \cong G^E_{\mu\nu}\partial_\pm x^\nu - 2(BG^{-1})_\mu{}^\nu\partial_\pm y_\nu\\ +8(\Pi_\pm G^{-1})_\mu{}^\nu\mathcal{A}_{\pm\nu}(V)\Delta(\sigma),\\ \pm\partial_\pm x^\mu \cong 2(G^{-1}B)^\mu{}_\nu\partial_\pm x^\nu + (G^{-1})^{\mu\nu}\partial_\pm y_\nu\\ -4G^{-1\mu\nu}\mathcal{A}_{\pm\nu}(x)\Delta(\sigma). \tag{3.1}$$

We can rewrite these T-duality relations in the simple form

$$\partial_\pm Z^M \cong \pm\Omega^{MN}[\mathcal{H}_{NK}\,\partial_\pm Z^K\\ -2(\mathcal{H} + \sigma_3\mathcal{H}\sigma_3)_{NK}A_\pm^K(\check{Z}_{arg})\Delta(\sigma)], \tag{3.2}$$

where we introduced the double coordinates Z^M and the corresponding arguments of the background fields $\check{Z}_{\text{arg}}$

$$Z^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix}, \qquad \check{Z}_{\text{arg}} = \begin{vmatrix} V^\mu \\ x^\mu \end{vmatrix}_D . \tag{3.3}$$

Note the different notation for arguments of the background fields, introduced in Ref. [3]. The double space coordinate Z^M has $2D$ rows; D components of initial coordinates x^μ in the upper D rows and D components of T-dual coordinates y_μ in the lower D rows. In the arguments of the background fields $\check{Z}_{\text{arg}}$ in each row there is the complete D dimensional vector. Rewritten in the form of one column the arguments of the background fields are a $2D^2$ dimensional vector.

Because the arguments of all background fields in $\mathcal{A}^M_\pm(\check{Z}_{\text{arg}})$ and ${}^\star\mathcal{A}^K_\pm({}^\star\check{Z}_{\text{arg}})$ (see (3.10)) are the same in the upper D rows as well as in the lower D rows we can write them in two component notation as in (3.3). We indicated this with the index D.

We also introduced the so-called $O(D, D)$ invariant metric Ω^{MN}, the generalized metric $\mathcal{H}_{MN}$ and constant matrix σ_3,

$$\Omega^{MN} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\mathcal{H}_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2\, B_{\mu\rho}(G^{-1})^{\rho\nu} \\ 2(G^{-1})^{\mu\rho}\, B_{\rho\nu} & (G^{-1})^{\mu\nu} \end{pmatrix}, \tag{3.4}$$

$$(\sigma_3)_M{}^N = \begin{pmatrix} 1_D & 0 \\ 0 & -1_D \end{pmatrix}, \tag{3.5}$$

and the double gauge fields

$$\mathcal{A}^M_\pm(\check{Z}_{\text{arg}}) = \begin{pmatrix} {}^\star\mathcal{A}^\mu_\pm(V) \\ \mathcal{A}_{\pm\mu}(x) \end{pmatrix} = \begin{pmatrix} \kappa\, \theta^{\mu\nu}_\pm \mathcal{A}_{\pm\nu}(V) \\ \mathcal{A}_{\pm\mu}(x) \end{pmatrix}. \tag{3.6}$$

Note that as in Refs. [2–5,14] all coordinates are doubled. It is easy to check that

$$\mathcal{H}^T \Omega \mathcal{H} = \Omega, \tag{3.7}$$

which shows that manifest $O(D, D)$ symmetry is automatically incorporated into theory.

3.1 T-duality in double space along all coordinates

Let us derive the expression for the T-dual generalized metric and T-dual double gauge fields following the approach of Ref. [3]. Then, besides the double space coordinates Z^M, we should also transform the extended coordinates of the arguments of the background fields $\check{Z}_{\text{arg}}$ (3.3). We will require that the T-duality transformations (3.2) are invariant under transformations of the double space coordinates Z^M and $\check{Z}_{\text{arg}}$

$${}^\star Z^M = {}^\star\mathcal{T}^M{}_N Z^N, \qquad {}^\star\check{Z}_{\text{arg}} = {}^\star\check{\mathcal{T}}\, \check{Z}_{\text{arg}}. \tag{3.8}$$

We want to offer an interpretation for the case where T-dualization has been performed along all coordinates. So, we are going to exchange all initial with all T-dual coordinates, which is described by the matrices ${}^\star\mathcal{T}$ and ${}^\star\check{\mathcal{T}}$ of the form

$${}^\star\mathcal{T} = \Omega_2 \otimes 1_D = \begin{pmatrix} 0 & 1_D \\ 1_D & 0 \end{pmatrix},$$
$${}^\star\check{\mathcal{T}} = \Omega_2 \otimes 1_{D^2} = \begin{pmatrix} 0 & 1_{D^2} \\ 1_{D^2} & 0 \end{pmatrix}. \tag{3.9}$$

The T-dual coordinates ${}^\star Z^M$ and ${}^\star\check{Z}_{\text{arg}}$ should satisfy the same relation as the initial one, Eq. (3.2), but in terms of the T-dual background fields:

$$\partial_\pm {}^\star Z^M \cong \pm\, \Omega^{MN}[{}^\star\mathcal{H}_{NK}\, \partial_\pm {}^\star Z^K - 2({}^\star\mathcal{H} + \sigma_3 {}^\star\mathcal{H}\sigma_3)_{NK}\, {}^\star\mathcal{A}^K_\pm({}^\star\check{Z}_{\text{arg}})\Delta(\sigma)]. \tag{3.10}$$

This produces the expression for the dual generalized metric and dual double gauge fields in terms of the initial ones:

$${}^\star\mathcal{H} \cong {}^\star\mathcal{T}\mathcal{H}{}^\star\mathcal{T}, \qquad {}^\star\mathcal{A}_\pm({}^\star\check{Z}_{\text{arg}}) \cong {}^\star\mathcal{T}\mathcal{A}_\pm(\check{Z}_{\text{arg}}). \tag{3.11}$$

It is well known [2,3] that the first relation gives the standard T-dual transformations of the metric and Kalb–Ramond fields (2.24). Rewriting the second relation in components, with the help of (2.24) and (3.6) we have

$${}^\star\mathcal{A}^\mu_\pm \cong \kappa\, \theta^{\mu\nu}_\pm \mathcal{A}_{\pm\nu}. \tag{3.12}$$

Using Eqs. (2.11) and the first relation of (2.23) we obtain

$${}^\star\mathcal{A}^a_\pm \cong \kappa\, \theta^{ab}_\pm \mathcal{A}_{\pm b} = 2\, {}^\star\Pi^{ab}_\mp A^N_b,$$
$${}^\star\mathcal{A}^i_\pm \cong \kappa\, \theta^{ij}_\pm \mathcal{A}_{\pm j} = 2\kappa\, \theta^{ij}_\pm \Pi_{\mp jk} G^{-1kq} A^D_q = G^{-1ij} A^D_j. \tag{3.13}$$

On the other hand, the T-dual effective fields should have the form (2.27)

$${}^\star\mathcal{A}^a_\pm = 2\, {}^\star\Pi^{ab}_\mp {}^\star G^{-1}_{bc}\, {}^\star A^c_D, \qquad {}^\star\mathcal{A}^i_\pm = {}^\star A^i_N. \tag{3.14}$$

From (3.13) and (3.14) with the help of (2.24) we have

$${}^\star A^a_D = G^{-1ab}_E A^N_b, \quad {}^\star A^i_N = G^{-1ij} A^D_j, \tag{3.15}$$

which is just the Buscher T-duality relation for vector fields (2.23).

So, inclusion of gauge fields does not change the interpretation of T-duality in double space. It is again a replacement of the initial and T-dual coordinates, which shows that these transformations are nonphysical.

3.2 Double space field strength

If in addition to (3.3) we introduce new double fields,

$$\tilde{Z}^M = \begin{pmatrix} \tilde{x}^\mu \\ \tilde{y}_\mu \end{pmatrix}, \quad \partial_M = \begin{pmatrix} \partial_{x^\mu} \\ \partial_y^\mu \end{pmatrix}, \quad \tilde{\partial}_M = \begin{pmatrix} \partial_{\tilde{x}^\mu} \\ \partial_{\tilde{y}}^\mu \end{pmatrix}, \tag{3.16}$$

we can reexpress the field strengths of both initial and T-dual case (see Eqs. (5.11) and (7.31) of Ref. [1]) in the form

$$\mathcal{F}^{MN} = \Omega^{MK}(\hat{\partial}_{+K}\mathcal{A}_+^N(\check{Z}_{\text{arg}}) - \hat{\partial}_{-K}\mathcal{A}^N(\check{Z}_{\text{arg}})) = \begin{pmatrix} {}^\star\mathcal{F}^{\mu\nu} & 0 \\ 0 & \mathcal{F}_{\mu\nu} \end{pmatrix}, \tag{3.17}$$

where we defined

$$\hat{\partial}_{\pm M} = \partial_M \pm \tilde{\partial}_M. \tag{3.18}$$

4 Example: three-torus with D_1-brane in double space

In this section the example of three-torus with a D_1-brane, considered in Ref. [1] will be presented in double space. We will show how to perform T-dualization along all coordinates in double space.

4.1 Initial theory in double space

We will start with a definition of the background fields of the initial theory in double space. Let us denote the coordinates of the $D = 3$ dimensional torus by x^0, x^1, x^2 and introduce non-trivial components of the background fields by

$$G_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & B & 0 \\ -B & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.1}$$

It is easy to find the corresponding effective metric and the non-commutativity parameter

$$G^E_{\mu\nu} = \begin{pmatrix} G_E & 0 & 0 \\ 0 & -G_E & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.2}$$

where as we defined in [1]

$$G_E \equiv 1 - 4B^2, \quad \theta \equiv \frac{2B}{\kappa G_E}. \tag{4.3}$$

We will also need an expression for the combination of the background fields

$$\theta_\pm^{\mu\nu} = \theta^{\mu\nu} \mp \frac{1}{\kappa} G_E^{-1\mu\nu} = \begin{pmatrix} \mp\frac{1}{\kappa G_E} & \theta & 0 \\ -\theta & \pm\frac{1}{\kappa G_E} & 0 \\ 0 & 0 & \pm\frac{1}{\kappa} \end{pmatrix}. \tag{4.4}$$

According to (3.4) it produces

$$\mathcal{H}_{MN} = \begin{pmatrix} G^E_{\mu\nu} & -2(BG^{-1})_\mu{}^\nu \\ 2(G^{-1}B)^\mu{}_\nu & (G^{-1})^{\mu\nu} \end{pmatrix} = \begin{pmatrix} G_E & 0 & 0 & 0 & 2B & 0 \\ 0 & -G_E & 0 & 2B & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2B & 0 & 1 & 0 & 0 \\ 2B & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.5}$$

Similarly, we have

$$\sigma_3\mathcal{H}\sigma_3 = \begin{pmatrix} G^E_{\mu\nu} & 2(BG^{-1})_\mu{}^\nu \\ -2(G^{-1}B)^\mu{}_\nu & (G^{-1})^{\mu\nu} \end{pmatrix} = \begin{pmatrix} G_E & 0 & 0 & 0 & -2B & 0 \\ 0 & -G_E & 0 & -2B & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2B & 0 & 1 & 0 & 0 \\ -2B & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.6}$$

The double space coordinates are

$$Z^M = \begin{pmatrix} x^\mu \\ y_\mu \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ y_0 \\ y_1 \\ y_2 \end{pmatrix}, \quad \check{Z}_{arg} = \begin{vmatrix} V^\mu \\ x^\mu \end{vmatrix}_{D=3} = \begin{vmatrix} V^\mu \\ V^\mu \\ V^\mu \\ x^\mu \\ x^\mu \\ x^\mu \end{vmatrix}, \tag{4.7}$$

while the double gauge field according to (3.6) takes the form

$$\mathcal{A}_\pm^M(\check{Z}_{\text{arg}}) = \begin{pmatrix} {}^\star\mathcal{A}_\pm^\mu(V) \\ \mathcal{A}_{\pm\mu}(x) \end{pmatrix} = \begin{pmatrix} \kappa\,\theta_\pm^{\mu\nu}\mathcal{A}_{\pm\nu}(V) \\ \mathcal{A}_{\pm\mu}(x) \end{pmatrix} = \begin{pmatrix} {}^\star\mathcal{A}_\pm^0(V) \\ {}^\star\mathcal{A}_\pm^1(V) \\ {}^\star\mathcal{A}_\pm^2(V) \\ \mathcal{A}_{\pm 0}(x) \\ \mathcal{A}_{\pm 1}(x) \\ \mathcal{A}_{\pm 2}(x) \end{pmatrix}. \tag{4.8}$$

Note that the dimension of $\check{Z}_{\text{arg}}$ is $2 \times D^2 = 2 \times 3^2 = 18$.

We will start with the D_1-brane defined with the Dirichlet boundary conditions $x^2(\tau,\sigma)/_{\sigma=0} = x^2(\tau,\sigma)/_{\sigma=\pi}$ =const. It means that we will work with Neumann background fields

A_N^0 and A_N^1 and the Dirichlet background field A_D^2 and according to our convention we will have $p=1, a, b \in \{0,1\}$ and $i, j \in \{2\}$.

In terms of initial Neumann and Dirichlet fields we obtain

$$\mathcal{A}_\pm^M(\check{Z}_{\rm arg}) = \begin{pmatrix} \mp\frac{1}{G_E}A_0^N(V)+\kappa\theta A_1^N(V) \\ -\kappa\theta A_0^N(V)\pm\frac{1}{G_E}A_1^N(V) \\ -A_2^D(V) \\ A_0^N(x) \\ A_1^N(x) \\ \mp A_2^D(x) \end{pmatrix}, \tag{4.9}$$

where we used the second expression of Eq. (4.2).

4.2 T-dual theory in double space

On the other hand, for the T-dual case we have

$$\begin{aligned} {}^\star\mathcal{A}_\pm^M(\check{Z}_{\rm arg}) &= \begin{pmatrix} \mathcal{A}_{\pm\mu}(V) \\ {}^\star\mathcal{A}_\pm^\mu(x)\end{pmatrix} = \begin{pmatrix} 2\Pi_{\mp\mu\nu}{}^\star\mathcal{A}_\pm^\nu(V) \\ {}^\star\mathcal{A}_\pm^\mu(x)\end{pmatrix} \\ &= \begin{pmatrix} \mp{}^\star\mathcal{A}_\pm^0(V)+2B\,{}^\star\mathcal{A}_\pm^1(V) \\ -2B\,{}^\star\mathcal{A}_\pm^0(V)\pm{}^\star\mathcal{A}_\pm^1(V) \\ \pm{}^\star\mathcal{A}_\pm^2(V) \\ {}^\star\mathcal{A}_{\pm0}(x) \\ {}^\star\mathcal{A}_{\pm1}(x) \\ {}^\star\mathcal{A}_{\pm2}(x) \end{pmatrix}, \end{aligned} \tag{4.10}$$

or with the help of (2.27), in terms of T-dual Neumann and Dirichlet fields,

$${}^\star\mathcal{A}_\pm^M(\check{Z}_{\rm arg}) = \begin{pmatrix} G_E{}^\star A_D^0(V) \\ -G_E{}^\star A_D^1(V) \\ \pm{}^\star A_N^2(V) \\ \mp{}^\star A_D^0(x)-2B\,{}^\star A_D^1(x) \\ -2B\,{}^\star A_D^0(x)\mp{}^\star A_D^1(x) \\ {}^\star A_N^2(x) \end{pmatrix}. \tag{4.11}$$

Using the second equation (3.11), with the help of (4.9) and (4.11) we obtain

$$\begin{aligned} {}^\star\mathcal{A}_\pm^M({}^\star\check{Z}_{\rm arg}) &= \begin{pmatrix} G_E{}^\star A_D^0(x) \\ -G_E{}^\star A_D^1(x) \\ \pm{}^\star A_N^2(x) \\ \mp{}^\star A_D^0(V)-2B\,{}^\star A_D^1(V) \\ -2B\,{}^\star A_D^0(V)\mp{}^\star A_D^1(V) \\ {}^\star A_N^2(V) \end{pmatrix} \\ &\cong {}^\star\mathcal{T}\mathcal{A}_\pm(\check{Z}_{arg}) = \begin{pmatrix} A_0^N(x) \\ A_1^N(x) \\ \mp A_2^D(x) \\ \mp\frac{1}{G_E}A_0^N(V)+\kappa\theta A_1^N(V) \\ -\kappa\theta A_0^N(V)\pm\frac{1}{G_E}A_1^N(V) \\ -A_2^D(V) \end{pmatrix}, \end{aligned} \tag{4.12}$$

where for this example we have

$${}^\star\mathcal{T} = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix}. \tag{4.13}$$

Note that the transition from $\check{Z}_{\rm arg}$ to ${}^\star\check{Z}_{\rm arg}$ changes $x^\mu \leftrightarrow V^\mu$, while the operator ${}^\star\mathcal{T}$ exchanges the first three with the last three rows from Eq. (4.9). Equation (4.12) produces just the T-duality relations

$${}^\star A_D^0 = \frac{1}{G_E}A_0^N, \quad {}^\star A_D^1 = -\frac{1}{G_E}A_1^N, \quad {}^\star A_N^2 = -A_2^D, \tag{4.14}$$

in accordance with (2.23), (4.1) and (4.2).

The same relation can be obtained with the help of the compact notation which produces ${}^\star\mathcal{A}_\pm^\mu \cong \kappa\,\theta_\pm^{\mu\nu}\mathcal{A}_{\pm\nu}$, (see Eq. (3.12)). According to (2.11) and (2.27) we have, respectively,

$$\mathcal{A}_{\pm0} = A_0^N, \quad \mathcal{A}_{\pm1} = A_1^N, \quad \mathcal{A}_{\pm2} = \mp A_2^D, \tag{4.15}$$

and

$$\begin{aligned} &{}^\star\mathcal{A}_\pm^0 = \mp{}^\star A_D^0 - 2B\,{}^\star A_D^1, \\ &{}^\star\mathcal{A}_\pm^1 = -2B\,{}^\star A_D^0 \mp {}^\star A_D^1, \ {}^\star\mathcal{A}_\pm^2 = {}^\star A_N^2. \end{aligned} \tag{4.16}$$

Then Eq. (3.12) takes the form

$$\begin{aligned} \begin{pmatrix} \mp{}^\star A_D^0 - 2B\,{}^\star A_D^1 \\ -2B\,{}^\star A_D^0 \mp {}^\star A_D^1 \\ {}^\star A_N^2 \end{pmatrix} &= \begin{pmatrix} \mp\frac{1}{G_E} & \kappa\theta & 0 \\ -\kappa\theta & \pm\frac{1}{G_E} & 0 \\ 0 & 0 & \pm1 \end{pmatrix} \begin{pmatrix} A_0^N \\ A_1^N \\ \mp A_2^D \end{pmatrix} \\ &= \begin{pmatrix} \mp\frac{1}{G_E}A_0^N + \kappa\theta A_1^N \\ -\kappa\theta A_0^N \pm \frac{1}{G_E}A_1^N \\ -A_2^D \end{pmatrix}, \end{aligned} \tag{4.17}$$

which again produces Eq. (4.14).

4.3 Double space field strength

The structure of our example produces $\gamma_2^{(0)} = \kappa x'^2$ and the action (2.10) takes the form

$$\begin{aligned} S_{\rm open}[x] &= \kappa\int_\Sigma d^2\xi\,\partial_+x^\mu\Pi_{+\mu\nu}\partial_-x^\nu \\ &+2\kappa\int_{\partial\Sigma} d\tau\,(A_0^N[x]\dot{x}^0 + A_1^N[x]\dot{x}^1 + A_2^D[x]x'^2). \end{aligned} \tag{4.18}$$

Note the unusual coupling of the Dirichlet part A_2^D with x'^2.

According with (2.18) the non-trivial vector background fields are

$$\begin{aligned} &A_0^N(x) = A_0^0 - \frac{1}{2}F^{(a)}x^1, \qquad A_1^N(x) = A_1^0 + \frac{1}{2}F^{(a)}x^0, \\ &A_2^D(x) = A_2^0 - \frac{1}{4}F^{(s)}x^2, \end{aligned} \tag{4.19}$$

where $F^{(a)} \equiv F^{(a)}_{01} = \partial_0 A^N_1 - \partial_1 A^N_0$ and $F^{(s)} \equiv F^{(s)}_{22} = -4\,\partial_2 A^D_2$. Consequently, the field strength of the initial theory is

$$F_{\mu\nu} = F^{(a)}_{\mu\nu} + \frac{1}{2}F^{(s)}_{\mu\nu} = \begin{pmatrix} 0 & F^{(a)} & 0 \\ -F^{(a)} & 0 & 0 \\ 0 & 0 & \frac{1}{2}F^{(s)} \end{pmatrix}. \qquad (4.20)$$

Note the unusual expression and the unusual appearance of the symmetric field strength $F^{(s)}$.

5 Conclusions

In the present article we extend the interpretation of T-duality in double space to the case of an open string. This includes consideration of T-duality for the vector gauge fields.

In string theory the gauge fields appear at the boundary of the open string. Their role is to enable complete local gauge symmetries. In fact, there are two important symmetries of the closed string theory: the local gauge symmetry of the Kalb–Ramond field and general coordinate transformations. In Ref. [1] we showed that "restricted general coordinate transformations" (transformations of the background fields without transformations of the coordinates) are T-dual to the local gauge symmetry of the Kalb–Ramond field. Both symmetries fail at the open string end-points. The function of the gauge fields is to restore these symmetries at the string end-points.

For each symmetry of string theory there is an appropriate gauge field. As a consequence of the boundary conditions only parts of these gauge fields survive. From the gauge field corresponding to the local gauge symmetry of the Kalb–Ramond field the components along the coordinates with Neumann boundary conditions A^N_a survive. From the gauge field corresponding to restricted general coordinate transformations the components along the coordinates with Dirichlet boundary conditions A^D_i survive. So, the complete vector field is $A_\mu = \{A^N_a, A^D_i\}$.

It is well-known that x'^μ is T-dual to π_μ. In Ref. [1] was shown that it produces chain of T-dualities between restricted general coordinate transformation and local gauge transformations; and vector fields with Neumann A^N_a and Dirichlet boundary conditions A^D_i.

In the present article we showed that all the above results have a simple interpretation in double space. The double space contains $2D$ coordinates, D initial x^μ and corresponding D T-dual y_μ. The T-dualization of the present article (along all coordinates) corresponds to the replacement of all initial coordinates x^μ with all T-dual coordinates y_μ and all initial arguments of the background fields x^μ with all T-dual ones V^μ. Such an operation reproduces all results described above. So, in the open string case a complete set of T-duality transformations form the same subgroup of the $2D$ permutation group as in the closed string case.

Let us stress that there is an essential difference between our approach and that of double field theories (DFT) [19,20]. In DFT there are two coordinates: the initial x^μ and its double, denoted $\tilde{x}_\mu$. The variable $\tilde{x}_\mu$ corresponds to our y_μ but we have an additional dual coordinate $\tilde{y}_\mu$ defined in Eq. (2.26). For a discussion of the boundary conditions in DFT see Ref. [21].

Consequently, in the double space we are able to represent the backgrounds of all T-dual open string theories in a unified manner as well as in the cases of bosonic [2,3] and type II superstring theories [4].

This step is an important ingredient in better understanding M-theory. We already explained the role of the double space in the interpretation of T-duality and consequently in the attempt to construct M-theory [4,5]. The present article is an extension of these considerations to the case of an open string.

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