

Hamiltonian approach to GR – Part 2: covariant theory of quantum gravity

Claudio Cremaschini^{1,a}, Massimo Tassarotto^{2,3}

¹ Institute of Physics and Research Center for Theoretical Physics and Astrophysics, Faculty of Philosophy and Science, Silesian University in Opava, Bezručovo nám.13, 74601 Opava, Czech Republic

² Department of Mathematics and Geosciences, University of Trieste, Via Valerio 12, 34127 Trieste, Italy

³ Institute of Physics, Faculty of Philosophy and Science, Silesian University in Opava, Bezručovo nám.13, 74601 Opava, Czech Republic

Received: 6 January 2017 / Accepted: 21 April 2017 / Published online: 19 May 2017

© The Author(s) 2017. This article is an open access publication

Abstract A non-perturbative quantum field theory of General Relativity is presented which leads to a new realization of the theory of covariant quantum gravity (CQG-theory). The treatment is founded on the recently identified Hamiltonian structure associated with the classical space-time, i.e., the corresponding manifestly covariant Hamilton equations and the related Hamilton–Jacobi theory. The quantum Hamiltonian operator and the CQG-wave equation for the corresponding CQG-state and wave function are realized in 4-scalar form. The new quantum wave equation is shown to be equivalent to a set of quantum hydrodynamic equations which warrant the consistency with the classical GR Hamilton–Jacobi equation in the semiclassical limit. A perturbative approximation scheme is developed, which permits the adoption of the harmonic oscillator approximation for the treatment of the Hamiltonian potential. As an application of the theory, the stationary vacuum CQG-wave equation is studied, yielding a stationary equation for the CQG-state in terms of the 4-scalar invariant-energy eigenvalue associated with the corresponding approximate quantum Hamiltonian operator. The conditions for the existence of a discrete invariant-energy spectrum are pointed out. This yields a possible estimate for the graviton mass together with a new interpretation about the quantum origin of the cosmological constant.

1 Introduction

This paper is part of a research project on the foundations of classical and quantum gravity. Following the theoretical premises presented in Ref. [1] (herein referred to as Part 1), in this paper the axiomatic setting of the theory of manifestly

covariant quantum gravity is established for the standard formulation of general relativity (briefly SF-GR), namely the Einstein field equations and the corresponding treatment of the gravitational field [2–5]. The new quantum theory, based on the introduction of massive gravitons and constructed in such a way to be consistent with SF-GR in the semiclassical limit, will be referred to here as the theory of covariant quantum gravity (CQG) or briefly *CQG-theory*.

Distinctive features of CQG-theory presented here are that, just like the CCG-theory (i.e., the theory of covariant classical gravity) developed in Part 1, realizing a canonical quantization approach for SF-GR, it satisfies the principles of general covariance and manifest covariance. This means that—in comparison with customary literature canonical quantization approaches to SF-GR [6,7]—the theory proposed here preserves its form under arbitrary local point transformations. As a consequence, in particular, it does not rely on the adoption of space-time coordinates involving $3+1$ or $2+2$ foliation schemes [6,8–12] (see also the related discussion in Part 1).

In addition, consistent with the same principles, first it is based on the adoption of 4-tensor continuum Lagrangian coordinates and canonical momentum operators and a manifestly covariant quantum wave equation referred to here as *CQG-wave equation*. Second, the same quantum wave equation satisfies the quantum unitarity principle and admits a closed set of equivalent quantum hydrodynamic equations. Third, its formulation is of general validity, i.e., it applies to arbitrary possible realizations of the underlying classical space-time.

The goal of the present paper is also to display its non-perturbative character which, nevertheless, allows for the development of perturbative approximation schemes for the analytical evaluation of quantum solutions of physical interest. The latter should include, in principle, also the inves-

^a e-mail: claudiocremaschini@gmail.com

tigation of particular quantum solutions which can be considered suitably “close”, i.e., localized, with respect to the background classical space-time field tensor.

The theoretical framework is provided by Part 1 where a realization inspired by the De Donder–Weyl manifestly covariant approach [13–15] was reached for the covariant Hamiltonian structure of SF-GR $\{x_R, H_R\}$ based on CCG-theory. Its crucial feature is that of relying upon the adoption of a new kind of variational approach called the *synchronous variational principle* earlier developed in Refs. [16,17]. According to the notations adopted in Part 1, this involves the introduction of a variational set $\{x_R, H_R\}$, referred to as the classical GR-Hamiltonian structure, formed, respectively, by a suitable reduced-dimensional classical canonical state x_R and a corresponding manifestly covariant Hamiltonian density H_R . More precisely, the variational canonical state x_R is identified with the set $x_R = \{g, \pi\}$, with $g \equiv \{g_{\mu\nu}\}$ and $\pi \equiv \{\pi_{\mu\nu}\}$, respectively, representing the continuum Lagrangian coordinate and conjugate canonical momentum, both realized by means of second-order 4-tensors. Instead, consistent with Ref. [17] and in formal analogy with the customary symbolic representation holding in relativistic particle dynamics, the Hamiltonian density H_R is taken to be of the form

$$H_R \equiv T_R + V, \quad (1)$$

with T_R and V denoting suitable 4-scalar effective “kinetic” and “potential” densities. As shown in Part 1, these fields, together with the state x_R , are then prescribed in such a way to provide an appropriate Hamiltonian variational formulation of the Einstein field equations, i.e., to yield a corresponding equivalent set of continuum Hamilton equations.

A characteristic physical requirement of the resulting Hamiltonian theory is that it should satisfy the *general covariance principle* (GCP) with respect to local point transformations [18] as well as its more restrictive manifest covariant form, namely the *principle of manifest covariance* (PMC). Accordingly it should always be possible to represent in 4-tensor form all the relevant field variables and operators, including the variational functional, the corresponding Lagrangian and Hamiltonian densities and operators, the canonical variables as well their variations and the Euler–Lagrange equations.

The prescription of the covariant Hamiltonian structure $\{x_R, H_R\}$ consistent with these properties and the construction of the corresponding manifestly covariant Hamilton–Jacobi theory for SF-GR developed in Part 1 are *mandatory physical prerequisites* for the establishment of the CQG-theory.

Meeting these physical requirements appears a priori as a difficult task despite the huge number of contributions to be found in the past literature and dealing with quan-

tum gravity. In fact, one has to notice that a common difficulty met by many of previous non-perturbative Hamiltonian approaches to GR is that they are based on the adoption of variational Lagrangian densities, Lagrangian coordinates and/or momenta which have a non-manifestly covariant character, i.e., they are not 4-scalars or 4-tensors. Incidentally, a strategy of this type is intrinsic in the construction of the original Einstein’s variational formulation for his namesake field equations which is based on the Einstein–Hilbert asynchronous variational principle (see related discussion in Ref. [16]). A choice of analogous type, for example, is typical of Dirac’s Hamiltonian approach to GR, which is based on the so-called Dirac constrained dynamics [19–23]. Its key principle, in fact, is that of singling out the “time” component of the 4-position in terms of which the generalized velocity is identified with $g_{\mu\nu,0}$. This lead one to identify the canonical momentum in terms of the manifestly non-tensorial quantity $\pi_{\text{Dirac}}^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{EH}}}{\partial g_{\mu\nu,0}}$, where \mathcal{L}_{EH} is the Einstein–Hilbert variational Lagrangian density. Such a choice corresponds to selecting a particular subset of GR-frames.

In the cases indicated above the possibility is prevented of establishing at the classical level a Hamiltonian theory of SF-GR in which the Euler–Lagrange equations (in particular the Hamilton equations) are manifestly covariant. On the other hand, the conjecture that a manifestly covariant Hamiltonian formulation must be possible for continuum systems is also suggested by the analogous theory holding for discrete classical particle systems. Indeed, its validity is fundamentally implied by the state-of-the-art theory of classical N -body systems subject to non-local electromagnetic (EM) interactions. The issue is exemplified by the Hamiltonian structure of the EM radiation-reaction problem in the case of classical extended particles as well as N -body EM interactions among particles of this type [24–28]. Nevertheless, it must be mentioned that in the case of continuum fields, the appropriate formalism is actually well established, being provided by the Weyl–De Donder Lagrangian and Hamiltonian treatments [13–15]. The need to adopt an analogous approach also in the context of classical GR, and in particular for the Einstein equation itself or its possible modifications, has been recognized before [29–33]. The fulfillment of the physical prerequisites indicated above in the context of a classical treatment of SF-GR and the definition of the related conceptual framework for GR has been provided recently by Part 1 and Refs. [16,17].

The viewpoint adopted in this paper for the development of the new approach to the covariant quantum gravity has analogies with the *first-quantization* approach developed in Ref. [34]. This pertains to the relativistic quantum theory of an extended charged particle subject to EM self-interaction, the so-called EM radiation reaction, and immersed in a flat Minkowski space-time. In fact, as shown in the same reference, the appropriate relativistic quantum wave equa-

tion advancing the quantum state of the same particle was achieved by the preliminary construction of the manifestly covariant classical Hamilton and Hamilton–Jacobi equation for the classical dynamical system [35]. Furthermore a classical treatment of external and self-interactions acting on the quantum system was utilized, both being expressed in terms of deterministic force fields, so as to leave the development of quantum theory only for the particle state dynamics.

As shown here an analogous procedure can be adopted in the case of covariant quantum gravity in order to achieve the proper form of the relativistic quantum wave equation corresponding to SF-GR. More precisely, consistent with the setting required by the adoption of synchronous variational principle, a so-called *background space-time picture* is adopted, so that the underlying classical space-time geometry $(Q^4, \widehat{g}(r))$ is considered as prescribed, in the sense that when it is parametrized in terms of arbitrary curvilinear coordinates $r \equiv \{r^\mu\}$ its metric tensor $\widehat{g}_{\mu\nu}(r)$ is regarded as a *prescribed classical field*, eventually to be identified with the metric tensor of the physical space-time. Such a choice is of crucial importance since it permits one to recover “habitual physical notions such as causality, time, scattering states, and black holes” [7].

For this purpose, the construction of the covariant quantum wave equation for the gravitational field reached in this paper is based on the classical GR-Hamilton–Jacobi equation reported in Part 1, in which the *prescribed field* $\widehat{g}_{\mu\nu}(r)$ is assumed to realize a particular smooth solution of the Einstein field equations and to determine in this way the *geometric structure* associated with the background space-time $(Q^4, \widehat{g}(r))$. Hence, $\widehat{g}_{\mu\nu}(r)$ establishes the tensor transformation laws by raising and lowering indices of all tensor quantities, together with the prescribed values of the invariant space-time volume element as well as of the standard connections (Christoffel symbols), covariant derivatives and Ricci tensor. It must be stressed that, despite the adoption of the prescribed field $\widehat{g}_{\mu\nu}(r)$ in the construction of the quantum theory of gravity reported here, the whole treatment remains a priori *exact*, i.e., non-asymptotic in character, and at the same time *background independent*, since the theory does not rely on a particular realization of $\widehat{g}_{\mu\nu}(r)$, which can be any solution of the Einstein field equations.

In the framework of quantum theory the prescription of the background geometric structure has also formal conceptual analogies with the so-called induced gravity (or emergent gravity), namely the conjecture that the geometrical properties of space-time reveal themselves as a suitable mean field description of microscopic stochastic or quantum degrees of freedom underlying the classical solution. In the present approach this is achieved by introducing in the Lagrangian and Hamiltonian operators themselves the notion of prescribed metric tensor $\widehat{g}_{\mu\nu}(r)$ which is held constant under the action of all the quantum operators and has therefore to

be distinguished from the quantum field $g_{\mu\nu}$. As a result, the classical variational field $g_{\mu\nu}$ is now interpreted as a quantum observable. The ensemble spanned by all possible values of $g_{\mu\nu}$ determines the configuration space U_g with respect to which the quantum-gravity state has to be prescribed, so that U_g can be identified with the real vector space $U_g \equiv \mathbb{R}^{16}$ (or $U_g \equiv \mathbb{R}^{10}$ if the quantum field $g_{\mu\nu}$ is assumed symmetric).

Taking into account these considerations the work-plan of the paper is as follows. In Sect. 2, the principles of the axiomatic formulation of manifestly covariant quantum gravity corresponding to the reduced-dimensional GR-Hamiltonian structure earlier reported in Ref. [1] are discussed, with the aim of addressing in particular the following Axioms of CQG:

CQG - Axiom 1: prescription of the quantum-gravity state (CQG-state) ψ , to be assumed a 4-scalar complex function of the form $\psi = \psi(g, \widehat{g}, s)$, with $g, \widehat{g} \in U_g \subseteq \mathbb{R}^{16}$ and $s \in I \equiv \mathbb{R}$, with ψ spanning a Hilbert space Γ_ψ , i.e., a finite-dimensional linear vector space endowed with a scalar product to be properly prescribed. Here s denotes the proper time along an arbitrary background geodetics, i.e., prescribed requiring that the line element ds satisfies the differential identity $ds^2 = \widehat{g}_{\mu\nu}(r(s))dr^\mu dr^\nu$, with dr^μ being the tangent displacement performed along the same geodetics.

CQG - Axiom 2: prescription of the expectation values of the quantum observables and of the related quantum probability density function (CQG-PDF).

CQG - Axiom 3: formulation of the quantum correspondence principle for the GR-Hamiltonian structure. This includes the prescription of the form of the quantum Hamiltonian operator generating the proper-time evolution of the CQG-state.

In Sect. 3 the problem is posed of the prescription of a quantum wave equation (CQG-wave equation), namely the formulation of the quantum wave equation advancing in proper time the same CQG-state (*CQG - Axiom 4*). As for the classical theory developed in Part 1, the covariant quantization of the gravitational field reached here is realized in a 4-dimensional space-time. Then it is shown that, upon introducing a Madelung representation for the quantum wave function and invoking validity of the quantum unitarity principle, the CQG-wave equation is equivalent to a couple of quantum hydrodynamic equations identified, respectively, with the continuity and quantum Hamilton–Jacobi equations (*CQG-Axiom 5*). In particular, given the validity of the semiclassical limit, the CQG-wave equation is proved to recover the classical Hamilton–Jacobi equation reported in Part 1, thus warranting the conceptual consistency between the two descriptions. In Sect. 4 the development of a perturbative approach to CQG-theory starting from the exact quantum representation is presented, a feature that allows for the implementation of the harmonic oscillator approximation for the analytical treatment of the quantum Hamiltonian potential.

A number of selected applications of CQG-theory are considered. In particular, in Sect. 5 the investigation of the stationary CQG-wave equation holding in the case of vacuum and subject to the assumption of having a non-vanishing cosmological constant is treated. Then, in Sect. 6 the proof of the existence of a discrete spectrum for the energy eigenvalues associated with the same vacuum CQG-wave equation is obtained. In Sect. 7, based on the identification of the minimum energy eigenvalue of the discrete spectrum for the vacuum CQG-wave equation, the quantum prescription of the rest mass m_o as well as of the corresponding characteristic scale length $L(m_o)$ entering the CQG-theory are discussed. This provides an estimate for the ground-state graviton mass of the vacuum solution, which is proved to be strictly positive under the same physical prescriptions, which warrant the existence of a discrete energy spectrum. At the same time it yields a new interpretation of the quantum origin of the cosmological constant, which is shown to be related to the Compton wavelength of the ground-state oscillation mode of the quantum of the gravitational field. In Sect. 8, the main differences and comparisons with the two main existing categories of literature with approaches to quantum gravity, which are usually referred to as canonical and covariant quantization approaches, respectively, are pointed out. Finally, in Sect. 9 the main conclusions and a summary of the investigation are presented.

2 Axiomatic foundations of CQG—axioms 1–3

In this section we start addressing the axiomatic formulation of CQG which is consistent with the physical prerequisites indicated above. In particular here the axioms are provided which permit to prescribe the key physical properties of the *quantum GR-Hamiltonian system* associated with the GR-Hamiltonian structure $\{x_R, H_R\}$, or equivalently its dimensional-normalized representation $\{\bar{x}_R, \bar{H}_R\}$, both specified in Part 1. These include:

- The functional setting of the quantum-gravity state (CQG-state) ψ .
- The definition of quantum expectation values and observables.
- The correspondence principle between the classical and quantum GR-Hamiltonian systems, to be established in terms of a mapping between the classical canonical state and the related classical GR-Hamiltonian density with the corresponding quantum observables.

2.1 The Hilbert space of the CQG-state ψ

The first axiom concerns the prescription of the *CQG-state* and its corresponding functional space to be identified with a Hilbert space.

CQG-Axiom 1—The Hilbert space of the CQG-state ψ

The CQG-state is identified with a single 4-scalar and complex function $\psi(s)$ (CQG-wave function) of the form

$$\psi(s) \equiv \psi(g, \widehat{g}(r), r(s), s). \tag{2}$$

This can be associated with a corresponding spin-2 quantum particle having a strictly positive invariant rest mass m_o . In fact, in the context of a first-quantization approach developed here, $\psi(s)$ can always be identified with the tensor product of the form $\psi(s) = \widehat{g}_{\mu\nu} \psi^{\mu\nu}$. Regarding the functional setting of $\psi(s)$ here it is assumed that:

(A₁) $\psi(s)$ is taken to depend smoothly on the tensor field $g \equiv \{g_{\mu\nu}\}$ spanning the configurations space U_g and in addition to admit a Lagrangian path (LP) parametrization in terms of the geodetics $r(s) \equiv \{r^\mu(s)\}$ associated locally with the prescribed classical tensor field $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$. It means that it may be smoothly dependent on the prescribed field in terms of the parametrization $\widehat{g}(r(s)) \equiv \{\widehat{g}_{\mu\nu}(r(s))\}$, on the s -parametrized geodetics $r(s) \equiv \{r^\mu(s)\}$ and on the classical proper time s associated with the same geodetics.

(B₁) The functions ψ defined by Eq. (2) span a Hilbert space Γ_ψ , i.e., a finite-dimensional linear vector space endowed with the scalar product

$$\langle \psi_a | \psi_b \rangle \equiv \int_{U_g} d(g) \psi_a^*(g, \widehat{g}(r), r(s), s) \psi_b(g, \widehat{g}(r), r(s), s), \tag{3}$$

with $d(g) \equiv \prod_{\mu,\nu=1,4} dg_{\mu\nu}$ denoting the canonical measure on U_g and $\psi_{a,b}(s) \equiv \psi_{a,b}(g, \widehat{g}(r), r(s), s)$ being arbitrary elements of the Hilbert space Γ_ψ , where as usual ψ_a^* denotes the complex conjugate of ψ_a .

(C₁) The real function $\rho(s) \equiv \rho(g, \widehat{g}(r), r(s), s)$ prescribed as

$$\rho(s) \equiv |\psi(s)|^2 \tag{4}$$

identifies on the configuration space U_g the quantum probability density function (CQG-PDF) associated with the CQG-state. Here by assumption $\rho(s)$ is the probability density of $g \equiv \{g_{\mu\nu}\}$ in the volume element $d(g)$ belonging to the configuration space U_g . Thus, if L_g is an arbitrary subset of U_g , its probability is defined as

$$P(L_g) = \int_{U_g} d(g)\rho(s)\Theta(L_g), \tag{5}$$

with $\Theta(L_g)$ denoting the characteristic function of L_g , namely such that $\Theta(L_g) = 1, 0$ if, respectively, $g \equiv \{g_{\mu\nu}\} \in U_g$ does or does not belong to L_g . In addition, by assumption the normalization

$$P(U_g) \equiv \langle \psi | \psi \rangle \equiv \int_{U_g} d(g)\rho(s) = 1 \tag{6}$$

is assumed to hold identically for arbitrary $(\widehat{g}(r), r(s), s)$.

(D₁) The real function $S^{(q)}(s) \equiv S^{(q)}(g, \widehat{g}(r), r(s), s)$ defined as

$$S^{(q)}(s) \equiv \arcsin h \left\{ \frac{\psi(s) - \psi^*(s)}{2\rho(s)} \right\} \tag{7}$$

identifies on the configuration space U_g the quantum phase-function associated with the same CQG-state $\psi(s)$.

In Eqs. (2), (3) and (4) the notations are as follows. First, $g \equiv \{g_{\mu\nu}\}$ is the continuum Lagrangian coordinate spanning the configuration space $U_g \subseteq \mathbb{R}^{16}$. Second, $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$ is the classical deterministic 4-tensor which for an arbitrary coordinate parametrization $r \equiv \{r^\mu\}$ identifies the prescribed metric tensor of the background space-time $(\mathbf{Q}^4, \widehat{g}(r))$, which lowers (and raises) the tensor indices of all tensor fields. Third, $r(s) \equiv \{r^\mu\}$ is the 4-position belonging to the local geodesics associated with the prescribed metric tensor $\widehat{g}(r)$ such that, for an arbitrary $r \in (\mathbf{Q}^4, \widehat{g}(r))$ locally occurs that $r \equiv r(s)$. Fourth, s is the proper time on the same local geodesics $\{r(s)\}$, which spans the time axis $I \equiv \mathbb{R}$.

2.2 Expectation values and observables

The second axiom deals with the prescription of the expectation values of tensor functions and CQG-observables, namely configuration-space 4-tensor functions or 4-tensor operators whose expectation values are expressed in terms of real tensor functions.

CQG-Axiom 2—Prescription of CQG-expectation values and CQG-observables

Given an arbitrary tensor function or local tensor operator $X(s) \equiv X(g, \widehat{g}(r), r(s), s)$, which acts on an arbitrary wave function $\psi(s) \equiv \psi(g, \widehat{g}(r), r(s), s)$ of the Hilbert space Γ_ψ , the weighted integral

$$\langle \psi | X \psi \rangle \equiv \int_{U_g} d(g)\psi^*(s)X(s)\psi(s), \tag{8}$$

which is assumed to exist, is called the CQG-expectation value of X . Then by construction $\langle \psi | X \psi \rangle$ is a 4-tensor field, generally of the form

$$\langle \psi | X \psi \rangle = G_X(\widehat{g}(r), r(s), s). \tag{9}$$

In the particular case in which $\langle \psi | X \psi \rangle$ is real, namely

$$\langle \psi | X \psi \rangle = \langle X^* \psi | \psi \rangle \equiv \int_{U_g} d(g)\psi(s)X^*(s)\psi^*(s), \tag{10}$$

with $X^*(s)$ denoting the complex conjugate of $X(s)$, then X identifies a CQG-observable.

The trivial example of observable is provided by the identification $X \equiv 1$. The normalization condition (6) then necessarily yields in such a case $G_X = 1$. Other examples of CQG-observables include:

- (A) The 4-tensor function $X \equiv g_{\mu\nu}$. Then, for all $r \equiv r(s) \in (\mathbf{Q}^4, \widehat{g}(r))$ the integral

$$\begin{aligned} \langle \psi | g_{\mu\nu} \psi \rangle &= \int_{U_g} d(g)g_{\mu\nu}\rho(g, \widehat{g}(r), r(s), s) \\ &= \widetilde{g}_{\mu\nu}(\widehat{g}(r), r(s), s) \end{aligned} \tag{11}$$

identifies the CQG-expectation value of $g_{\mu\nu}$ at $r \equiv r(s)$. Here $\widetilde{g}_{\mu\nu}(s) \equiv \widetilde{g}_{\mu\nu}(\widehat{g}(r), r(s), s)$ is by construction a real tensor field to be considered generally different from the prescribed metric tensor $\widehat{g}(r) (\equiv \widehat{g}(s)) \equiv \{\widehat{g}_{\mu\nu}(r(s))\}$, while $r \equiv r(s) \equiv \{r^\mu(s)\}$ is again the 4-position of the background space-time $(\mathbf{Q}^4, \widehat{g}(r))$. Thus, letting $\delta g_{\mu\nu}(r) = \widehat{g}_{\mu\nu}(r) - \widetilde{g}_{\mu\nu}(r)$ it follows that the CQG-expectation value of the tensor function $X \equiv g_{\mu\nu} + \delta g_{\mu\nu}(r)$ is just

$$\begin{aligned} \langle \psi | (g_{\mu\nu} + \delta g_{\mu\nu}(r)) \psi \rangle &= \int_{U_g} d(g)(g_{\mu\nu} + \delta g_{\mu\nu}(r)) \\ &\times \rho(g, \widehat{g}(r), r(s), s) = \widehat{g}_{\mu\nu}(r), \end{aligned} \tag{12}$$

i.e., it coincides identically with the deterministic classical metric tensor which at the 4-position $r \equiv r(s)$ is associated with $(\mathbf{Q}^4, \widehat{g}(r))$. In the light of the classical theory developed in Part 1, the quantum expectation values provided by Eqs. (11) and (12), i.e., respectively, $\widetilde{g}_{\mu\nu}(\widehat{g}(r), r(s), s)$ and $\widehat{g}_{\mu\nu}(r)$ should be suitably related. This point will be discussed elsewhere.

- (B) The CQG-expectation value of the *momentum CQG-operator* $\pi^{(q)\mu\nu} \equiv -i\hbar \frac{\partial}{\partial g_{\mu\nu}}$ is prescribed in such a way that the integral

$$\begin{aligned} \langle \psi | \pi^{(q)\mu\nu} \psi \rangle &= \int_{U_g} d(g) \psi^*(g, \widehat{g}(r), r(s), s) \\ &\times \left(-i\hbar \frac{\partial}{\partial g_{\mu\nu}} \right) \psi^*(g, \widehat{g}(r), r(s), s) \\ &\equiv \widetilde{\pi}^{(q)\mu\nu}(\widehat{g}(r), r(s), s) \end{aligned} \tag{13}$$

always exists, with $\widetilde{\pi}^{(q)\mu\nu}$ being a real tensor field. Thus the CQG-operator $\pi^{(q)\mu\nu}$ is necessarily a CQG-observable. Then, introducing the CQG-operator $T \equiv \pi^{(q)\mu\nu} \pi_{\mu\nu}^{(q)}$, its CQG-expectation value is just

$$\begin{aligned} \langle \psi | T \psi \rangle &= \int_{U_g} d(g) \psi^*(g, \widehat{g}(r), r(s), s) T \psi^* \\ &\times (g, \widehat{g}(r), r(s), s) \\ &\equiv \widetilde{T}(\widehat{g}(r), r(s), s), \end{aligned} \tag{14}$$

which is assumed to exist, with \widetilde{T} manifestly identifying a real scalar field. Therefore the CQG-operator T is necessarily a CQG-observable too.

2.3 Prescription of the CQG-correspondence principle

The classical dimensionally normalized Hamiltonian structure $\{\bar{x}_R, \bar{H}_R\}$ determined in Part 1 is defined in terms of the canonical state $\bar{x}_R \equiv \{\bar{g}_{\mu\nu}, \bar{\pi}_{\mu\nu}\}$ and the Hamiltonian \bar{H}_R . More precisely, $\bar{g}_{\mu\nu} \equiv g_{\mu\nu}$ and $\bar{\pi}_{\mu\nu} = \frac{\alpha L}{\kappa} \pi_{\mu\nu}$ is the normalized conjugate momentum, where $\kappa = \frac{c^3}{16\pi G}$, L is a 4-scalar scale length and α is a suitable dimensional 4-scalar, both to be defined below. Instead, \bar{H}_R is defined as the real 4-scalar field

$$\bar{H}_R(\bar{x}_R, \widehat{g}, r, s) = \bar{T}_R(\bar{g}, \widehat{g}, r, s) + \bar{V}(\bar{g}, \widehat{g}, r, s), \tag{15}$$

with $\bar{T}_R(\bar{g}, \widehat{g}, r, s) \equiv \frac{1}{f(h)} \frac{\bar{\pi}^{\mu\nu} \bar{\pi}_{\mu\nu}}{2\alpha L}$ and $\bar{V}(\bar{g}, \widehat{g}, r, s) \equiv \sigma \bar{V}_o(\bar{g}, \widehat{g}, r, s) + \sigma \bar{V}_F(\bar{g}, \widehat{g}, r, s)$ being the normalized effective kinetic and potential densities. Here $\bar{V}_o \equiv h\alpha L [g^{\mu\nu} \widehat{R}_{\mu\nu} - 2\Lambda]$ and $V_F \equiv \frac{h\alpha L}{2k} L_F$ represent, respectively, the vacuum and external field contributions (see the definitions in Part 1), with $\widehat{R}_{\mu\nu}$ being the background Ricci tensor and Λ being the cosmological constant. Finally, $f(h)$ and $\sigma = \pm 1$ are suitable *multiplicative gauges*, i.e., real 4-scalar fields which remain in principle still arbitrary at the classical level, where $h = (2 - \frac{1}{4} g^{\alpha\beta}(r) g_{\alpha\beta}(r))$ and $f(h)$ satisfies by construction the constraint $f(\widehat{g}(r)) = 1$.

Given these premises, we can now introduce the core canonical quantization rules in the context of CQG-theory. These are based on a suitable correspondence principle between the classical and the relevant quantum functions and operators. This is given as follows.

CQG-Axiom 3—CQG-correspondence principle for the GR-Hamiltonian structure.

Given the classical GR-Hamiltonian structure $\{\bar{x}_R, \bar{H}_R\}$, the CQG-correspondence principle is realized by the map

$$\begin{cases} \bar{g}_{\mu\nu} \equiv g_{\mu\nu} \rightarrow g_{\mu\nu}^{(q)} \equiv g_{\mu\nu}, \\ \bar{\pi}_{\mu\nu} \rightarrow \pi_{\mu\nu}^{(q)} \equiv -i\hbar \frac{\partial}{\partial g^{\mu\nu}}, \\ \bar{H}_R \rightarrow \bar{H}_R^{(q)} = \frac{1}{f(h)} \bar{T}_R^{(q)}(\bar{\pi}) + \bar{V}, \end{cases} \tag{16}$$

where $\bar{H}_R^{(q)}$ is the CQG-Hamiltonian operator, $x^{(q)} \equiv \{g_{\mu\nu}^{(q)}, \pi_{\mu\nu}^{(q)}\}$ is the quantum canonical state and $\pi_{\mu\nu}^{(q)}$ is the quantum momentum operator prescribed so that the commutator $[g_{\mu\nu}^{(q)}, \pi^{(q)\alpha\beta}] = i\hbar \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta}$ exactly. In addition, $\bar{T}_R^{(q)}(\bar{\pi})$ is the kinetic density quantum operator

$$\bar{T}_R^{(q)}(\bar{\pi}) = \frac{\pi^{(q)\mu\nu} \pi_{\mu\nu}^{(q)}}{2\alpha L} \equiv \frac{T}{2\alpha L}, \tag{17}$$

where $T \equiv \pi^{(q)\mu\nu} \pi_{\mu\nu}^{(q)}$ is the 4-scalar operator introduced above (see Eq. (14)).

Hence, Eq. (16) mutually map in each other, respectively, the classical canonical state \bar{x}_R and the GR-Hamiltonian density \bar{H}_R onto the corresponding quantum variables/operators $x^{(q)}$ and $\bar{H}_R^{(q)}$. Given the prescriptions (16), key consequence (of CQG-Axiom 3) is therefore the validity of the canonical commutation rule at the basis of the canonical quantization formalism of CQG theory, namely

$$[\pi_{\mu\nu}^{(q)}, g_{\alpha\beta}^{(q)}] = -i\hbar \delta_{\mu\alpha} \delta_{\nu\beta}. \tag{18}$$

It is worth pointing out that the same correspondence principle (16) also prescribes the gauge properties of the quantum Hamiltonian operator $\bar{H}_R^{(q)}$ and canonical momentum $\pi_{\mu\nu}^{(q)}$ provided by gauge transformations of the corresponding classical fields \bar{H}_R and $\bar{\pi}_{\mu\nu}$ given in Part 1. In particular, this means that $\bar{H}_R^{(q)}$ and the canonical momentum $\pi_{\mu\nu}^{(q)}$ are endowed, respectively, with the gauge transformations

$$\begin{cases} \pi_{\mu\nu}^{(q)} \rightarrow \pi'^{(q)}_{\mu\nu} = f(h) \pi_{\mu\nu}^{(q)}, \\ \bar{T}_R^{(q)}(\bar{\pi}) \rightarrow \frac{1}{f(h)} \bar{T}_R^{(q)}(\bar{\pi}), \end{cases} \tag{19}$$

where $f(h)$ denotes in principle an arbitrary, non-vanishing and smoothly differentiable 4-scalar function, whose precise value is determined below by requiring validity of quantum hydrodynamic equations in conservative form.

3 Axiomatic foundations of CQG—axioms 4 and 5

In this section additional constitutive aspects of CQG are formulated which concern the prescription of:

- The generic form of the CQG-wave equation advancing in proper time ψ itself (CQG-Axiom 4).
- The realization of the corresponding quantum hydrodynamic equations in conservative form (CQG-Axiom 5).

3.1 The quantum-gravity wave equation

We first introduce the quantum wave equation which in the framework of CQG realizes an evolution equation for the quantum state $\psi(s)$. This is provided by the following axiom.

CQG-Axiom 4—Prescription of the CQG-wave equation for ψ

The evolution equation advancing in time the CQG-state $\psi(s)$ is assumed to be provided by the CQG-quantum wave equation (CQG-QWE)

$$i\hbar \frac{\partial}{\partial s} \psi(s) + [\psi(s), \overline{H}_R^{(q)}] = 0, \tag{20}$$

where $[A, B] \equiv AB - BA$ denotes the quantum commutator, i.e.,

$$[\psi(s), \overline{H}_R^{(q)}] \equiv -\overline{H}_R^{(q)} \psi(s). \tag{21}$$

The CQG-wave equation (20) uniquely prescribes the evolution of the quantum state $\psi(s)$ along the geodetics of the prescribed metric tensor $\widehat{g}_{\mu\nu}(r)$, which is associated with the background curved space-time $(\mathbf{Q}^4, \widehat{g}_{\mu\nu}(r))$. Equation (20) is a first order partial differential equation, to be supplemented by suitable initial conditions, namely prescribing for all $r(s_o) = r_o \in (\mathbf{Q}^4, \widehat{g}(r))$ the condition $\psi(s_o) = \psi_o(g, \widehat{g}(r_o), r_o)$, as well as boundary conditions at infinity on the improper boundary of configuration space U_g , i.e., letting $\lim_{g \rightarrow \infty} \psi(g, \widehat{g}(r), r(s), s) = 0$.

Nevertheless, $\psi(s)$ and in particular $\psi(s_o)$ are manifestly non-unique. This is due to the gauge property indicated above (see Eq. (19)) which characterizes the Hamiltonian and canonical momentum operators $\overline{H}_R^{(q)}$ and $\pi_{\mu\nu}^{(q)}$. However, in this regard, a potential consistency issue arises for Axiom 4. More precisely, this refers to the compatibility of Eq. (20) with the normalization condition (6) earlier set at the basis of Axiom 1. In fact it is not obvious whether Eq. (6) may actually hold identically for all $s \in I \equiv \mathbb{R}$ for arbitrary choices of the Hamiltonian operator $\overline{H}_R^{(q)}$ and in particular arbitrary choices of the undetermined function $f(h)$ (see the rhs of the third equation in Eq. (16)). In fact, Eq. (6) demands in such a case also the validity of the additional requirement

$$\frac{\partial}{\partial s} \int_{U_g} d(g) \rho(s) \equiv \int_{U_g} d(g) \frac{\partial}{\partial s} \rho(s) \equiv 0 \tag{22}$$

to hold identically for all $s \in I$. The issue will be addressed in Sect. 3.2.

3.2 The quantum hydrodynamic equations

Let us now investigate whether and under which conditions the CQG-wave equation introduced in Axiom 4 (see Eq. (20)) may be equivalent to a prescribed set of quantum hydrodynamic equations (QHE) written in conservative form, i.e., in such a way to conserve quantum probability. In fact, in analogy with the Schroedinger equation and the generalized Klein–Gordon equation reported in Ref. [34] for the radiation-reaction problem, the QHE should be realized, respectively, by: (a) a continuity equation for the quantum PDF $\rho(s)$; (b) a quantum Hamilton–Jacobi equation for the quantum phase-function $S^{(q)}(s) = S^{(q)}(g, \widehat{g}(r), r(s), s)$. We remark that the derivation of QHE is required since it provides a theoretical framework for the physical prescription of the gauge indeterminacy on $f(h)$ characterizing the CQG-wave equation and the logical consistency of the CQG-theory with the classical Hamilton–Jacobi equation determined in Part I.

We notice preliminarily that the CQG-state defined by the complex function $\psi(s)$ (see Eq. (2)) can always be cast in the form of an exponential representation of the type realized by the Madelung representation as

$$\psi(s) = \sqrt{\rho(s)} \exp \left\{ \frac{i}{\hbar} S^{(q)}(s) \right\}, \tag{23}$$

$\rho(s)$ and $S^{(q)}(s)$ being the real 4-scalar field functions prescribed, respectively, by Eqs. (4) and (7). The following additional axiom is then introduced.

CQG-Axiom 5—Quantum hydrodynamics equations

Given the validity of the Madelung representation (23) in terms of the Hamiltonian operator $\overline{H}_R^{(q)}$, provided the constraint condition

$$f(h) = 1 \tag{24}$$

is fulfilled in order to satisfy the quantum unitarity principle, the CQG-wave (20) is equivalent to the following set of real PDEs:

$$\frac{\partial \rho(s)}{\partial s} + \frac{\partial}{\partial g_{\mu\nu}} (\rho(s) V_{\mu\nu}(s)) = 0, \tag{25}$$

$$\frac{\partial S^{(q)}(s)}{\partial s} + \overline{H}_c^{(q)} = 0, \tag{26}$$

referred to here as CQG-quantum continuity equation and CQG-quantum Hamilton–Jacobi equation advancing in proper time, respectively, $\rho(s)$ and $S^{(q)}(s)$. Here the notation is as follows. The quantum hydrodynamics fields $\rho(s) \equiv \rho(g, \widehat{g}, s)$ and $S^{(q)}(s) \equiv S^{(q)}(g, \widehat{g}, s)$ are assumed to depend smoothly on the tensor field $g \equiv \{g_{\mu\nu}\}$ spanning the configurations space U_g and in addition to admit a LP-parametrization in terms of the geodetics $r(s) \equiv \{r^\mu(s)\}$

associated locally with the prescribed field $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$. The 4-tensor $V_{\mu\nu}(s)$ is prescribed as $V_{\mu\nu}(s) = \frac{1}{\alpha L} \frac{\partial S^{(q)}}{\partial g^{\mu\nu}}$. Then $\overline{H}_c^{(q)}$ identifies up to an arbitrary multiplicative gauge transformation (see Eq. (19)) the effective quantum Hamiltonian density

$$\overline{H}_c^{(q)} = \frac{1}{2\alpha L} \frac{1}{f(h)} \frac{\partial S^{(q)}}{\partial g^{\mu\nu}} \frac{\partial S^{(q)}}{\partial g_{\mu\nu}} + V_{QM} + \overline{V}, \tag{27}$$

with $V \equiv \overline{V}(g, \widehat{g}(r), r, s)$ being the effective potential density and V_{QM} a potential density called a Bohm-like effective quantum potential, which is prescribed as

$$V_{QM}(g, \widehat{g}(r), r, s) = \frac{\hbar^2}{8\alpha L} \frac{\partial \ln \rho}{\partial g^{\mu\nu}} \frac{\partial \ln \rho}{\partial g_{\mu\nu}} - \frac{\hbar^2}{4\alpha L} \frac{\partial^2 \rho}{\rho \partial g_{\mu\nu} \partial g^{\mu\nu}}. \tag{28}$$

Then one can show that the validity of Eqs. (25) and (26) necessarily requires one to uniquely fix the arbitrary multiplicative gauge function $f(h)$ in Eq. (16) so that identically in the prescription of the function $\overline{H}_c^{(q)}$ given above Eq. (24) must be fulfilled.

The proof of the statement follows from elementary algebra. One notices in fact that, upon substituting Eq. (23) in Eq. (20), explicit evaluation yields, respectively, for arbitrary $s \in I \equiv \mathbb{R}$:

$$\begin{aligned} \frac{\partial \rho(s)}{\partial s} + \frac{1}{f(h)} \frac{\partial}{\partial g_{\mu\nu}} (\rho(s) V_{\mu\nu}(s)) &= 0, \tag{29} \\ \frac{\partial S^{(q)}(s)}{\partial s} + \frac{1}{f(h)} \frac{1}{2\alpha L} \frac{\partial S^{(q)}(s)}{\partial g^{\mu\nu}} \frac{\partial S^{(q)}(s)}{\partial g_{\mu\nu}} + V_{QM}(s) \\ + \overline{V}_R(s) &= 0, \tag{30} \end{aligned}$$

where the first one coincides with Eq. (25) if Eq. (24) is satisfied. Hence this implies necessarily that also in Eq. (28) which defines $\overline{H}_c^{(q)}$, $f(h)$ must be determined in the same way. Incidentally one notices also that the prescription for $f(h)$ given above is also consistent with the normalization condition (6) holding for the CQG-PDF and in particular with Eq. (22) too. Indeed, integration of the continuity equation (25) manifestly recovers identically Eq. (22). Hence the constraint condition of Eq. (24) is actually required to warrant the *quantum unitarity principle*, namely the conservation of quantum probability, i.e., the validity of Eq. (6) for all $s \in I$. In addition, this uniquely determines also the Hamiltonian structure holding at the classical level, i.e., the precise form of the variational Hamiltonian density H_R .

Important theoretical results follow from the CQG-quantum Hamilton–Jacobi equation determined here. The first one is that the same equation generalizes the classical GR-Hamilton–Jacobi equation earlier determined in Part 1

(see Eq. (32) in Part 1), so that—in analogy to the same equation and Ref. [51]—it must imply the validity of the corresponding Hamilton equations to be expressed in terms of the effective quantum Hamiltonian density $\overline{H}_c^{(q)}$ (see Eq. (27)). It follows that, due to the presence of the Bohm-like effective quantum potential $V_{QM}(g, \widehat{g}(r), r, s)$, the latter now generally must depend explicitly on the proper time s (see also the related discussion in Part 1, Subsection 2D). Detailed implications, involving the construction of time-dependent solutions of the non-stationary-dependent CQG-wave equation (20), will be discussed elsewhere.

The second outcome concerns the validity of the so-called semiclassical limit of CQG, to be prescribed letting $\hbar \rightarrow 0$. By requiring that in the same limit both α and $L(m_o)$ reduce to their classical definitions and that the real limit function $\lim_{\hbar \rightarrow 0} \frac{S^{(q)}(s)}{\hbar} = \frac{S(s)}{\alpha}$ exists for arbitrary $s \in I \equiv \mathbb{R}$, with $S(s)$ identifying the classical reduced Hamilton principal function (see Part 1), one can show that the quantum Hamilton–Jacobi equation (26) reduces to the analogous classical Hamilton–Jacobi equation (see Part 1), while the limit $\lim_{\hbar \rightarrow 0} \frac{V_{QM}(s)}{\hbar} = 0$ holds identically. In fact, considering without loss of generality the case of vacuum, the semiclassical limit of Eq. (30) yields

$$\frac{1}{\alpha} \frac{\partial S(s)}{\partial s} + \frac{1}{2\alpha^2 L} \frac{\partial S(s)}{\partial g^{\mu\nu}} \frac{\partial S(s)}{\partial g_{\mu\nu}} + \lim_{\hbar \rightarrow 0} \frac{V_{QM}(s)}{\hbar} = 0, \tag{31}$$

where by construction the last term on the lhs vanishes identically. As a consequence the effective quantum Hamiltonian density $\overline{H}_c^{(q)}$ necessarily must reduce to the limit function

$$\overline{H}_o = \frac{1}{2\alpha L} \frac{\partial S(s)}{\partial g^{\mu\nu}} \frac{\partial S(s)}{\partial g_{\mu\nu}}. \tag{32}$$

This coincides in form with the classical normalized Hamiltonian density given above by Eq. (15), while Eq. (31) reduces to the classical GR-Hamilton–Jacobi equation. Hence, this proves that the derivation of the quantum hydrodynamic equations is a fundamental theoretical result to be established for the validity of CQG-theory. Indeed, the CQG-quantum continuity equation prescribes the expression of the gauge function $f(h)$, while the CQG-quantum Hamilton–Jacobi equation establishes the connection of the CQG-wave equation with the classical Hamilton–Jacobi theory determined in Part 1. This issue represents a necessary conceptual consistency aspect of quantum theory to be ascertained to hold between classical and quantum descriptions of gravitational field dynamics.

Finally, one notices that the effective potential $V_{QM}(s)$ introduced here (see Eq. (28)) is analogous to the well-known Bohm potential met in non-relativistic quantum mechanics (see for example Refs. [50,51]), its physical origin being similar and arising due to the non-uniformity of the quantum

PDF. In the present case the non-uniformity occurs because generally we must have $\frac{\partial}{\partial g_{\mu\nu}}\rho(s) \neq 0$, with consequent non-vanishing contributions arising in Eqs. (25) and (26).

4 Perturbative approximation scheme

In this section a theoretical feature related to the CQG-wave equation determined above is analyzed. The issue here is whether—based on suitable asymptotic orderings—a perturbative scheme can be developed both for the classical GR-Hamiltonian theory and the corresponding CQG-theory indicated above, in order to allow the adoption of a harmonic oscillator approximation for the analytical treatment of the Hamiltonian potential. To prove how this goal can be reached we start by considering the decomposition

$$g_{\mu\nu} = \widehat{g}_{\mu\nu}(r) + \delta g_{\mu\nu}, \tag{33}$$

with $\delta g^{\mu\nu}$, referred to here as the *displacement 4-tensor field*, to be assumed suitably small. Concerning the notation, we remark that the symbol $\delta g^{\mu\nu}$ is used to indicate displacement of the 4-tensor field $g_{\mu\nu}$ and must be distinguished from the similar notation adopted in Part 1, which refers instead to the synchronous variations of tensorial fields. Then it is obvious that $\delta g_{\mu\nu}$ identifies, both in the context of classical and quantum theories, an equivalent possible realization of the Lagrangian coordinates, which is alternative to $g_{\mu\nu}$. To make further progress, however, we need also an approximation scheme. For this purpose $g_{\mu\nu}$ is required to belong to a suitable infinitesimal neighborhood of $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$, i.e., the subset of U_g denoted

$$U_g(\widehat{g}(r), \varepsilon) = \{g_{\mu\nu} \equiv \widehat{g}_{\mu\nu}(r) + \delta g_{\mu\nu}, \delta g_{\mu\nu} \lesssim O(\varepsilon), g_{\mu\nu} \in U_g\}, \tag{34}$$

such that for all displacements $\delta g_{\mu\nu}$ the asymptotic ordering

$$\delta g_{\mu\nu} \lesssim O(\varepsilon) \tag{35}$$

holds. Here ε is a suitable infinitesimal real parameter, while by construction in such a set all components of $\delta g_{\mu\nu}$ are of order $O(\varepsilon)$ or higher. Let us consider the implications of Eqs. (33) and (35) in the two cases and applying them—for definiteness—as regards the validity of the prescription $f(h) \equiv 1$.

First, in the case of the classical Hamiltonian theory one notices that in $U_g(\widehat{g}(r), \varepsilon)$ the normalized GR-Hamilton equations (see Part 1) can be equivalently represented as

$$\begin{cases} \frac{D\delta\widehat{g}_{\mu\nu}}{Ds} = \frac{\overline{\pi}_{\mu\nu}}{\alpha L}, \\ \frac{D\overline{\pi}_{\mu\nu}}{Ds} = -\frac{\partial\overline{V}^{(a)}(g, \widehat{g})}{\partial\delta\widehat{g}^{\mu\nu}}, \end{cases} \tag{36}$$

with $\overline{V}^{(a)}(g, \widehat{g})$ being the potential to be conveniently approximated. When the identity $\overline{V}(g, \widehat{g}) \equiv \overline{V}_o(g, \widehat{g})$ holds, elementary algebra yields up to an additive constant gauge the asymptotic approximation

$$\begin{aligned} \overline{V}(g, \widehat{g}) \cong & \frac{\sigma\alpha L}{4} \{-[\delta g_{\alpha\beta}\delta g^{\alpha\beta}\widehat{g}^{\mu\nu} + 2\widehat{g}_{\alpha\beta}\delta g^{\alpha\beta}\delta g^{\mu\nu}]\widehat{R}_{\mu\nu} \\ & + 2\Lambda\delta g_{\mu\nu}\delta g^{\mu\nu}\} \equiv \overline{V}_R^{(a)}(g, \widehat{g}). \end{aligned} \tag{37}$$

Then, thanks to the vacuum solution with non-vanishing cosmological constant discussed in Part 1, for which $\widehat{R}_{\mu\nu} = \Lambda\widehat{g}_{\mu\nu}$, the previous equation yields

$$\overline{V}_R^{(a)}(g, \widehat{g}) \equiv -\frac{\sigma\alpha L\Lambda}{2} \{\delta g_{\alpha\beta}\delta g^{\alpha\beta} + \widehat{g}_{\alpha\beta}\delta g^{\alpha\beta}\delta g^{\mu\nu}\widehat{g}_{\mu\nu}\}, \tag{38}$$

$\overline{V}_R^{(a)}(g, \widehat{g})$ representing the vacuum normalized effective potential density in the same infinitesimal neighborhood as indicated above. An analogous approximation holding for the non-vacuum case can readily be obtained. It must be remarked that in all cases the conceptual consistency underlying the harmonic expansion of the Hamiltonian potential is granted by the structural stability analysis of the classical Hamiltonian theory performed in Part 1 and the related determination of the conditions for the occurrence of stable classical solutions.

In the context of CQG-theory the transformation of the Lagrangian coordinates $g_{\mu\nu} \rightarrow \delta g_{\mu\nu}$ is manifestly obtained replacing the correspondence principle realized by means of Eq. (16) with the equivalent map

$$\begin{cases} \delta g_{\mu\nu} \rightarrow \delta g_{\mu\nu}^{(q)} \equiv \delta g_{\mu\nu}, \\ \overline{\pi}_{\mu\nu} \rightarrow \pi_{\mu\nu}^{(q)} = -i\hbar\frac{\partial}{\partial\delta g^{\mu\nu}} \equiv \delta\pi_{\mu\nu}, \\ \overline{H}_R \rightarrow \overline{H}_R^{(q)} = \frac{\pi^{(q)\mu\nu}\pi_{\mu\nu}^{(q)}}{2\alpha L} + \overline{V}. \end{cases} \tag{39}$$

Notice that in principle the asymptotic ordering (35) may affect, in some sense, also the behavior of the quantum PDF $\rho(s) = \rho(\widehat{g}(r) + \delta g, \widehat{g}(r), r(s), s)$. Indeed in the limit $\delta g_{\mu\nu} \rightarrow 0$, and consequently also if $\delta g_{\mu\nu}$ is considered as an infinitesimal of $O(\varepsilon)$, it means that the probability density $\rho(s)$ should be suitably localized in the set $U_g(\widehat{g}(r), \varepsilon)$ indicated above. Nevertheless, due to the arbitrariness of the solutions which the CQG-wave equation may have, the possibility of prescribing a priori its precise asymptotic behavior seems unlikely.

5 Application #1 – Construction of the stationary vacuum CQG-wave equation

The CQG-wave equation (20) admits generally non-stationary solutions $\psi(s) \equiv \psi(g, \widehat{g}, r, s)$, i.e., in which the proper-

time dependence cannot be simply factored out. This happens because the CQG-wave equation admits generally solutions which are far from the classical one, i.e., the prescribed background solution $\widehat{g}_{\mu\nu}(r)$ (which is stationary by assumption; see also Part 1). For sure they are important for applications/developments of the theory, although the proper treatment of such issues must be forcibly left to a separate investigation.

However, an equally important issue is to investigate the possible existence of a discrete spectrum associated with the stationary CQG-wave equation following from Eq. (20). Let us consider for this purpose the case of vacuum in which by construction $\overline{V} = \overline{V}_o$, requiring that the cosmological constant Λ and the still arbitrary multiplicative gauge constant σ are such that $-\sigma\Lambda > 0$, and hence either ($\Lambda > 0, \sigma = -1$) or ($\Lambda < 0, \sigma = 1$). For definiteness, let us assume that the CQG-Hamiltonian operator $\overline{H}_R^{(q)}$ defined by the quantum correspondence principle (see Eq. (16)) does not depend on s , at least in an asymptotic sense, so that the CQG-wave equation admits exact (or asymptotic) separable particular solutions of the form

$$i\hbar \frac{\partial}{\partial s} \psi(s) = \frac{E}{c} \psi(s), \tag{40}$$

with E being a real constant 4-scalar independent of the proper time s . It follows that $\psi(s)$ is of the form

$$\psi(s) = \exp\left\{-\frac{i}{\hbar c} E(s - s_o)\right\} \psi_o(g, \widehat{g}, r), \tag{41}$$

$\psi_o(g) \equiv \psi_o(g, \widehat{g}, r)$ being a solution of the asymptotic proper-time-independent quantum wave equation

$$\overline{H}_R^{(q)} \psi_o(g) = \frac{E}{c} \psi_o(g), \tag{42}$$

to be referred to as the *stationary vacuum CQG-wave equation*. Notice, in particular, that $\psi_o(g, \widehat{g}, r)$ here is assumed to be suitably localized in the neighborhood of the background equilibrium solution $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$ so that possible additional classical stationary solutions can be effectively ignored (see Part 1).

For definiteness, let us now invoke the perturbative approximation scheme indicated above requiring in addition that the normalized effective quantum potential density holding in the case of vacuum $\overline{V} \equiv \overline{V}_o$ can be considered, at least in a suitable asymptotic sense, as independent of s . It follows that in the subset of the configuration-space set defined above, i.e., the infinitesimal neighborhood $U_g(\widehat{g}(r), \varepsilon)$ (see Eq. (34))—upon ignoring constant additive gauge contributions (to $\overline{H}_R^{(q)}$)—Eq. (42) when expressed in terms of the field variables $\delta g^{\mu\nu}$ takes the form

$$\frac{1}{2\alpha L} \left(-i\hbar \frac{\partial}{\partial \delta g_{\mu\nu}}\right) \left(-i\hbar \frac{\partial}{\partial \delta g^{\mu\nu}}\right) \psi(s) - \frac{\sigma\alpha L\Lambda}{2} \{\delta g_{\alpha\beta} \delta g^{\alpha\beta} + \widehat{g}_{\alpha\beta} \delta g^{\alpha\beta} \delta g^{\mu\nu} \widehat{g}_{\mu\nu}\} \psi(s) = 0, \tag{43}$$

where by construction $-\frac{\sigma\alpha L\Lambda}{2} > 0$. Next, let us introduce the notations:

$$\begin{cases} \delta g^2 \equiv \delta g_{\mu\nu} \delta g^{\mu\nu}, \\ \delta \pi^2 \equiv \delta \pi_{\mu\nu} \delta \pi^{\mu\nu}, \end{cases} \tag{44}$$

with $\delta \pi^{\mu\nu}$ identifying now the normalized quantum canonical momentum operator $\delta \pi^{\mu\nu} \equiv -\frac{i\hbar}{L} \frac{\partial}{\partial \delta g_{\mu\nu}}$ conjugate to the displacement field $\delta g^{\mu\nu}$. Hence the same quantum wave equation (43) can be equivalently written in the form

$$\left[\frac{\delta \pi^2}{2M} + \frac{1}{2} M \omega^2 L^2 (\delta g^2 + \widehat{g}_{\alpha\beta} \delta g^{\alpha\beta} \delta g^{\mu\nu} \widehat{g}_{\mu\nu}) \right] \psi(s) = 0, \tag{45}$$

to be referred to as the *quantum-oscillator quantum wave equation*, with the operator

$$H \equiv \frac{\delta \pi^2}{2M} + \frac{1}{2} M \omega^2 L^2 (\delta g^2 + \widehat{g}_{\alpha\beta} \delta g^{\alpha\beta} \delta g^{\mu\nu} \widehat{g}_{\mu\nu}) \tag{46}$$

being referred to as the (quantum) *invariant-energy operator*. Moreover, here M and ω are the real 4-scalars which, respectively, identify the *effective mass* and *characteristic frequency* defined as

$$\begin{cases} M = \frac{\alpha}{cL} \equiv m_o, \\ \omega = c\sqrt{-\sigma\Lambda}. \end{cases} \tag{47}$$

We conclude, therefore, that in the case of vacuum Eq. (45) realizes in the same infinitesimal neighborhood $U_g(\widehat{g}(r), \varepsilon)$ the stationary CQG-wave equation indicated above (see Eq. (42)).

6 Application #2 – The discrete spectrum of the stationary CQG-wave equation

The eigenvalue equation (45) is qualitatively similar in form to the analogous quantum wave equation holding for the quantum harmonic oscillator (QHO) in the case of ordinary quantum mechanics. In this respect it must be clarified that the quadratic expansion of the potential determined by Eq. (38) applies in the neighborhood of the extremum set by the condition $\widehat{g}_{\mu\nu}(r) = g_{\mu\nu}(r)$. The validity of Eq. (45) is physically supported by the fact that:

- (a) In the framework of the first-quantization approach adopted here, quantum solutions are defined with respect

to the classical background represented by $\widehat{g}_{\mu\nu}(r)$ only if the condition $\widehat{g}_{\mu\nu}(r) = g_{\mu\nu}(r)$ is set to determine the extremum of the potential. If this is not the case, one would have a harmonic expansion around a solution which is not the stationary one, namely it is not the classical solution of the Einstein equation. Other alternative extrema of the potential remain necessarily excluded on such basis.

- (b) The condition $\widehat{g}_{\mu\nu}(r) = g_{\mu\nu}(r)$ to set the extremum of the harmonic expansion is the only physically acceptable one in the conceptual framework developed here also because it yields the stationary solution, thus motivating the search of eigenstates of the stationary quantum harmonic oscillator given above.
- (c) The harmonic solution must be intended as a perturbative solution, i.e., it is an approximate local analytical solution of the stationary quantum wave equation, and therefore it applies in a neighborhood of the extremum point set by the condition $\widehat{g}_{\mu\nu}(r) = g_{\mu\nu}(r)$.

The question which arises now is whether Eq. (45) actually admits a discrete spectrum for its energy eigenvalues like QHO. In this section we intend to prove that in validity of the prescription $-\sigma\Lambda > 0$ and based on Dirac’s ladder operator approach, Eq. (45) can be shown to admit indeed a discrete spectrum of eigenvalues. For this purpose, let us introduce the creation and annihilation operators for a spin-2 particle, i.e., represented by second-order 4-tensors. These can be identified, respectively, with the operators $a_{\mu\nu}$ and $a_{\mu\nu}^\dagger$:

$$a_{\mu\nu} = \sqrt{\frac{M\omega}{2\hbar}} \left(L\delta g_{\mu\nu} + \frac{L}{K_o} \widehat{g}_{\mu\nu} \delta g^{\alpha\beta} \widehat{g}_{\alpha\beta} - \frac{i}{M\omega} \delta\pi_{\mu\nu} \right), \tag{48}$$

$$a_{\mu\nu}^\dagger = \sqrt{\frac{M\omega}{2\hbar}} \left(L\delta g_{\mu\nu} + \frac{L}{K_o} \widehat{g}_{\mu\nu} \delta g^{\alpha\beta} \widehat{g}_{\alpha\beta} + \frac{i}{M\omega} \delta\pi_{\mu\nu} \right), \tag{49}$$

with K_o denoting a suitable real number identified with one of the two roots of the algebraic equation $K_o^2 - 2K_o - 4 = 0$, namely $K_o = 1 \pm \sqrt{5}$. Then one can show that in terms of the operator products $a_{\mu\nu} a^{\dagger\mu\nu}$ and $a_{\mu\nu}^\dagger a^{\mu\nu}$ the identities

$$H = \hbar\omega(a_{\mu\nu} a^{\dagger\mu\nu} + \gamma), \tag{50}$$

$$H = \hbar\omega(a_{\mu\nu}^\dagger a^{\mu\nu} - \gamma), \tag{51}$$

hold, H being the invariant-energy operator (46) and γ the constant real parameter $\gamma = 8 + \frac{2}{K_o}$. The proof of both identities (50) and (51) follows from elementary algebra. Consider for example the proof of the first one (i.e., Eq. (50)). The product $a_{\mu\nu} a^{\dagger\mu\nu}$ gives in fact

$$a_{\mu\nu} a^{\dagger\mu\nu} = \frac{M\omega}{2\hbar} \left(L^2 \delta g^2 + L^2 \left(\frac{4}{K_o^2} + \frac{2}{K_o} \right) \widehat{g}_{\alpha\beta} \delta g^{\alpha\beta} \delta g^{\mu\nu} \right. \\ \left. \times \widehat{g}_{\mu\nu} - \frac{\hbar}{M\omega} \left(16 + \frac{4}{K_o} \right) + \frac{1}{(M\omega)^2} \delta\pi^2 \right). \tag{52}$$

Hence, requiring that $\frac{4}{K_o^2} + \frac{2}{K_o} = 1$ one obtains

$$a_{\mu\nu} a^{\dagger\mu\nu} = \frac{1}{2\hbar} \left(M\omega L^2 \delta g^2 + M\omega L^2 \widehat{g}_{\alpha\beta} \delta g^{\alpha\beta} \delta g^{\mu\nu} \widehat{g}_{\mu\nu} \right. \\ \left. + \frac{1}{M\omega} \delta\pi^2 \right) - \gamma, \tag{53}$$

so that Eq. (50) manifestly applies. Furthermore, it is immediate to obtain the following commutator relations:

$$[a_{\mu\nu}, a^{\dagger\mu\nu}] \equiv -2\gamma, \tag{54}$$

$$[a_{\mu\nu}, a^{\dagger\alpha\beta}] \equiv - \left(\delta_\mu^\alpha \delta_\nu^\beta + \frac{1}{K_o} \widehat{g}^{\alpha\beta} \widehat{g}_{\mu\nu} \right), \tag{55}$$

so that the operator H can equivalently be represented in the form indicated by (51). Next, in terms of the operator $\mathcal{N} = a_{\mu\nu}^\dagger a^{\mu\nu}$, elementary algebra shows that

$$[\mathcal{N}, a_{\mu\nu}] = a_{\mu\nu} + \frac{1}{K_o} \widehat{g}_{\mu\nu} \widehat{g}^{\alpha\beta} a_{\alpha\beta}, \tag{56}$$

$$[\mathcal{N}, a_{\mu\nu}^\dagger] = -a_{\mu\nu}^\dagger - \frac{1}{K_o} \widehat{g}_{\mu\nu} \widehat{g}^{\alpha\beta} a_{\alpha\beta}^\dagger. \tag{57}$$

In analogy with the axioms given in Sects. 2 and 3 and the adoption of a 4-scalar wave function, it is possible to introduce also the 4-scalar operators $a = a_{\mu\nu} \widehat{g}^{\mu\nu}$ and $a^\dagger = a_{\mu\nu}^\dagger \widehat{g}^{\mu\nu}$ representing the projections of the tensor operators $a_{\mu\nu}$ and $a_{\mu\nu}^\dagger$ along the prescribed metric tensor $\widehat{g}^{\mu\nu}$. Then, defining the normalized operator $\widehat{\mathcal{N}} = \mathcal{N}/\beta$, with $\beta \equiv 1 + \frac{4}{K_o}$, it follows

$$[\widehat{\mathcal{N}}, a] = a, \tag{58}$$

$$[\widehat{\mathcal{N}}, a^\dagger] = -a^\dagger, \tag{59}$$

while the identity (51) yields for the invariant-energy operator the representation

$$H = \hbar\omega(\beta\widehat{\mathcal{N}} - \gamma). \tag{60}$$

Therefore let us denote by $|n\rangle$, n , respectively, the eigenstate and real eigenvalue of the same operator $\widehat{\mathcal{N}}$, i.e., such that $\widehat{\mathcal{N}}|n\rangle = n|n\rangle$, one notices that $|n\rangle$ is necessarily also an eigenvector of the quantum energy operator H , with n being generally not an integer number. In addition, as a consequence of the commutator identities (56) and (57), it follows

$$\widehat{\mathcal{N}}a|n\rangle = (a\widehat{\mathcal{N}} + [\widehat{\mathcal{N}}, a])|n\rangle \\ = (a\widehat{\mathcal{N}} + a)|n\rangle = (n + 1)a|n\rangle, \tag{61}$$

$$\begin{aligned} \widehat{N}a^\dagger|n\rangle &= (a^\dagger\widehat{N} + [\widehat{N}, a^\dagger])|n\rangle \\ &= (a^\dagger\widehat{N} - a^\dagger)|n\rangle = (n - 1)a^\dagger|n\rangle, \end{aligned} \tag{62}$$

which proves that the 4-tensor operators $a_{\mu\nu}$ and $a_{\mu\nu}^\dagger$ act indeed as creation and annihilation operators. Therefore the eigenvalues of the transformed states $a|n\rangle$ and $a^\dagger|n\rangle$ are, respectively, $n + 1$ and $n - 1$, so that introducing the corresponding eigenstates of the same operator, namely such that

$$\widehat{N}|n + 1\rangle = (n + 1)|n + 1\rangle, \tag{63}$$

$$\widehat{N}|n - 1\rangle = (n - 1)|n - 1\rangle, \tag{64}$$

one expects that $a^\dagger|n\rangle = K_{(n+1)}|n + 1\rangle$ and $a|n\rangle = K_{(n-1)}|n - 1\rangle$, $K_{(n+1)}$ and $K_{(n-1)}$ being suitable 4-scalars. These results show that the spectrum of the non-negative eigenvalues corresponding to the set of eigenstates

$$|n - s\rangle, |n - s + 1\rangle, \dots |n\rangle, |n + 1\rangle, |n + 2\rangle, \dots \tag{65}$$

is discrete and numerable, with the integer $k \equiv \min(s)$ to be suitably prescribed. It is interesting to notice that in terms of the operator \widehat{N} defined above also the so-called number operator can be prescribed, which by construction has only integer eigenvalues. In fact, denoting by n_o the minimum positive integer such that $n_o \equiv \text{int}(n) \geq n$, then $N = \widehat{N} \frac{n_o}{n}$ identifies the so-called *number operator*. By construction N has only integer eigenvalues, namely it is such that $N|n\rangle = n_o|n\rangle$, while being $N|n - s\rangle = (n_o - s)|n - s\rangle$. Now we notice that for all relative integers in the set $\{-s, -s + 1, \dots + \infty\}$ the state $|n - s\rangle$ is also an eigenstate of the invariant-energy operator H . Its eigenvalue, referred to as an invariant-energy eigenvalue, is manifestly

$$E_{n-s} \equiv \hbar\omega(\beta(n - s) - \gamma). \tag{66}$$

Then, the positive integer k in the set of eigenstates (65) can be prescribed in such a way that E_{n-s} has for $s = k$ the minimum positive value, and hence it identifies the *minimum invariant-energy eigenvalue*,

$$E_{\min} \equiv E_{n-k} = \hbar\omega(\beta(n - k) - \gamma). \tag{67}$$

In terms of the integer n_o indicated above one obtains for the minimum positive eigenvalue of H , namely $E_{\min} \equiv E_{n-k}$ (see Eq. (67)), the upper estimate

$$E_{\min} < \hbar\omega(\beta(n_o - k) - \gamma) \equiv \gamma_o \hbar\omega, \tag{68}$$

where γ_o is the positive real number $\gamma_o = \text{int}(\gamma) - \gamma > 0$, and $\text{int}(\gamma) \equiv n_o - k$ is the minimum positive integer such that $\beta(n_o - k) > \gamma$. Since $K_o = 1 \pm \sqrt{5}$, one can show that the only admissible root is the one associated with the positive-root, namely $K_{o1} \equiv 3.236$. Hence, $\gamma = 8 + \frac{2}{1+\sqrt{5}} \equiv$

$\gamma_1 \cong 8.618$ and $\beta = 1 + \frac{4}{1+\sqrt{5}} \equiv \beta_1 \cong 2.236$. This means also that $\gamma_o \cong 0.326$, so that the majorization (68) actually requires the *weak upper bound* $E_{\min} \lesssim 0.326\hbar\omega$ to hold for the minimum invariant-energy E_{\min} . In view of these considerations it follows therefore that the stationary CQG-wave equation (45) admits a spectrum of invariant-energy eigenvalues E_n associated with the quantum energy operator H (see Eq. (46)). The minimum invariant energy E_{\min} is non-vanishing as it is proportional to the characteristic frequency ω (see Eq. (68)). Finally, provided the cosmological constant Λ is non-vanishing and $-\sigma\Lambda > 0$, the spectrum indicated above is discrete and numerable.

7 Application #3 – Quantum prescription of the characteristic scale length $L(m_o)$ and the graviton rest mass m_o

An interesting issue is related to the physical prescription at the quantum level for the invariant parameter $L(m_o)$ and consequently for the rest mass m_o and the invariant parameter α appearing in the quantum Hamiltonian operator $\overline{H}_R^{(q)}$ (see Eq. (16)). In principle these quantities are not necessarily the same as those entering the corresponding classical normalized Hamiltonian structure $\{\overline{x}_R, \overline{H}_R\}$.

It should be stressed in fact that the precise prescription of the invariant mass m_o as well of $L(m_o)$ should follow from the theory of CQG itself and be consistent with the physical interpretation of m_o in terms of the graviton mass. A possible solution of the task can be achieved based on the asymptotic treatment developed above in the case of vanishing external fields and non-vanishing cosmological constant. It follows that, in the perturbative framework considered here, both the mass prediction and the invariant length therefore depend on actual (experimental or theoretical) estimates for a possibly non-vanishing cosmological constant. We notice, in fact, that the eigenvalue stationary quantum wave equation (45) and the related quantum energy operator (46) still depend both on the invariant mass m_o and the characteristic scale length $L \equiv L(m_o)$. It must be remarked, however, that the minimum energy prediction (68) provides a possible prescription for the invariant mass m_o . The minimum energy, or ground-state, eigenvalue E_{\min} , which is associated with the quantum-oscillator quantum wave equation (45), yields in fact the corresponding *ground-state mass estimate* $m_o = \frac{E_{\min}}{c^2}$.

In view of Eq. (47) and the expression for E_{\min} considered above, the upper bound estimate

$$m_o \lesssim 0.326 \frac{\hbar\sqrt{-\sigma\Lambda}}{c} \tag{69}$$

must apply, with the invariant rest mass m_o depending accordingly on the cosmological constant. In case $-\sigma\Lambda > 0$,

the model established by the stationary CQG-wave equation (45) thus provides a tentative candidate for the identification of the rest mass of the massive graviton. Such a particle would therefore necessarily be endowed with a sub-luminal speed of propagation. For the example considered above one can estimate the numerical value of E_{\min} and m_o . Adopting for Λ the current astrophysically estimated value $\Lambda \cong 1.2 \times 10^{-52} \text{ m}^{-2}$ [36], it is found that $E_{\min} \cong 1.1 \times 10^{-52} \text{ J} \sim 7 \times 10^{-34} \text{ eV}$, while $m_o \cong 1.26 \times 10^{-69} \text{ kg} \sim 7 \times 10^{-34} \text{ eV}/c^2$, so that the resulting graviton–electron mass ratio is

$$\frac{m_o}{m_e} \cong 1.38 \times 10^{-39}, \tag{70}$$

with m_e denoting the electron rest mass.

The final problem to address pertains to the identification of the invariant length $L(m_o)$ in the CQG-theory. If the graviton is considered as a point particle this can be identified either with the classical Schwarzschild radius $L_{\text{Sch}} \equiv \frac{2Gm_o}{c^2}$ associated with the graviton rest mass m_o (see Part 1), or the Compton wavelength $\lambda_C \equiv \frac{\hbar}{m_o c}$. In the first case upon invoking Eq. (69) one finds that $L_{\text{Sch}} \ll L_{\text{Planck}}$, where $L_{\text{Planck}} \cong 10^{-35} \text{ m}$ is the Planck length. On the other hand it is well known that the same Planck length provides, at least in order of magnitude, the minimum physically admissible quantum length. Thus, a fortiori, it necessarily must realize a lower bound for the same characteristic length $L(m_o)$. This means that in the quantum regime the classical prescription for $L(m_o)$ based on the Schwarzschild radius is not physically acceptable, thus implying that quantum phenomena for the graviton are dominant with respect to classical ones. Therefore the prescription of $L(m_o)$ must be realized by means of the Compton wavelength. In terms of Eq. (69) this yields in the present case

$$\lambda_C \equiv \frac{1}{\gamma_o} \frac{1}{\sqrt{-\sigma \Lambda}}, \tag{71}$$

while a numerical evaluation of Eq. (71) gives $\lambda_C \cong 2.8 \times 10^{26} \text{ m}$. In view of the estimate for E_{\min} this shows that $L(m_o)$ necessarily coincides up to a factor of order unity with the invariant characteristic length $L_\Lambda \equiv \frac{1}{\sqrt{-\sigma \Lambda}}$ associated with the cosmological constant Λ , suggesting at the same time also the possible quantum origin of the cosmological constant [37–39]. The result is based on the analytical estimate of the minimum eigenvalue of the discrete spectrum associated with the invariant-energy operator (the precise calculation of the same eigenvalue can in principle be performed numerically). The interpretation of L_Λ in terms of the graviton Compton wavelength follows therefore on physical grounds and not simply on dimensional analysis arguments.

In this framework, the cosmological constant Λ is associated with the Compton wavelength of the ground-state oscillation mode of the quantum of the gravitational field, i.e., the graviton mass m_o . As a final comment, it must be stressed that the estimate given here for m_o refers to the ground-state eigenvalue of the discrete spectrum corresponding to the vacuum CQG-equation (45). However, each eigenvalue of the same discrete spectrum should give rise to its corresponding mass value, so that the discrete energy spectrum is sided by a discrete mass spectrum. On similar grounds, quantum wave equations different from the vacuum one (45) studied here should generate a corresponding different mass spectrum. Hence, the value of m_o is non-unique and depends both on the physical properties of the background space-time as well as the solution spectrum of the CQG-wave equation to be solved (e.g., stationary or non-stationary equation, vacuum or non-vacuum equation, etc.).

In connection with this we notice that, in the first-quantization approach developed here, the metric tensor of the background space-time $\widehat{g}_{\mu\nu}(r)$, the Ricci curvature 4-tensor $R_{\mu\nu}(\widehat{g}(r))$ as well as the cosmological constant Λ are all considered to be in principle arbitrarily prescribed quantities. The theory turns out to be intrinsically background independent, i.e., it holds for any realization of the space-time $\widehat{g}_{\mu\nu}(r)$. Nevertheless the stationary approximation (40) might not be generally applicable, while even in the case in which the Hamiltonian operator $\overline{H}_R^{(q)}$ does not depend explicitly on s , the proper-time-dependent equation (20) may still admit non-trivial explicitly time-dependent solutions. As a consequence a more general class of solutions with respect to that considered above might occur for the CQG-wave equation, as corresponds to complex physical phenomenologies characterized by non-uniform behavior of both the prescribed metric tensor $\widehat{g}_{\mu\nu}(r)$, of the corresponding Ricci tensor $R_{\mu\nu}(\widehat{g}(r))$, as well as of the cosmological constant and the non-vanishing external fields.

As a final point, a peculiar connection exists between the classical GR-Hamiltonian structure developed in Part 1 and the corresponding quantum one represented by the quantum state $x^{(q)} \equiv (g_{\mu\nu}^{(q)}, \pi_{\mu\nu}^{(q)})$ and the Hamiltonian operator $\overline{H}_R^{(q)}$. As pointed out in Part 1, an arbitrary solution of the GR-Hamilton equation, in particular the stationary solution $\widehat{g}_{\mu\nu}(r)$ determined by the vacuum Einstein field equations, is stable with respect to infinitesimal perturbations provided suitable physical conditions are met. As shown here, the same requirement applies in the context of CQG-theory when the stationary quantum wave equation (42) is reduced to a quantum harmonic oscillator. The interesting consequence which emerges is therefore the validity of the following mutual logical implication: the existence of a stable stationary solution of the classical Hamiltonian structure of GR appears effectively, at the same time, as a prerequisite and a consequence

of the existence of a discrete energy spectrum for the stationary CQG-wave equation, and hence also the existence of a finite graviton rest mass m_o .

8 Discussion and comparisons with literature

Quantization methods, both in quantum mechanics and quantum gravity, are usually classified in two approaches, the canonical and the covariant ones [7]. However, while for quantum mechanics the same approaches are equivalent, this is not so in the case of quantum gravity. The reason is the radically different approach taken in the two cases for the treatment of the quantum state, of the causality principle and of space-time itself.

In the canonical framework a canonical quantization approach is developed which leaves formally arbitrary the space-time. Key ingredients usually adopted for this purpose are, first, the introduction of $(3 + 1)$ - or $(2 + 2)$ -decompositions [8–12] for the representation of the same space-time and, second, the adoption of a quantum state represented in terms of non-4-tensor continuum fields. In the covariant approaches, instead, all physical quantities including the quantum state are represented exclusively by means of 4-tensor fields so that the property of manifest covariance is automatically fulfilled. As a consequence covariant quantization necessarily involves the assumption of some sort of classical background space-time structure, for example identified with the flat Minkowski space-time. In order to realize such a strategy, however, it turns out that the quantum state is typically described by means of superabundant variables. As a consequence covariant quantization usually also requires the treatment of suitable constraint conditions.

Let us consider first the canonical approach. A choice of this type, for example, is the one adopted by Dirac and based on the Dirac constrained dynamics [19–23]. By construction such an approach is not manifestly covariant. We stress that this refers in principle both to transformation properties with respect to local point transformations, i.e., LPT-theory, as well as the theory of non-local point transformations (NLPT) developed in Ref. [18]. It is immediate to realize that this is indeed the case for the Dirac Hamiltonian approach. In this picture in fact the field variable is identified with the metric tensor $g_{\mu\nu}$, but the corresponding generalized velocity is defined as $g_{\mu\nu,0}$, namely with respect to the “time” component of the 4-position. Consequently, in Dirac’s canonical theory, the canonical momentum remains identified with the manifestly non-tensorial quantity $\pi_{\text{Dirac}}^{\mu\nu} = \frac{\partial L_{\text{EH}}}{\partial g_{\mu\nu,0}}$, where L_{EH} is the Einstein–Hilbert variational Lagrangian density. Hence, such a choice necessarily violates the principle of manifest covariance [16,17].

The same kind of strategy was adopted in the approach developed later by Arnowitt, Deser and Misner (ADM theory, 1959–1962 [6]). In this case manifest covariance is lost specifically because of the adoption, inherent in the ADM approach, of Lagrangian and Hamiltonian variables which are not 4-tensors. In fact this is based on the introduction of the so-called $3 + 1$ decomposition of space-time, which by construction is foliation dependent, in the sense that it relies on a peculiar choice of a family of GR-frames for which time and space transform separately so that space-time is effectively split into the direct product of a 1-dimensional time and a 3-dimensional space subset, respectively [9]. For the same reason, the quantum wave equation (20) proposed in this research is intrinsically different from the Wheeler–DeWitt wave equation [40]. In fact, Eq. (20) yields a dynamical evolution with respect to the invariant proper time s defined on the background space-time, while the Wheeler–DeWitt equation follows from the ADM foliation theory and is expressed as an evolution Schrödinger-like equation advancing the dynamics of the wave function with respect to the coordinate-time t , which is not an invariant parameter. In addition, in the absence of a background space-time, the same equation carries a conceptual problem related to the definition of coordinate time, which is simultaneously the dynamical parameter and a component of space-time which must be quantized by solving the same equation. This problem, however, does not arise in the theory of CQG proposed above.

Another interesting example worth to be mentioned is the one exemplified by the so-called Ashtekar variables, originally identified, respectively, with a suitable self-dual spinorial connection (the generalized coordinates) and their conjugate momenta (see Refs. [41,42]). It is well known that Ashtekar variables provide an alternative canonical representation of SF-GR. Such a choice is at the basis of the so-called “loop representation of quantum general relativity” [43] usually referred to as “loop quantum gravity” (LQG) and first introduced by Rovelli and Smolin in 1988–1990 [44,45] (see also Ref. [46]). Nevertheless, also the Ashtekar variables can be shown to be by construction intrinsically non-tensorial in character either with respect to the LPT or NLPT-groups. The basic consequence is that also the canonical representation of Einstein field equations based on these variables, as well as ultimately also LQG itself, violates the principle of manifest covariance.

Despite these considerations, it must be stressed that, as far as the classical Hamiltonian formulation of GR is concerned, the canonical approach and the manifestly covariant theory proposed in Paper 1 and in the present work are complementary, in that they exhibit distinctive physical properties associated with two canonical Hamiltonian structures underlying GR itself. The corresponding Hamiltonian flows, however, are different, being referred to an appropriate coordinate-

time of space-time foliation in the canonical approach, and to a suitable invariant proper time in the present theory. As a consequence, the physical interpretation of quantum theories of GR built upon these Hamiltonian structures remain distinctive. The CQG-theory developed here in fact reveals the possible existence of a discrete spectrum of metric tensors having non-vanishing momenta at the quantum level, but whose realization at the classical level remains excluded for the extremal field equations, i.e., when the Hamiltonian theory is required to recover the Einstein field equations (see also Part 1 and further discussions on the issue reported below in Sect. 9).

Let us now consider the covariant approaches to quantum gravity [47–49]. In this case the usual strategy is to split the space-time metric tensor $g_{\mu\nu}$ into two parts according to the decomposition of the type $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the background metric tensor defining the space-time geometry (usually identified with the flat background), and $h_{\mu\nu}$ is the dynamical field (deviation field) for which quantization applies. From the conceptual point of view there are some similarities between the literature covariant approaches and the manifestly covariant quantum-gravity theory developed here. The main points of contact are: (1) the adoption of 4-tensor variables, without invoking any space-time foliation; (2) the implementation of a first-quantization approach, in the sense that there exists by assumption a continuum classical background space-time with a geometric connotation, over which the relevant quantum fields are dynamically evolving; (3) the adoption of superabundant variables, which in the two approaches are identified with the sets $(\eta_{\mu\nu}, h_{\mu\nu})$ and $(\widehat{g}_{\mu\nu}, g_{\mu\nu})$, respectively.

Nevertheless, important differences must be pointed out as well. In fact, first of all, the CQG-theory developed here (and the CCG theory reported in Part 1 on which it is based) is intrinsically non-perturbative in character. It means, in other words, that the background metric tensor can be identified with an arbitrary continuum solution of the Einstein equations, while a priori the canonical variable $g_{\mu\nu}$ is not required to be a perturbation field. On the other hand, a decomposition of the type (33) resembling the one invoked in covariant literature approaches can always be introduced a posteriori for the implementation of appropriate perturbative schemes. This occurs in particular for an analytical evaluation of discrete-spectrum quantum solutions (see Sects. 4–6). Second, the present theory is constructed starting from the De Donder–Weyl manifestly covariant approach. As a consequence the present approach is based on a variational formulation, which relies on the introduction of a synchronous variational principle for the Einstein equations first reported in Ref. [16]. Such a feature is unique since all previous literature is actually based on the adoption of asynchronous variational principles, i.e., principles in which the invariant volume element is considered variational rather than prescribed. Indeed, as shown

in the same reference it is precisely the synchronous principle which makes possible the distinction between variational and extremal (or prescribed) metric tensors, and the consequent introduction of non-vanishing canonical momenta. As a result, manifestly covariant classical Lagrangian, Hamiltonian and Hamilton–Jacobi theories of GR have been formulated as pointed out in Refs. [16, 17] and Part 1. In particular, based on the GR-Hamiltonian structure determined in Part 1 such a feature allows for the adoption of the canonical quantization represented by Eq. (18). Third, in the present approach superabundant variables are implemented, while the same covariant quantization holds with respect to a 4-dimensional space-time, with no extra-dimensions being required for its prescription. Fourth, the CQG-theory proposed in this paper is obtained from the preliminary derivation of the reduced Hamiltonian theory given in Paper 1, which establishes a connection between field theory and particle dynamics, although without requiring specification of any additional degrees of freedom for the dynamical system predicted by Hamilton–Jacobi wave theory [50, 51]. As a consequence the Hamiltonian structure which CCG-theory is built on is free of constraints, a feature which permits the implementation of the canonical quantization rule represented by the same equation (see again Eq. (18)).

Finally, regarding covariant quantization, a further interesting comparison concerns the Batalin–Vilkovisky (BV) formalism originally developed in Refs. [52–55]. Such a method is usually implemented for the quantization of gauge field theories and topological field theories in Lagrangian formulation [56–58]. Nevertheless also a corresponding Hamiltonian formulation is available [59, 60]. Further critical aspects of the BV formalism can be found for example in Ref. [61]. In the case of the gravitational field it has been formerly applied in the context of perturbative quantum gravity to treat constraints arising from initial metric decomposition (i.e., in reference with the so-called gauge-fixing and ghost terms). Its basic features are the adoption of an asynchronous Lagrangian variational principle of GR [16], the use of superabundant canonical variables and the consequent introduction of constraints. These features mark the main differences with the present manifestly covariant Hamiltonian approach, which is non-perturbative, which follows from the synchronous Lagrangian variational principle defined in Ref. [16] and is constraint-free. The key implication, as indicated in Part 1, is that of permitting the construction of both the corresponding Hamiltonian and Hamilton–Jacobi classical theories. These features are crucial also for the validity of the CQG-wave equation presented here (see Eq. (20)).

In conclusion, CQG-theory realizes at the same time a canonical and a manifestly covariant quantization method, in this way establishing a connection both with former canonical and covariant approaches. Nevertheless, the emerging new features of the present theory depart in several ways from

previous literature and might/should hopefully help shading further insight into the long-standing problem of quantization of gravity.

9 Conclusions

The paper has been devoted to the formulation of a new theory of covariant quantum gravity (CQG), referred to here as CQG-theory. The theoretical foundations of the research presented here are based on the manifestly covariant Hamiltonian theory for the Einstein field equations earlier developed in Refs. [1, 16, 17].

The quantum theory of gravitational field developed here distinguishes itself from previous literature with approaches to the problem. In fact, from one side the present theory satisfies the principle of manifest covariance, while at the same time the validity of the classical GR field equations is preserved identically. Therefore, the realization of the CQG-theory does not rely on the violation of manifest covariance in order to attempt a quantization of the space-time through a discretization of its geometric properties, nor it requires a modification of the Einstein field equations at the variational level or the assumption a priori of the implementation of perturbative treatments from the start. The present theory respects the canonical procedure well known in the foundations of quantum field theory, which requires one to follow the logical path consisting in: (a) the identification of the appropriate classical Lagrangian density in 4-scalar form; (b) the subsequent definition of conjugate momenta and realization of a corresponding classical Hamiltonian theory holding for a canonical state; (c) the introduction of canonical transformations and development of Hamilton–Jacobi theory; (d) the canonical quantization method relying on classical Poisson brackets and the prescription of quantum wave equation.

The development of the present CQG-theory is made possible by the adoption of the new type of variational principle for the Einstein field equations, for the first time pointed out in Ref. [16]. The synchronous variational formulation is characterized by distinguishing variational ($g_{\mu\nu}(r)$) and prescribed ($\widehat{g}_{\mu\nu}(r)$) tensor fields in such a way that the variational ones are allowed to possess different physical properties with respect to the prescribed fields, while preserving at the same time the correct validity of the prescribed equations. In the realm of the classical theory the physical behavior of variational fields provide the mathematical background for the establishment of a manifestly covariant Hamiltonian theory of GR. The background metric tensor $\widehat{g}_{\mu\nu}(r)$ is purely classical and has a geometric connotation, raising/lowering tensor indices and defining the Christoffel symbols. At the classical level it must be $\widehat{\nabla}_\alpha \widehat{g}_{\mu\nu}(r) = 0$, namely the covariant derivative of the prescribed metric tensor is identically vanishing. Adopting the language of clas-

sical dynamics, we can say by analogy that $\widehat{g}_{\mu\nu}(r)$ does not possess a “kinetic energy”, since the corresponding generalized “velocity field” $\widehat{\nabla}_\alpha \widehat{g}_{\mu\nu}(r)$ is null by definition. However, the advantage of the synchronous variational principle lies in the possibility of having variational metric tensor fields $g_{\mu\nu}(r)$ for which the covariant derivative defined with respect to the background space-time can be non-vanishing, so that $\widehat{\nabla}_\alpha g_{\mu\nu}(r) \neq 0$. We stress that this feature remains a property of variational (and therefore virtual) fields $g_{\mu\nu}(r)$, which therefore acquire a non-null generalized kinetic energy. This permits the identification of canonical momenta and the construction of corresponding covariant Hamiltonian theory holding for the Hamiltonian structure $\{x_R, H_R\}$. When passing to the covariant quantum theory variational fields become quantum observables and inherit the corresponding tensor transformation laws of classical fields together with the mentioned physical properties. It is then found that the quantum observable corresponding to $g_{\mu\nu}(r)$ is endowed with non-vanishing momenta having a quantum probability density. The resulting physical interpretation of the present theory is straightforward. In actual classical theory the physical field $g_{\mu\nu}(r)$ is “frozen-in” with the prescribed field $\widehat{g}_{\mu\nu}(r)$ which has a geometrical connotation. Violation of the condition $\widehat{\nabla}_\alpha \widehat{g}_{\mu\nu}(r) = 0$ is only allowed for variational fields. In the realm of quantum theory the prescribed field $\widehat{g}_{\mu\nu}(r)$ keeps on retaining its meaning consistent with the picture of GR, while the field $g_{\mu\nu}(r)$ acquires the physical meaning of a quantum field which is permitted to deviate from $\widehat{g}_{\mu\nu}(r)$ and to “oscillate” over the background space-time, thus violating at the quantum level the frozen-in condition $\widehat{\nabla}_\alpha \widehat{g}_{\mu\nu}(r) = 0$. These features are exemplified by the structure of the Hamiltonian density determined above, which can be expressed as the sum of kinetic and potential density terms, in full analogy with standard quantum theory of fields, as well as the possibility of recovering (at least in a proper asymptotic treatment) the peculiar structure of the Hamiltonian characteristic of the harmonic oscillator having a discrete spectrum of eigenvalues.

The theory proposed here is believed to be susceptible of applications to a wide range of quantum physics, theoretical physics and astrophysics-related problems and to provide also new insight to the axiomatic foundations of Quantum Gravity. Among the applications of CQG-theory special mention deserve those which have been investigated in the paper. These include in particular the proof of the existence of discrete energy spectra for the stationary CQG-wave equation for solutions which are close to the classical prescribed one $\widehat{g}_{\mu\nu}(r)$ and the related quantum prescription of the free parameters which characterize the classical Hamiltonian structure, namely the estimate for the graviton rest mass m_o and the determination of the characteristic scale length $L(m_o)$. Nevertheless, important issues concern also the search of more general solutions pertaining to

the non-stationary CQG-wave equation as well as second-quantization effects such as the possible quantum modification of the prescribed metric tensor associated with the background space-time. Such tasks will be undertaken in forthcoming investigations.

Acknowledgements Work developed within the research projects of the Czech Science Foundation GAČR Grant No. 14-07753P (C.C.) and the Albert Einstein Center for Gravitation and Astrophysics, Czech Science Foundation No. 14-37086G (M.T.).

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP³.

References

- C. Cremaschini, M. Tassarotto, Hamiltonian approach to GR – Part 1: covariant theory of classical gravity. *Eur. Phys. J. C* (2017). doi:[10.1140/epjc/s10052-017-4854-1](https://doi.org/10.1140/epjc/s10052-017-4854-1)
- A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 2004)
- L.D. Landau, E.M. Lifschitz, *Field Theory, Theoretical Physics*, vol. 2 (Addison-Wesley, New York, 1957)
- C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, 1st edn. (W.H. Freeman, New York, 1973)
- R.M. Wald, *General Relativity*, 1st edn. (University of Chicago Press, Chicago, 1984)
- R. Arnowitt, S. Deser, C.W. Misner, in *Gravitation: An Introduction to Current Research*, ed. by L. Witten (Wiley, New York, 1962)
- A. Ashtekar, *New J. Phys.* **7**, 198 (2005)
- Z.B. Etienne, Y.T. Liu, S.L. Shapiro, *Phys. Rev. D* **82**, 084031 (2010)
- M. Alcubierre, *Introduction to 3+1 Numerical Relativity* (Oxford University Press, Oxford, 2008)
- S. Vacaru, *J. Math. Phys.* **46**, 042503 (2005)
- T. Gheorghiu, O. Vacaru, S. Vacaru, *Eur. Phys. J. C* **74**, 3152 (2014)
- V. Ruchin, O. Vacaru, S. Vacaru, *Eur. Phys. J. C* **77**, 184 (2017)
- Th. De Donder, *Théorie Invariantive Du Calcul des Variations* (Gauthier-Villars & Cia, Paris, 1930)
- H. Weyl, *Ann. Math.* **36**, 607 (1935)
- J. Struckmeier, A. Redelbach, *Int. J. Mod. Phys. E* **17**, 435 (2008)
- C. Cremaschini, M. Tassarotto, Synchronous Lagrangian variational principles in general relativity. *Eur. Phys. J. Plus* **130**, 123 (2015)
- C. Cremaschini, M. Tassarotto, Manifest covariant Hamiltonian theory of general relativity. *Appl. Phys. Res.* **8**, 2 (2016). doi:[10.5539/apr.v8n2p60](https://doi.org/10.5539/apr.v8n2p60)
- M. Tassarotto, C. Cremaschini, *Adv. Math. Phys.* **2016**, 9619326 (2016). doi:[10.1155/2016/9619326](https://doi.org/10.1155/2016/9619326)
- P.A.M. Dirac, *Can. J. Math.* **2**, 129 (1950)
- K. Sundermeyer, *Constrained Dynamics. Lecture Notes in Physics* (Springer, Berlin, 1982)
- E.C.G. Sudarshan, N. Mukunda, *Classical Dynamics—A Modern Perspective* (Wiley-Interscience Publication, New York, 1964)
- N. Mukunda, *Phys. Scr.* **21**, 783 (1980)
- L. Castellani, *Ann. Phys.* **143**, 357 (1982)
- C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **126**, 42 (2011)
- C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **126**, 63 (2011)
- C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **127**, 4 (2012)
- C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **127**, 103 (2012)
- C. Cremaschini, M. Tassarotto, *Phys. Rev. E* **87**, 032107 (2013)
- G. Esposito, G. Gionti, C. Stornaiolo, *Il Nuovo Cimento B* **110**, 1137 (1995)
- I.V. Kanatchikov, *Rep. Math. Phys.* **41**, 49 (1998)
- I.V. Kanatchikov, *Adv. Theor. Math. Phys.* **20**, 1377–1396 (2016)
- M. Reisenberger, C. Rovelli, *Phys. Rev. D* **65**, 125016 (2002)
- C. Rovelli, *Lecture Notes in Physics, in Decoherence and Entropy in Complex Systems*, ed. by H.-T. Elze (Springer, Berlin, 2003), pp. 36–62
- C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **130**, 166 (2015)
- C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **129**, 247 (2014)
- Planck Collaboration, P.A.R. Ade, N. Aghanim, C. Armitage-Caplan, M. Arnaud et al., Planck 2015 results. XIII. Cosmological parameters (2015). arXiv preprint [arXiv:1502.1589v2](https://arxiv.org/abs/1502.1589v2)
- A. Einstein, *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie (Cosmological Considerations in the General Theory of Relativity)*, Königlich Preußische Akademie der Wissenschaften, Sitzungsberichte (Berlin) (1917), pp. 142–152
- S. Weinberg, The cosmological constant problem. *Rev. Mod. Phys.* **61**, 1–23 (1989)
- S. Carroll, *Spacetime and Geometry* (Addison Wesley, San Francisco, 2004)
- B.S. DeWitt, *Phys. Rev.* **160**, 1113–1148 (1967)
- A. Ashtekar, New variables for classical and quantum gravity. *Phys. Rev. Lett.* **57**, 2244 (1986)
- A. Ashtekar, New Hamiltonian formulation of general relativity. *Phys. Rev. D* **36**, 1587 (1987)
- T. Jacobson, L. Smolin, Nonperturbative quantum geometries. *Nucl. Phys. B* **299**, 295 (1988)
- C. Rovelli, L. Smolin, Knot theory and quantum gravity. *Phys. Rev. Lett.* **61**, 1155 (1988)
- C. Rovelli, L. Smolin, Loop space representation of quantum general relativity. *Nucl. Phys. B* **331**, 80 (1990)
- C. Rovelli, Ashtekar formulation of general relativity and loop space nonperturbative quantum gravity: a report. *Class. Quantum Gravity* **8**, 1613 (1991)
- A. Ashtekar, R. Geroch, *Rep. Prog. Phys.* **37**, 1211 (1974)
- S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972)
- B.S. DeWitt, Covariant quantum geometrodynamics, in *Magic Without Magic*, ed. by J.A. Wheeler, J.R. Klauder (W. H. Freeman, San Francisco, 1972)
- M. Tassarotto, C. Cremaschini, *Found. Phys.* **46**(8), 1022–1061 (2016)
- M. Tassarotto, M. Mond, D. Batic, *Found. Phys.* **46**(9), 1127–1167 (2016)
- I.A. Batalin, G.A. Vilkovisky, *Phys. Lett. B* **69**, 309 (1977)
- I.A. Batalin, G.A. Vilkovisky, *Phys. Lett. B* **102**, 27 (1981)
- I.A. Batalin, G.A. Vilkovisky, *Phys. Lett. B* **120**, 166 (1983)
- I.A. Batalin, G.A. Vilkovisky, *Phys. Rev. D* **28**, 2567 (1983)
- B.P. Mandal, S.K. Rai, S. Upadhyay, *Eur. Phys. Lett.* **92**, 21001 (2010)
- S. Upadhyay, B.P. Mandal, *Eur. Phys. J. C* **72**, 2059 (2012)
- S. Upadhyay, *Phys. Lett. B* **723**, 470–474 (2013)
- E.S. Fradkin, G.A. Vilkovisky, CERN Report No. TH-2332 (1977)
- E.S. Fradkin, G.A. Vilkovisky, *Phys. Lett. B* **69**, 309 (1977)
- K. Fredenhagen, K. Rejzner, *Commun. Math. Phys.* **314**, 93–127 (2012)