

Hamiltonian approach to GR – Part 1: covariant theory of classical gravity

Claudio Cremaschini^{1,a}, Massimo Tessorotto^{2,3}

¹ Faculty of Philosophy and Science, Institute of Physics and Research Center for Theoretical Physics and Astrophysics, Silesian University in Opava, Bezručovo nám.13, 74601 Opava, Czech Republic

² Department of Mathematics and Geosciences, University of Trieste, Via Valerio 12, 34127 Trieste, Italy

³ Faculty of Philosophy and Science, Institute of Physics, Silesian University in Opava, Bezručovo nám.13, 74601 Opava, Czech Republic

Received: 6 January 2017 / Accepted: 21 April 2017 / Published online: 19 May 2017

© The Author(s) 2017. This article is an open access publication

Abstract A challenging issue in General Relativity concerns the determination of the manifestly covariant continuum Hamiltonian structure underlying the Einstein field equations and the related formulation of the corresponding covariant Hamilton–Jacobi theory. The task is achieved by adopting a synchronous variational principle requiring distinction between the prescribed deterministic metric tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$ solution of the Einstein field equations which determines the geometry of the background spacetime and suitable variational fields $x \equiv \{g, \pi\}$ obeying an appropriate set of continuum Hamilton equations, referred to here as GR-Hamilton equations. It is shown that a prerequisite for reaching such a goal is that of casting the same equations in evolutionary form by means of a Lagrangian parametrization for a suitably reduced canonical state. As a result, the corresponding Hamilton–Jacobi theory is established in manifestly covariant form. Physical implications of the theory are discussed. These include the investigation of the structural stability of the GR-Hamilton equations with respect to vacuum solutions of the Einstein equations, assuming that wave-like perturbations are governed by the canonical evolution equations.

1 Introduction

This is the first paper of a two-part investigation dealing with the Hamiltonian theory of the gravitational field, and more precisely, the one which is associated with the so-called Standard Formulation of General Relativity (SF-GR) [1–4], i.e., the Einstein field equations. In the second paper the corresponding quantum formulation will be presented. For this purpose, in the two papers new manifestly covariant Hamil-

tonian (respectively classical and quantum) approaches are developed. The two formulations will be referred to as theories of Covariant Classical and, respectively, Quantum Gravity (CCG/CQG) or briefly *CCG-* and *CQG-theory*.

As shown below *CCG-theory* is built upon the results presented in Refs. [5,6] about the variational formulation of GR achieved in the context of a *DeDonder–Weyl*-type approach and the corresponding possible realization of a super-dimensional and manifestly covariant Hamiltonian theory. In particular, based on a suitable identification of the effective kinetic energy and the related Hamiltonian density 4-scalars adopted in Ref. [6], the aim here is to prescribe a reduced-dimensional continuum Hamiltonian structure of SF-GR, to be referred to here as *Classical Hamiltonian Structure* (CHS). The crucial goal of the paper is to show that CHS can be associated with arbitrary possible solutions of the Einstein field equations corresponding either to vacuum or non-vacuum conditions. In other words, this means that the same Hamiltonian structure is coordinate-independent and occurs for arbitrary external source terms which may appear in the variational potential density.

Despite being intimately related to the one earlier considered in Ref. [6], the new Hamiltonian structure is achieved in fact by means of the parametrization of the corresponding canonical state in terms of the proper time determined along arbitrary geodetics of the background metric field tensor. This feature turns out to be of paramount importance for the establishment of the corresponding canonical transformation and covariant Hamilton–Jacobi theories. The CHS determined in this way is shown to be realized by the ensemble $\{x_R, H_R\}$ represented by an appropriate variational canonical state $x_R = \{g, \pi\}$, with both g and π being suitably identified tensor fields representing appropriate continuum Lagrangian coordinates and conjugate momenta and H_R a corresponding variational Hamiltonian density.

^a e-mail: claudiocremaschini@gmail.com

Its basic feature is that of being based, in analogy with Ref. [6], on the adoption of the *synchronous Hamiltonian variational principle* for the variational formulation of GR. However, in difference with the same reference, new features are added which, as we intend to show, are mandatory for the construction of the corresponding canonical transformation and Hamilton–Jacobi theories. In particular for this purpose, first, a reduced-dimensional representation is introduced for the canonical momenta, which, however, leaves formally unchanged the corresponding variational Hamiltonian density. Second, an appropriate parametrization by means of a suitably defined proper time s is introduced so that the resulting Euler–Lagrange equations are now realized by means of Hamilton equations in evolution form.

Accordingly, variational and prescribed tensor fields are introduced, with the prescribed ones, in contrast to the variational fields, being left invariant by the synchronous variations. In particular, the same Hamilton equations may reduce identically to the Einstein field equations, which are fulfilled by the prescribed fields, if suitable initial conditions are set. This occurs provided the Poisson bracket of the Hamiltonian density is a local function, i.e., it does not depend explicitly on proper time. In the realm of the classical theory the physical behavior of variational fields provide the mathematical background for the establishment of a manifestly covariant Hamiltonian theory of GR, and in particular the CCG-theory realized here. When passing to the corresponding covariant quantum theory (i.e., in the present case the CQG-theory to be developed in the subsequent paper) variational fields become quantum fields and inherit the corresponding tensor transformation laws of classical fields. Thanks to its intrinsic consistency with the principles of covariance and manifest covariance, the synchronous variational setting developed in Refs. [5,6] provides at the same time:

- the natural framework for a Hamiltonian theory of classical gravity which is consistent with SF-GR;
- the prerequisite for the establishment of a covariant quantum theory of gravitational field which is in turn consistent with classical theory and SF-GR (see Ref. [7], herein Part 2).

According to such an approach the 4-scalar, i.e., invariant, 4-volume element of the space-time ($d\Omega$) entering the action functional is considered independent of the functional class of variations, so that it must be defined in terms of a *prescribed metric tensor field* $\widehat{g}(r)$, represented equivalently either in terms of its covariant or counter variant component, i.e., either $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$ or $\widehat{g}(r) \equiv \{\widehat{g}^{\mu\nu}(r)\}$. Here $r \equiv \{r^\mu\}$ and $\widehat{g}(r)$ denote, respectively, an arbitrary GR-frame parametrization and an arbitrary particular solution of the Einstein field equations. This is obtained therefore upon

identifying in the action functional $d\Omega \equiv d^4r \sqrt{-|\widehat{g}|}$, with d^4r being the corresponding canonical measure expressed in terms of the said parametrization and $|\widehat{g}|$ denoting as usual the determinant of the metric tensor $\widehat{g}(r)$. In the context of the synchronous variational principle to GR a further requirement is actually included which demands that the prescribed field $\widehat{g}_{\mu\nu}(r)$ must determine, besides $d\Omega$, also the *geometric properties* of space-time. This means that $\widehat{g}_{\mu\nu}(r)$ should uniquely prescribe the tensor transformation laws of arbitrary tensor fields, which may depend, in principle, besides $\widehat{g}_{\mu\nu}(r)$, both on the variational state x_R and the 4-position $r \equiv \{r^\mu\}$. This requires in particular that $\widehat{g}_{\mu\nu}(r)$ and $\widehat{g}^{\mu\nu}(r)$, respectively, lower and raise tensor indices of the same tensor fields. In a similar way $\widehat{g}_{\mu\nu}(r)$ uniquely determines also the standard Christoffel connections which enter both the Ricci tensor $\widehat{R}_{\mu\nu}$ and the covariant derivatives of arbitrary variational tensor fields. Therefore, in the context of synchronous variational principle to GR the approach known in the literature as “background space-time picture” [8–10] is adopted, whereby the background space-time ($\mathbf{Q}^4, \widehat{g}(r)$) is considered defined “a priori” in terms of $\widehat{g}_{\mu\nu}(r)$, while leaving unconstrained all the variational fields $x_R = \{g, \pi\}$ and in particular the Lagrangian coordinates $g(r) \equiv \{g_{\mu\nu}(r)\}$.

Indeed, consistent with Ref. [6], the physical interpretation which arises from CCG-theory exhibits a connection also with the so-called *induced gravity* (or *emergent gravity*) [11,12], namely the conjecture that the geometrical properties of space-time should reveal themselves as a mean field description of microscopic stochastic or quantum degrees of freedom underlying the classical solution. In the present approach this is achieved by introducing the prescribed metric tensor $\widehat{g}_{\mu\nu}(r)$ in the Lagrangian and Hamiltonian action functionals, which is held constant in the variational principles when performing synchronous variations and has to be distinguished from the variational field $g_{\mu\nu}(r)$. In this picture, $\widehat{g}_{\mu\nu}(r)$ should arise as a macroscopic prescribed mean field emerging from a background of variational fields $g_{\mu\nu}(r)$, all belonging to a suitable functional class. This permits one to introduce a new representation for the action functional in superabundant variables, depending both on $g_{\mu\nu}(r)$ and $\widehat{g}_{\mu\nu}(r)$. Such a feature, as explained above, is found to be instrumental for the identification of the covariant Hamiltonian structure associated with the classical gravitational field and provides a promising physical scenario where to develop a covariant quantum treatment of GR.

In this reference, one has to acknowledge the fact that the Hamiltonian description of classical systems is a mandatory conceptual prerequisite for achieving a corresponding quantum description [14,15], i.e., in the case of continuum systems, the related relativistic quantum field theory. This task involves the identification of the appropriate Hamiltonian representation of the continuum field, to be realized by means of the following steps:

Step #1 : Establishment of underlying Lagrangian and Hamiltonian variational action principles.

Step #2 : Construction of the corresponding Euler–Lagrange equations, realized, respectively, in terms of appropriate continuum Lagrangian and Hamiltonian equations.

Step #3 : Determination of the corresponding set of continuum canonical transformations and formulation of the related Hamilton–Jacobi theory.

The proper realization of these steps remains crucial. In actual fact, the last target appears as a prerequisite of foremost importance for being able to reach a consistent formulation of relativistic quantum field theory for General Relativity, i.e., the so-called Quantum Gravity. The conclusion follows by analogy with Electrodynamics. In fact, as it emerges from the recent investigation concerning the axiomatic foundations of relativistic quantum field theory for the radiation-reaction problem associated with classical relativistic extended charged particles (see Refs. [16–21]), it is the Hamilton–Jacobi theory which naturally provides the formal axiomatic connection between classical and quantum theory, to be established by means of a suitable realization of the quantum correspondence principle.

Prerequisite for reaching such goals in the context of relativistic quantum field theory is the establishment of a theory fulfilling at all levels both the *Einstein general covariance principle* and the *principle of manifest covariance*. Such a viewpoint is mandatory in order that the axiomatic construction method of SF-GR makes any sense at all [22]. Indeed, in order that physical laws have an objective physical character they cannot depend on the choice of the GR reference frame. This requisite can only be met provided all classical physical observables and the corresponding mathematical relationships holding among them, i.e., the physical laws, can actually be expressed in tensorial form with respect to the group of transformations indicated above. In the context of SF-GR the adoption of the same strategy requires therefore the realization of *Steps #1–#3* in manifest covariant form. As far as the actual identification of *Steps #1* and *#2* for SF-GR is concerned, the candidate is represented by the variational theory reported in Refs. [5, 6]. The distinctive features of such a variational theory, which sets it apart from previous Hamiltonian formulations in the literature [23–28], lie in its consistency with the criteria indicated above and the DeDonder–Weyl classical field theory approach [29–37].

Nevertheless, well-known alternative approaches exist in the literature which are based on non-manifestly covariant approaches. For the purpose of formal comparison let us briefly mention some of them, a detail analysis being left to future developments. For definiteness, approaches can be considered which are built upon space-time foliations, namely based on so-called 3 + 1 and/or 2 + 2 splitting

schemes. In fact, GR can be formulated in any GR-frame (i.e., coordinate system) by introducing a suitable local point transformation $r^\mu \rightarrow r'^\mu = f^\mu(r)$ leading to a decomposition of this type. In particular, the 3 + 1 approach is convenient for purposes related, for example, to the definition of conventional energy-momentum tensors, thermodynamic and kinetic values, and to provide corresponding methods of quantization [38–40]. The latter are exemplified by the well-known approach developed by Arnowitt, Deser and Misner (1959–1962 [28]), usually referred to as ADM theory in the literature. The same theory is based on the introduction of the so-called 3 + 1 decomposition of space-time, which by construction is foliation dependent, in the sense that it relies on a peculiar choice of a family of GR frames for which time and space transform separately so that space-time is effectively split into the direct product of a 1-dimensional time and a 3-dimensional space subsets, respectively (ADM-foliation) [41]. Instead, different types of 2 + 2 splitting (or with double 3 + 1 and 2 + 2 splitting) are considered, for instance, to find new classes of GR exact solutions [42–44], to develop the theory of geometric flows related to classical gravity, quantum gravity and geometric thermodynamics [45, 46], or to elaborate some approaches based on deformation quantization of GR and modified gravity theories [47, 48]. In comparison with these approaches, the manifestly covariant Lagrangian and Hamiltonian formulations of GR reported in Refs. [5, 6] and developed below mainly differ because, first, there is no introduction of foliation of space-time, so that the 4-tensor formalism is preserved at all stages of investigation. Second, in contrast to the Hamiltonian theory of GR obtained from the ADM decomposition [4], both Lagrangian and Hamiltonian dynamical variables and the canonical state are expressed in 4-tensor notation and satisfy as well the manifest covariance principle. Third, in the context of CCG-theory the Hamiltonian flow associated with the Hamiltonian structure $\{x_R, H_R\}$ (see Eq. (4) below) is defined with respect to an invariant proper time s , and not a coordinate-time as in ADM theory. Finally, it must be stressed that, in such a context for the proper implementation of the DeDonder–Weyl formalism, besides the customary 4-scalar curvature term of the Einstein–Hilbert Lagrangian, 4-tensor (i.e., manifestly covariant) momenta must be adopted in the action functional. This property which can be fulfilled only adopting a synchronous variational principle is missing in the ADM Hamiltonian theory, where field variables and conjugate momenta are identified only after performing the 3 + 1 foliation on the Einstein–Hilbert Lagrangian of the associated asynchronous variational principle. Despite this difference the two approaches are complementary in the sense that they exhibit distinctive physical properties associated with the two canonical Hamiltonian structures underlying SF-GR (see again Ref. [6]).

1.1 Goals of the paper

Going beyond the considerations discussed above, the construction of CCG-theory involves a number of questions, closely related to the continuum Hamiltonian theory reported in Ref. [6], which remain to be addressed. This involves posing the following distinct goals:

- **GOAL #1: Reduced continuum Hamiltonian theory for SF-GR** – The search for a reduced-dimensional realization of the continuum Hamiltonian theory for the Einstein field equations, which still satisfies the principle of manifest covariance. In fact, as a characteristic feature of the DeDonder–Weyl approach, in the Hamiltonian theory given in Ref. [6] the canonical variables defining the canonical state have different tensorial orders, with the momenta being realized by third-order 4-tensors. In contrast, the new approach to be envisaged here should provide a realization of the canonical state $x_R \equiv \{g_{\mu\nu}, \pi_{\mu\nu}\}$ in which both generalized coordinates and corresponding momenta have the same tensorial dimension and are represented by second-order 4-tensor fields.
- **GOAL #2: Evolution form of the reduced continuum Hamilton equations** – A further problem is whether the same reduced continuum Hamilton equations can be given a causal evolution form, namely they can be cast as *canonical evolution equations*. Since originally the continuum Hamilton equations are realized by PDE, this means that some sort of Lagrangian representation should be determined. Hence, by introducing a suitable Lagrangian Path (LP) parametrization of the canonical state $x_R \equiv \{g_{\mu\nu}, \pi_{\mu\nu}\}$ in terms of the proper time associated with the prescribed tensor field $\widehat{g}_{\mu\nu}(r)$ indicated above, the corresponding continuum canonical equations are found to be realized by means of evolution equations advancing in proper time the canonical state. These will be referred to as *GR-Hamilton equations* of CCG-theory: they generate the evolution of the corresponding canonical fields by means of a suitable canonical flow.
- **GOAL #3: Realization of manifestly covariant continuum Hamilton–Jacobi theory** – A related question which arises involves, in particular, the determination of the canonical transformation, which generates the flow corresponding to the continuum canonical evolution equations. This concerns, more precisely, the development of a corresponding Hamilton–Jacobi theory applicable in the context of CCG-theory and the investigation of the canonical transformation generated by the corresponding Hamilton principal function.
- **GOAL #4: Global prescription and regularity properties of the corresponding GR-Lagrangian and Hamiltonian densities.** The Lagrangian and Hamiltonian formulations should be globally prescribed in the appropriate phase

spaces. The global prescription should include also the validity of suitable *regularity properties* of the corresponding Hamiltonian density H_R .

- **GOAL #5: Identification of the gauge properties of the classical GR-Lagrangian and Hamiltonian densities.** The related issue concerns the identification of the possible gauge indeterminacies, in terms of suitable *gauge functions*, characterizing the Lagrangian and Hamiltonian densities.
- **GOAL #6: Dimensionally-normalized form of CHS.** In particular, the goal here is to show that a suitable dimensional normalization of the Hamiltonian structure $\{x_R, H_R\}$ can be reached so that the canonical momenta acquire the physical dimensions of an action, a feature required for the establishment of a quantum theory of GR in terms of Hamilton–Jacobi theory. More precisely, this involves the construction of a non-symplectic canonical transformation for the GR-Hamiltonian density H_R . The issue is to show that this can be taken to be of the form

$$\begin{cases} g^{\mu\nu} \rightarrow \overline{g}^{\mu\nu} = g^{\mu\nu}, \\ \pi^{\mu\nu} \rightarrow \overline{\pi}^{\mu\nu} = \frac{\alpha L}{\kappa} \pi^{\mu\nu}, \\ H_R \rightarrow \overline{H}_R \equiv \overline{T}_R + \overline{V} = \frac{\alpha L}{\kappa} H_R, \end{cases}, \quad (1)$$

where κ is the dimensional constant $\kappa = \frac{c^3}{16\pi G}$, L is a 4-scalar scale length to be defined, α is a suitable dimensional 4-scalar, while \overline{T}_R , and \overline{V} denote the corresponding transformed effective kinetic and potential densities defining the transformed Hamiltonian density \overline{H}_R . Then the question arises whether α can be prescribed in such a way that the transformed canonical momentum $\overline{\pi}^{\mu\nu}$ has the dimensions of an action.

- **GOAL #7: Structural stability of the GR-Hamilton equations of CCG-theory** – The final issue concerns the study of the structural stability which in the framework of CCG-theory the canonical equations exhibit with respect to their stationary solutions, i.e., the solutions of the Einstein equations. In fact, depending on the specific realization of CHS considered here, infinitesimal perturbations whose dynamics is governed by the said canonical evolution equations may exhibit different stability behaviors, i.e., be stable/unstable or marginally stable, with respect to arbitrary solutions of the Einstein field equations. For definiteness, the case of vacuum solutions with a non-vanishing cosmological constant Λ is treated. It is shown that the stability analysis provides a prescriptions for the gauge functions indicated above which characterize the GR-Hamiltonian density.

In view of these considerations and of the results already achieved in Refs. [5,6], in this paper the attention will be focused on the investigation of *GOALS #1–#7*. These topics,

together with the continuum Lagrangian and Hamiltonian theories proposed in Refs. [5,6], have potential impact in the context of both classical and quantum theories of General Relativity.

2 Evolution form of Hamilton equations for SF-GR

In this section the problem of the determination of a *reduced continuum Hamiltonian theory* for GR is addressed for a prescribed Hamiltonian system. This is represented by the CHS $\{x_R, H_R\}$ which is formed by an appropriate 4-tensor canonical state x_R and a suitable 4-scalar Hamiltonian density $H_R(x_R, \widehat{x}_R(r), r, s)$. In particular, the target requires one to find a realization of the variational canonical momentum in such a way that, in the corresponding reduced canonical state, fields and reduced momenta form a couple of second-rank conjugate 4-tensors. The requisite is that such a Hamiltonian theory should warrant the validity of the non-vacuum Einstein field equations to be achieved as realizations of suitable reduced continuum Hamilton equations set in evolution form and referred to as *GR-Hamilton equations* of CCG-theory. More precisely, these are realized by the initial-value problem represented by the canonical equations:

$$\begin{cases} \frac{Dg_{\mu\nu}(s)}{Ds} = \frac{\partial H_R(x_R, \widehat{x}_R(r), r, s)}{\partial \pi^{\mu\nu}(s)}, \\ \frac{D\pi_{\mu\nu}(s)}{Ds} = -\frac{\partial H_R(x_R, \widehat{x}_R(r), r, s)}{\partial g^{\mu\nu}(s)}, \end{cases} \tag{2}$$

and the initial conditions of the type

$$\begin{cases} g_{\mu\nu}(s_o) \equiv g_{\mu\nu}^{(o)}(s_o), \\ \pi_{\mu\nu}(s_o) \equiv \pi_{\mu\nu}^{(o)}(s_o). \end{cases} \tag{3}$$

Then the solution of the initial-value problem (2) and (3) generates the Hamiltonian flow

$$x_R(s_o) \rightarrow x_R(s), \tag{4}$$

which is associated with the Hamiltonian structure $\{x_R, H_R\}$. Here the notation is as follows. First, s denotes the proper time prescribed along an arbitrary geodesic curve $r(s) \equiv \{r^\mu(s)\}$. This is associated with the prescribed metric tensor $\widehat{g}_{\mu\nu}(r)$ of the background space-time. Second,

$$x_R(s) \equiv \{g_{\mu\nu}(r(s)), \pi_{\mu\nu}(r(s))\} \tag{5}$$

identifies the s -parametrized *reduced-dimensional variational canonical state*, with $g_{\mu\nu}(r)$ and $\pi_{\mu\nu}(r)$ being the corresponding continuum Lagrangian coordinates and the conjugate momenta, $\widehat{x}_R(s) \equiv \{\widehat{g}_{\mu\nu}(r(s)), \widehat{\pi}_{\mu\nu}(r(s)) \equiv 0\}$ being the corresponding prescribed state and $H_R(x_R, \widehat{x}_R(r), r, s)$

the variational Hamiltonian 4-scalar density to be suitably determined. Finally, $\frac{D}{Ds}$ is the covariant s -derivative

$$\frac{D}{Ds} = \frac{\partial}{\partial s} + t^\alpha(s) \widehat{\nabla}_\alpha, \tag{6}$$

while $t^\alpha(s)$ and $\widehat{\nabla}_\alpha$ are respectively the tangent 4-vector to the geodesics $r(s) \equiv \{r^\mu(s)\}$ and the covariant derivative evaluated at the same position in terms of the prescribed metric tensor $\widehat{g}_{\mu\nu}(r)$.

The GR-Hamilton equations are covariant with respect to arbitrary canonical transformations. This property implies that the same equations are covariant with respect to an arbitrary local point transformation (LPT) which leaves invariant a given space-time represented by the differential manifold of the type $(\mathbf{Q}^4, \widehat{g}(r))$, so that the general covariance principle and the principle of manifest covariance are necessarily both fulfilled by construction.

The realization of the evolution form of the GR-Hamilton equations represents a requirement for the construction of a corresponding manifestly covariant Hamilton–Jacobi theory of GR. The construction of these equations is based on the following steps.

2A–Step #1: prescription of the reduced-dimensional Hamiltonian density

In the first step the Hamiltonian density $H_R(x_R, \widehat{x}_R(r), r, s)$ is identified, extending the treatment given in Ref. [6]. In terms of the reduced canonical variables this yields

$$H_R(x, \widehat{x}, r) \equiv T_R(x_R, \widehat{x}_R) + V(g, \widehat{x}, r), \tag{7}$$

where the effective kinetic and potential densities $T_R(x_R, \widehat{x}_R)$ and $V(g, \widehat{x}, r, s)$ can be taken, respectively, of the general form

$$\begin{cases} T_R(x_R, \widehat{x}_R) \equiv \frac{1}{2\kappa f(h)} \pi_{\mu\nu} \pi^{\mu\nu}, \\ V(g, \widehat{x}, r, s) \equiv \sigma V_o(g, \widehat{x}) + \sigma V_F(g, \widehat{x}, r, s). \end{cases} \tag{8}$$

Here $f(h)$ and σ denote suitable *multiplicative gauge* functions which remain in principle still arbitrary at this point. More precisely, $f(h)$ identifies an “a priori” arbitrary non-vanishing and smoothly-differentiable real gauge function depending on the variational weight-factor $h = (2 - \frac{1}{4} g^{\alpha\beta} g_{\alpha\beta})$ introduced in Ref. [5] and prescribed in such a way that

$$f(\widehat{g}^{\mu\nu}(r)) = 1. \tag{9}$$

We anticipate here that the function $f(h)$ will be shown in Part 2 to be identically $f(h) = 1$, as required by quantum

theory of GR. Furthermore, σ denotes the additional constant gauge function $\sigma = \pm 1$. Finally, in Eq. (8) the two 4-scalars

$$\begin{aligned} V_o(g, \widehat{x}) &\equiv \kappa h [g^{\mu\nu} \widehat{R}_{\mu\nu} - 2\Lambda], \\ V_F(g, \widehat{x}, r) &\equiv h L_F(g, \widehat{x}, r), \end{aligned} \tag{10}$$

identify, respectively, the gravitational and external-field source contributions defined in Ref. [5], with L_F being associated with a non-vanishing stress-energy tensor.

2B–Step #2: Lagrangian path parametrization

In the second step we introduce the notion of Lagrangian path (LP) [14, 15]. For this purpose, preliminarily the real 4-tensor $t^\gamma(\widehat{g}(r), r)$ is introduced such that identically

$$\begin{cases} t^\alpha(\widehat{g}(r), r) \widehat{\nabla}_\alpha t^\gamma(\widehat{g}(r), r) = 0, \\ \widehat{g}_{\gamma\delta}(r) t^\gamma(\widehat{g}(r), r) t^\delta(\widehat{g}(r), r) = 1, \end{cases} \tag{11}$$

so that by construction $t^\gamma(\widehat{g}(r), r)$ is tangent to an arbitrary geodetics belonging to an arbitrary 4-position $r \equiv \{r^\mu\}$ of the space-time $(\mathbf{Q}^4, \widehat{g}(r))$ [2]. Then the LP is identified with the geodetic curve

$$\{r^\mu(s)\} \equiv \{r^\mu(s) \mid \forall s \in \mathbb{R}, r^\mu(s_o) = r_o^\mu\}, \tag{12}$$

which is solution of the initial-value problem

$$\begin{cases} \frac{dr^\mu(s)}{ds} = t^\mu(s), \\ r^\mu(s_o) = r_o^\mu. \end{cases} \tag{13}$$

Here the 4-scalar proper time s is defined along the same curve $\{r^\mu(s)\}$ so that $ds^2 = \widehat{g}_{\mu\nu}(r) dr^\mu(s) dr^\nu(s)$. Furthermore, $t^\mu(s)$ identifies the LP-parametrized 4-vector $t^\mu(s) \equiv t^\mu(\widehat{g}(r(s)), r(s))$. In Eq. (13) $\frac{d}{ds} \equiv \frac{\partial}{\partial s}$ identifies the ordinary derivative with respect to s . In the following we shall call *implicit s-dependences* the dependences on the proper time s appearing in the variational fields through the LP parametrization of the fields. In contrast, we shall denote as *explicit s-dependences* the proper-time dependences which enter either explicitly on s itself or through the dependence on $r(s) \equiv \{r^\mu(s)\}$.

Let us now introduce the parametrization obtained replacing everywhere, in all the relevant tensor fields, $r \equiv \{r^\mu\}$ with $r(s) \equiv \{r^\mu(s)\}$, namely obtained identifying

$$\begin{cases} g^{\mu\nu}(s) \equiv g^{\mu\nu}(r(s)), \\ \pi^{\mu\nu}(s) \equiv \pi^{\mu\nu}(r(s)), \\ \widehat{x}(r) \equiv \widehat{x}(r(s)). \end{cases} \tag{14}$$

This yields for the Hamiltonian density H_R the so-called LP-parametrization, in terms of which the reduced Hamilton equations (2) can in turn be represented. In the remainder,

for greater generality, such a representation shall be taken of the form

$$H_R(s) \equiv H_R(x_R(s), \widehat{x}_R(r), r(s), s), \tag{15}$$

i.e., including also a possible explicit dependence in terms of the proper time s . Specific examples in which explicit s -dependences may occur in the theory include:

- (1) Continuum canonical transformations and in particular canonical transformations generating local or non local point transformations (see Ref. [13]). In this case explicit s -dependences may arise in the transformed Hamiltonian density due to explicit s -dependent generating functions.
- (2) Hamilton–Jacobi theory (see Sect. 3), where in a similar way the explicit s -dependence in the Hamiltonian density may be generated by the canonical flow.
- (3) Stability theory for wave-like perturbations where explicit s -dependences may appear in the variational fields $x_R = x_R(s)$ (see Sect. 5).

2C–Step #3: the reduced Hamiltonian variational principle

Given these premises, in the context of CCG-theory the explicit construction of the GR-Hamilton equations (2) follows in analogy with the extended Hamiltonian theory achieved in Ref. [6]. The goal also in the present case is in fact the development of a manifestly covariant variational approach, i.e., in which at all levels all variational fields, including the canonical variables, the Hamiltonian density, as well as their synchronous variations and the related Euler–Lagrange equations, are expressed in 4-tensor form. To this end in the framework of the synchronous variational principle developed there – and in agreement with the DeDonder–Weyl approach – the variational functional is identified with a real 4-scalar

$$S_R(x, \widehat{x}) \equiv \int d\Omega L_R(x, \widehat{x}, r, s), \tag{16}$$

with $L_R(x, \widehat{x}, r, s)$ being the variational Lagrangian density

$$L_R(x, \widehat{x}, r, s) \equiv \pi_{\mu\nu} \frac{D}{Ds} g^{\mu\nu} - H_R(x, \widehat{x}, r, s). \tag{17}$$

Thus, $L_R(x, \widehat{x}, r, s)$ is identified with the Legendre transform of the corresponding variational Hamiltonian density $H_R(x, \widehat{x}, r, s)$ defined above, with $\pi_{\mu\nu} \frac{D}{Ds} g^{\mu\nu}$ denoting the so-called exchange term. Then the variational principle associated with the functional $S_R(x, \widehat{x})$ is prescribed in terms of the synchronous-variation operator δ (i.e., identified with the Frechet derivative according to Ref. [5]), i.e., by means of the synchronous variational principle

$$\delta S_R(x, \hat{x}) = 0 \tag{18}$$

obtained keeping constant both the prescribed state \hat{x} and the 4-scalar volume element $d\Omega$. This yields the 4-tensor Euler–Lagrange equations cast in symbolic form

$$\begin{cases} \frac{\delta S_R(x, \hat{x})}{\delta g^{\mu\nu}} = 0, \\ \frac{\delta S_R(x, \hat{x})}{\delta \pi_{\mu\nu}} = 0, \end{cases} \tag{19}$$

which are manifestly equivalent to the Hamilton equations (2). These equations can be written in the equivalent Poisson-bracket representation

$$\frac{D}{Ds} x_R(s) = [x_R, H_R(x_R, \hat{x}_R(r), r, s)]_{(x_R)}, \tag{20}$$

with $[\cdot, \cdot]_{(x_R)}$ denoting the Poisson bracket evaluated with respect to the canonical variables x_R , namely

$$\begin{aligned} [x_R, H_R(x_R, \hat{x}_R(r), r, s)]_{(x_R)} \\ = \frac{\partial x_R}{\partial g^{\mu\nu}} \frac{\partial H_R(s)}{\partial \pi_{\mu\nu}} - \frac{\partial x_R}{\partial \pi_{\mu\nu}} \frac{\partial H_R(s)}{\partial g^{\mu\nu}}. \end{aligned} \tag{21}$$

Then, after elementary algebra, the PDE’s (20) yield the GR-Hamilton equations in evolution form given above by Eq. (2). In particular, invoking Eqs. (8)–(10) it follows that

$$\begin{aligned} \frac{\partial V(g, \hat{x}_R(r), r, s)}{\partial g^{\mu\nu}(s)} &= \sigma \kappa h(s) \hat{R}_{\mu\nu} \\ &- \sigma \kappa g_{\mu\nu}(s) \frac{1}{2} (g^{\alpha\beta}(s) \hat{R}_{\alpha\beta} - 2\Lambda) - \sigma \kappa \frac{8\pi G}{c^2} T_{\mu\nu}, \end{aligned} \tag{22}$$

where $\hat{R}_{\mu\nu} \equiv \hat{R}_{\mu\nu}(s)$ and $T_{\mu\nu} \equiv T_{\mu\nu}(s)$ denote the LP-parametrizations of the Ricci and stress-energy tensors. Hence, in the case the gauge function $f(h)$ is prescribed as $f(h) = 1$, the canonical equations (2) reduce to the single equivalent *Lagrangian evolution equation* for the variational field $g_{\mu\nu}(s)$ in the LP-parametrization:

$$\begin{aligned} \frac{D}{Ds} \left[\frac{D}{Ds} g_{\mu\nu}(s) \right] + \sigma h(s) \hat{R}_{\mu\nu} \\ - \sigma g_{\mu\nu}(s) \frac{1}{2} [g^{\alpha\beta}(s) \hat{R}_{\alpha\beta} - 2\Lambda] - \sigma \frac{8\pi G}{c^2} T_{\mu\nu} = 0. \end{aligned} \tag{23}$$

This concludes the proof that the GR-Hamilton equations (2), as well as the equivalent Lagrangian equation (23) are – as expected – both variational.

2D–Step #4: connection with Einstein field equations

The connection of the canonical equations (2) with the Einstein theory of GR can be obtained under the assumption that the Hamiltonian density does not depend explicitly on proper time s , i.e., it is actually of the form

$$H_R = H_R(x_R, \hat{x}_R(r), r). \tag{24}$$

In this case, one furthermore notices that the identities $\hat{g}_{\mu\nu}(s) \hat{g}^{\mu\nu}(s) = \delta^\mu_\mu$ and $\frac{D}{Ds} \hat{g}_{\mu\nu}(s) \equiv 0$ hold, so that by construction $\hat{\pi}_{\mu\nu}(s) \equiv 0$ and hence the canonical equation for $\hat{\pi}_{\mu\nu}(s)$ (or equivalently Eq. (23)) yields for the prescribed fields

$$\hat{R}_{\mu\nu} - \hat{g}_{\mu\nu}(s) \frac{1}{2} [\hat{g}^{\alpha\beta}(s) \hat{R}_{\alpha\beta} - 2\Lambda] = \frac{8\pi G}{c^2} \hat{T}_{\mu\nu}, \tag{25}$$

which coincides with the Einstein field equations. Therefore, in this framework the latter are obtained by looking for a stationary solution of the GR-Hamilton equation (2), i.e., requiring the initial conditions

$$\begin{cases} g_{\mu\nu}(s_0) \equiv \hat{g}_{\mu\nu}(s_0), \\ \pi_{\mu\nu}(s_0) \equiv \hat{\pi}_{\mu\nu}(s_0) = 0, \end{cases} \tag{26}$$

while requiring furthermore for all $s \in I$

$$\hat{\pi}_{\mu\nu}(s) = 0. \tag{27}$$

Notice that, in principle, additional extrema may exist for the effective potential, i.e. such that $\frac{\partial V(g, \hat{x}_R(r), r, s)}{\partial g^{\mu\nu}(s)} = 0$. One can show that this indeed happens, for example, in the case of vacuum, namely letting $\hat{T}_{\mu\nu} \equiv 0$. Thus, besides $g_{\mu\nu}(s) \equiv \hat{g}_{\mu\nu}$ additional extrema include $g_{\mu\nu}(s) \equiv -\frac{2}{3} \hat{g}_{\mu\nu}$ and the case in which $g_{\mu\nu}(s)$ satisfies identically the constraint equations $h(s) = 0$ and $1 - \frac{1}{2} g_{\mu\nu}(s) \hat{g}^{\mu\nu} = 0$. However, once the initial conditions (26) are set the stationary solution is unique. The prerequisite for the existence of such a particular solution is, however, the validity of the constraint condition (24), i.e., the requirement that the GR-Hamilton equations (2) are autonomous. Such a property is non-trivial. In fact, it might be in principle violated if non-local effects are taken into account (see for example Refs. [17, 21]). Analogous circumstance might arise due to possible quantum effects. The issue will be further discussed in Part 2.

Finally, for completeness, we mention also the connection between the reduced Hamiltonian system $\{x_R, H_R\}$ defined according to Eqs. (5) and (7) and the representation given in Ref. [6] in terms of the “extended” Hamiltonian system $\{x, H\}$ and based on the adoption of the “extended” canonical state $x \equiv \{g^{\mu\nu}, \Pi^\alpha_{\mu\nu}\}$. More precisely, the connection is obtained, first, by the prescription $H = H_R$, and, second, upon identifying $\Pi^\alpha_{\mu\nu} = t^\alpha \pi_{\mu\nu}$. In fact it then follows that $\pi_{\mu\nu} = t_\alpha \Pi^\alpha_{\mu\nu}$, so that $\pi_{\mu\nu}(r)$ represents the projection of $\Pi^\alpha_{\mu\nu}(r)$ along the tangent vector $t_\alpha(s)$ to the background geodesic curve.

2E–Step #5: alternative Hamiltonian structures

As indicated above, Eq. (22) together with the GR-Hamilton equations (2) provides the required connection with the Ein-

stein field equations. In practice this means that any suitably-smooth 4-scalar function such that

$$\begin{aligned} & \left. \frac{\partial V(g, \widehat{x}_R(r), r, s)}{\partial g^{\mu\nu}(s)} \right|_{g^{\mu\nu}(s)=\widehat{g}^{\mu\nu}(s)} \\ &= \sigma\kappa \widehat{R}_{\mu\nu} - \sigma\kappa \widehat{g}_{\mu\nu}(s) \frac{1}{2} (\widehat{g}^{\alpha\beta}(s) \widehat{R}_{\alpha\beta} - 2\Lambda) \\ & - \sigma\kappa \frac{8\pi G}{c^2} T_{\mu\nu} = 0 \end{aligned} \tag{28}$$

realizes an admissible Hamiltonian structure of GR. The choice corresponding to Eq. (8) with the functions $V_o(g, \widehat{x})$ and $V_F(g, \widehat{x}, r)$ prescribed according to Eq. (10) corresponds to the lowest-order polynomial representation (but it is still non-linear, and thus non-trivial) in terms of the variational field $g_{\mu\nu}(s)$ for the variational Hamiltonian.

However, alternative possible realizations of the Hamiltonian structure $\{x_R, H_R\}$ can be readily identified. In fact, once the initial conditions (26) are set, alternative possible realizations of the GR-Hamilton equations (2), leading to the correct realization of the Einstein field equations, can be achieved. These are obtained introducing a transformation of the type

$$\begin{cases} g_{\mu\nu}(s) \rightarrow g_{\mu\nu}(s), \\ \pi^{\mu\nu}(s) \rightarrow \pi^{\mu\nu}(s) - (s - s_o) P^{\mu\nu}(\widehat{x}_R), \\ V(g, \widehat{x}, r, s) \rightarrow V_1(g, \widehat{x}, r, s) + U_o(\widehat{g}, \widehat{x}, s). \end{cases} \tag{29}$$

Notice that here the function $U_o(\widehat{g}, \widehat{x}, s)$ remains in principle arbitrary, so that it can always be determined so that the extremal value of the potential density is preserved, namely $V(\widehat{g}, \widehat{x}, r, s) = V_1(\widehat{g}, \widehat{x}, r, s) + U_o(\widehat{g}, \widehat{x}, s)$. However, $\widehat{\pi}^{\mu\nu}(s)$, $P_{\mu\nu}(\widehat{x}_R)$ and $V_1(g, \widehat{x}, r, s)$ can always be determined so that:

- (1) the extremal momentum $\widehat{\pi}^{\mu\nu}(s)$ is prescribed so that

$$\widehat{\pi}^{\mu\nu}(s) = (s - s_o) P^{\mu\nu}(\widehat{x}_R); \tag{30}$$

- (2) $P^{\mu\nu}(\widehat{x}_R)$ and $V_1(g, \widehat{x}, r, s)$ are such that

$$\begin{aligned} & \left. \frac{\partial V_1(g, \widehat{x}_R(r), r, s)}{\partial g^{\mu\nu}(s)} \right|_{g^{\mu\nu}(s)=\widehat{g}^{\mu\nu}(s)} - P_{\mu\nu}(\widehat{x}_R) \\ &= \sigma\kappa \widehat{R}_{\mu\nu} - \sigma\kappa \widehat{g}_{\mu\nu}(s) \frac{1}{2} (\widehat{R} - 2\Lambda) - \sigma\kappa \frac{8\pi G}{c^2} T_{\mu\nu}. \end{aligned} \tag{31}$$

Hence, a particular possible realization which leads to a functionally-different prescription of the potential density, and hence of the same Hamiltonian structure, is provided, for example, by the setting

$$\begin{cases} V_1(g, \widehat{x}_R(r), r, s) \equiv \kappa h g^{\mu\nu} \widehat{R}_{\mu\nu} + V_F(g, \widehat{x}, r), \\ U_o(\widehat{g}, \widehat{x}, s) = -2\kappa\Lambda, \\ P_{\mu\nu}(\widehat{x}_R) = -\sigma\kappa \widehat{g}_{\mu\nu}(s)\Lambda, \end{cases} \tag{32}$$

with $V_F(g, \widehat{x}, r)$ being given by Eq. (10). The present example means that the contribution of the cosmological constant in the Einstein field equations can also be interpreted as arising due to a non-vanishing canonical momentum of the form given by Eq. (30). Alternatively, a realization of the Einstein field equations with vanishing cosmological constant ($\Lambda \equiv 0$) can be achieved in terms of the potential density $V(g, \widehat{x}_R(r), r, s)$ of the form given above by Eq. (10), while setting at the same time

$$P_{\mu\nu}(\widehat{x}_R) = \sigma\kappa \widehat{g}_{\mu\nu}(s)\Lambda, \tag{33}$$

where now Λ can be interpreted as an arbitrary real 4-scalar.

From the previous considerations it follows, however, that if a solution of the type (30) is permitted the actual identification of the variational potential density remains essentially undetermined. It is, however, obvious that once the requirement $P_{\mu\nu}(\widehat{x}_R) \equiv 0$, or equivalently the constraint condition introduced above (27) are set, the transformations (29) reduce necessarily to the trivial one, namely

$$\begin{cases} g_{\mu\nu}(s) \rightarrow g_{\mu\nu}(s), \\ \pi^{\mu\nu}(s) \rightarrow \pi^{\mu\nu}(s), \\ V(g, \widehat{x}, r, s) \rightarrow V_1(g, \widehat{x}, r, s) + U_o(\widehat{g}, \widehat{x}, s), \end{cases} \tag{34}$$

which leaves unaffected the CHS. As a consequence, in validity of the constraint (27), the same Hamiltonian structure remains uniquely determined.

3 Manifestly covariant Hamilton–Jacobi theory

From the results established in the previous section, it follows that, thanks to the realization introduced here for the GR-Hamilton equations of CCG-theory (i.e. Eq. (2)), the same take the form of dynamical evolution equations. This follows as a consequence of the parametrization in terms of the proper time s adopted for all geodetics belonging to the background space-time. This feature permits one to develop in a standard way, in close analogy with classical Hamiltonian mechanics, the theory of canonical transformations. Given these premises, in this section the problem of constructing a Hamilton–Jacobi theory of GR is addressed. Such a theory should describe a dynamical flow connecting a generic phase-space state with a suitable initial phase-space state charac-

terized by identically-vanishing (i.e., stationary with respect to s) coordinate fields and momenta. In view of the similarity of the LP formalism for GR with classical mechanics, it is expected that also in the present context the Hamilton–Jacobi theory follows from constructing a symplectic canonical transformation associated with a mixed-variable generating function of type $S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)$.

Accordingly, the transformed canonical state $X_R \equiv \{G_{\mu\nu}, P_{\mu\nu}\}$ must satisfy the constraint equations

$$\frac{D}{Ds} P_{\mu\nu}(s_0) = 0, \tag{35}$$

$$\frac{D}{Ds} G^{\mu\nu}(s_0) = 0, \tag{36}$$

which imply the Hamilton equations

$$0 = [P_{\mu\nu}, K_R(X_R, \widehat{X}_R, r, s)]_{(X_R)}, \tag{37}$$

$$0 = [G^{\mu\nu}, K_R(X_R, \widehat{X}_R, r, s)]_{(X_R)}, \tag{38}$$

where $K_R(X_R, \widehat{X}_R, r, s)$ is the transformed Hamiltonian given by

$$K_R(X_R, \widehat{X}_R, r) = H_R(x_R, \widehat{x}_R, r, s) + \frac{\partial}{\partial s} S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s). \tag{39}$$

Thanks to Eqs. (37) and (38), the transformed Hamiltonian is necessarily independent of X_R . As a consequence, K_R identifies an arbitrary gauge function, i.e., in actual fact $K_R = K_R(\widehat{x}_R, r)$, which can always be set equal to zero ($K_R = 0$). On the other hand, canonical transformation theory requires that it must be

$$\pi_{i\xi} = \frac{\partial S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)}{\partial g^{i\xi}}, \tag{40}$$

$$G^{i\xi} = \frac{\partial S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)}{\partial P_{i\xi}}. \tag{41}$$

Then, introducing the s -parametrization, it follows that Eq. (39) yields

$$H_R\left(g^{\beta\gamma}, \frac{\partial S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)}{\partial g^{i\xi}}, \widehat{x}_R, r, s\right) + \frac{\partial}{\partial s} S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s) = 0, \tag{42}$$

which realizes the GR-Hamilton–Jacobi equation for the mixed-variable generating function $S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)$. Due to its similarity with the customary Hamilton–Jacobi equation, well known in Hamiltonian classical dynamics, in the following S will refer to the (classical) GR-Hamilton principal function. The canonical transformations generated by $S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, s)$ are then obtained by the set of Eqs. (40)–(42).

Now we notice, in view of the discussions given above, that the inverse canonical transformation $X_R \rightarrow x_R$ locally exists provided the invertibility condition on the Hessian

determinant $\det \left[\left[\frac{\partial^2 S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)}{\partial g^{\rho\sigma} \partial P_{i\xi}} \right]_{X_R = \widehat{x}_R} \right] \neq 0$ is met. Under such a condition the direct canonical equation (41) determines $g^{\beta\gamma}$ as an implicit function of the form $g^{\beta\gamma} = g^{\beta\gamma}(G^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)$.

The following statement holds on the relationship between the GR-Hamilton–Jacobi and the GR-Hamilton equations.

THM.1 – equivalence of GR-Hamilton and GR-Hamilton–Jacobi equations

The GR-Hamilton–Jacobi equation (42) subject to the constraint (40) is equivalent to the set of GR-Hamilton equations expressed in terms of the initial canonical variables, as given by Eq. (2).

Proof Without loss of generality and avoiding possible misunderstandings, the compact notation $S(g, P, \widehat{x}_R, r, s)$ will be used in the following proof to denote the GR-Hamilton principal function. To start with, we evaluate first the partial derivative of Eq. (42) with respect to g^{ik} , keeping both $\frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{i\xi}}$ and r^μ constant. This gives

$$\frac{\partial}{\partial g^{ik}} H_R\left(g^{\beta\gamma}, \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{i\xi}}, \widehat{x}_R, r, s\right) + \frac{\partial}{\partial s} \left[\frac{\partial}{\partial g^{ik}} S(g, P, \widehat{x}_R, r, s) \right]_{(g, P)} = 0. \tag{43}$$

Then let us evaluate in a similar manner the partial derivative with respect to $\frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{ik}}$, keeping $g^{\mu\nu}$ and r^μ constant. This gives

$$\frac{\partial}{\partial \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{ik}}} H_R\left(g^{\beta\gamma}, \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{i\xi}}, \widehat{x}_R, r, s\right) + \left[\frac{\partial}{\partial \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{ik}}} \frac{\partial}{\partial s} S(g, P, \widehat{x}_R, r, s) \right]_{(g, P)} = 0. \tag{44}$$

With the identification $\pi_{i\xi} = \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{i\xi}}$ provided by Eq. (40) it follows that Eq. (43) becomes

$$\frac{\partial}{\partial g^{ik}} H_R(g^{\beta\gamma}, \pi_{i\xi}, \widehat{x}_R, r) + \frac{D}{Ds} \pi_{ik} = 0, \tag{45}$$

which coincides with the second Hamilton equation in (2). To prove also the validity of the Hamilton equation for $g^{\beta\gamma}$ we first invoke the following identity:

$$\left[\frac{\partial}{\partial \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{ik}}} \frac{\partial}{\partial s} S(g, P, \widehat{x}_R, r, s) \right]_{(g, P)} = \frac{\partial}{\partial \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{ik}}} \frac{\partial}{\partial s} S(g, P, \widehat{x}_R, r, s) - \frac{D}{Ds} g^{\beta\gamma} \frac{\partial}{\partial \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{ik}}} \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{\beta\gamma}}, \tag{46}$$

where

$$\frac{\partial}{\partial \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{ik}}} \frac{\partial S(g, P, \widehat{x}_R, r, s)}{\partial g^{\beta\gamma}} = \delta^i_\beta \delta^k_\gamma. \tag{47}$$

The first term on the r.h.s. of Eq. (46) vanishes identically because $\frac{\partial}{\partial s} S(g, P, \widehat{x}_R, r, s)$ must be considered as independent of π_{ik} . Therefore, Eq. (44) gives

$$\frac{\partial}{\partial \pi_{ik}} H_R(g^{\beta\gamma}, \pi_{i\xi}, \widehat{x}_R, r, s) - \frac{D}{Ds} g^{ik} = 0, \tag{48}$$

which coincides with the Hamilton equation for g^{ik} and gives also the relationship of the generalized velocity $\frac{D}{Ds} g^{ik}$ with the canonical momentum, since here no explicit s -dependence appears. This proves the equivalence between the GR-Hamilton–Jacobi and GR-Hamilton equations, both expressed in manifestly covariant form. \square

This conclusion recovers the relationship between Hamilton and Hamilton–Jacobi equations holding in Hamiltonian Classical Mechanics for discrete dynamical systems. The connection is established also in the present case for the continuum gravitational field thanks to the manifestly covariant LP-parametrization of the theory and the representation of the Hamiltonian and Hamilton–Jacobi equations as dynamical evolution equations with respect to the proper time s characterizing background geodetics. The physical interpretation which follows from the validity of THM.1 is remarkable. This concerns the meaning of the Hamilton–Jacobi theory in providing a wave mechanics description of the continuum Hamiltonian dynamics. This follows also in the present context by comparing the mathematical structure of the Hamilton–Jacobi equation (42) with the well-known eikonal equation of geometrical optics. In fact, Eq. (42) contains the squared of the derivative $\frac{\partial S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)}{\partial g^{i\xi}}$, so that the Hamilton principal function $S(g^{\beta\gamma}, P_{\mu\nu}, \widehat{x}_R, r, s)$ is associated with the eikonal (i.e., the phase of the wave), while the remaining contributions due to the geometrical and physical properties of the curved space-time formally play the role of a non-uniform index of refraction in geometrical optics [49]. The outcome pointed out here proves that the dynamics of the field $g_{\mu\nu}(s)$ in the virtual domain of variational fields where the Hamiltonian structure is defined and the Hamilton–Jacobi theory (42) applies must be characterized by a wave-like behavior and can therefore be given a geometrical optics interpretation. This feature is expected to be crucial for the establishment of the corresponding manifestly covariant quantum theory of the gravitational field.

An important qualitative feature must be pointed out regarding the Hamilton–Jacobi theory developed here. This

refers to a formal difference arising between the Hamilton–Jacobi theory for continuum fields built on the DeDonder–Weyl covariant approach and the Hamilton–Jacobi theory holding in classical mechanics for particle dynamics. This concerns, more precisely, the dimensional units to be adopted for the Hamilton principal function S and hence also the canonical momentum $\pi^{\mu\nu}$. Indeed, as is well known, in particle dynamics S retains the dimension of an action (and therefore of the action functional), so that $[S] = [\hbar]$. In the present case instead (see Eq. (7)) one has $[S] = [\hbar L^{-3}]$, namely the dimension of S differs from that of an action by the cubic length L^{-3} . This arises because for continuum fields the action functional is an integral over the 4-volume element of the Hamiltonian density, while for particle mechanics it is expressed as a line integral over the proper-time length. One has to notice, however, that, first, the dimensions of S may be changed by the introduction of a non-symplectic canonical transformation. This means that, by a suitable choice of the same transformation, S can actually recover the dimension of an action. Second, the relationship between the Hamilton principal function S and the Hamiltonian function itself remains in all cases the same, with the two functions differing by the dimension of a length (see also Sect. 4 below).

Before concluding, the following additional remarks are in order:

- (1) The GR-Hamilton–Jacobi description permits one to construct explicitly canonical transformations mapping in each other the physical and virtual domains. The generating function determined by the GR-Hamilton–Jacobi equation is a real 4-scalar field.
- (2) The generating function obtained in this way realizes the particular subset of canonical transformations which map the physically-observable state \widehat{x}_R into a neighboring admissible virtual canonical state x_R .
- (3) The virtue of the approach is that it preserves the validity of the Einstein equation in the physical domain. In other words, the canonical transformations do not affect the physical behavior.
- (4) A further issue concerns the connection between the same prescribed metric tensor $\widehat{g}(r) \equiv \{\widehat{g}_{\mu\nu}(r)\}$ and the variational/extremal state $x_R(s) = \{g(s), \pi(s)\}$. This can be obtained by establishing a proper statistical theory achieved by considering the initial state

$$x_R(s_o) = \widehat{x}(s_o) + \delta x_R(s_o) \tag{49}$$

as a stochastic tensor and thus endowed with a suitable phase-space probability density. The topic can be developed in the framework of a statistical description of classical gravity to be discussed elsewhere in detail.

4 Properties of CHS

In this section some properties are discussed which characterize the manifestly covariant Hamiltonian theory of GR.

4A – global prescription and regularity

This refers, first of all, to the global prescription and regularity of the GR-Lagrangian and GR-Hamiltonian densities $L_R \equiv L_R(y, \widehat{g}, r, s)$ and $H_R \equiv H_R(x, \widehat{g}, r, s)$ defined according to Eqs. (17) and (7) which are associated with the corresponding Hamiltonian structure $\{x_R, H_R\}$ indicated above. For this purpose we notice that in Eq. (17) the effective kinetic and potential densities expressed in terms of the Lagrangian state $y = \{g_{\mu\nu}, \frac{D}{Ds}g^{\mu\nu}\}$ and the LP-parametrization (20) can be taken to be of the general type

$$\begin{cases} T_R(y, \widehat{g}) = \frac{f(h)}{2\kappa} \frac{D}{Ds}g_{\mu\nu} \frac{D}{Ds}g^{\mu\nu}, \\ V(g, \widehat{g}, r, s) \equiv \sigma V_o(g, \widehat{g}) + \sigma V_F(g, \widehat{g}, r, s). \end{cases} \quad (50)$$

Here, $f(h)$ and σ identify the two distinct multiplicative gauge functions introduced above (see Sect. 2), κ is the dimensional constant $\kappa = \frac{c^3}{16\pi G}$, while the rest of the notations is expressed in standard form according to Refs. [5,6]. More precisely, in the second equation $V_o(g, \widehat{g})$ and $V_F(g, \widehat{g}, r, s) \equiv hL_F(g, \widehat{g}, r, s)$ are defined as in Eq. (10) and must be expressed here in Lagrangian variables, with the field Lagrangian $L_F(g, \widehat{g}, r, s)$ being prescribed according to Ref. [5]. Furthermore, $T_R(y, \widehat{g})$ identifies the *generic form* of the effective kinetic density. It follows that a sufficient condition for the global prescription of the canonical state, i.e., the existence of a smooth bijection connecting the Lagrangian and Hamiltonian states is the so-called *regularity condition* of the GR-Hamiltonian (and corresponding GR-Lagrangian) density. This requires more precisely that in the whole Hamiltonian phase space

$$\left| \frac{\partial^2 H_R}{\partial \pi_{\mu\nu} \partial \pi^{\alpha\beta}} \right| \equiv \left| \frac{\partial^2 T_R}{\partial \pi_{\mu\nu} \partial \pi^{\alpha\beta}} \right| = \frac{1}{\kappa f(h)} \neq 0. \quad (51)$$

4B – gauge indeterminacies of CHS

As shown in Ref. [6] at the classical level the Hamiltonian structure $\{x_R, H_R\}$ of SF-GR remains intrinsically non-unique, with the Hamiltonian density H_R being characterized by suitable gauge indeterminacies. Leaving aside the treatment of additive gauge functions earlier discussed in Refs. [5,6], these refer more precisely to the following properties:

- (A) *The first one is the so-called multiplicative gauge transformation of the effective kinetic density.* To identify it we notice that the scalar factor $f(h)$ appearing in the prescriptions of the effective kinetic density (see first

equation in (50)) remains in principle essentially indeterminate. In fact the regularity condition (51) requires only that

$$\left| \frac{\partial^2 T_R}{\partial \pi_{\mu\nu} \partial \pi^{\alpha\beta}} \right| = \frac{1}{\kappa f(h)} \neq 0, \quad (52)$$

implying that the function $f(h)$ can be realized by an arbitrary non-vanishing (for example, strictly positive) and suitably smooth dimensionless real function. In addition, in order that both $T_R(y, \widehat{g})$ and $V(y, \widehat{g})$ (see again Eq. (50)) are realized by means of integrable functions in the configurations space U_g , accordingly the functions $f(h)$ and $1/f(h)$ should be summable too. As a consequence the prescription of $f(h)$ remains in principle still free within the classical theory of SF-GR developed in Sect. 2. This means that $f(h)$ should be intended in such a context as a gauge indeterminacy affecting the Hamiltonian density H_R , i.e., with respect to the *multiplicative gauge transformation*

$$\begin{cases} T_R(x_R, \widehat{g}) \equiv \frac{1}{2\kappa} \pi_{\mu\nu} \pi^{\mu\nu} \rightarrow T'_R(x_R, \widehat{g}) \\ \equiv \frac{1}{2\kappa f(h)} \pi'_{\mu\nu} \pi'^{\mu\nu} = \frac{1}{f(h)} T_R(x_R, \widehat{g}), \\ \pi_{\mu\nu} \rightarrow \pi'_{\mu\nu} = f(h) \pi_{\mu\nu}. \end{cases} \quad (53)$$

- (B) *The second indeterminacy is related to the so-called multiplicative gauge transformation of the effective potential density $V(g, \widehat{g}, r, s)$* (see again Sect. 2). More precisely the indeterminacy is related to the constant gauge factor $\sigma = \pm 1$.
- (C) *The third indeterminacy is related to the so-called additive gauge transformation.* Indeed, one can readily show (see Ref. [6]) that $L_R(y, \widehat{g}, s)$ is prescribed up to an arbitrary additive gauge transformation of the type

$$L_R(y, \widehat{g}, r, s) \rightarrow L_R(y, \widehat{g}, r, s) + \Delta V \quad (54)$$

being ΔV a gauge scalar field of the form $\Delta V = \frac{DF(g, \widehat{g}, r, s)}{Ds}$, with $F(g, \widehat{g}, s)$ being an arbitrary, suitably-smooth real gauge function of class $C^{(2)}$ with respect to the variables (g, s) (see also related discussion in Ref. [5]).

4C – dimensional normalization of CHS

In this section it is shown that the Hamiltonian structure $\{x_R, H_R\}$ can be equivalently realized in such a way that x_R , and consequently also H_R , can be suitably normalized, i.e., so that to achieve prescribed physical dimensions. Granted the non-symplectic canonical nature of the transformation indicated above (i.e., Eq. (1)) one can always identify the 4-scalar α with a classical invariant parameter, i.e., both frame-independent and space-time independent. In particular, in the

framework of a classical treatment it should be identified with the classical parameter $\alpha \equiv \alpha_{\text{Classical}}$, being

$$\alpha_{\text{Classical}} = m_o c L > 0, \tag{55}$$

with c being the speed of light in vacuum, m_o a suitable rest-mass (to be later identified with the non-vanishing graviton mass in the framework of quantum theory of GR) and L a characteristic scale length to be considered as an invariant non-null 4-scalar. Without loss of generality this can always be assumed of the form

$$L = L(m_o), \tag{56}$$

with m_o itself being regarded as an invariant 4-scalar. Here L is regarded as a classical invariant parameter, so that it should remain independent of all quantum parameters, i.e., in particular \hbar . In addition, in view of the covariance property of the theory, whereby the choice of the background space-time $(\mathbf{Q}^4, \widehat{g}(r))$ is in principle arbitrary, the 4-scalars m_o and L should be *universal constants*, namely also invariant with respect to the action of local and non-local point transformations [13]. As an example, a possible consistent choice for the invariant function $L(m_o)$ is realized by means of the so-called Schwarzschild radius, i.e.,

$$L(m_o) = \frac{2m_o G}{c^2}. \tag{57}$$

The invariant rest-mass m_o remains, however, still arbitrary at this level, its prescription being left to quantum theory. It follows that the transformed GR-Hamilton equations (2) can always be cast in the *dimensional normalized form*

$$\begin{cases} \frac{Dg_{\mu\nu}}{Ds} = \frac{\partial \overline{H}_R}{\partial \overline{\pi}^{\mu\nu}} \equiv \frac{\partial H_R}{\partial \pi^{\mu\nu}}, \\ \frac{D\overline{\pi}^{\mu\nu}}{Ds} = -\frac{\partial \overline{H}_R}{\partial g^{\mu\nu}} \equiv -\frac{\alpha L}{k} \frac{\partial H_R}{\partial g^{\mu\nu}}, \end{cases} \tag{58}$$

where the transformed Hamiltonian \overline{H}_R identifies the *normalized GR-Hamiltonian density*

$$\overline{H}_R(\overline{x}_R, \widehat{g}, r, s) = \frac{1}{f(\hbar)} \overline{T}_R(\overline{x}_R, \widehat{g}, r, s) + \overline{V}(\widehat{g}, \widehat{g}, r, s). \tag{59}$$

Here the notation is as follows. First, for an arbitrary curved space-time $(\mathbf{Q}^4, \widehat{g}(r))$ the functions \overline{T}_R and \overline{V} now are identified with

$$\begin{cases} \overline{T}_R(\overline{x}_R, \widehat{g}, r, s) = \frac{\overline{\pi}^{\mu\nu} \overline{\pi}_{\mu\nu}}{2\alpha L}, \\ \overline{V}(g, \widehat{g}, r, s) \equiv \sigma \overline{V}_o(g, \widehat{g}, r, s) + \sigma \overline{V}_F(g, \widehat{g}, r, s), \end{cases} \tag{60}$$

so that $\overline{T}_R(\overline{x}_R, \widehat{g}, r, s)$ identifies the normalized effective kinetic density and \overline{V} by analogy is the corresponding normalized effective potential density, with $\overline{V}_o(g, \widehat{g})$ and $\overline{V}_F(g, \widehat{g}, r, s)$ now being prescribed, respectively, in terms of $V_o(g, \widehat{g})$ and $V_F(g, \widehat{g}, r, s)$ as

$$\begin{cases} \overline{V}_o(g, \widehat{g}) \equiv \hbar \alpha L [g^{\mu\nu} \widehat{R}_{\mu\nu} - 2\Lambda], \\ \overline{V}_F(g, \widehat{g}, r, s) \equiv \frac{\hbar \alpha L}{2k} L_F(g, \widehat{g}, r, s). \end{cases} \tag{61}$$

From the canonical equations (58) it is obvious that by construction the transformed canonical momentum $\overline{\pi}^{\mu\nu}$ takes the dimensions of an action, i.e., $[\overline{\pi}^{\mu\nu}] = [\hbar]$. The set $\{\overline{x}_R, \overline{H}_R\}$ thus provides an admissible representation of CHS. In particular it follows that the GR-Hamilton equations in normalized form become, respectively,

$$\begin{cases} \frac{Dg_{\mu\nu}}{Ds} = \frac{\overline{\pi}_{\mu\nu}}{\alpha L}, \\ \frac{D\overline{\pi}^{\mu\nu}}{Ds} = -\frac{\partial \overline{V}(g, \widehat{g}, r, s)}{\partial g^{\mu\nu}}, \end{cases} \tag{62}$$

with the operator D/Ds being prescribed again in terms of the corresponding prescribed metric tensors $\widehat{g}_{\mu\nu}(r)$.

For completeness, we mention here also the normalized Hamilton–Jacobi equation corresponding to the canonical equations (62). This is reached introducing the corresponding normalized Hamilton principal function $S(g, P, \widehat{g}, r, s)$, i.e., the mixed-variable generating function for the canonical transformation

$$x_R(s_o) \equiv (\mathbf{G}_{\mu\nu}, \mathbf{P}^{\mu\nu}) \Leftrightarrow x(s) \equiv (g_{\mu\nu}(s), \pi^{\mu\nu}(s)), \tag{63}$$

with $x_R(s_o) \equiv (\mathbf{G}_{\mu\nu}, \mathbf{P}^{\mu\nu})$ denoting the initial canonical GR-state. Then $S(g, \mathbf{P}, \widehat{g}, r, s)$ is prescribed in such a way that the normalized canonical momentum $\pi^{\mu\nu}(s)$ is given by $\overline{\pi}_{\mu\nu} = \frac{\partial S(g, \mathbf{P}, \widehat{g}, r, s)}{\partial g^{\mu\nu}}$, while the initial canonical coordinate $\mathbf{G}_{\mu\nu}$ is determined by the inverse canonical transformation $\mathbf{G}_{\mu\nu} = \frac{\partial S(g, \mathbf{P}, \widehat{g}, r, s)}{\partial \mathbf{P}^{\mu\nu}}$. It follows that the corresponding dimensionally normalized Hamilton–Jacobi equation which is equivalent to Eqs. (62) is provided by

$$\frac{\partial S(g, \mathbf{P}, \widehat{g}, r, s)}{\partial s} + \overline{H}_R \left(g, \overline{\pi} \equiv \frac{\partial S(g, \mathbf{P}, \widehat{g}, r, s)}{\partial g}, \widehat{g}, r, s \right) = 0, \tag{64}$$

with \overline{H}_R being prescribed by Eq. (59).

5 Structural stability of the GR-Hamilton equations

In this section we present an application of the GR-Hamiltonian theory for the Einstein field equations developed in this paper, which is represented by Eq. (2) or equivalently by the Hamilton–Jacobi equation (42). This refers to the stability of the GR-Hamilton equations with respect to their

stationary solution. As shown above the latter realizes by construction a solution of the Einstein field equations (25) in its most general form, i.e., in the presence of arbitrary external sources. Therefore the task to be addressed concerns the so-called *structural stability* of the GR-Hamilton equations (with respect to the Einstein field equations), namely the stability of stationary solutions of the GR-Hamilton equations assuming that the perturbed fields realize particular solutions of the same GR-Hamilton equations.

As a first illustration of the problem, here we consider the case of arbitrary vacuum solutions realized by setting a vanishing stress-energy tensor ($\widehat{T}_{\mu\nu} = 0$) and possibly retaining also a non-vanishing cosmological constant as corresponds to de Sitter ($\Lambda > 0$) or anti-de Sitter ($\Lambda < 0$) space-times.

Let us address the problem in the context of the reduced continuum manifestly covariant Hamiltonian theory. The study is supported by the conclusions concerning the wave mechanics interpretation of the reduced continuum Hamiltonian dynamics of the gravitational field determined by the Hamilton–Jacobi theory. For this purpose we shall consider perturbations of the reduced canonical state $x_R(s)$ which are suitably close to $\widehat{x}_R(s)$, namely of the form

$$g^{\mu\nu}(s) = \widehat{g}^{\mu\nu}(s) + \varepsilon \delta g^{\mu\nu}(s), \tag{65}$$

$$\pi_{\mu\nu}(s) = \varepsilon \delta \pi_{\mu\nu}(s), \tag{66}$$

where $\varepsilon \ll 1$ is an infinitesimal dimensionless parameter identifying the perturbations of the canonical fields. In particular, consistent with the existence of a Hamilton–Jacobi theory and its physical interpretation pointed out above, we are authorized to consider a wave-like form of the perturbations. These are assumed to propagate along field geodesics, namely the same perturbations of the canonical fields ($\delta g^{\mu\nu}(s), \delta \pi_{\mu\nu}(s)$) are taken of the form

$$\delta g^{\mu\nu}(s) = \delta \widehat{g}^{\mu\nu}(\widehat{g}(s)) \exp\{G(s)\}, \tag{67}$$

$$\delta \pi_{\mu\nu}(s) = \delta \widehat{\pi}_{\mu\nu}(\widehat{g}(s)) \exp\{G(s)\}. \tag{68}$$

Here $G(s)$ denotes the eikonal

$$G(s) = i \frac{\omega}{c} s - i K r^\mu(s) t_\mu(s), \tag{69}$$

with ω and K being 4-scalar parameters which, by construction, have, respectively, the dimensions of a frequency and that of a wave number (i.e., the inverse of a length). Therefore, denoting $K \equiv 1/\lambda$, according to the representation (69), ω and λ identify the invariant frequency and wave-length of the wave-like perturbations of the canonical fields. The invariant character of ω and λ is a characteristic feature of the manifestly covariant Hamiltonian theory.

It is then immediate to show that, in terms of the canonical evolution equations (20), the following set of linear differential equations advancing in proper time the perturbed fields

$\delta g^{\mu\nu}(s)$ and $\delta \pi^{\mu\nu}(s)$ and accurate through $O(\varepsilon)$ is obtained thanks also to the requirement (9):

$$\frac{D}{Ds} \delta g^{\mu\nu}(s) = \frac{1}{\kappa} \delta \pi^{\mu\nu}(s), \tag{70}$$

$$\begin{aligned} \frac{D}{Ds} \delta \pi_{\mu\nu}(s) &= \frac{\sigma}{2} \kappa (\widehat{R} + 2\Lambda) \delta g_{\mu\nu}(s) \\ &+ \frac{\sigma}{2} \kappa (\widehat{g}_{\mu\nu}(s) \widehat{R}_{\alpha\beta} + \widehat{R}_{\mu\nu} \widehat{g}_{\alpha\beta}(s)) \delta g^{\alpha\beta}. \end{aligned} \tag{71}$$

In particular, introducing the representation (69) and recalling the definition of the differential operator $\frac{D}{Ds}$, the first equation yields a unique relationship between $\delta \pi^{\mu\nu}(s)$ and $\delta g^{\mu\nu}(s)$, namely $\delta \pi^{\mu\nu}(s) = i \kappa [\frac{\omega}{c} - K] \delta g^{\mu\nu}(s)$. Then Eq. (71) determines the algebraic linear equation for $\delta g_{\mu\nu}(s)$:

$$\begin{aligned} &\left(-\kappa \left[\frac{\omega}{c} - K \right]^2 - \frac{\sigma}{2} \kappa [\widehat{R} + 2\Lambda] \right) \delta g_{\mu\nu}(s) \\ &= \frac{\sigma \kappa}{2} (\widehat{g}_{\mu\nu}(s) \widehat{R}_{\alpha\beta} + \widehat{R}_{\mu\nu} \widehat{g}_{\alpha\beta}(s)) \delta g^{\alpha\beta}(s). \end{aligned} \tag{72}$$

To solve it explicitly one needs to determine the corresponding algebraic equations holding for the independent tensor products $\widehat{g}_{\mu\nu}(s) \delta g^{\alpha\beta} \widehat{R}_{\alpha\beta}$ and $\widehat{g}_{\alpha\beta}(s) \delta g^{\alpha\beta} \widehat{R}_{\mu\nu}$ appearing on the r.h.s. of the same equation. Thus, by first multiplying tensorially term by term Eq. (72) by $\widehat{R}^{\mu\nu}$ it follows

$$\begin{aligned} &\left(-\left[\frac{\omega}{c} - K \right]^2 - \sigma [\widehat{R} + \Lambda] \right) \widehat{R}^{\mu\nu} \delta g_{\mu\nu}(s) \\ &= \frac{\sigma}{2} \widehat{R}_{\mu\nu} \widehat{R}^{\mu\nu} \widehat{g}_{\alpha\beta}(s) \delta g^{\alpha\beta}(s). \end{aligned} \tag{73}$$

Next, multiplying tensorially term by term Eq. (72) by $\widehat{g}^{\mu\nu}(s)$ one obtains instead

$$\begin{aligned} &\left(-\left[\frac{\omega}{c} - K \right]^2 - \sigma [\widehat{R} + \Lambda] \right) \widehat{g}^{\mu\nu}(s) \delta g_{\mu\nu}(s) \\ &= 2\sigma \widehat{R}_{\alpha\beta} \delta g^{\alpha\beta}(s). \end{aligned} \tag{74}$$

Combining together Eqs. (73) and (74) one is finally left with the equation

$$\left(-\left[\frac{\omega}{c} - K \right]^2 - \sigma [\widehat{R} + \Lambda] \right)^2 - \widehat{R}_{\mu\nu} \widehat{R}^{\mu\nu} = 0, \tag{75}$$

which identifies the dispersion relation between ω and K , i.e., the condition under which in the context of the reduced continuum Hamiltonian theory the infinitesimal perturbations (67) and (68) can occur.

To analyze in terms of Eq. (75) the conditions for the existence of stable, marginally stable or unstable oscillatory solutions for the canonical fields ($\delta g^{\mu\nu}(s), \delta \pi_{\mu\nu}(s)$), it is convenient to classify the possible complex roots for ω which can locally occur. Such a classification has therefore necessarily a local character. More precisely, single roots of this equation such that locally (a) $Im(\omega) < 0$, (b) $Im(\omega) = 0$ or (c) $Im(\omega) > 0$ will be referred to as (locally) stable,

marginally stable and unstable, respectively. Correspondingly, the perturbation $(\delta g^{\mu\nu}(s), \delta \pi_{\mu\nu}(s))$ will be classified to be locally (a) decaying, (b) oscillatory, (c) growing. Therefore, the equilibrium solution $\widehat{g}(r)$ will be denoted as (a) locally stable, (b) locally marginally stable and (c) locally unstable, respectively, if:

- (a) all roots of ω are locally stable: namely for them $Im(\omega) < 0$;
- (b) there is at least one root of ω which is locally marginally stable, namely such that $Im(\omega) = 0$;
- (c) there is at least one locally unstable root of ω , namely such that $Im(\omega) > 0$.

To investigate the stability problem we consider the vacuum configuration in which $T_{\mu\nu} \equiv 0$, but still $\Lambda \neq 0$, so that $\widehat{R}_{\mu\nu} = \Lambda \widehat{g}_{\mu\nu}$ and $\widehat{R} = 4\Lambda$ is the constant Ricci scalar. Then Eq. (75) yields the dispersion relation in the explicit form

$$\left(-\left[\frac{\omega}{c} - K\right]^2 - 5\sigma\Lambda\right)^2 - 4(-\sigma\Lambda)^2 = 0, \tag{76}$$

which yields the roots,

$$\frac{\omega}{c} - K = \left\{ \begin{array}{l} \pm\sqrt{-7\sigma\Lambda} \\ \pm\sqrt{-3\sigma\Lambda} \end{array} \right. . \tag{77}$$

Therefore, two possible alternatives can be distinguished in which respectively:

- (A) *First case:* $-\sigma\Lambda \geq 0$. Then the equilibrium solution $\widehat{g}(r)$ is marginally stable since all roots of the dispersion relation (76) have vanishing imaginary part.
- (B) *Second case:* $-\sigma\Lambda < 0$. Then $\widehat{g}(r)$ is necessarily unstable (there exists always an unstable root of the same Eq. (76)).

Hence, both for the case of de Sitter and anti-de Sitter space-times (Λ , respectively, > 0 or < 0) the possible occurrence of stability or instability depends on the multiplicative gauge parameter σ appearing in the definition of the effective potential density V (see the second equation given above in (60)). However, a physically admissible Hamiltonian theory of GR should predict stable solutions, i.e., which are structurally stable in the sense indicated above. This should occur in principle not just for special realizations of the background space-time but – at least in the case of vacuum – *for arbitrary vacuum background space-times* $(\mathbf{Q}^4, \widehat{g}(r))$. In particular, if $\Lambda > 0$ – as most frequently invoked in the literature (see for example [50,51]) – this happens provided the gauge factor σ is uniquely identified with $\sigma = -1$. Although the rigorous proof of the validity of such a choice still remains a mere conjecture at this point, its full justification should

ultimately emerge from quantum theory. Nevertheless, the stability property pointed out here can be viewed as a prerequisite for the consistent development of a covariant quantum gravity theory. For this reason the issue will be further discussed in Part 2.

6 Conclusions

A common fundamental theoretical aspect laying at the foundation of both General Relativity (GR) and classical field theory is the variational character of the fundamental dynamical laws which identify these disciplines. This concerns both the representation of the Einstein field equations and the covariant dynamics of classical fields as well as of discrete (e.g., test particles) or continuum systems in curved space-time. Issues related to the variational formulation of the Einstein equations have been treated in Refs. [5,6], where the existence of a new type of Lagrangian and Hamiltonian variational approaches has been identified in terms of synchronous variational principles realized in the framework of the DeDonder–Weyl formalism. As shown in Ref. [6], this leads to the realization of a manifestly covariant Hamiltonian theory for the Einstein equations.

In this paper new aspects of the Hamiltonian structure of GR have been displayed which is referred to here as CCG-theory.

In particular, we have shown that a reduced-dimensional realization of the continuum Hamiltonian theory for the Einstein field equations, denoted here as Classical Hamiltonian Structure of GR (CHS), actually exists in which both generalized coordinates and corresponding conjugate momenta are realized by means of second-order 4-tensors. The virtue of such an approach lies precisely in its general validity. This means in fact that the same Hamiltonian structure holds for arbitrary particular solutions of the Einstein field equations and arbitrary realizations of the external source terms appearing in the variational potential density. As a result, a causal form has been obtained for the corresponding continuum Hamilton equations by introducing a suitable Lagrangian parametrization prescribed in terms of the proper time s defined along field geodetics of the curved space-time. As a result, the same equations are cast in the equivalent form of an initial-value problem for suitable canonical evolution equations, referred to here as GR-Hamilton equations. This provides a physical interpretation for the reduced Hamiltonian theory, according to which an arbitrary initial canonical state is dynamically advanced by means of the canonical flow generated by the same Hamilton equations.

Given validity to such Hamiltonian theory, the case of canonical transformations which generate the flow corresponding to the continuum canonical equations has been considered. This has been obtained by introducing the appro-

appropriate mixed-variable generating function – the so-called Hamilton principal function – and by developing the corresponding Hamilton–Jacobi theory in manifestly covariant form. As a result, the same generating function has been shown to obey a 4-scalar continuum Hamilton–Jacobi equation which has been proved to be equivalent to the corresponding canonical evolution equations of the Hamiltonian theory. The global prescription and regularity of the Hamiltonian structure have been analyzed and the gauge transformation properties of the reduced Hamiltonian density H_R have been pointed out.

Finally, as an application of the Hamiltonian formulation developed here, the structural stability of the Hamiltonian theory has been investigated, a feature which is required for a consistent development of a corresponding quantum theory of GR based on the same canonical representation. In this paper we have studied the stability of perturbed fields which realize particular solutions of the GR-Hamilton equations with respect to stationary solutions, i.e., metric tensor solutions of the Einstein field equations. The case of background vacuum solutions having vanishing stress-energy tensor and non-null cosmological constant has been analyzed, determining the conditions for the occurrence of stable and unstable roots, adopting an eikonal representation for the perturbed fields.

These conclusions highlight the key features of the reduced Hamiltonian theory and corresponding Hamilton–Jacobi equation determined here. The new theory, besides being a mandatory prerequisite for the covariant theory of quantum gravity to be established in Part 2, is believed to be susceptible of applications to physics and astrophysics-related problems and to provide at the same time new insights in the axiomatic foundations of General Relativity.

Acknowledgements Work developed within the research projects of the Czech Science Foundation GACR Grant No. 14-07753P (C.C.) and the Albert Einstein Center for Gravitation and Astrophysics, Czech Science Foundation No. 14-37086G (M.T.).

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made. Funded by SCOAP³.

References

1. A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 2004)
2. L.D. Landau, E.M. Lifschitz, *Field Theory, Theoretical Physics*, vol. 2 (Addison-Wesley, New York, 1957)
3. C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, 1st edn (W.H. Freeman, 1973)
4. R.M. Wald, *General Relativity*, 1st edn (University of Chicago Press, 1984)
5. C. Cremaschini, M. Tassarotto, Synchronous Lagrangian variational principles in general relativity. *Eur. Phys. J. Plus* **130**, 123 (2015)
6. C. Cremaschini, M. Tassarotto, Manifest covariant Hamiltonian theory of general relativity. *Appl. Phys. Res.* **8**, 2 (2016). doi:10.5539/apr.v8n2p60
7. C. Cremaschini, M. Tassarotto, Hamiltonian approach to GR – Part 2: covariant theory of quantum gravity. *Eur. Phys. J. C* (2017). doi:10.1140/epjc/s10052-017-4855-0
8. I.T. Drummond, *Phys. Rev. D* **63**, 043503 (2001)
9. J.W. Moffat, *Int. J. Mod. Phys. D* **12**, 281–298 (2003)
10. S. Hossenfelder, *Phys. Rev. D* **78**, 044015 (2008)
11. T. Padmanabhan, *Mod. Phys. Lett. A* **30**, 1540007 (2015)
12. S. Bhattacharya, S. Shankaranarayanan, *Int. J. Mod. Phys. D* **24**, 1544005 (2015)
13. M. Tassarotto, C. Cremaschini, Theory of nonlocal point transformations in general relativity. *Adv. Math. Phys.* **2016**, 9619326 (2016). doi:10.1155/2016/9619326
14. M. Tassarotto, C. Cremaschini, Generalized Lagrangian-path representation of non-relativistic quantum mechanics. *Found. Phys.* **46**(8), 1022–1061 (2016)
15. M. Tassarotto, M. Mond, D. Batic, Hamiltonian structure of the Schrödinger classical dynamical system. *Found. Phys.* **46**(9), 1127–1167 (2016)
16. C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **126**, 42 (2011)
17. C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **126**, 63 (2011)
18. C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **127**, 4 (2012)
19. C. Cremaschini, M. Tassarotto, *Phys. Rev. E* **87**, 032107 (2013)
20. C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **129**, 247 (2014)
21. C. Cremaschini, M. Tassarotto, *Eur. Phys. J. Plus* **130**, 166 (2015)
22. S.W. Hawking, *General Relativity*, in *An Einstein Centenary Survey*, ed. by S.W. Hawking, W. Israel (Cambridge University Press, Cambridge, 1979)
23. B.S. Dewitt, *Phys. Rev.* **162**, 1195 (1967)
24. P.G. Bergmann, R. Penfield, R. Schiller, H. Zatzkis, *Phys. Rev.* **80**, 81 (1950)
25. F.A.E. Pirani, A. Schild, R. Skinner, *Phys. Rev.* **87**, 452 (1952)
26. P.A.M. Dirac, *Proc. R. Soc. (London)* **A246**, 333 (1958)
27. P.A.M. Dirac, *Phys. Rev.* **114**, 924 (1959)
28. R. Arnowitt, S. Deser, C.W. Misner, *Gravitation: An Introduction to Current Research*, Witten edn. (Wiley, New York, 1962)
29. Th De Donder, *Théorie Invariante Du Calcul des Variations* (Gauthier-Villars & Cia, Paris, 1930)
30. H. Weyl, *Ann. Math.* **36**, 607 (1935)
31. D.J. Saunders, *The Geometry of Jet Bundles* (Cambridge University Press, Cambridge, 1989)
32. G. Sardanashvily, *Generalized Hamiltonian Formalism for Field Theory* (World Scientific, Singapore, 1995)
33. A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, *Fortschr. Phys.* **44**, 235 (1996)
34. I.V. Kanatchikov, *Rep. Math. Phys.* **41**, 49 (1998)
35. M. Forger, C. Paufler, H. Romer, *Rev. Math. Phys.* **15**, 705 (2003)
36. V.V. Kisil, *J. Phys. A Math. Gen.* **37**, 183 (2004)
37. J. Struckmeier, A. Redelbach, *Int. J. Mod. Phys. E* **17**, 435 (2008)
38. M. Marklund, P.K.S. Dunsby, G. Betschart, M. Servin, C.G. Tsagas, *Class. Quantum Gravity* **20**, 1823 (2003)
39. Z.B. Etienne, Y.T. Liu, S.L. Shapiro, *Phys. Rev. D* **82**, 084031 (2010)
40. F.A. Asenjo, S.M. Mahajan, A. Qadir, *Phys. Plasmas* **20**, 022901 (2013)
41. M. Alcubierre, *Introduction to 3 + 1 Numerical Relativity* (Oxford University Press, Oxford, 2008)
42. S. Vacaru, *J. Math. Phys.* **46**, 042503 (2005)
43. S. Vacaru, *Int. J. Geom. Methods Mod. Phys.* **4**, 1285–1334 (2007)

44. S. Vacaru, Int. J. Geom. Methods Mod. Phys. **8**, 9–21 (2011)
45. T. Clifton, G.F.R. Ellis, R. Tavakol, Class. Quantum Gravity **30**, 125009 (2013)
46. V. Ruchin, O. Vacaru, S. Vacaru, Eur. Phys. J. C **77**, 184 (2017)
47. S. Vacaru, J. Phys. Conf. Ser. **543**, 012021 (2013)
48. T. Gheorghiu, O. Vacaru, S. Vacaru, Eur. Phys. J. C **74**, 3152 (2014)
49. H. Goldstein, *Classical Mechanics*, 2nd edn (Addison-Wesley, New York, 1980)
50. S. Weinberg, The cosmological constant problem. Rev. Mod. Phys. **61**, 1–23 (1989)
51. S. Carroll, *Spacetime and Geometry* (Addison Wesley, San Francisco, 2004), pp. 171–174