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Reentrant phase transitions of higher-dimensional AdS black holes in dRGT massive gravity

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Abstract We study the P–V criticality and phase transition in the extended phase space of anti-de Sitter (AdS) black holes in higher-dimensional de Rham, Gabadadze and Tolley (dRGT) massive gravity, treating the cosmological constant as pressure and the corresponding conjugate quantity is interpreted as thermodynamic volume. Besides the usual small/large black hole phase transitions, the interesting thermodynamic phenomena of reentrant phase transitions (RPTs) are observed for black holes in all $d \geq 6$-dimensional spacetime when the coupling coefficients $c_i m^2$ of massive potential satisfy some certain conditions.

1 Introduction

Einstein's general relativity (GR) is a relativistic theory of gravity where the graviton is a massless spin-2 particle [1–3]. It is also the current description of gravitation in modern physics and has significant astrophysical implications. Nevertheless, whether there exist a consistent extension of GR by a mass term is a basic challenge of classical field theory, since the open questions such as the old cosmological constant problem and the origin of the late-time acceleration of the Universe remain behind the puzzles at the interface between gravity/cosmology and particle physics. In general, by adding generic mass terms for the gravitons on the given background usually brings about various instabilities for the gravitational theories, sometimes on the nonlinear level. A new nonlinear massive gravity theory was proposed by de Rham, Gabadadze and Tolley (dRGT) [4–6], where the Boulware–Deser ghost [7] was eliminated by introducing higher order interaction terms in the action. Until now, a nontrivial black hole solution with a Ricci flat horizon has

been constructed by Vegh [8,9] in four-dimensional dRGT massive gravity. Later, spherically symmetric solutions were also addressed in [10–12], and the corresponding charged black hole solution was found in [13], including its bi-gravity extension [14,15]. Moreover, the charged AdS black hole solution in higher-dimensional dRGT massive gravity, and its corresponding thermodynamics and phase structure in the grand canonical and canonical ensembles were also presented in [16]. Ge et al. [17] examined the relations between dynamical instabilities and thermodynamic instabilities in the dRGT massive gravity.

Recently, the study of thermodynamics in AdS black holes has been generalized to the extended phase space, where the cosmological constant is regarded as a variable and also identified with thermodynamic pressure [18,19]

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2} \quad (1)$$

in geometric units $G_N = \hbar = c = k = 1$. Here d stands for the number of spacetime dimensions and l denotes the AdS radius. In this case, the variation of the cosmological constant is included in the first law of black hole thermodynamics, which ensures the consistency between the first law of black hole thermodynamics and the Smarr formula. In [20], it was pointed out that the extended phase space can be interpreted as an RG-flow in the space of field theories, where isotherm curves codify how the number of degrees of freedom N (or the central charge c) runs with the energy scale. Moreover, the variation of cosmological constant could be corresponded to variation of number of the colors in Yang–Mills theory residing on the boundary spacetime [21,22]. In the extended phase space, the charged AdS black hole admits a more direct and precise coincidence between the first order small/large black holes (SBH/LBH) phase transition and Van der Waals liquid-gas phase transition, and both systems share the same critical exponents near the critical point [23]. As a result, the analogy between the charged AdS black hole and the Van der Waals

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system becomes more complete. More discussions in various gravity theories can be found in [24–47]. In this direction, some investigations for thermodynamics of AdS black holes in the dRGT massive gravity have been generalized to the extended phase space [48–53], which revealed the existence of Van der Waals-like SBH/LBH phase transition. In addition, a link between the Van der Waals-like SBH/LBH phase transition and quasinormal modes (QNMs) has established in four [54] and higher-dimensional Reissner–Nordström AdS black hole [55], including time-domain profiles [56], and higher-dimensional charged black hole in the presence of Weyl coupling [57]. In terms of AdS/CFT, holographic entanglement entropy (HEE), Wilson loop, and two point correlation function also provide useful tools to probe the Van der Waals-like SBH/LBH phase transition [58–63].

Recently, Ref. [64] firstly recovered the existence of intermediate/small/large phase transitions in the four- dimensional Born–Infeld–AdS black hole, which is reminiscent of reentrant phase transitions (RPTs) observed for multicomponent fluid systems, ferroelectrics, gels, liquid crystals, and binary gases, e.g., [65]. A system undergoes an RPT if a monotonic variation of any thermodynamic quantity results in two (or more) phase transitions such that the final state is macroscopically similar to the initial state. Moreover, this RPT also appears in the higher-dimensional rotating AdS black holes [66,67], five-dimensional hairy AdS black hole [68], and higher-dimensional Gauss–Bonnet AdS black hole [69,70]. It is interesting to generalize the discussion to the black holes in the dRGT massive gravity. In this paper, we will report the finding of interesting RPTs in all $d \geq 6$-dimensional black holes when the coupling coefficients $c_i m^2$ of massive potential satisfy some certain conditions.

This paper is organized as follows. In Sect. 2, we review the thermodynamics of massive gravity black holes in the extended phase space. In Sect. 3, we study the critical behavior of higher-dimensional AdS black hole in context of $P-V$ criticality and phase diagrams. We end the paper with closing remarks in Sect. 4.

2 Thermodynamics of higher-dimensional AdS black hole in dRGT massive gravity

We start with the action of higher-dimensional dRGT massive gravity in the presence of a negative cosmological constant

$$\mathscr{I} = \frac{1}{16\pi} \int d^d x \sqrt{-g} \left[R - 2\Lambda + m^2 \sum_{i=1}^{4} c_i \mathscr{U}_i(g, f) \right],$$

(2)

where the last four terms are the massive potential associate with graviton mass m, c_i are constants and f is a fixed rank-2 symmetric tensor. Moreover, $\mathscr{U}_i$ are symmetric polynomials

of the eigenvalues of the $d \times d$ matrix $\mathscr{K}^\mu_\nu \equiv \sqrt{g^{\mu\alpha} f_{\alpha\nu}}$

$$\mathscr{U}_1 = [\mathscr{K}],$$
$$\mathscr{U}_2 = [\mathscr{K}]^2 - [\mathscr{K}^2],$$
$$\mathscr{U}_3 = [\mathscr{K}]^3 - 3[\mathscr{K}][\mathscr{K}^2] + 2[\mathscr{K}^3],$$
$$\mathscr{U}_4 = [\mathscr{K}]^4 - 6[\mathscr{K}^2][\mathscr{K}]^2 + 8[\mathscr{K}^3][\mathscr{K}]$$
$$\quad + 3[\mathscr{K}^2]^2 - 6[\mathscr{K}^4].$$

(3)

The square root in $\mathscr{K}$ is understood as the matrix square root, i.e., $(\sqrt{A})^\mu_\nu (\sqrt{A})^\nu_\lambda = A^\mu_\lambda$, and the rectangular brackets denote traces $[\mathscr{K}] = \mathscr{K}^\mu_\mu$.

Consider the metric of d-dimensional spacetime in the following form:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 h_{ij}dx^i dx^j,$$

(4)

where $h_{ij}dx^i dx^j$ is the line element for an Einstein space with constant curvature $(d-2)(d-3)k$. The constant k characterizes the geometric property of black hole horizon hypersurface, which takes values $k = 0$ for flat, $k = -1$ for negative curvature and $k = 1$ for positive curvature, respectively.

By using the reference metric

$$f_{\mu\nu} = \text{diag}(0, 0, c_0^2 h_{ij}),$$

(5)

the metric function $f(r)$ is obtained [16]:

$$f(r) = k + c_0^2 c_2 m^2 + \frac{16\pi P}{(d-1)(d-2)} r^2 + \frac{c_0 c_1 m^2}{d-2} r$$
$$\quad - \frac{16\pi M}{(d-2) V_{d-2} r^{d-3}} + \frac{(d-3) c_0^3 c_3 m^2}{r}$$
$$\quad + \frac{(d-3)(d-4) c_0^4 c_4 m^2}{r^2}.$$

(6)

Here c_0 is a positive constant, V_{d-2} is the volume of space spanned by coordinates x^i, and M is the black hole mass. It is necessary to point out that the terms $c_3 m^2$ and $c_4 m^2$ only appear in the black hole solutions for $d \geq 5$ and $d \geq 6$, respectively [16]. When $m \to 0$, namely, without the massive potential, Eq. (6) reduces to the d-dimensional Schwarzschild AdS (SAdS) black hole solution.

In terms of the radius of the horizon r_+, the mass M, Hawking temperature T and entropy S of black holes can be written as

$$M = \frac{(d-2) V_{d-2} r_+^{d-3}}{16\pi} \left[k + \frac{16\pi P}{(d-1)(d-2)} r_+^2 + \frac{c_0 c_1 m^2 r_+}{d-2} \right.$$
$$\quad \left. + c_0 c_2 m^2 + \frac{(d-3) c_0^3 c_3 m^2}{r_+} + \frac{(d-3)(d-4) c_0^4 c_4 m^2}{r_+^2} \right],$$

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi r_+} \left[(d-3)k + \frac{16\pi P}{d-2} r_+^2 + c_0 c_1 m^2 r_+ \right.$$
$$\quad \left. + (d-3) c_0^2 c_2 m^2 + \frac{(d-3)(d-4) c_0^3 c_3 m^2}{r_+} \right.$$

$$+ \frac{(d-3)(d-4)(d-5)c_0^4 c_4 m^2}{r_+^2}\Bigg],$$

$$S = \frac{V_{d-2}}{4} r_+^{d-2}. \tag{7}$$

The black hole mass M can be considered as the enthalpy rather than the internal energy of the gravitational system. Moreover, the first law of black hole thermodynamics and the Smarr relation are given by [48]

$$dM = T dS + V dP + \frac{c_0 m^2 V_{d-2} r_+^{d-2}}{16\pi} dc_1$$
$$+ \frac{(d-2)c_0^2 m^2 V_{d-2} r_+^{d-3}}{16\pi} dc_2$$
$$+ \frac{(d-2)(d-3)c_0^3 m^2 V_{d-2} r_+^{d-4}}{16\pi} dc_3$$
$$+ \frac{(d-2)(d-3)(d-4)c_0^4 m^2 V_{d-2} r_+^{d-5}}{16\pi} dc_4, \tag{8}$$

$$(d-3)M = (d-2)TS - 2VP - \frac{c_0 c_1 m^2 V_{d-2}}{16\pi} r_+^{d-2}$$
$$+ \frac{(d-2)(d-3)c_0^3 c_3 m^2 V_{d-2}}{16\pi} r_+^{d-4}$$
$$+ \frac{(d-2)(d-3)(d-4)c_0^4 c_4 m^2 V_{d-2} r_+^{d-5}}{8\pi}, \tag{9}$$

where V_{d-2} denotes the thermodynamic volume and equals $\frac{V_{d-2}}{d-1} r_+^{d-1}$.

3 Critical behaviors of higher-dimensional AdS black holes

3.1 Equation of state

For further convenience, we denote

$$\hat{T} = T - \frac{c_0 c_1 m^2}{4\pi}, \quad w_2 = -\frac{k + c_0^2 c_2 m^2}{8\pi},$$
$$w_3 = -\frac{c_0^3 c_3 m^2}{8\pi}, \quad w_4 = -\frac{c_0^4 c_4 m^2}{8\pi}. \tag{10}$$

Here $\hat{T}$ denotes the shifted temperature and can be negative according to the value of $c_0 c_1 m^2$. Then the equation of state of the black hole can be obtained from Eq. (7)

$$P = \frac{d-2}{4r_+}\Bigg[\hat{T} + \frac{2(d-3)w_2}{r_+} + \frac{2(d-3)(d-4)w_3}{r_+^2}$$
$$+ \frac{2(d-3)(d-4)(d-5)w_4}{r_+^3}\Bigg]. \tag{11}$$

To compare with the Van der Waals fluid equation, we can translate the "geometric" equation of state to a physical one by identifying the specific volume v of the fluid with the radius of the horizon of the black hole as $v = \frac{4r_+}{d-2}$. Evidently, the specific volume v is proportional to the radius of the horizon r_+, therefore we will just use the radius of the horizon in the equation of state for the black hole hereafter in this paper.

We know that the critical point occurs when P has an inflection point,

$$\frac{\partial P}{\partial r_+}\Bigg|_{\hat{T}=\hat{T}_c, r_+=r_c} = \frac{\partial^2 P}{\partial r_+^2}\Bigg|_{\hat{T}=\hat{T}_c, r_+=r_c} = 0, \tag{12}$$

where the subscript stands for the quantities at the critical point. The critical shifted temperature is obtained:

$$\hat{T}_c = -\frac{2(d-3)}{r_c}\Bigg[2w_2 + \frac{3(d-4)w_3}{r_c} + \frac{4(d-4)(d-5)w_4}{r_c^2}\Bigg], \tag{13}$$

and the equation for the critical radius of the horizon r_c is given by

$$6(d-4)(d-5)w_4 + 3(d-4)w_3 r_c + w_2 r_c^2 = 0. \tag{14}$$

One can easily find that in four-dimensional spacetime ($d = 4$, $w_3 = w_4 = 0$), the absence of positive solution of Eq. (14) indicates that no criticality can occur [48]. A similar situation also occurs in the d-dimensional Schwarzschild AdS black hole ($m \to 0$), since there does not exist any real root of Eq. (14) with $w_3 = w_4 = 0$.

We further discuss the critical behaviors of the higher-dimensional ($d \geq 5$) AdS black hole when $w_2 \neq 0$ and $w_3 \neq 0$. When setting $w_4 = 0$, one has

$$r_c = -\frac{3(d-4)w_3}{w_2}, \quad \hat{T}_c = \frac{2(d-3)w_2^2}{3(d-4)w_3},$$
$$P_c = -\frac{(d-2)(d-3)w_2^3}{54(d-4)^2 w_3^2}. \tag{15}$$

Note that the critical behavior occurs only when $w_2 < 0$ and $w_3 > 0$. We can easily find an interesting relation among the critical pressure P_c, temperature $\hat{T}_c$ and radius of the horizon r_c:

$$\frac{P_c r_c}{\hat{T}_c} = \frac{d-2}{12}. \tag{16}$$

For $d = 5$, Eqs. (15) and (16) reduce to the equations described in [48].

With regard to the case of $w_4 \neq 0$, which only appears for $d \geq 6$, the direct solution of Eq. (14) reads

$$r_{c1,2} = \frac{1}{2w_2}\Bigg[\pm\sqrt{9(d-4)^2 w_3^2 - 24(d-4)(d-5)w_2 w_4}$$
$$- 3(d-4)w_3\Bigg] \equiv \frac{\chi_\pm}{2w_2} \tag{17}$$

if $3(d-4)w_3^2 \geq 8(d-5)w_2 w_4$. In this case, r_{c1} and r_{c2} correspond to the "−" and "+" branches, respectively. The

condition of $r_{c1,2} > 0$ crucially depends on the dimension of the spacetime and values of w_2, w_3 and w_4.

For $w_2 < 0$, the positivity of solution r_{c1} leads to $w_3 < 0$ and $w_4 > 0$ or $w_3 > 0$. In order for r_{c2} to be positive, it requires an additional constraint: $w_3 > 0$ and $w_4 < 0$. By substituting the solutions $r_{c1,2}$ (17) into Eqs. (11) and (13), we obtain

$$\hat{T}_{c1,2} = \frac{8(d-3)(d-4)w_2^2 \left[16(d-5)w_2 w_4 + 3w_3 \chi_\pm\right]}{\chi_\pm^3}, \quad (18)$$

$$P_{c1,2} = \frac{\pm 4(d-2)(d-3)(d-4)w_2^3 \left[6(d-5)w_2 w_4 + w_3 \chi_\pm\right]}{\chi_\pm^4}. \quad (19)$$

Due to the shifted temperature $\hat{T}_{c1,2}$ can be negative; here we only evaluate the results of $P_{c1,2}$. When $P_{c1} > 0$, we shall keep $w_3 < 0$ and $w_4 > 0$ or $w_3 > 0$. On the other hand, taking $w_3 > 0$, $w_4 < 0$ and $\frac{3(d-4)w_3^2}{8(d-5)w_4} < w_2 < \frac{3(d-4)w_3^2}{9(d-5)w_4}$ lead to $P_{c2} > 0$. As a result, two critical points ($r_{c1,2} > 0$ and $P_{c1,2} > 0$) will appear in the range of $\frac{3(d-4)w_3^2}{8(d-5)w_4} < w_2 < \frac{3(d-4)w_3^2}{9(d-5)w_4}$ with $w_3 > 0$ and $w_4 < 0$.

In case of $w_2 > 0$, there is only one critical point ($r_{c1} > 0$ and $P_{c1} > 0$) for $w_3 < 0$, $w_4 > 0$ and $w_2 < \frac{(d-4)w_3^2}{3(d-5)w_4}$. We summarize the critical points in Table 1. The corresponding P–r_+ diagrams for $d = 6$ are displayed in Figs. 1 and 2.

To study the possible phase transitions in the system, let us now turn to the expression for the Gibbs free energy.

3.2 Gibbs free energy

The behavior of the free energy G is important to determine the thermodynamic phase transition. The free energy G obeys the thermodynamic relation

$$G = M - TS = -V_{d-2} r_+^{d-3} \left[\frac{Pr_+^2}{(d-1)(d-2)} + \frac{w_2}{2} \right.$$

$$\left. + \frac{(d-3)w_3}{r_+} + \frac{3(d-3)(d-4)w_4}{2r_+^2} \right]. \quad (20)$$

Here r_+ is understood as a function of pressure and temperature, $r_+ = r_+(P, \hat{T})$, via equation of state (11).

In the range of $\frac{3(d-4)w_3^2}{8(d-5)w_4} < w_2 < \frac{3(d-4)w_3^2}{9(d-5)w_4}$ with $w_3 > 0$ and $w_4 < 0$, the behavior of G in the six-dimensional spacetime is depicted in Fig. 3a. We have one physical (with posi-

Table 1 The behavior of critical points for different values of coupling constants when $d \geq 6$

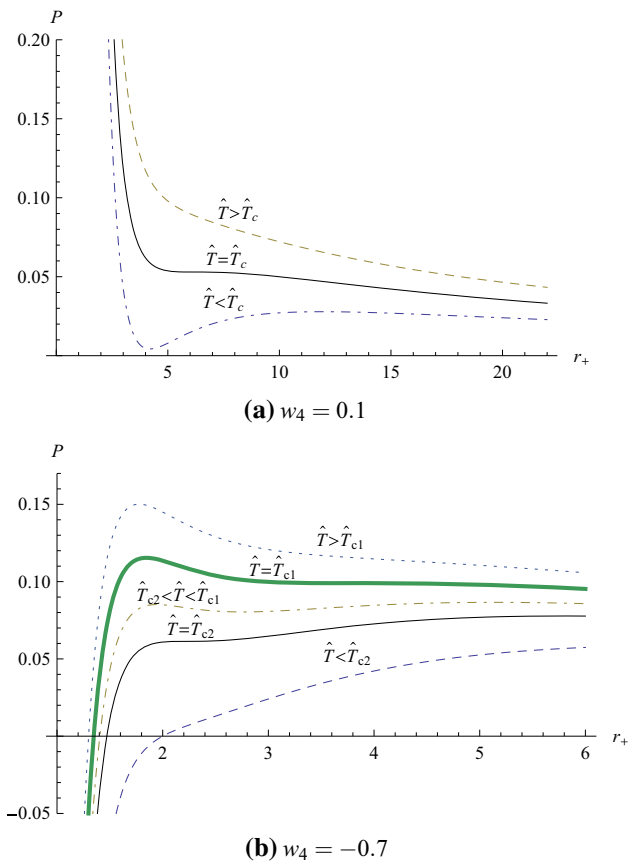| parameters	$w_2 < 0$		$w_2 > 0$
w_3	$w_3 > 0$	$w_3 < 0$	$w_3 < 0$
w_4	$w_4 \geq 0$	$w_4 < 0$	$w_4 \geq 0$	$w_4 > 0$
Number of critical point	one	two	one	one

(a) $w_4 = 0.1$

(b) $w_4 = -0.7$

Fig. 1 The P–r_+ diagrams of six-dimensional AdS black holes for $w_2 = -1$ and $w_3 = 1$. **a** The *upper dashed line* corresponds to the idea gas phase behavior for $\hat{T} > \hat{T}_c$. The critical temperature case $\hat{T} = \hat{T}_c$ is denoted by the *solid line*. The *line below* is with temperatures smaller than the critical temperature. We have $\hat{T}_c = 0.9788$ in **a**. In **b**, we have now two critical points at positive pressure. The *upper one* has higher radius, temperature, and mass. We have $\hat{T}_{c1} = 1.27718$ and $\hat{T}_{c2} = 1.1718$ in **b**

tive pressure) critical point and the corresponding first order SBH/LBH phase transition. This phase transition occurs for $\hat{T} < \hat{T}_{c1}$ and terminates at $\hat{T} = \hat{T}_t$. In particular, there also exists a certain range of temperatures, $\hat{T} \in (\hat{T}_t, \hat{T}_z)$, for which the global minimum of G is discontinuous; see Fig. 3b. In this range of temperatures, two separate branches of intermediate size and small size black holes co-exist. They are separated by a finite jump in G, which is so-called "zeroth-order phase transition". This phenomenon is also seen in superfluidity and superconductivity [71].

This novel situation can also clearly be illustrated in the P–$\hat{T}$ diagrams in Fig. 4. There is the expected SBH/LBH line of coexistence, which initiates from the critical point $(\hat{T}_{c1}, P_{c1})$ and terminates at $(\hat{T}_t, P_t)$. Especially, a "triple point" between the small, intermediate, and large black holes appears in the point $(\hat{T}_t, P_t)$. For $\hat{T} \in (\hat{T}_t, \hat{T}_z)$, a new IBH/SBH line of coexistence appears and then it terminates in another critical point $(\hat{T}_z, P_z)$. The range for the RPT is
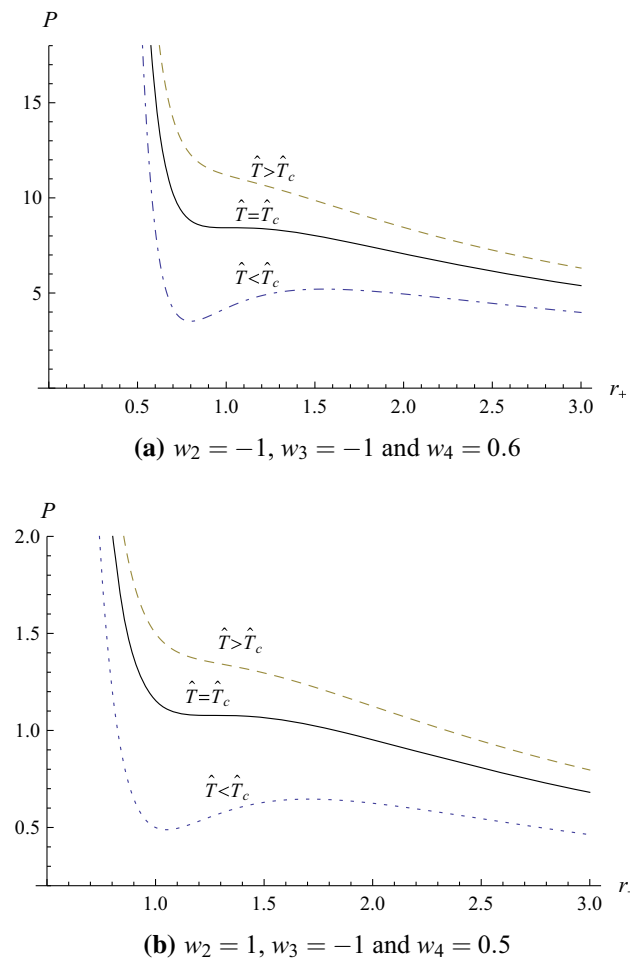
(a) $w_2 = -1$, $w_3 = -1$ and $w_4 = 0.6$

(b) $w_2 = 1$, $w_3 = -1$ and $w_4 = 0.5$

Fig. 2 The $P-r_+$ diagrams of six-dimensional AdS black holes. The *upper dashed line* corresponds to the idea gas phase behavior for $\hat{T} > \hat{T}_c$. The critical temperature case $\hat{T} = \hat{T}_c$ is denoted by the *solid line*. The *lines below* is with temperatures smaller than the critical temperature. We have $\hat{T}_c = 19.2290$ in **a** and $\hat{T}_c = 1.1547$ in **b**

(a)

(b)

Fig. 3 The $G-\hat{T}$ diagrams of six-dimensional AdS black holes for $w_2 = -1$, $w_3 = 1$ and $w_4 = -0.7$. For $P \in (P_t, P_z)$, we observe a "zeroth-order phase transition" signifying the onset of an RPT in **a**. In **b** with $P = 0.0869 \in (P_t, P_z)$, a close-up of **a** illustrates the discontinuity in the global minimum of G at $\hat{T} = \hat{T}_0 \approx 1.223398 \in (\hat{T}_t, \hat{T}_z)$ and the so-called Van der Waals-like phase transition at $\hat{T} = \hat{T}_1 \approx 1.22554$

quite narrow and must be determined numerically. Taking $w_2 = -1$, $w_3 = 1$, $w_4 = -0.7$ and $d = 6$, we obtain

$$(\hat{T}_t, \hat{T}_z, \hat{T}_{c1}) \approx (1.22194, 1.22459, 1.27718),$$
$$(P_t, P_z, P_{c1}) \approx (0.08615, 0.08747, 0.09899). \qquad (21)$$

In Fig. 5, we also plot the behavior of Gibbs free energy of six-dimensional AdS black hole for three other cases, showed in Table 1. One can see that the G surface demonstrates the characteristic "swallow tail" behavior, which indicates the occurrence of Van der Waals-like SBH/LBH phase transition when $P < P_c$ in the corresponding system. Moreover, the corresponding $P-\hat{T}$ diagram (not shown) is reminiscent of what was observed for charged black holes in [23] and is analogous to the Van der Waals $P-\hat{T}$ diagram in each case.
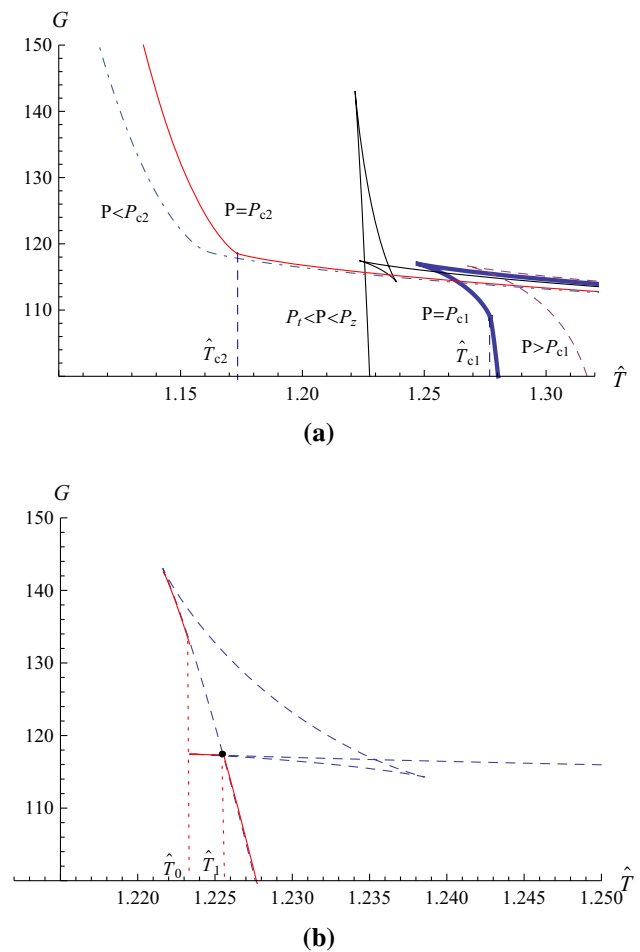
When $m \to 0$, namely, $w_3 = w_4 = 0$ and $w_2 = -\frac{k}{8\pi}$, we find that the free energy G always maintains negative in cases of $k = 0$ and $k = -1$, which correspond to a Ricci flat and hyperbolic topology of the black hole horizon of d-dimensional Schwarzschild AdS black hole, respectively. It is of great interest to discuss the d-dimensional Schwarzschild AdS black hole with spherical horizon ($k = 1$). In Fig. 6, it is shown that the temperature T has a minimal value $T_{\min}$, below which no black hole solution exists. When the temperature drops to a certain value larger than $T_{\min}$, the Gibbs free energy G will be larger than zero, and then a more stable vacuum will be found. At $T = T_{HP}$, there is a first order Hawking–Page [72] phase transition between thermal radiation and black hole phase. This phase transition can be interpreted as a confinement/deconfinement phase transition in the dual quark–gluon plasma [73].
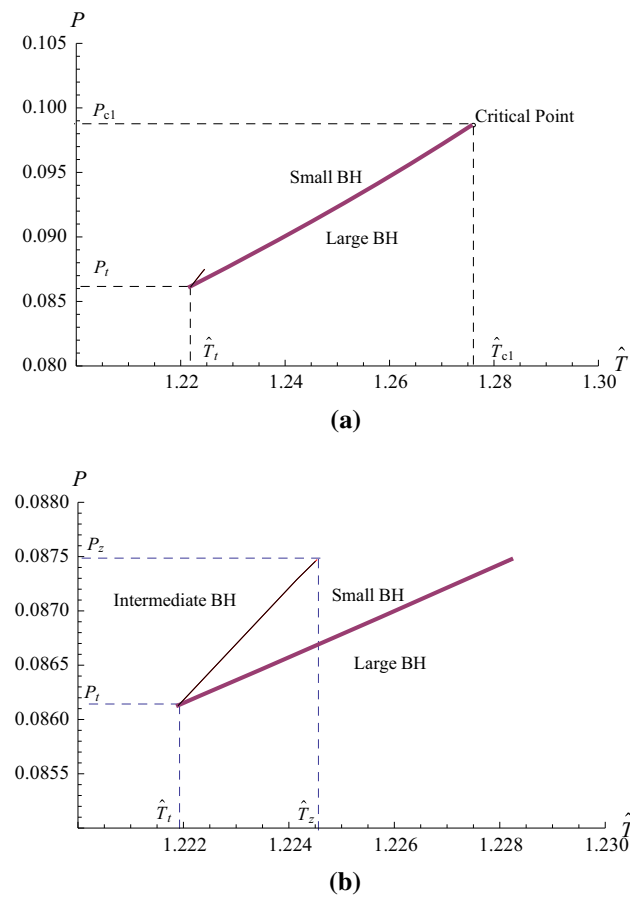
(a)

(b)

Fig. 4 The $P-\hat{T}$ diagram of six-dimensional AdS black holes for $w_2 = -1$, $w_3 = 1$ and $w_4 = -0.7$. The coexistence line of Van der Waals-like phase transition is depicted by a *thick solid line*, which initiates from the critical point $(P_{c1}, \hat{T}_{c1})$ and terminates at $(P_t, \hat{T}_t)$. The *solid line* in the *inset* indicates the coexistence line of small and intermediate black holes, separated by a finite gap in G, indicating the RPT. It commences from $(P_z, \hat{T}_z)$ and terminates at $(P_t, \hat{T}_t)$

4 Closing remarks

In this paper we have studied the thermodynamic behavior of higher-dimensional AdS black hole in the dRGT massive gravity. We discussed this issue in the extended phase space where the cosmological constant appears as the pressure of the thermodynamic system and its conjugate quantity is the thermodynamic volume of the black holes. Following the standard thermodynamic techniques, we have written out the equations of state and examined the phase structures. When the coupling coefficients of massive potential satisfy $\frac{3(d-4)w_3^2}{8(d-5)w_4} < w_2 < \frac{3(d-4)w_3^2}{9(d-5)w_4}$ with $w_3 > 0$ and $w_4 < 0$, we found that a monotonic lowering of the temperature yields a large-small-large black hole transition, where we refer to the latter "large" state as an intermediate black hole (IBH), which is reminiscent of reentrant phase transitions. Moreover, this process is also accompanied by a discontinuity in the global minimum of the Gibbs free energy, referred
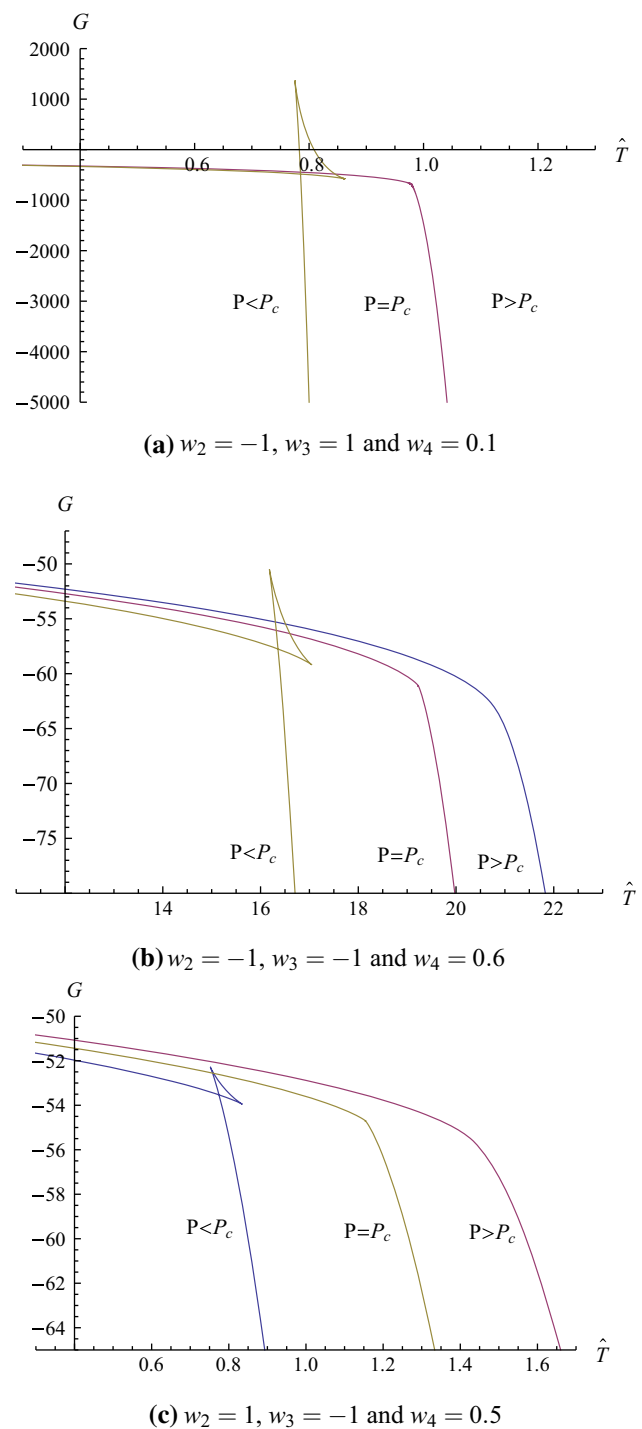

(a) $w_2 = -1$, $w_3 = 1$ and $w_4 = 0.1$

(b) $w_2 = -1$, $w_3 = -1$ and $w_4 = 0.6$

(c) $w_2 = 1$, $w_3 = -1$ and $w_4 = 0.5$

Fig. 5 The $G-\hat{T}$ diagrams of six-dimensional AdS black holes. The behavior of the Gibbs free energy is depicted as a function of temperature for fixed pressure. There is one critical point and the corresponding Van der Waals-like phase transition for $\hat{T} < \hat{T}_c$

to as a zeroth-order phase transition. For three other cases in Table 1, the usual Van der Waals-like small/large black hole phase transition occurred when coupling coefficients of massive potential adopt some proper values in the higher-
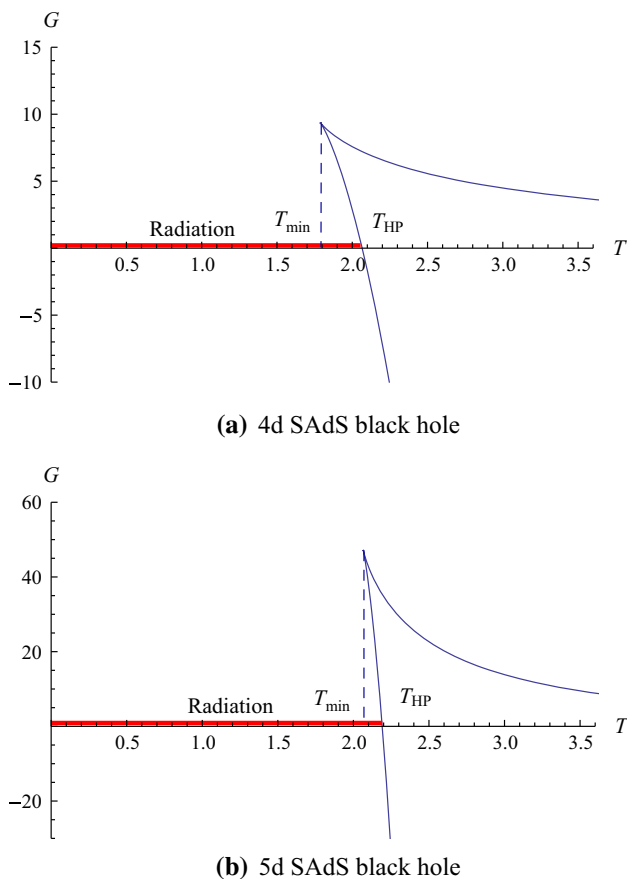
(a) 4d SAdS black hole

(b) 5d SAdS black hole

Fig. 6 The $G-T$ diagrams of four- and five-dimensional Schwarzschild AdS black holes for $P = 0.2$ and $k = 1$. The radiation phase is displayed by a *horizontal magenta line*. The Hawking–Page phase transition between thermal radiation and black holes occurs at $T = T_{HP}$. For $T > T_{HP}$, this branch has a negative Gibbs free energy and the corresponding black holes represent the globally thermodynamically preferred state

dimensional spacetime. In addition, the solution (6) recovered d-dimensional Schwarzschild AdS black holes in the case of $m \rightarrow 0$. It demonstrated the existence of a so-called Hawking–Page phase transition between SAdS black hole with spherical horizon and vacuum if $d \geq 4$.

It is necessary to point out that the charged black hole [74], Born–Infeld black hole [75], and black hole in the Maxwell and Yang–Mills fields [76] have been recently constructed in Gauss–Bonnet massive gravity. It has also showed the existence of Van der Waals like first order SBH/LBH phase transition in these models. It would be interesting to extend our discussion to these black hole solutions and see whether the reentrant phase transition can appear.

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