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Higher derivative extensions of 3d Chern–Simons models: conservation laws and stability

D. S. Kaparulin[a], I. Yu. Karataeva[b], S. L. Lyakhovich[c]

Physics Faculty, Tomsk State University, Tomsk 634050, Russia

Abstract We consider the class of higher derivative 3d vector field models with the field equation operator being a polynomial of the Chern–Simons operator. For the nth-order theory of this type, we provide a general recipe for constructing n-parameter family of conserved second rank tensors. The family includes the canonical energy-momentum tensor, which is unbounded, while there are bounded conserved tensors that provide classical stability of the system for certain combinations of the parameters in the Lagrangian. We also demonstrate the examples of consistent interactions which are compatible with the requirement of stability.

1 Introduction

In this paper we consider a class of 1-form field $A = A_\mu \mathrm{d}x^\mu$ models on 3d Minkowski space with the action

$$
\begin{aligned}
S = \frac{m^2}{2} \\
\times \int A \wedge \Big(-a_0 * A + a_1 \frac{2}{m} dA + a_2 \frac{4}{m^2} d * dA \\
+ a_3 \frac{8}{m^3} d * d * dA + a_4 \frac{16}{m^4} d * d * d * dA + \dots \Big),
\end{aligned}
\tag{1}
$$

where m is a constant with dimension of mass, $a_0, a_1, a_2, a_3, \dots$ are some real dimensionless coefficients, $*$ is Hodge conjugation, and the signature is $(+, -, -)$. The coefficient $a_0 m^2$ corresponds to the usual mass term, $a_1 m$ is the Chern–Simons mass, a_2 is a coefficient for the Maxwell's Lagrangian, a_3 corresponds to the extended Chern–Simons Lagrangian [1] and the fourth-order term appears in the Podolsky electrodynamics Lagrangian [2]. With appropri-

ate choice of the coefficients $a_k, k = 0, 1, 2, \dots$, this action reproduces various known 3d models, including the Chern–Simons–Proca [3,4], Maxwell–Chern–Simons [5,6], Maxwell–Chern–Simons–Proca [7,8] and the other previously studied higher derivative models [9,10].

In any dimension, inclusion of the higher derivative terms results in the unbounded canonical energy, so classical stability becomes the issue. It is also known that the ghost poles can emerge in the propagator once higher derivatives are included in the action.

The specifics of higher-order terms in three dimensions is that they can be viewed as derived from the Chern–Simons term by the repeated shift of the field by its strength: $A \mapsto A + 2m^{-1} * dA$. As a result, the operator of field equations is a polynomial in the first-order operator $W = 2m^{-1} * dA$. This special structure allows us to make some conclusions concerning conservation laws and stability. The observation is that the nth-order theory of the class (1) admits n parametric family of conserved second rank tensors whenever $a_0 \neq 0$. Once $a_0 = 0$ (the theory is gauge invariant in this case), there exists an $n - 1$ parametric family of conserved tensors. The canonical energy-momentum is included in the family in every instance. We provide the general recipe for constructing these conservation laws and the related symmetries. The construction in fact applies to any system (of any field A, not necessarily a 1-form) with the operator of field equations being a polynomial in another operator,

$$
MA = 0, \quad M = m^2 \sum_{k=0}^{n} a_k W^k,
\tag{2}
$$

where W can be any self-adjoint[1] differential operator, a_k are real constants, and $a_n \neq 0$. We term the models of the type (2) *derived* from the theory with equations $WA = 0$. For the case (1), when $W = 2m^{-1} * d$, we apply the general procedure to explicitly deduce the conserved

[a] e-mail: dsc@phys.tsu.ru

[b] e-mail: karin@phys.tsu.ru

[c] e-mail: sll@phys.tsu.ru

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[1] The conjugation rule is explained in the next section.

tensors for the third-order actions of this class. As we see, the bounded conserved tensors are contained in the family, once the polynomial $M(2m^{-1} * d)$ has only simple roots, or at most one double zero root. In this generic case, the theory is classically stable even though the canonical energy-momentum is unbounded. As we shall explain, these models can admit certain interactions such that the stability survives at nonlinear level. The case of multiple roots is special. It also admits a family of n conserved tensors, including the canonical energy-momentum, though there are no bounded conserved quantities in this family. As we see, the corresponding representations of the Poincaré group are non-unitary, while in the generic case, the representation decomposes into unitary ones.

The article is organized in the following way. In the next section we describe the general structure of field equations in the higher derivative models that fall into the class of derived theories (2). For the generic derived system of order n we suggest a procedure of constructing n-parametric family of conserved tensors whose structure depends on the coefficients a_k in the field equations (2). In Sect. 2, we explicitly construct the families of conserved tensors for the theory (1) involving terms up to third order. As we see, four different cases are possible from the viewpoint of existence the bounded representative in the family of conserved quantities. These cases are distinguished by the structure of roots in the polynomial (2). Once the positive conserved quantity exists, the theory is stable at the classical level, even though the canonical energy is unbounded. In Sect. 3, we demonstrate the example of the self-interaction such that the nonlinear theory remains stable. In conclusion, we summarize the results and comment on the stability of the theory (1) at the quantum level.

2 Derived theories, higher symmetries, and conservation laws

In this section, we consider the field equations of general structure (2). We demonstrate that combining the space-time translations with the powers of operator W, one can construct non-trivial higher-order symmetries and find related conserved tensors. The construction is quite general, it applies to any system of the form (2). The explicit details for the extension of Chern–Simons theory (1) are provided in Sect. 3.

2.1 Derived theories

Consider a set of fields $A^J(x)$ on d-dimensional Minkowski space with local coordinates x^μ. The multi-index J accommodates all the tensor, spinor, isotopic indices labeling the field components. Here, we suppose that the theory admits appropriate constant metrics that can be used to raise and lower the multi-indices. In this setting, any local linear sys-

tem of field equations can be represented in the following form:

$$M_{IJ}(\partial)A^J = 0, \qquad (3)$$

where $M_{IJ}(\partial)$ is a square matrix whose entries are polynomials in the formal variables ∂_μ. If ∂_μ are understood as the partial derivatives in Minkowski space coordinates x^μ, (3) will be a linear PDE system. The formal adjoint to the operator M is defined by

$$M^\dagger_{IJ}(\partial) = M_{JI}(-\partial). \qquad (4)$$

The field equations (3) are variational whenever $M = M^\dagger$, in which case the action reads

$$S = \int d^d x L, \quad L = \frac{1}{2} A^I M_{IJ}(\partial) A^J. \qquad (5)$$

Let us further suppose that the self-adjoint linear differential operator $W_{IJ}(\partial)$ exists (cf. (2)) such that the operator of the field equations is polynomial in W:

$$M(W) = m^2 \sum_{k=0}^n a_k W^k = a_n m^2 \prod_{i=1}^r (W - \lambda_i)^{p_i}$$
$$\times \prod_{j=1}^s (W^2 - (\omega_j + \overline{\omega}_j)W + \omega_j \overline{\omega}_j)^{q_j}. \qquad (6)$$

The real numbers λ_i and the complex conjugate numbers $\omega_j, \overline{\omega}_j$ are the roots of the polynomial $M(W)$ with multiplicities p_i and q_j, respectively. The multiplicity of the roots is connected with the total degree of the polynomial,

$$\sum_{i=1}^r p_i + 2 \sum_{j=1}^s q_j = n.$$

If W is a differential operator of finite order, n_W, the order of the PDE system (3) will not exceed $n \times n_W$.

Once the field equation operator $M(\partial)$ is a polynomial of another self-adjoint operator $W(\partial)$, we say that the theory is a derived model. In [11], the special case of the factorization (6) was studied, where M has two different simple real roots in W. This simple assumption has far-reaching consequences. In particular, each of the factors defines its own Lagrangian theory whose order is lower than that of the derived theory. Let us mention some of these consequences noticed in [11]. Once the two lower-order theories are translation invariant, the derived higher derivative theory has a two-parameter family of independent conserved tensors. This family includes the canonical energy-momentum tensor of the derived theory. The canonical energy is unbounded in general, as it should be in the higher derivative system, while some other conserved quantities can be bounded in this family. The existence of the bounded conserved quantities guaranties the classical stability of dynamics. As demonstrated in Ref. [11], every conserved tensor in this family can be connected to translation

invariance of the system by appropriate Lagrange anchor.[2] As we will see in this section, for any derived system (2), one can construct n-parameter family of conserved tensors, where n is the order of polynomial $M(W)$.

We consider two ways of constructing conserved tensors in the derived theories. First, we notice that the symmetry algebra of the derived theory includes higher-order symmetries generated by the operator W, and translations, once W is translation invariant. Then we derive the conserved tensors from these symmetries by the Noether theorem. Another option employs the procedure of reducing the order of higher derivative theory (2) by assigning a lower-order system to every irreducible factor in the decomposition of the polynomial (6). Then, making use of the canonical conserved tensors for the lower-order systems, we get the family of the conserved tensors for the original theory (2). Although the Noether theorem provides a uniform way for deducing conservation laws from given symmetries, the conserved tensors obtained from the lower-order equivalent system appear in a more convenient form in this case, and we will use them for further analysis of stability.

2.2 Higher-order symmetries and conservation laws.

Provided the operator W is translation invariant, the action (5) admits the following symmetry transformations:

$$\delta_\varepsilon A^J = -\varepsilon^\alpha \partial_\alpha (W^k A)^J, \quad k = 0, \ldots, n-1. \tag{7}$$

The space-time translations correspond to $k = 0$. The higher-order transformations with $k = n, n+1, \ldots$ are equivalent to the lower-order ones taking account of the equations of motion (3), while for $k < n$ one has independent symmetries. By the Noether theorem one can link the symmetries (7) with the conserved tensors

$$(T^k)^\mu{}_\nu (A), \quad \partial_\mu (T^k)^\mu{}_\nu$$
$$= -(\partial_\nu (W^k A)^J)(MA)_J, \quad k = 0, 1, \ldots, n-1. \tag{8}$$

Here, $k = 0$ corresponds to the usual energy-momentum tensor. There are n independent tensors in the set (8).

2.3 Conservation laws by the reduction of order.

Consider the polynomial (6). Denote the cofactors to the real roots λ_i and the complex roots ω_j by Λ_i and Ω_j, respectively,

$$\Lambda_i = \prod_{k \neq i} (W - \lambda_k)^{p_k} \prod_{j=1}^{s} (W^2 - (\omega_j + \overline{\omega}_j)W + \omega_j \overline{\omega}_j)^{q_j},$$

$$\Omega_j = \prod_{i=1}^{r} (W - \lambda_i)^{p_i} \prod_{k \neq j} (W^2 - (\omega_k + \overline{\omega}_k)W + \omega_k \overline{\omega}_k)^{q_k}. \tag{9}$$

By definition, the polynomials $\Lambda_i(W)$ and $\Omega_j(W)$ are coprime. Obviously,

$$M = a_n m^2 (W - \lambda_i)^{p_i} \Lambda_i$$
$$= a_n m^2 (W^2 - (\omega_j + \overline{\omega}_j)W + \omega_j \overline{\omega}_j)^{q_j} \Omega_j$$
$$\text{(no summation in } i, j).$$

For each cofactor, we introduce the new set of fields,

$$(\xi_i)^J = (\Lambda_i A)^J, \quad i = 1, \ldots, r,$$
$$(\zeta_j)^J = (\Omega_j A)^J, \quad j = 1, \ldots, s, \tag{10}$$

called *components*. Once the original fields A are subject to the original field equations (3), the components satisfy the lower-order derived equations

$$a_n m^2 (W - \lambda_i)^{p_i} \xi_i = 0,$$
$$a_n m^2 (W^2 - (\omega_j + \overline{\omega}_j)W + \omega_j \overline{\omega}_j)^{q_j} \zeta_j = 0,$$
$$\text{(no summation in } i, j) \tag{11}$$

where p_i, q_j are the multiplicities of the roots λ_i, ω_j in the operator of the original equations (6).

The one-to-one correspondence between solutions of these equations and the original system (3) is easy to see. The inverse transformation to (10) is established by the relations

$$A^J = \sum_{i=1}^{r} (B_i \xi_i)^J + \sum_{j=1}^{s} (C_j \zeta_j)^J,$$

$$B_i = \sum_{p=0}^{p_i-1} b_i^p W^p, \quad C_j = \sum_{q=0}^{2q_j-1} c_j^q W^q, \tag{12}$$

where the polynomials $B_i(W)$ and $C_j(W)$ can be found by the method of undetermined coefficients. The coefficients b_i^p, c_j^q are defined by the relation

$$\sum_{i=1}^{r} B_i \Lambda_i + \sum_{j=1}^{s} C_j \Omega_j = 1. \tag{13}$$

The last equality is just Bezout's identity for the coprime univariate polynomials $\Lambda_i(W)$ and $\Omega_j(W)$.

Whenever the equivalent formulation (11) is known, the conserved tensors can be obtained by applying Eq. (8) sepa-

rately to every component and then summarizing the results. We denote the conserved tensors for the components by

$$(\tau_i^p)^\mu{}_\nu(\xi_i), \quad p = 0, \ldots, p_i - 1,$$
$$(\sigma_j^q)^\mu{}_\nu(\zeta_j), \quad q = 0, \ldots, 2q_j - 1, \tag{14}$$

where the indices i, j label the corresponding components (10) while p_i, q_j are the multiplicities of the corresponding roots (6). The conserved tensors of the original derived theory are obtained by substitution (10):

$$(T_i^p)^\mu{}_\nu(\Lambda_i A) = (\tau_i^p)^\mu{}_\nu(\xi_i)\Big|_{\xi_i = \Lambda_i A},$$
$$(U_j^q)^\mu{}_\nu(\Omega_j A) = (\sigma_j^q)^\mu{}_\nu(\zeta_j)\Big|_{\zeta_j = \Omega_j A}. \tag{15}$$

By construction,

$$\partial_\mu (T_i^p)^\mu{}_\nu(\Lambda_i A) = -(\partial_\nu (W^p \Lambda_i A)^J)(MA)_J,$$
$$\partial_\mu (U_j^q)^\mu{}_\nu(\Omega_j A) = -(\partial_\nu (W^q \Omega_j A)^J)(MA)_J. \tag{16}$$

There are n conserved tensors (15). The relationship between "new" and "old" conserved tensors is established by comparing their divergences (8) and (16). In particular, the canonical energy-momentum tensor of the derived theory (2) has the following representation:

$$(T^0)^\mu{}_\nu(A) = \sum_{i=1}^{r} \sum_{p=0}^{p_i - 1} b_i^p (T_i^p)^\mu{}_\nu(\Lambda_i A)$$
$$+ \sum_{j=1}^{r} \sum_{q=0}^{2q_j - 1} c_j^q (U_j^q)^\mu{}_\nu(\Omega_j A),$$

with the coefficients of linear combination being defined by Eq. (13).

Notice that some combinations of the conserved tensors (8) or (15) may be trivial. A conserved tensor is said to be trivial if it is given by the divergence of an antisymmetric tensor modulo the equations of motion, i.e.,

$$T^\mu{}_\nu(A)\Big|_{MA=0} = \partial_\alpha \Sigma^{\alpha\mu}{}_\nu, \quad \Sigma^{\alpha\mu}{}_\nu = -\Sigma^{\mu\alpha}{}_\nu.$$

The trivial conserved tensors do not result in any conserved quantity and have to be systematically ignored. However, we provide the expressions for the conserved tensors modulo divergence terms, but keep the contributions from the equations of motion. Consistency of the computations can then be verified by taking the divergence; see (8) and (16).

As the issue of stability is concerned, the *positive* conserved tensors are relevant. By a positive tensor we mean the one whose 00-component is positive for any solution which is not a pure gauge. We consider the ansatz for the general conserved tensor of the derived theory (3) in the form

$$T^\mu{}_\nu(A) = \sum_{i=1}^{r} \sum_{p=0}^{p_i - 1} \beta_i^p (T_i^p)^\mu{}_\nu(\Lambda_i A)$$
$$+ \sum_{j=1}^{s} \sum_{q=0}^{2q_j - 1} \gamma_j^q (U_j^q)^\mu{}_\nu(\Omega_j A). \tag{17}$$

The ansatz means that we consider the conserved tensors being additive in the contributions from bilinear combinations of $\Lambda_i A$ and $\Omega_j A$, where Λ_i, Ω_j are the cofactors (9) to the real roots λ_i and complex roots ω_j in the decomposition (6). The quadratic forms (T_i^p) and (U_j^q) are defined by Eqs. (14) and (15). In fact, they represent the conserved tensors (8) of the component fields ξ_i, ζ_j subject to equations (11) in terms of the original field A. Here, $(T_i^0), (U_j^0)$ are just the energy-momentum tensors for the component fields ξ_i, ζ_j expressed in terms of A by substitution (10), while $p, q > 0$ correspond to the higher-order symmetries (7) of the component fields.

As far as the components (10) are independent, the conserved tensor (17) is positive if and only if the tensors

$$\sum_{p=0}^{p_i} \beta_i^p (T_i^p)^\mu{}_\nu(\xi_i), \quad \sum_{q=0}^{q_j - 1} \gamma_j^q (U_j^q)^\mu{}_\nu(\zeta_j)$$

are. In other words, the derived theory (2) is stable if and only if all the components (10) are stable.

Below, we examine the third-order extension of the Chern–Simons theory from the viewpoint of the existence of bounded 00-components of the conserved tensors we found above.

3 Conserved tensors in the third-order extension of the Chern–Simons theory

The field equations of the higher derivative extension of the Chern–Simons model (1) fall into the class of derived theories (2), with W being the composition of the Hodge and de Rham operators:

$$(W)_\mu{}^\nu = (2m^{-1} * d)_\mu{}^\nu,$$
$$W_\mu{}^\nu A_\nu = m^{-1} \varepsilon_\mu{}^{\alpha\nu} \partial_\alpha A_\nu, \quad \varepsilon_{012} = \varepsilon^{012} = 1. \tag{18}$$

The nth-order theory (1) has n degrees of freedom if there are no zero roots in the polynomial (6). If the zero root exists of any multiplicity (including a simple zero root) one degree of freedom is gauged out by the transformation $\delta_\chi A = d\chi(x)$, so the theory has $n - 1$ DoF. The theory (1) describes a (decomposable) representation of the proper Poincaré group. Its indecomposable sub-representations are described by the components (10). In particular, the field content of the theory with simple real roots includes n massive vector fields that satisfy the Chern–Simons–Proca equations ($n - 1$ massive fields and one Chern–Simons field in

the gauge case).[3] A double zero root describes Maxwell's theory. A pair of complex conjugate roots results in the theory with tachyons. The representations related to multiple nonzero roots and zero roots of multiplicity higher than 2 are non-unitary. The case of multiple roots is special because the set of conserved tensors (14) includes a number of terms corresponding to the multiplicity of root. One of the terms corresponds to the energy-momentum tensor of the component, while the others are connected to the higher-order symmetries of the components

$$\delta_\varepsilon \xi_i = -\varepsilon^\alpha \partial_\alpha (W^p \xi_i), \quad p = 1, \ldots, p_i - 1,$$
$$\delta_\varepsilon \zeta_j = -\varepsilon^\alpha \partial_\alpha (W^q \zeta_j), \quad q = 1, \ldots, 2q_j - 1,$$

where p_i, q_j are the multiplicities of real and complex roots. Below we will observe that the equations do not have positive conserved quantities in the family (19) once they involve tachyon or non-unitary representations (which corresponds to complex, double or higher multiplicity nonzero real or triple or higher multiplicity zero roots). The models leading to the unitary representations [which corresponds to simple roots, or at most one double zero root in (6)] admit the conserved tensors with bounded 00-component, even though the canonical energy is unbounded in all the instances.

The conserved tensors (8) and (15) of the higher derivative extension of the Chern–Simons model are given by

$$(T^k)^\mu{}_\nu = -\frac{m^2}{2} \left\{ \frac{1}{m^2} \delta^\mu{}_\nu (W^k A)^\alpha (MA)_\alpha \right.$$
$$\left. + \sum_{l=0}^n a_l (t^{k,l})^\mu{}_\nu (A) \right\},$$

$$(T_i^p)^\mu{}_\nu = -\frac{m^2}{2} \left\{ \frac{1}{m^2} \delta^\mu{}_\nu (W^p A)^\alpha (MA)_\alpha \right.$$
$$\left. + a_n \sum_{l=0}^{p_i} \frac{p_i! (-\lambda_i)^{p_i-l}}{l!(p_i-l)!} (t^{p,l})^\mu{}_\nu (\Lambda_i A) \right\},$$

$$(U_j^q)^\mu{}_\nu = -\frac{m^2}{2} \left\{ \frac{1}{m^2} \delta^\mu{}_\nu (W^q A)^\alpha (MA)_\alpha + a_n \sum_{l=0}^{q_j} \sum_{k=0}^{q_j-l} \right.$$
$$\times \frac{q_j! (-\omega_j - \overline{\omega}_j)^k (\omega_j \overline{\omega}_j)^{q_j-l-k}}{l! k! (q_j - l - k)!}$$
$$\left. \times (t^{q,2l+k})^\mu{}_\nu (\Omega_j A) \right\}, \quad k = 0, 1, \ldots, n-1,$$
$$p = 0, 1, \ldots, p_i - 1, \quad q = 0, 1, \ldots, 2q_j - 1,$$
$$(19)$$

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[3] The irreducible massive vector corresponds to the massive representation of the proper Poincaré group. Being subject to the self-duality equation proposed in [3,4], it has one physical polarization. On the generalities of the Poincaré group unitary irreducible representations in $d = 3$ we refer to [14–16].

where the notation is used

$$(t^{k,l})^\mu{}_\nu$$
$$= \frac{1}{m} \varepsilon^{\mu\alpha\beta} \left[\sum_{s=1}^{l-k} (W^{k+s-1}A)_\alpha \partial_\nu (W^{l-s}A)_\beta \right.$$
$$\left. - \sum_{s=1}^{k-l} (W^{k-s}A)_\alpha \partial_\nu (W^{l+s-1}A)_\beta \right], \quad (t^{k,l})^\mu{}_\nu \Big|_{l=k} = 0.$$

The expressions for the conserved tensors (19) can be simplified making use of the identity

$$\frac{1}{m} \varepsilon^{\mu\alpha\beta} (W^k A)_\alpha \partial_\nu (W^l A)_\beta$$
$$= (W^{k+1}A)^\mu (W^l A)_\nu + (W^{l+1}A)^\mu (W^k)_\nu$$
$$- \delta^\mu{}_\nu (W^{l+1}A)^\alpha (W^k A)_\alpha$$
$$- \frac{1}{m} \partial_\alpha (\varepsilon^{\mu\alpha\beta} (W^l A)_\nu (W^k A)_\beta), \quad k, l \geq 0. \quad (20)$$

Applying this formula one can express all the conserved tensors in terms of $W^k A$, $k = 0, \ldots, n-1$.

For $a_0 = 0$, only $n - 1$ of n conserved tensors (19) are non-trivial. The trivial conserved tensor reads

$$(T_i^{p_i-1})^\mu{}_\nu \equiv \sum_{k=p_i}^n a_k (T^{k-1})^\mu{}_\nu = -(MA)^\mu (W^{p_i-1}\Lambda_i A)_\nu$$
$$+ \frac{m}{2} \partial_\alpha (\varepsilon^{\mu\alpha\beta} (W^{p_i-1}\Lambda_i A)_\nu (W^{p_i-1}\Lambda_i A)_\beta),$$
$$(21)$$

with $\lambda_i = 0$. The simplest example of that kind is provided by the energy-momentum tensor for the Chern–Simons theory, where $n = 1$, $\lambda_1 = 0$, $p_1 = 1$.

Taking account of (21), we consider the following ansatz for the general conserved tensor of the derived theory (3):

$$T^\mu{}_\nu (W^{n-1}A, \ldots, WA, A) = \sum_{i=1}^r \sum_{p=0}^{\widetilde{p}_i-1} \beta_i^p (T_i^p)^\mu{}_\nu (\Lambda_i A)$$
$$+ \sum_{j=1}^s \sum_{q=0}^{2q_j-1} \gamma_j^q (U_j^q)^\mu{}_\nu (\Omega_j A), \quad (22)$$

where $\widetilde{p}_i = p_i$ if $\lambda_i \neq 0$ and $\widetilde{p}_i = p_i - 1$ otherwise. Here, its 00-component is given by the quadratic form in $W^k A$, $k = 0, \ldots, n-1$ ($k = 1, \ldots, n-1$ in the case $a_0 = 0$). Identification of the range of the parameters β and γ that satisfy the positivity condition is a well-known problem of linear algebra. It can always be solved in various ways, for example, by the Silvester criterion.

Let us turn to the case when $a_k = 0$ for $k > 3$ and $a_3 = 1$. This is the most general case of the third-order derived theory. The equations of motion (2) read

$$M_\mu{}^\nu A_\nu = 0, \quad M_\mu{}^\nu = m^2(W_\mu{}^\alpha W_\alpha{}^\beta W_\beta{}^\nu$$
$$+ a_2 W_\mu{}^\beta W_\beta{}^\nu + a_1 W_\mu{}^\nu + a_0 \delta_\mu{}^\nu)$$
$$= -\frac{1}{m}\Box\varepsilon_\mu{}^{\alpha\nu}\partial_\alpha - a_2(\Box\delta_\mu{}^\nu - \partial_\mu\partial^\nu)$$
$$+ a_1 m\varepsilon_\mu{}^{\alpha\nu}\partial_\alpha + a_0 m^2 \delta_\mu{}^\nu. \tag{23}$$

This model has a three-parameter family of conserved tensors if $a_0 \neq 0$ and a two-parameter family if $a_0 = 0$. Depending on the structure of the roots in the decomposition (6) for the third-order equations (23), the four different cases are seen with different behavior of the 00-component of the conserved tensors.

Case A: Three different real roots. The family of conserved tensors includes the one with positive 00-component.

Case B: Simple real root and real root of multiplicity 2. The conserved tensor exists with the positive 00-component if the double root is zero, otherwise the conserved quantity is unbounded.

Case C: Simple real root and pair of complex conjugate roots. The conserved tensor with the positive 00-component does not exist.

Case D: Real root of multiplicity 3. The conserved tensor with the positive 00-component does not exist.

Below we elaborate on each case separately.

3.1 Case A

The coefficients a_0, a_1, a_2 are defined by three real roots $\lambda_1 < \lambda_2 < \lambda_3$ of the polynomial (6),

$$a_2 = -(\lambda_1 + \lambda_2 + \lambda_3), \quad a_1 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3,$$
$$a_0 = -\lambda_1\lambda_2\lambda_3.$$

The factorization (6) for the equations of motion (3) reads

$$M = m^2 \prod_{i=1}^{3}(W - \lambda_i),$$

which corresponds to $r = 3$, $s = 0$, $p_i = 1$.

The general solution to the theory (23) is decomposed into three components (10),

$$\xi_i = \Lambda_i A, \quad \Lambda_i = \prod_{j\neq i}(W - \lambda_j), \quad i = 1, 2, 3, \tag{24}$$

that satisfy the Chern–Simons–Proca equations

$$m^2(W - \lambda_i)\xi_i = 0. \tag{25}$$

Each of the equations describes the massive vector field with the mass $m|\lambda_i|$. Thus, the third-order theory describes a collection of three massive fields with different masses. At the level of the propagator, the decomposition into irreducible

components has been noticed already in the original paper [1], where the third-order extension was proposed for the Chern–Simons theory. In this paper we see the decomposition at the level of solutions to the equations of motion and elaborate on conserved tensors. In the case of second-order theory, $n = 2$, the decomposition into components was noticed [17]. The solution (12) to the original theory (23) is reconstructed by the formula

$$A = \sum_{i=1}^{3} B_i\xi_i, \quad B_i \equiv b_i^0 = \prod_{j\neq i}(\lambda_i - \lambda_j)^{-1}. \tag{26}$$

The conserved tensors (14) are labeled by the indices $i = 1, 2, 3$, $p = 0$ and have the form

$$(T_i^0)^\mu{}_\nu(\Lambda_i A)$$
$$= -\frac{m^2}{2}\left\{2\lambda_i(\Lambda_i A)^\mu(\Lambda_i A)_\nu - \lambda_i\delta^\mu{}_\nu(\Lambda_i A)^\alpha(\Lambda_i A)_\alpha\right\}$$
$$- (MA)^\mu(\Lambda_i A)_\nu. \tag{27}$$

The sign of the corresponding 00-component coincides with the sign of $-\lambda_i$,

$$(T_i^0)^0{}_0(\Lambda_i A) = -\frac{m^2}{2}\lambda_i(\Lambda_i A, \Lambda_i A) - (MA)^0(\Lambda_i A)_0. \tag{28}$$

Here, the Euclidean scalar product is used,

$$(\Lambda_i A, \Lambda_i A) = ((\Lambda_i A)_0)^2 + ((\Lambda_i A)_1)^2 + ((\Lambda_i A)_2)^2 > 0.$$

The conserved tensors (27) can be combined into the tensor

$$T^\mu{}_\nu(A) = \sum_{i=1}^{3} \beta_i^0 (T_i^0)^\mu{}_\nu(\Lambda_i A) \tag{29}$$

with the positive 00-component if and only if three conditions $-\beta_i^0\lambda_i > 0$ are simultaneously satisfied. This result admits a simple physical interpretation. Each of the tensors (27) has the sense of the energy-momentum tensor of the component ξ_i. The 00-component of the general conserved tensor (29) is bounded if the contributions of all the components have the same sign. In contrast, the 00-component of the canonical energy-momentum tensor with $\beta_i^0 = b_i^0$ is always unbounded because the different components contribute with different signs.

Finally, there is an option when one of the roots is zero. In this case, the corresponding conserved tensor becomes trivial and the positivity of the 00-component of the general conserved tensor (22) is ensured by imposing the condition $-\beta_i^0\lambda_i > 0$ for the nonzero roots. The 00-component of the canonical energy-momentum tensor is again unbounded.

3.2 Cases B and C

We deduce the explicit expressions for the conserved quantities in Case C. The corresponding expressions for Case B follow from the ones of the Case C by setting the imaginary part of the complex root to zero.

The polynomial (6) has the simple real root λ_1 and the simple complex root ω_1, i.e.,

$$M = m^2(W - \lambda_1)(W^2 - (\omega_1 + \overline{\omega}_1)W + \omega_1\overline{\omega}_1).$$

Here, $r = 1$ and $s = 1$, so the indices i, j numerating real and complex roots take a single value, $i = j = 1$. The parametrization for the coefficients a_0, a_1, a_2 of the polynomial (2) reads

$$a_2 = -(\lambda_1 + \omega_1 + \overline{\omega}_1), \quad a_1 = \lambda_1(\omega_1 + \overline{\omega}_1) + \omega_1\overline{\omega}_1,$$
$$a_0 = -\lambda_1\omega_1\overline{\omega}_1.$$

The general solution to the theory (23) decomposes into the pair of components (10),

$$\xi_1 = \Lambda_1 A, \quad \zeta_1 = \Omega_1 A,$$
$$\Lambda_1 = W^2 - (\omega_1 + \overline{\omega}_1)W + \omega_1\overline{\omega}_1, \quad \Omega_1 = W - \lambda_1$$

that satisfy the first-order and the second-order equations (11),

$$m^2(W - \lambda_1)\xi_1 = 0,$$
$$m^2(W^2 - (\omega_1 + \overline{\omega}_1)W + \omega_1\overline{\omega}_1)\zeta_1 = 0, \tag{30}$$

respectively. The equations for the ξ-component correspond to the Chern–Simons–Proca theory [3,4] with mass $m|\lambda_1|$. The ζ-field satisfies the (tachyon) Maxwell–Chern–Simons–Proca equations [7,8]. The solution (12) to the original theory (23) is reconstructed by the formula

$$A = \frac{1}{(\lambda_1 - \omega_1)(\lambda_1 - \overline{\omega}_1)}\xi_1$$
$$+ \left[\frac{W - \omega_1}{(\overline{\omega}_1 - \lambda_1)(\overline{\omega}_1 - \omega_1)} + \frac{W - \overline{\omega}_1}{(\omega_1 - \lambda_1)(\omega_1 - \overline{\omega}_1)}\right]\zeta_1.$$

The conserved tensors (19) of the theory are parameterized by the indices $p = 0$ and $q = 0, 1$. The expressions for the tensors have the form

$$(T_1^0)^\mu_{\ \nu} = -\frac{m^2}{2}\left\{2\lambda_1(\Lambda_1 A)^\mu(\Lambda_1 A)_\nu - \lambda_1\delta^\mu_{\ \nu}(\Lambda_1 A)^\alpha(\Lambda_1 A)_\alpha\right\}$$
$$- (MA)^\mu(\Lambda_1 A)_\nu,$$
$$(U_1^0)^\mu_{\ \nu} = -\frac{m^2}{2}\left\{2(W\Omega_1 A)^\mu(W\Omega_1 A)_\nu\right.$$
$$- 2\omega_1\overline{\omega}_1(\Omega_1 A)^\mu(\Omega_1 A)_\nu$$
$$- \delta^\mu_{\ \nu}[(W\Omega_1 A)^\alpha(W\Omega_1 A)_\alpha$$
$$\left.-\omega_1\overline{\omega}_1(\Omega_1 A)^\alpha(\Omega_1 A)_\alpha]\right\} - (MA)^\mu(\Omega_1 A)_\nu,$$

$$(U_1^1)^\mu_{\ \nu} = -\frac{m^2}{2}\left\{2(\omega_1 + \overline{\omega}_1)(W\Omega_1 A)^\mu(W\Omega_1 A)_\nu\right.$$
$$- 2\omega_1\overline{\omega}_1\big((\Omega_1 A)^\mu(W\Omega_1 A)_\nu + (W\Omega_1 A)^\mu(\Omega_1 A)_\nu\big)$$
$$- \delta^\mu_{\ \nu}[(\omega_1 + \overline{\omega}_1)(W\Omega_1 A)^\alpha(W\Omega_1 A)_\alpha$$
$$\left.- 2\omega_1\overline{\omega}_1(W\Omega_1 A)^\alpha(\Omega_1 A)_\alpha]\right\}$$
$$- (MA)^\mu(W\Omega_1 A)_\nu. \tag{31}$$

The 00-components read

$$(T_1^0)^0_{\ 0} = -\frac{m^2}{2}\lambda_1(\Lambda_1 A, \Lambda_1 A) - (MA)^0(\Lambda_1 A)_0,$$
$$(U_1^0)^0_{\ 0} = -\frac{m^2}{2}\left\{(W\Omega_1 A, W\Omega_1 A) - \omega_1\overline{\omega}_1(\Omega_1 A, \Omega_1 A)\right\}$$
$$- (MA)^0(\Omega_1 A)_0,$$
$$(U_1^1)^0_{\ 0} = -\frac{m^2}{2}\left\{(\omega_1 + \overline{\omega}_1)(W\Omega_1 A, W\Omega_1 A)\right.$$
$$\left.- 2\omega_1\overline{\omega}_1(W\Omega_1 A, \Omega_1 A)\right\}$$
$$- (MA)^0(W\Omega_1 A)_0. \tag{32}$$

The sign of $(T_1^0)^0_{\ 0}$ coincides with the sign of $-\lambda_1$; see (28). The linear combination of (U_1^0) and (U_1^1) does give rise to a positive conserved tensor unless $\omega_1\overline{\omega}_1 = 0$ (Case B, $\lambda_2 = 0$). Thus, Cases B and C of theory (23) are unstable unless the decomposition (6) has one simple nonzero root and a double zero root. The degrees of freedom of the stable theory include one massive and one massless vector mode.

3.3 Case D

The polynomial (6) has the simple real root λ_1 multiplicity 3, i.e.,

$$M = m^2(W - \lambda_1)^3 = m^2(W^3 - 3\lambda_1 W^2 + 3\lambda_1^2 W - \lambda_1^3).$$

The comparison with (6) brings us to the identification $r = 1$ and $s = 0$. In this case, the index i can take a single value $i = 1$ and $p_1 = 3$. The parametrization for the coefficients a_0, a_1, a_2 of the polynomial (6) reads

$$a_2 = -3\lambda_1, \quad a_1 = 3\lambda_1^2, \quad a_0 = -\lambda_1^3.$$

The general solution to the theory (23) consists of one component. The new variables (10) are not introduced.

The conserved tensors are constructed by the general rule (19) and parameterized by the indices $i = 1$ and $p = 0, 1, 2$. The expressions for the tensors have the form

$$(T_1^0)^\mu_{\ \nu} = -\frac{m^2}{2}\left\{2(w^2 A)^\mu(WA)_\nu + 2(WA)^\mu(w^2 A)_\nu\right.$$
$$+ 2\lambda_1(wA)^\mu(wA)_\nu - \delta^\mu_{\ \nu}[2(w^2 A)^\alpha(WA)_\alpha$$
$$\left.+ \lambda_1(wA)^\alpha(wA)_\alpha]\right\} - (MA)^\mu A_\nu,$$

$$(T_1^1)^\mu{}_\nu = -\frac{m^2}{2}\Big\{2(wWA)^\mu(wWA)_\nu + 2\lambda_1(w^2A)^\mu(WA)_\nu$$
$$+ 2\lambda_1(WA)^\mu(w^2A)_\nu - \delta^\mu{}_\nu\big[(wWA)^\alpha(wWA)_\alpha$$
$$+ 2\lambda_1(w^2A)^\alpha(WA)_\alpha\big]\Big\} - (MA)^\mu(WA)_\nu,$$

$$(T_1^2)^\mu{}_\nu = -\frac{m^2}{2}\Big\{2\lambda_1(w^2A)^\mu(w^2A)_\nu + 4\lambda_1(wWA)^\mu(wWA)_\nu$$
$$+ 2\lambda_1^2(w^2A)^\mu(WA)_\nu + 2\lambda_1^2(WA)^\mu(w^2A)_\nu$$
$$- 2\lambda_1^3(wA)^\mu(wA)_\nu - \delta^\mu{}_\nu\big[\lambda_1(w^2A)^\alpha(w^2A)_\alpha$$
$$+ 2\lambda_1(wWA)^\alpha(wWA)_\alpha + 2\lambda_1^2(w^2A)^\alpha(WA)_\alpha$$
$$- \lambda_1^3(wA)^\alpha(wA)_\alpha\big]\Big\} - (MA)^\mu(W^2A)_\nu, \tag{33}$$

where the notation $w = W - \lambda_1$ is used.

The 00-components read

$$(T_1^0)^0{}_0 = -\frac{m^2}{2}\Big\{2(WA, W^2A) - 3\lambda_1(WA, WA)$$
$$+ \lambda_1^3(A, A)\Big\} - (MA)^0 A_0,$$

$$(T_1^1)^0{}_0 = -\frac{m^2}{2}\Big\{(W^2A, W^2A) - 3\lambda_1^2(WA, WA)$$
$$+ 2\lambda_1^3(WA, A)\Big\} - (MA)^0(WA)_0,$$

$$(T_1^2)^0{}_0 = -\frac{m^2}{2}\lambda_1\Big\{3(W^2A, W^2A) - 6\lambda_1(W^2A, WA)$$
$$+ \lambda_1^2(WA, WA) + 2\lambda_1^2(W^2A, A)\Big\}$$
$$- (MA)^0(W^2A)_0. \tag{34}$$

One can check that the quantities (33) are not combined into a positive tensor. This result also applies to the case $\lambda_1 = 0$. The theory with a root of multiplicity three has to be considered as unstable anyway.

4 An example of stable self-interactions

As we have seen, some of the higher derivative extensions of the Chern–Simons theory admit positive conserved tensors at the free level. In this section, we provide an example of an interaction in Case A, such that the theory still has a positive conserved tensor and remains therefore classically stable. The equations of motion read

$$\mathcal{M}A \equiv m^2(W-\lambda_1)(W-\lambda_2)(W-\lambda_3)A - U'(\xi^\alpha\xi_\alpha)\xi = 0,$$
$$\xi = \sum_{i=1}^3 \beta_i^0 \Lambda_i A, \tag{35}$$

where $U(s)$ can be any scalar function; $U'(s) = \frac{dU(s)}{ds}$ and β_i^0 are treated as the parameters of interactions. The interaction could be constructed by the factorization method of Refs. [11,18], which ensures survival of the selected conservation law of the free theory at the interacting level. Here, we do not

elaborate on the procedure for constructing the interaction, we just examine consistency and stability of the interacting model.

The theory admits the conserved tensor

$$T^\mu{}_\nu(A) = \sum_{i=1}^3 \beta_i^0 (T_i^0)^\mu{}_\nu(\Lambda_i A) + \frac{1}{2}\delta^\mu{}_\nu U(\xi^\alpha\xi_\alpha),$$
$$\partial_\mu T^\mu{}_\nu = -\partial_\nu \xi^\alpha(\mathcal{M}A)_\alpha. \tag{36}$$

Taking account of the equations of motion it can be rewritten as

$$T^\mu{}_\nu = -\sum_{i=1}^3 \frac{m^2\beta_i^0\lambda_i}{2}\Big\{2(\Lambda_iA)^\mu(\Lambda_iA)_\nu - \delta^\mu{}_\nu(\Lambda_iA)^\alpha(\Lambda_iA)_\alpha\Big\}$$
$$- U'(\xi^\alpha\xi_\alpha)\xi^\mu\xi_\nu + \frac{1}{2}\delta^\mu{}_\nu U(\xi^\alpha\xi_\alpha) - (MA)^\mu\xi_\nu. \tag{37}$$

The conserved tensor is positive if $-\beta_i^0\lambda_i > 0$ and $U > 0, U' < 0$. The latter property is not satisfied by the polynomial interactions. The admissible choice can be $U(s) = \pi/2 - \text{arctg}(s)$, for example.

The consistent inclusion of interactions should not change the degree of freedom number. The interaction (35) is consistent. This fact can be seen from decomposition of solution into components (24). The equations of motion for the components take the form

$$\mathcal{M}_i\xi_i \equiv m^2(W-\lambda_i)\xi_i - U'(\xi^\alpha\xi_\alpha)\xi = 0,$$
$$\xi = \sum_{j=1}^3 \beta_j^0\xi_j, \quad i = 1, 2, 3. \tag{38}$$

At the free level, elimination of longitudinal degree of freedom is ensured by the transversality conditions $\partial^\alpha(\xi_i)_\alpha = 0, i = 1, 2, 3$. In nonlinear theory, the transversality conditions are modified but still remain the first-order constraints,

$$\partial^\mu(\mathcal{M}_i\xi_i)_\mu = \partial^\mu\Big\{m^2\lambda_i(\xi_i)_\mu + U'(\xi^\alpha\xi_\alpha)\xi_\mu\Big\} = 0. \tag{39}$$

The degree of freedom number can also be covariantly computed without depressing the order, e.g. by bringing the original higher derivative equations into the involutive form as explained in [19]. Anyway, Eq. (38) still describe three degrees of freedom, so the interaction (35) is stable (if $U > 0, U' < 0$) and consistent.

Rare examples are known of stable interactions in the higher derivative systems. The best known example is $f(R)$-gravity [20,21] where the canonical energy is bounded at the linearized level. This exceptional phenomenon happens because the theory is strongly constrained. In Ref. [22], the stability of some interactions is demonstrated for the Pais–Uhlenbeck oscillator (whose canonical energy is unbounded) by numerical simulations. The stable interactions were recently proposed for Podolsky electrodynamics [11] and for the higher-order Pais–Uhlenbeck oscillator [18].

The example of this section extends the limited list of known stable interactions in higher derivative models.

Concluding remarks

Let us summarize the results. In this paper, we suggest a simple general procedure of constructing a family of higher-order symmetries and related conservation laws for the derived theories whose equations are polynomial in certain operator (2). For the higher-order extensions of the Chern–Simons theory (1), being an example of derived theory, we explicitly deduce the conserved tensors. In some cases, depending on the structure of the roots in the polynomial (6), the positive tensors exist among the conserved quantities, while in the other cases, none of the conserved quantities is positive. Once a positive conserved tensor exists, the theory is classically stable, even though the canonical energy is unbounded. In the third-order examples of the theory (1) we notice that the stable theories realize the irreducible unitary representations of the Poincaré group, while the models admitting only unbounded conserved tensors correspond to non-unitary representations. We also demonstrate that a stable free theory can admit consistent interactions that do not break the stability.

Finally, we make remarks on the stability at the quantum level. Let us mention that the derived theories (2) admit nontrivial Lagrange anchors that can be constructed as polynomials in W of order lower than n. The construction of the anchor for the case $n = 2$ is demonstrated in [11]. This may allow one to quantize the classically stable theory without loss of stability. As established in [23–25], every Lagrange anchor leads to a Poisson bracket and Hamiltonian in the first-order formalism. The inequivalent Lagrange anchors lead to the canonically inequivalent Poisson brackets, so the theory will be a multi-Hamiltonian in the first-order formulation once it admits different Lagrange anchors. As demonstrated in [13], the Lagrange anchor maps conservation laws to symmetries. In the examples of classically stable higher derivative systems admitting the different Lagrange anchors [11,18], the anchor exists such that it maps the positive conserved quantity to the time shift. This means that in the corresponding Hamiltonian formalism (which is not unique, once there exist inequivalent Lagrange anchors) the positive quantity can serve as a Hamiltonian. As an example, let us mention that for the Pais–Uhlenbeck oscillator positive Hamiltonians are known [26,27], also at interacting level [18]. As the Hamiltonian is bounded at the classical level, we can hope to have a bounded spectrum of energy in the quantum theory. In view of these reasons, we may expect that classically stable higher derivative extensions of the Chern–Simons model can remain stable at the quantum level once an appropriate Lagrange anchor is applied to quantize the theory.

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