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Twenty-five years of random asset exchange modeling

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Abstract. The last 25 years have seen the development of a significant literature within the subfield of econophysics which attempts to model economic inequality as the emergent property of systems of stochastically interacting agents. In this article, the literature surrounding this approach to the study of wealth and income distributions, henceforth the "random asset exchange" literature following the terminology of Sinha (Phys Scr 2003(T106):59, 2003), is thoroughly reviewed for the first time. The foundational papers of Drăgulescu and Yakovenko (Eur Phys J B 17(4):723–729, 2000), Chakraborti and Chakrabarti (Eur Phys J B 17(1):167–170, 2000), and Bouchaud and Mézard (Physica A 282(3):536–545, 2000) are discussed in detail, and principal canonical models within the random asset exchange literature are established. The most common variations upon these canonical models are enumerated and the successes and limitations of such models are discussed. The paper concludes with an argument that the literature should move in the direction of more explicit representations of economic structure and processes to acquire greater explanatory power.

1 Introduction

Over the last 16 years, the problem of economic inequality has become the epicenter of one of the most intense political debates in the United States. Concern about the widening gap between rich and poor had been steadily growing since the 2008 U.S. bank bailouts and the 2010 *Citizens United v. FEC* Supreme Court decision, and the outbreak of the Occupy Wall Street movement in September 2011 decisively pushed the inequality question to the forefront of American politics. Though the Occupy movement, which introduced the dichotomy of "the 1%" versus "the 99%" to public discourse, did not immediately produce anything by way of practical politics, it nonetheless laid the foundation for U.S. Senator Bernie Sanders' two campaigns for president, in which he used the language of Occupy to reframe economic inequality as the result of pro-business policy choices which could be rectified through a social-democratic "political revolution."

However, the idea that economic inequality is an issue at all is by no means an uncontroversial one. A majority of Republican party voters do not believe that the current level of economic inequality in the United States is excessive [5]. The legacy of "Reaganomics"—the economic policy pursued by the Federal government of the United States under the tenure of former President Ronald Reagan, which was characterized by cuts to tax rates and other concessions to proponents of "supply-side" economic theory—remains contentious. And within the Republican delegation to the United States House of Representatives, a proposal to eliminate the current bracketed income tax system and to replace it with a much higher nationwide flat sales tax—as a way to dramatically lessen the tax burden on the wealthy—has gained traction [6].

Controversies surrounding the nature of economic inequality are just as longstanding and intense within the realm of academic economics. On one hand, economists tend to be more skeptical than other social scientists of government intervention into economic affairs, as a great deal of emphasis is placed on the fact that, within the discipline's canonical models, free markets have no trouble arriving at socially optimal allocations of resources all on their own. On the other hand, French economist Thomas Piketty's 2013 magnum opus *Capital in the 21st Century*, which proposes the imposition of a global, progressive tax on wealth in order to rein in inequality, has become greatly influential in popular-academic debates concerning the issue [7]. The intense political and academic disputes around the question of inequality therefore show no sign of abating anytime soon.

1.1 The universality of economic inequality

Whatever one believes about the moral question of inequality, it is indisputable that, in nearly every single

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developed market economy, the degree of stratification between the rich and everyone else is both staggering and increasing. In the United States, the share of household wealth owned by the top 1% of the population by net worth grew from 29.9% in 1989 to 35.5% in 2013; meanwhile, the share of wealth owned by the bottom 50% of the population shrunk from 3 to 1.1% over the same period [8]. In Germany, individuals at the 90th percentile of net assets possess 13 times as much wealth as the median individual and over a quarter of individuals possess liabilities equal to or greater than their assets, resulting in a negative net worth [9]. While there are many countries where the degree of wealth inequality is not this extreme—the United States has one of the most unequal distributions of wealth in the world—the overall structure is strikingly similar across almost all countries [10]. In every market economy for which data exists, many possess very little wealth and a few possess much.

Great inequality also governs the distribution of incomes in market economies. As reported by Horowitz et al. for the Pew Research Center, the share of aggregate income possessed by high-income households[1] has grown from 29% in 1970 to 48% today [5]. In the same period, the share of aggregate income possessed by low-income households[2] fell from 10 to 9% over the same period. The incomes of those who are already in the top 5% of the population in terms of earnings have likewise grown the faster than the incomes of all other earners over the past 40 years.

1.2 Measuring inequality: the Gini coefficient

One of the most popular metrics used to quantify the degree of inequality present in a given wealth or income distribution—or, indeed, any density distribution over a non-negative domain—is the Gini coefficient, named for Italian statistician Corrado Gini. The Gini coefficient of a distribution $f(x)$ is defined by reference to the Lorenz curve, itself defined as the function:

$$L\left(F(x)\right) = \frac{1}{\mu}\int_0^x s \cdot p(s)ds \tag{1}$$

where $F(x) = \int_0^x f(s)ds$ is the cumulative density distribution of $f(x)$ and $\mu = \int_0^\infty sf(s)ds$ is the mean of $f(x)$ [11]. Intuitively, this integral represents the share of some asset—say, income—held by the bottom $100x\%$ of the population, normalized by the mean of the distribution. The Gini coefficient is then given by twice the difference between the area under Lorenz curve of a perfectly egalitarian distribution—a straight line with a slope of 1—and the Lorenz curve of the distribution in question [12]. Thus, the canonical formula used to calculate the Gini coefficient is:

$$G = 1 - 2\int_0^1 L(x)dx \tag{2}$$

The Gini coefficient can take on any value between 0—perfect equality—and 1—perfect inequality. To extend this statistic to describe dispersion within a finite population $\{x_i\}_{i=1}^N$, however, it is more convenient to leverage the alternative definition of the Gini coefficient as half the relative mean absolute difference of a distribution:

$$G = \frac{1}{2}\frac{\left(\sum_{i=1}^N \sum_{j=1}^N |x_i - x_j|\right)/N^2}{\mu} \tag{3}$$

Note that this definition of the Gini coefficient over a discrete population does not perfectly correspond to its continuous counterpart, however, as the former has an upper bound of $1 - 1/N$ [13].

On its own, the Gini coefficient is not a faultless measure of inequality. It has been criticized on the basis that distributions with very different levels of concentration in the right tail can produce identical indices; there is therefore significant information lost when using it to represent an entire distribution with a single scalar value [14]. Nonetheless, the Gini coefficient serves as a useful and widely used statistic for summarizing the degree of dispersion present in a given wealth or income distribution.

Underscoring the universality of steep economic inequality in both wealth and income distributions, Table 1 displays the Gini coefficients for the wealth and income distributions of ten countries. One observes that wealth distributions are almost always "more unequal" than income distributions: Gini coefficients for wealth distributions tend to range between 0.5 and 0.8, while Gini coefficients for income distributions tend to range from 0.25 to 0.45. Furthermore, there is no obvious correlation between the Gini coefficients for wealth distributions and for income distributions: some countries, such as China, have Gini coefficients relatively close in value, while other countries, such as France, have Gini coefficients for wealth over twice as high as the corresponding value for income.

1.3 Pareto, Gibrat, and the econophysicists

The universality of the phenomenon of extreme inequality across market economies is highly significant and demands investigation. Different countries have dramatically different approaches to welfare programs, tax structures, and economic policy of all sorts. Yet the distributions of wealth and income present in these countries are remarkably similar in form. It follows that there is likely some shared set of characteristics that account for this common structure of wealth distribution. This line of questioning points one to an often overlooked and still poorly understood aspect of inequality: its cause.

[1] Defined as households with incomes greater than twice the national median.

[2] Defined as households with incomes less than two-thirds the national median.

Table 1 Gini coefficients of wealth and income inequality for ten countries, all major world economies, based on data from the year 2000

Gini Coefficients of Wealth and Income for Ten Countries

Country	Gini Coefficient of Wealth	Gini Coefficient of Income
United States	0.801	0.401
France	0.730	0.311
United Kingdom	0.697	0.396
India	0.669	0.344*
Germany	0.667	0.289
Netherlands	0.650	0.298*
Australia	0.622	0.331*
Italy	0.609	0.353
Spain	0.570	0.343
China	0.550	0.420*

Asterisks represent values for which the Gini coefficients of income for the year 2000 are unavailable in the relevant data set; Gini coefficients for the closest available year are provided instead

Data for Gini coefficients of wealth are taken from Davies et al. (2009), while data for Gini coefficients of income are taken from the World Bank, accessed via the FRED database hosted by the Federal Reserve Bank of St. Louis. Gini coefficients of income for India, the Netherlands, and Australia are from 2004. Gini coefficient of income for China is from 2002 [15]

The nature and origin of economic inequality has been an open problem in economics for more than a century. In 1897, the Italian civil engineer-turned-economist Vilfredo Pareto attempted to provide an answer after noticing a striking pattern in data for land-ownership rates in Italy. Specifically, Pareto posited that income in every society is distributed according to a decreasing power law; namely:

$$p(x) \propto x^{-1-\alpha} \tag{4}$$

where $p(x)$ represents the probability density function of income and α represents the "Pareto index," with smaller values producing fatter tails and thus representing more unequal distributions. This observation has come to be known as the "weak Pareto law," with its strong counterpart including the additional claim that the Pareto index possesses a value in the range 1.5 ± 0.5 [16]. But not long thereafter it became apparent that this law did not actually well characterize the entire income distribution. Instead, when low- and middle-income strata were taken into account, the data seemed to be much better fit by a right-skewed lognormal distribution:

$$p(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(x)-\mu)^2}{2\sigma^2}\right) \tag{5}$$

This fact was first noticed by economist Robert Gibrat, who posited as an explanation that growth rates (of firm size, personal wealth, etc.) are independent of current size [17]. More formally, Gibrat's law holds that the logarithm of growth rates $X_t = \ln(Y_t)$ follow the stochastic process:

$$\begin{aligned} Y_t &= \exp(X_t) \\ X_t &= \alpha t + \beta B_t \end{aligned} \tag{6}$$

where B_t is a Brownian motion and for which it is easily shown that $Y_t \sim \text{Lognormal}(\alpha t, \beta^2 t)$ [18]. It is now well established that, in fact, both Pareto and Gibrat were partially correct: a lognormal-like distribution tends to characterize the bulk of incomes, while the Pareto distribution tends to characterize the highest 2–3% of incomes [19].[3]

Since these discoveries, mainstream economic theory has, broadly speaking, shied away from further attempts to posit a universal form for these distributions or to explain the processes responsible for their emergence. There are both normative and methodological reasons for this aversion. The normative reason, as voiced by Piketty, manifests as skepticism that universal laws governing income distributions exist at all [7]. The methodological reason, on the other hand, stems from the fact that prototypical macroeconomic models make use of single, representative agents, an approach ill-suited for describing distributions over populations. More sophisticated tools capable of addressing such questions, such as Heterogeneous Agent New Keynesian (HANK) models, exist, but they are still relatively new to the scene [20]. This gap drew the attention of physicists interested in applying methods developed for the study of the natural sciences to questions in the social sciences in the late 1990s.

The aim of this group of researchers, who became known as *econophysicists*,[4] was to capture the characteristic features of empirical wealth and income distri-

[3] A similar double regime has been observed in other well-studied size distributions, e.g., city populations [18].

[4] According to Yakovenko and Rosser (2009), the term *econophysics*—and its derivative *econophysicist*—was coined in 1995 by American physicist H. E. Stanley to describe the utilization of techniques developed in statistical physics to explore phenomena normally considered to fall under the purview of economics [21].

butions, as made known by extensive statistical analyses. There is now substantial evidence that the bulk of the income distribution in all capitalist countries follows an exponential distribution [22]. The right tail of the income distribution follows the aforementioned Pareto law and the left tail follows Gibrat's law. The exponential bulk and the log-normal left tail are sometimes unified in the form of the closely related Gamma distribution:

$$p(x) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \tag{7}$$

where α is called the "shape parameter" and β the "rate parameter." However, the existing data are insufficient to conclude whether the Gamma or log-normal distribution provides the better empirical fit in general [23].

Wealth distributions are unfortunately much less well understood as there is a dearth of publicly available data. Rough estimates of wealth distributions in pre-capitalist societies, such as the New Kingdom of Egypt and medieval Hungary, provide some evidence that power-law distributions of wealth prevailed even then, but these results are far from conclusive [24,25]. Drăgulescu and Yakovenko (2001b) use inheritance tax data to study the wealth distribution in the modern United Kingdom, which is found to have a similar structure to the UK's income distribution [26]. Further supporting this conclusion, Sinha (2006), among others, find evidence that the very wealthiest stratum of society, as measured by published "rich lists," follows a power law distribution [27]. These features appear to emerge even in artificial economies, with a wealth distribution characterized by an exponential bulk and power-law tail being observed across players of a massively multiplayer online game with inbuilt systems of production and trade [28]. Thus, early exchange models in the econophysics literature sought to generate distributions exhibiting both the exponential bulk and power-law tail observed in data by means of symmetric, often binary, interactions.

The first paper in this lineage was Ispolatov et al. (1998), and shortly thereafter two papers which would ultimately become the cornerstones of the random asset exchange modeling literature—Drăgulescu and Yakovenko (2000) and Bouchaud and Mézard (2000)—emerged [2,4,29]. But as it turned out, the econophysicists were not the first to approach the question of inequality in this way. The sociologist John Angle had actually published a series of papers studying a model extremely similar to Ispolatov et al.'s more than a decade earlier [30–32], though the literature had no knowledge of this fact until it was pointed out by Lux (2005) [33]. Likewise, it was noticed by Patriarca et al. (2005) that Drăgulescu and Yakovenko's model was anticipated by a series of papers by Eleonora Bennati, which had been published in Italian and had not yet been translated into English [34–36].

Nonetheless, in the 25 years since Ispolatov et al.'s original paper, a sizeable literature on this class of models has emerged, with countless variations having been defined and investigated. The literature has also become much more diverse in that time: though this subject was initially solely the domain of a subset of physicists interested in exploring economic questions, it has since drawn attention from researchers with backgrounds in mathematics, economics, information and systems science, and more.

This article provides, for the first time, a comprehensive review of this literature, which, following the terminology of Sinha (2003), will be referred to as the "random asset exchange" literature. While many excellent partial reviews do already exist (see Chatterjee and Chakrabarti (2007), Yakovenko and Rosser (2009), Patriarca et al. (2010), Patriarca and Chakraborti (2013), and Boghosian's 2019 article in *Scientific American*, just to name a few [21,37–41]), all either have since become dated or have focused only on select parts of the literature. This review is the first to the authors' knowledge that not only discusses all of the most significant econophysical models of wealth and income inequality, but fully enumerates the most common variations upon the literature's benchmark models as well.

The remainder of the paper is structured as follows. Section 2 introduces the two principal categories of random asset exchange model and discusses a handful of the most significant ("canonical") papers of each type. Section 3 identifies six major themes in the literature—types of variations upon the canonical models meant to study the effect of a certain economic phenomenon on wealth distributions—and summarizes key papers relating to each theme. Section 4 concludes with a discussion of the overall significance and explanatory value of the random asset exchange literature, and makes the argument for moving in the direction of more concrete economic modeling.

2 The taxonomy of random asset exchange models

Most random asset exchange models fall into one of two categories. The first of these is conventionally called the "kinetic wealth exchange" (KWE) category of model, which was popularized by Drăgulescu and Yakovenko (2000). Named such because of the similarity of such models to thermodynamic models derived from the kinetic theory of gasses, KWE models are typically characterized by the following properties:

1. Pairwise exchange between agents is the primary system state transition function;
2. Total money present in the system is conserved; and
3. Total money present between all pairs of agents engaged in exchange is conserved.

These features are analogous to the role of particle collisions, conservation of energy, and conservation of momentum in the kinetic theory of gasses, respectively.

The second prominent category of model, inspired by models of directed polymers rather than ideal gasses, is the Bouchaud-Mézard (BM) type model, first introduced by Bouchaud and Mézard (2000) [4]. In contrast to KWE-style models, BM-style models tend to be characterized instead by static wealth flows between "adjacent" pairs of agents, as defined by an exogenously given adjacency network, being the main mechanism of system evolution. Furthermore, each agent's wealth is subject to endogenous stochastic variation, leading to systemic non-conservation of wealth.

While other formulations exist, models belonging to one of these two categories represent the great majority of the literature.

2.1 Kinetic wealth exchange

Though their work was anticipated by Angle (1986, 1992, 1993), Bennati (1988, 1993), and Ispolatov et al. (1998), it is Drăgulescu and Yakovenko (2000) who are credited with first formalizing and thoroughly studying the KWE model [2,29–32,35,36]. Their original formulation considers a system of $N \gg 0$ agents with $M \gg N$ units of wealth between them. Agents then engage in random pairwise exchanges, with a winner and loser being randomly selected in each pair and a transfer of wealth occurring, following some exchange rule:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} w_i(t) + \Delta w_{ij} \\ w_j(t) - \Delta w_{ij} \end{bmatrix} \tag{8}$$

with $\Delta w_{ij} > 0$ if agent i is the winner of the exchange and $\Delta w_{ij} < 0$ if j is victorious instead. Since KWE models almost always feature exclusively linear, pairwise exchanges, it is often convenient to represent the model's exchange rule as a 2×2 matrix $\mathbf{M}$, such that:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \mathbf{M} \begin{bmatrix} w_i(t) \\ w_j(t) \end{bmatrix} \tag{9}$$

Drăgulescu and Yakovenko demonstrate that, so long as Δw_{ij} is chosen such that the exchange process was time-reversal symmetric, then the distribution of money among agents converges to the entropy-maximizing exponential distribution:

$$p(w) = \frac{1}{T} \exp\left(-\frac{w}{T}\right) \tag{10}$$

where $T = \bar{w} = M/N$ represents the average wealth held by agents—analogous to temperature in the equivalent thermodynamic system. This result proves to be extremely robust, not varying with one's choice of time-reversal symmetric exchange rule or underlying adjacency network [42].

The differences between Drăgulescu and Yakovenko's model and those of Angle, Ispolatov et al., and Bennati are subtle. In both Angle's initial model (the "one-parameter inequality process," or OPIP) and Ispolatov et al.'s "multiplicative-random" exchange model, $\Delta w_{ij} = \varepsilon w_{loser}$, such that exchanges are of the form:

$$\begin{bmatrix} w_i(t+1) \\ w_j(t+1) \end{bmatrix} = \begin{bmatrix} w_i(t) + \varepsilon w_j(t) \\ (1-\varepsilon) w_j(t) \end{bmatrix} \tag{11}$$

if agent i wins the exchange. The sole difference between these two models is that Angle (1986) draws ε from a uniform distribution before each exchange, whereas Ispolatov et al. (1998) define ε as a fixed, global parameter. In contrast to the exchange rules investigated by Drăgulescu and Yakovenko (2000), both of these models break time symmetry and produce identical distributions which are very well-approximated by, but not exactly given by, Gamma distributions [32].

In both Ispolatov et al.'s additive-random exchange model and Bennati's model, agents exchange constant, quantized amounts of wealth, equivalent under rescaling to $\Delta w_{ij} = 1$. In Ispolatov et al. (1998), agents with 0 wealth are removed from the system entirely, causing all the wealth in the system to eventually be accumulated by a single agent (a phenomenon termed "condensation"). In Bennati (1988), however, agents with 0 wealth are permitted to win, but not to lose, exchanges, identical to the provision in the constant exchange rule discussed by Drăgulescu and Yakovenko. For that reason, the KWE-style model with time reversal-symmetric exchange rule is sometimes referred to as the Bennati-Drăgulescu-Yakovenko (BDY) model of wealth distribution. [21].

An extension of Drăgulescu and Yakovenko's model studied contemporaneously with its initial publication was that of Chakraborti and Chakrabarti (2000), who introduce a global "saving propensity" parameter λ [3]. Called the CC model (or, more rarely, the "saved wealth" [SW] model), its system dynamics are characterized by the fact that, for $\lambda \in [0, 1)$, each agent engages in multiplicative exchange with only a fraction $1 - \lambda$ of his total wealth. The exchange rule in such models can thus be described as:

$$\mathbf{M} = \begin{bmatrix} \lambda + \varepsilon(1-\lambda) & \varepsilon(1-\lambda) \\ (1-\varepsilon)(1-\lambda) & \lambda + (1-\varepsilon)(1-\lambda) \end{bmatrix} \tag{12}$$

where ε is drawn from a uniform distribution on $[0, 1]$ at every exchange.

Curiously, this slight modification dramatically alters the equilibrium distribution of money among agents within the system, with the mode of the distribution (the "most likely agent wealth") becoming non-zero for any value of $\lambda > 0$. More specifically, the mode approaches $T = \frac{M}{N}$ (an egalitarian distribution) as λ approaches 1. Gupta (2006) observes that this departure from the entropy-maximizing distribution is a consequence of the fact that the introduction of the saving propensity parameter λ causes the system transition matrix to become non-singular [43]. Patriarca et al. (2004a, b) demonstrate that the resultant distribution is extremely well fit by a scaled Gamma distribution:

$$p(w) = \frac{1}{\Gamma(n)} \left(n \cdot \frac{w}{T} \right)^{n-1} \exp\left(-n \cdot \frac{w}{T} \right) \tag{13}$$

where $n = 1 + 3\lambda/(1-\lambda)$ [44,45]. The fit is not exact, however, as the distributions differ in their fourth moments [46].

The CC model is extremely influential in the random asset exchange literature, and it itself has two major variations which must be mentioned. The first, introduced by Chatterjee et al. (2003), defines the saving propensity parameter as heterogeneously distributed throughout the population: instead of having identical saving propensities, each agent i has his own individual saving propensity $\lambda_i \in [0,1)$ drawn from the uniform distribution during model initialization [47]. The exchange rule of this model, called the CCM model, is thus:

$$\mathbf{M} = \begin{bmatrix} \lambda_i + \varepsilon(1-\lambda_i) & \varepsilon(1-\lambda_j) \\ (1-\varepsilon)(1-\lambda_i) & \lambda_j + (1-\varepsilon)(1-\lambda_j) \end{bmatrix} \tag{14}$$

The steady-state distribution exhibits a Gamma-like bulk, as in the CC model, as well as a right tail well fit by a power law with Pareto parameter $\alpha = 1$. This apparent power law is observed for any distribution of saving propensity of the form $\rho(\lambda) \approx |\lambda_0 - \lambda|^\alpha$, or for uniform distributions within a restricted range $\lambda_i \in [a,b] \subset [0,1)$ [48].

One well-known (and arguably unrealistic) aspect of the CCM model is that agents' wealth is highly correlated with the value of agents' saving parameter, such that the agents who save nearly all of their money in every transaction invariably become the wealthiest. This remains the case even if a significant bias in favor of poorer agents is introduced, because thrifty agents in the CCM model always stand to gain much more than they lose from every transaction [49]. This aspect of the model also explains the surprising appearance of the Pareto tail, which is actually somewhat illusory: the right tail of the equilibrium distribution of the CCM model is constituted by the overlapping exponential tails of the distributions corresponding to the subpopulations with the highest saving parameters [34] (Fig. 1).

Another significant drawback of the CCM model is that, while the right tails of the steady state distributions change from approximately Pareto with index 1 to exponential as the distribution of λ_i narrows, the empirical value of $\alpha \approx 1.5$ is never reached [50]. However, there are a number of ways to modify the CCM model and recover such a regime: Repetowicz et al. (2006), for instance, note that introducing modified wealth parameters with memory—$\hat{w}_i(t) = w_i(t) + \gamma w_i(q)$, with $\gamma \in (0,1)$ and $q < t$—before each transaction and applying the CC exchange rule thereto does permit Pareto tails with indices $\alpha > 1$ to be obtained [51]. Likewise, Bisi (2017) demonstrates that replacing the saving propensity parameter with a bounded, global function of an agent's wealth $\gamma(w_i)$ also permits superunitary Pareto indices [52].

The second variation, introduced by Cordier et al. (2005) and often called the CPT model, combines the CC model with an additional stochastic growth term:

$$\mathbf{M} = \begin{bmatrix} (1-\lambda) + \eta_i & \lambda \\ \lambda & (1-\lambda) + \eta_j \end{bmatrix} \tag{15}$$

where η_i and η_j are independent and identically distributed variables with mean 0 and variance σ^2 [53]. As in the BDY, CC, and CCM models, debts are not permitted, so a transaction only takes place so long as neither agent is reduced to a negative level of wealth. Because η_i and η_j are uncorrelated, total wealth is now only preserved in the mean. The CPT model has an inverse-Gamma equilibrium, with shape parameter $\alpha = 1 + \frac{2\lambda}{\sigma^2}$ and scale parameter $\beta = \alpha - 1$:

$$p(w) = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot w^{-1-\alpha} \cdot \exp\left(-\frac{\beta}{w} \right) \tag{16}$$

In this case, the shape parameter α may be interpreted as the "Pareto index" of the approximately power-law right tail.

The CPT model, like the CCM model, is a very flexible modeling framework and has consequently been studied in a variety of other contexts. Düring and Toscani (2008) employ a CPT model with quenched saving propensities to study international transactions, representing countries as subpopulations with different saving propensities [54]. Bisi and Spiga (2010) consider a variation on the CPT model wherein the amount of wealth an agent receives from his trading partner is also subject to stochastic fluctuations [55]. More recently, Zhou et al. (2021) investigate the effect of introducing a non-Maxwellian (i.e., wealth-varying) collision kernel in the CPT model [56].

2.1.1 Theft, fraud, and yard sales

Following the terminology of Hayes (2002) [57], binary exchange models in which the quantity of wealth transferred is proportional to the wealth of the loser, such as most of the exchange rules studied in Drăgulescu & Yakovenko (2000), are commonly referred to as "theft and fraud" (TF) models. In contrast, those in which the quantity of wealth transferred is proportional to the wealth of the poorer agent involved in a given exchange are referred to as "yard sale" (YS) models. YS-style models were first studied by Chakraborti (2002), who simulates an ensemble with the exchange rule:

$$\Delta w_{ij} = 2\varepsilon \min\{w_i, w_j\} \tag{17}$$

where ε is a uniform random variable with mean 0.5 [58].

The advantage of YS-style models over TF-style models is that, from a strategic perspective, agents

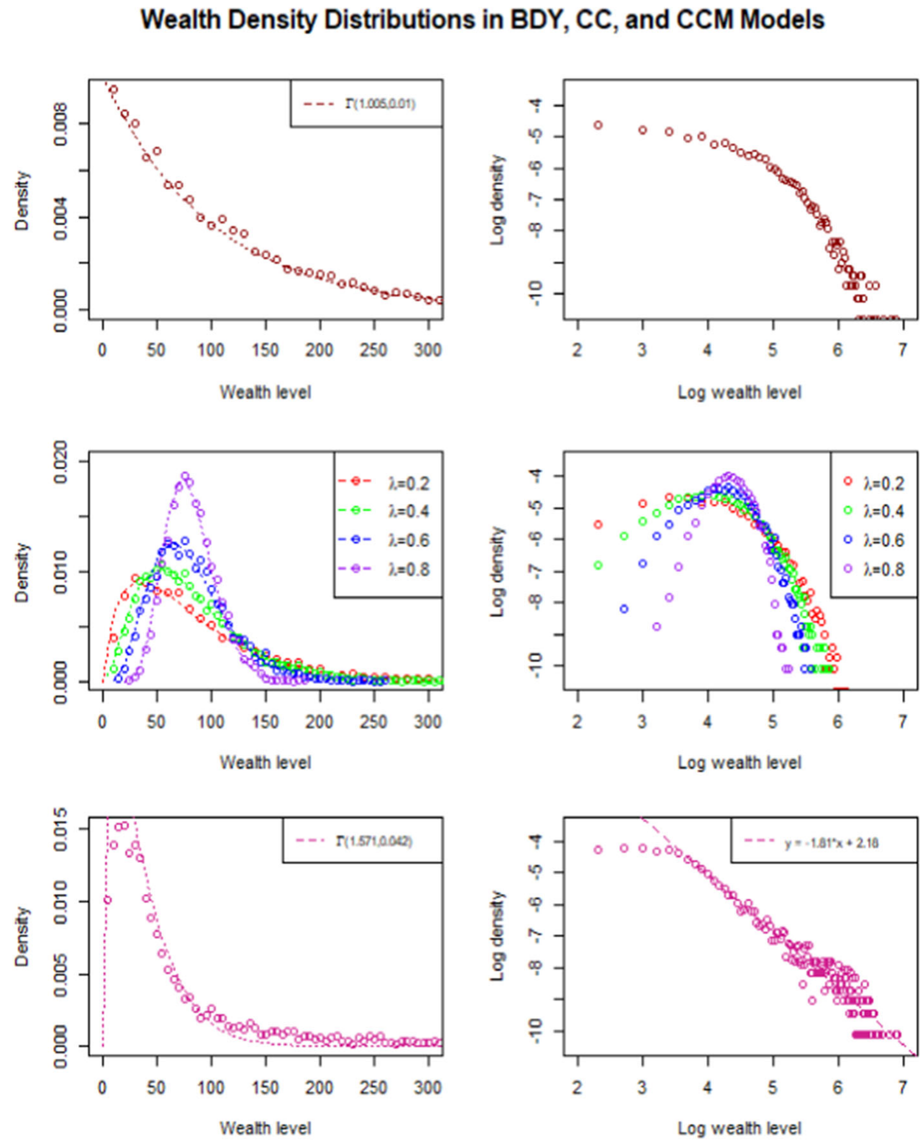

Fig. 1 Stationary distributions produced by the BDY (top), CC (middle), and CCM (bottom) models, with best-fit Gamma curves shown. All simulations were performed with the parameterization $N = 5000$ and $\bar{w} = 100$ over 10^5 iterations

are not disincentivized from engaging in trade, as the expected value of an exchange is always 0. This is in contrast to the TF model, which is so-named precisely because the expected value of an exchange is always negative for the richer agent. If risk-neutral agents were allowed to choose whether or not to engage in a given exchange, a TF economy would immediately freeze as soon as a wealth differential appears. The principal drawback of YS-style models is that, lacking modification, they always result in the condensation of wealth in the hands of a small number of agents [59, 60]. However, various studies have since demonstrated a number of different ways a non-degenerate equilibrium can be recovered, such as introducing a probabilistic bias in favor of the poorer agent [1], modeling the redistribution of wealth from rich to poor [59], employing extremal dynamics [61], or mixing in TF-style interactions with some non-vanishing probability [62].

Moukarzel et al. (2007) demonstrate that, in the case of a YS-style model where the proportion of the poorer agent's wealth at stake in each transaction is a fixed constant f, a sufficient bias of the probability p toward the poorer agent alone was sufficient to avoid condensation [63]. In particular, the critical probability p^* above which the system does not condense was found to be:

$$p^* = \frac{\log\left(\frac{1}{1-f}\right)}{\log\left(\frac{1+f}{1-f}\right)} \tag{18}$$

Based on this result, Bustos-Guajardo and Moukarzel (2012) study an extension of a YS-style model on an adjacency network, such that exchanges may only take place between adjacent agents; they find that the value of the critical probability remains the same regardless of the choice of network [64]. In fact, most system dynamics in the stable phase of the system are independent of the choice of network. However, certain dynamical aspects of the system (such as time required for the system to fully condense) do differ from the fully-connected case in the unstable (i.e., condensing) region. This is not entirely surprising, seeing as the number of agents to whom the wealth will condense is determined solely by the geometry of the underlying network; instead of one agent accumulating all the money, the distribution condenses to a set of "locally rich agents," sometimes termed the "oligarchy."

Redistribution in a YS-style model is examined by Boghosian (2014a), who introduces a mechanism by which, at each time step, a fraction χ of each agent's wealth is confiscated and subsequently redistributed uniformly among the population [59]. Introducing this mechanism not only prevents condensation, but also produces a Gamma-like steady state distribution with a Pareto-like tail. The dynamics of this mechanism are studied in more detail by Boghosian et al. (2017) and Devitt-Lee et al. (2018), in which it is combined with an bias in exchanges in favor of the wealthy, called the "wealth-attained advantage" (WAA) [65,66]. In this variation, termed the "extended yard sale" (EYS) model, the wealthier agent wins a given exchange with probability $p = rT(w_i - w_j)$, where $T = \bar{w}$ is the average wealth of the system and w_i is the wealth of the richer agent. The wealth-attained advantage formally acts as a net tax on the non-oligarchy, while the redistribution acts as a net tax on the oligarchy; once the rich-to-poor flux of the redistributive mechanism is eclipsed by the poor-to-rich flux of the wealthy agents' advantage (the system enters a "supercritical" state), the inequality of the resulting wealth distribution, as measured by the Gini coefficient, begins increasing rapidly. Boghosian and co-authors show that, if the poor-to-rich flux dominates, a "partial oligarchy" ultimately obtains a finite fraction of the total wealth of the system in the infinite-time limit—explicitly given by the ratio between χ (or $\lim_{w\to\infty} \chi(w) = \chi_\infty$ if nonlinear redistributive schemes are permitted) and ζ, the "scale" of the WAA in a given economy.

A generalization of the EYS model, the "affine wealth" (AW) model, is studied in a series of papers beginning with Li et al. (2019) [67]. The AW model permits negative wealth by defining a debt limit Δ, adding Δ to the wealths of both agents before each exchange, and subtracting Δ once the exchange is complete. Li et al. observe that the AW model provides a remarkably good fit to the U.S. wealth distribution, as reported by the U.S. Survey of Consumer Finances. Polk and Boghosian (2021) find in the context of the AWM model that, while the ratio between sub- and supercritical states of the AWM remains the ratio χ_∞/ζ, the fraction of wealth ultimately accumulated by the oligarchy also increases with the debt limit Δ [68]. Moreover, it is found that many developed capitalist economies appear to be very near criticality—the point below which a Gaussian distribution of wealth prevails, above which a partial oligarchy emerges, and at which an exponential distribution of wealth serves as an unstable stationary distribution [66,68]. Since the regime one finds oneself in near criticality is clearly sensitive to small changes in redistributive policy, Boghosian and co-authors assert that, if the AWM is to be believed, there is no universal form attributable to the right tail of wealth distributions; rather, policy is paramount.[5]

Also significant is the variation on the YS-style model first formulated by Iglesias et al. (2004) [71]. This model, sometimes referred to as the IGAV model, sees agents with wealths w_i and w_j and saving parameters λ_i and λ_j exchange quantities $\Delta w_{ij} = \min\{(1-\lambda_i)w_i, (1-\lambda_j)w_j\}$ with bias. The bias in favor of the poorer agent j is defined following Scafetta et al. (2002) [72]:

$$p = \frac{1}{2} + f \cdot \frac{w_i - w_j}{w_i + w_j} \tag{19}$$

The asymmetry flux index $f \in [0, 1/2]$ may be interpreted as the degree of social protection offered to the poor. A number of similar models which modify Iglesias et al.'s exchange rule are studied by Caon et al. (2007) [73]. Finally, Neñer and Laguna (2021a) show that, in the IGAV model, the richest agents are not necessarily the thriftiest [49]. Instead, the saving propensity λ_i^* that maximizes equilibrium average wealth lies in the interval $(0, 1)$, increasing with f (Fig. 2).

Heinsalu and Patriarca (2014) introduce a variation of the BDY model meant to more explicitly model the dynamics of barter economies [74]. While Drăgulescu and Yakovenko examined, among others, the TF exchange rule:

$$\mathbf{M} = \begin{bmatrix} \varepsilon & \varepsilon \\ 1-\varepsilon & 1-\varepsilon \end{bmatrix} \tag{20}$$

where ε is a uniform random variable with mean 0.5, Heinsalu and Patriarca consider the rule:

$$\mathbf{M} = \begin{bmatrix} 1-\varepsilon_i & \varepsilon_j \\ \varepsilon_i & 1-\varepsilon_j \end{bmatrix} \tag{21}$$

where ε_i and ε_j are i.i.d. uniform random variables with mean 0.5. This modification, called the "immediate exchange" (IE) model, changes the system from a pure TF one, where wealth flows unidirectionally and, on average, from richer to poorer agents, to one in which wealth flows bidirectionally. In the general IE model, transactions have some probability μ of occurring unidirectionally in the manner of Angle (1986). The pure IE model with $\mu = 0$ has a steady state distribution $p(w)$ which is an exact Gamma distribution with a shape parameter of 2, which implies that $\lim_{x\to 0} p(x) = 0$,

[5] The implications of these works are clearly relevant in modern economic contexts. Empirical work has shown both that, on average, wealth in the United States has flowed from poor to rich over the last forty years and that, simultaneously, the degree of inequality in labor incomes has dramatically increased [69,70]. If we are indeed tending toward a partial oligarchy characterized by extreme inequality, what kinds of interventions must be considered to prevent inequality from reaching economically and politically destabilizing levels?

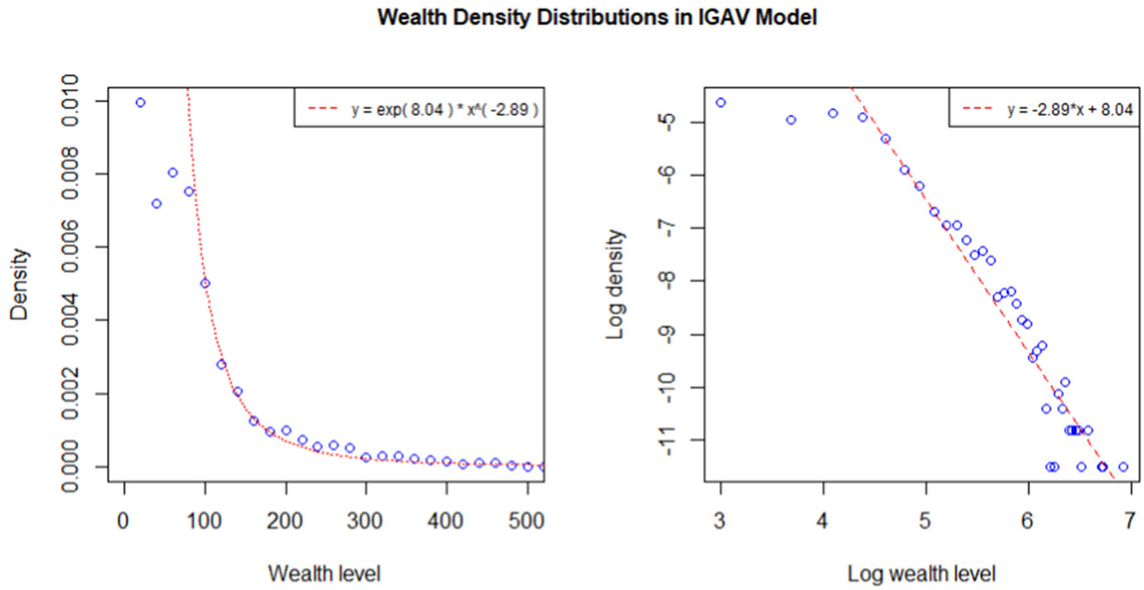

Fig. 2 Stationary distribution produced by the IGAV model with quenched savings propensities and a bias function per Eq. (19). Simulation was performed with the parameterization $N = 5000$ and $\bar{w} = 100$ over 10^6 iterations

$p(x)$ has a non-zero mode, and the right tail is well-approximated by a Pareto distribution with $\alpha = 1$ [75].

2.2 Bouchaud-Mézard models

Unlike KWE-style models, BM-style models do not make use of agents pairing up and engaging in binary transactions with a winner and a loser; rather, the rate of exchange between agents is defined by a fixed adjacency matrix $\mathbf{J}$, each entry of which J_{ij} represents the "cash flow rate" from agent j to agent i. In Bouchaud and Mézard (2000), each agent in the population of size N has two sources of income—stochastic returns from investments and sales of a product to other agents—and one source of expenses—purchases of products from other agents. Thus, the income of agent i is given by the Langevin equation:

$$\dot{w}_i = \eta_i(t) w_i(t) + \sum_{j \neq i} J_{ij} w_j(t) - \sum_{j \neq i} J_{ji} w_i(t) \qquad (22)$$

where the stochastic growth terms η_i are i.i.d. Gaussian random variables with mean μ and variance $2\sigma^2$. Similar stochastic processes have been studied in economics as well; notably, letting $J \to 0$ allows Eq. (22) to be rewritten (following Stratonovich's interpretation):

$$dw_i = \mu w_i dt + \sqrt{2}\sigma w_i \circ dB_{it} \qquad (23)$$

which is equivalent to the model of city growth (with an inverse power law stationary distribution) studied by Gabaix (1999) [76].

In stark contrast to KWE-style models, BM-style models have no restriction on total wealth being conserved. The simplest case, in which all rates of exchange are equalized such that $J_{ij} = \frac{\bar{J}}{N}$, is found by way of a mean-field approximation to produce an inverse-Gamma equilibrium distribution with shape parameter $\alpha = 1 + \frac{J}{\sigma^2}$ and scale parameter $\beta = \alpha - 1$—strikingly similar in form to the equilibrium distribution of the CPT model. However, this approximation is time-limited; for any finite number of agents, the BM model on a complete graph will eventually exhibit wealth condensation, with the probability that a given agent will have wealth less than any finite fraction of total wealth growing to 1 [77].

Further investigation into this category of model has shown that the resulting distribution is also sensitive to the nature of the underlying network defining the non-zero entries of the transaction matrix $\mathbf{J}$. Souma et al. (2001) demonstrate through simulation that defining $\mathbf{J}$ on a small-world network—where each agent neighbors only 0.1% of the population—leads to distributions which are best fit by a combination of log-normal and power-law distributions [78]. Garlaschelli and Loffredo (2004, 2008) likewise show that it is possible to retrieve a realistic mixed log-normal/power law distribution by simulating the model on a simple heterogeneous network with a small number of "hub" agents, and that the BM model on a homogeneous network is able to reproduce either a log-normal or a power law distribution—but not both—depending on the average number of adjacencies per agent [79,80]. Ma et al. (2013) simulate the BM model on a partially connected network and find the generalized inverse Gamma (GIGa) distribution provided the best fit to the steady state [81].

Though Bouchaud and Mézard's original model is written in continuous-time, a number of authors have studied similar models written in discrete time. Di Matteo et al. (2003), for example, consider the variation [82]:

$$w_i(t+1) = A_i(t) + (1 + B_i(t))\, w_t + \sum_{j \neq i} Q_{j \to i}(t) w_j(t) - \sum_{j \neq i} Q_{i \to j}(t) w_i(t) \tag{24}$$

For the purposes of their analysis, additive noise $A_i(t)$ is assumed to be Gaussian with mean zero, and multiplicative noise $B_i(t) = 0$. Additionally, each agent i is assumed to split a fixed share q_0 of his wealth evenly with all of his neighbors $j \in \mathcal{I}_i$, where $|\mathcal{I}| = z_i$. Thus, $Q_{i \to j}(t) = \frac{q_0}{z_i}$ if $j \in \mathcal{I}_i$ and 0 otherwise. The restricted system dynamics become:

$$w_i(t+1) = A_i(t) + (1 - q_0)\, w_i(t) + \sum_{j \in \mathcal{I}_i} \frac{q_0}{z_j} w_j(t) \tag{25}$$

The results this model produces depend on the choice of adjacency network: interestingly, but perhaps not surprisingly, scale-free networks produce power-law distributions. In this case, the equilibrium wealth level of a given node is nearly perfectly correlated with the number of neighbors it has in the specified network, as shown in Fig. 3.

Scafetta et al. (2004) propose another discrete-time variation of Bouchaud and Mézard's model, motivated by a dissatisfaction with the fact that the original BM model produces a constant wealth flux from rich to poor [83]. This is not necessarily realistic as wealth should only be transferred in exchange if an agent buys an asset for a price different than its value; such a model cannot explain wealth inequality under the assumption of perfect pricing. Thus, Scafetta et al. propose a model in which the wealth of agent i is given by:

$$w_i(t+1) = w_i(t) + r_i \xi(t) w_i(t) + \sum_{j \neq i} w_{ij}(t) \tag{26}$$

where $r_i > 0$ is the "individual investment index," given as the product of the global investment index and the proportion of wealth actually invested by agent i, $\xi(t)$ is a Gaussian random variable representing return on investment, and $w_{ij}(t)$ represents the flow of wealth between agents j to agent i in period t, which is assumed to be Gaussian with mean $\mu = fh\frac{w_i - w_j}{w_i + w_j} \min\{w_i, w_j\}$ and standard deviation $\sigma = h \min\{w_i, w_j\}$.

By varying f, h, and r, the authors produce wildly different system dynamics. If $h > 0$ and $f = r = 0$ (the symmetric trade-only model), wealth condensation occurs. If $f, h > 0$ and $r = 0$ (the asymmetric trade-only model), a Gamma-like distribution is observed. Finally, if $f, h, r > 0$ (the asymmetric trade-investment model), a Gamma-like distribution with a power-law tail is observed.

Various other modifications to the BM model have been studied as well. Huang (2004) extends the BM model to negative wealth levels, and Torregrossa and Toscani (2017) prove analytically that a unique steady state with support on the entire real number line exists [84,85]. Johnston et al. (2005) impose the additional restriction of conservation of wealth, finding that wealth condensation still occurs for high values of μ [86]. Finally Ichinomiya (2012a, b) relaxes Bouchaud and Mézard's mean field assumption to adiabatic and independent assumptions, drawn from quantum mechanics [87,88]. The power law-like tail is reproduced and condensation is seen to take place at a higher $\bar{J}$ than the mean-field case would indicate, though the Pareto index obtained remains smaller than empirical values [89].

2.3 Other formulations

While KWE-style models and BM-style models remain by far the most popular categories of random asset wealth exchange models, a variety of other formulations exist and deserve mention.

A simple model which has nonetheless proved influential is the multiplicative stochastic process (MSP), which was studied by the economists Robert Gibrat and D.G. Champernowne [80]. It would however be a stretch to say that MSP-style models are truly a type of random asset *exchange* model as they are principally characterized by the lack of exchange or any other sort of interaction between agents. Such models essentially represent agents' levels of wealth in terms of independent random walks, but nonetheless are able to capture some essential characteristics of observed distributions. The simplest model in this vein is the pure MSP $w(t+1) = \lambda(t) w(t)$, where $\lambda(t)$ is any Gaussian random variable. It is straightforward to show that the distribution of wealth among an ensemble of agents whose wealth evolution is governed by a pure MSP will follow a log-normal distribution, though variations which include additive noise (such as the Kesten process from the biological sciences) and "minimum wage"-style boundary constraints can also reproduce power law tails [90]. For examples of such models, see Biham et al. (1998), Huang and Solomon (2001), Souma and Nirei (2005), and Basu and Mohanty (2008) [91–94].

Another category of model from the early days of the RAE literature is the Generalized Lotka-Volterra (GLV) model, which also has its origins in the biological sciences. The original Lotka-Volterra process, studied by Biham et al. (1998), is given by:

$$w_i(t+1) = \lambda(t) w_i(t) + a\bar{w}(t) - b w_i(t) \bar{w}(t) \tag{27}$$

where λ is a time-dependent random variable and $\bar{w}(t)$ is the average wealth in the system [91]. The inclusion of the $\bar{w}(t)$ terms represents a form of indirect interaction between agents: much like in the mean-field approximation of the BM model, instead of including specific interaction terms $b_{ij} w_i(t) w_j(t)$, all interactions are assumed to be symmetrical: $b_{ij} = b/N$.

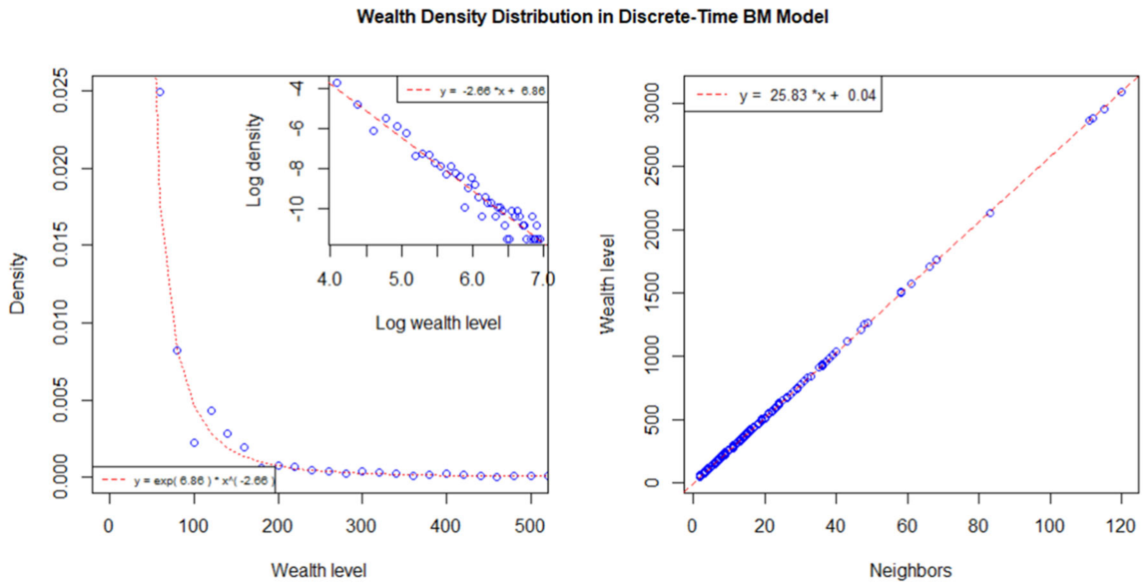

Fig. 3 Stationary distribution produced by the discrete-time BM model on a Barabási-Albert scale-free network, as described by Di Matteo et al. (2003). Simulation was performed with the parameterization $N = 5000$, $\bar{w} = 100$, $q_0 = 0.1$, and $E\left[A_i(t)^2\right] = 1$ over 10^5 iterations

The generalized form of this model is defined by Solomon and Richmond (2001, 2002) [95,96]:

$$w_i(t+1) = w_i(t) + (\varepsilon_i(t)\sigma_i + c_i(w_1, w_2, \ldots, w_N, t))\, w_i(t) + a_i \sum_j b_j w_j(t) \tag{28}$$

where ε_i is a Gaussian random variable with mean 0 and standard deviation 1, c_i represents endogenous and exogenous dynamics in returns, and a_i and b_i represent redistribution dynamics among agents. Under certain assumptions, this model also produces mixed exponential-Pareto distributions. However, this model ultimately faded in popularity due to the difficulty it has accurately representing the left tail of income distributions, as well as the lack of economic justification for some of its terms [50].

3 Prominent themes in the literature

While the papers discussed above (Fig. 4) serve as the foundation for the random asset exchange literature, the flexibility of the underlying modeling framework have allowed a vast number of featural variations upon these canonical models to have proliferated. This section provides an overview of the most significant of these trends and summarizes a few key papers of each.

3.1 Non-conservation of wealth

One of the criticisms leveled most consistently against the first generation of KWE-style models is that the assumption of total conservation of wealth, made by analogy with the physical principle of the conservation of energy, is economically unrealistic. In real economies, wealth is constantly being created and destroyed by means of production, consumption, and credit. Thus, a number of variations upon KWE-style models have attempted to represent this fact. Most such models can be further classified into one of two types: models which, like the CPT model, conserve wealth in the mean, and models which tie the global wealth level to a fixed influx rate.

Bisi et al. (2009) and Bassetti and Toscani (2010) both consider models of the first type [97,98]. The latter considers the non-conservative exchange rule:

$$\mathbf{M} = \begin{bmatrix} \varepsilon_i & \varepsilon_i \\ \varepsilon_j & \varepsilon_j \end{bmatrix} \tag{29}$$

where ε_i and ε_i are i.i.d. and $E[\varepsilon_i + \varepsilon_j] = 1$. Bassetti et al. (2014) consider a class of similar lotteries and demonstrates they tend to produce inverse-Gamma steady states [99].

Slanina (2004) was the first to consider a non-conservative model of the second type, in which a constant inflow of wealth from outside the system of interacting agents is permitted [100]. As in other formulations, the model sees pairs of agents i and j chosen at random to engage in a transfer of wealth, defined by the dynamics:

$$\mathbf{M} = \begin{bmatrix} 1-\lambda+\epsilon & \lambda \\ \lambda & 1-\lambda+\epsilon \end{bmatrix} \tag{30}$$

where $\lambda \in [0, 1)$ represents agents' propensity to save and $\epsilon > 0$ represents the rate at which exogenous wealth flows into the system. Slanina's model produces

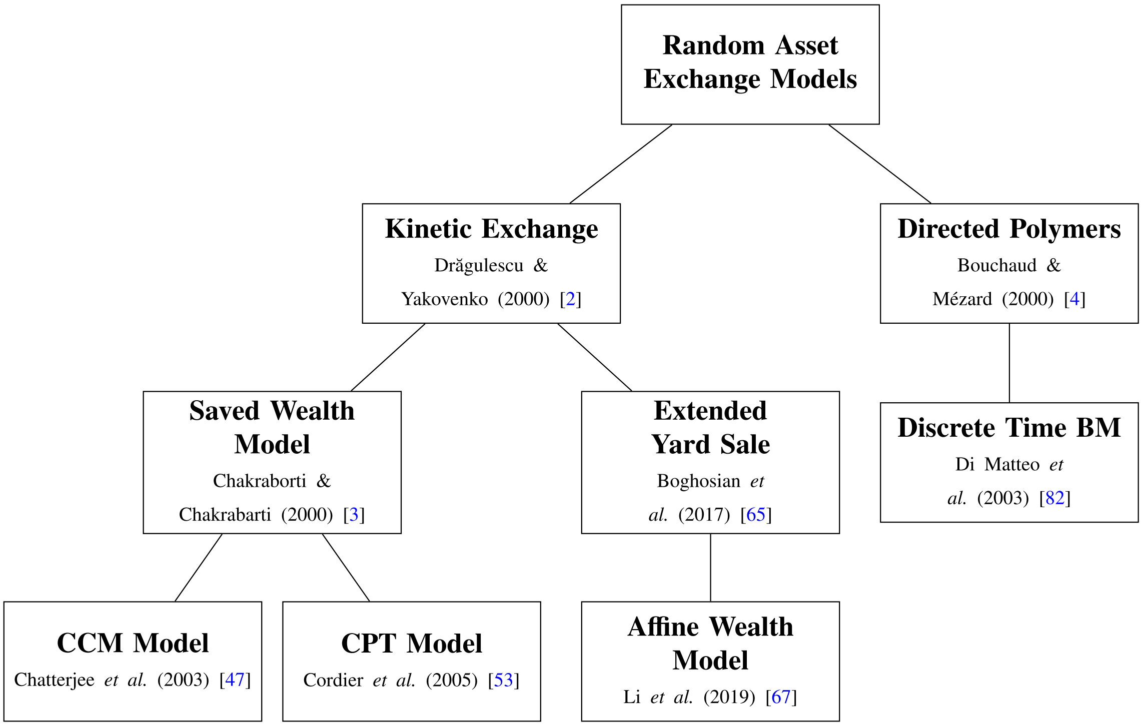

Fig. 4 "Family tree" of some significant papers in the RAE literature, where a "child" node represents a model formulated as a variation upon the "parent" node

a Gamma-like equilibrium distribution with a Pareto tail with an index $\alpha \sim 1 + \frac{2\lambda}{\epsilon^2}$. Coelho et al. (2008) extend this model, redefining $\lambda(w_i)$ as a piecewise function taking on two different values depending on which side of a pre-specified wealth threshold $n\bar{w}(t)$ an agent's wealth $w_i(t)$ fell:

$$\mathbf{M} = \begin{bmatrix} 1 - \lambda(w_i(t)) + \epsilon & \lambda(w_j(t)) \\ \lambda(w_i(t)) & 1 - \lambda(w_j(t)) + \epsilon \end{bmatrix} \tag{31}$$

This modification reproduced a double power-law regime, a phenomenon observed when comparing the right tail of income from tax data to estimates for the capital gains of a country's very wealthiest individuals [101].

A number of non-conservative models have dynamics which attempt to more directly model the process of money creation through borrowing. For example, Chen et al. (2013) consider a random exchange model in which agents who would otherwise reach zero wealth are permitted to borrow money from a central bank, which in turn can issue loans with no interest up to a certain global debt limit [102]. This process of money creation (issue of loans) and annihilation (paying back of loans) leads to a system in which the money supply grows logarithmically. Schmitt et al. (2014) introduce a similar system of money creation and analyzes the non-local effect that issuing credit has on the rest of the system; though the recipient of the loan clearly benefits, the effects of the increase in the money supply quickly propagate and all agents suffer the resultant inflationary effects [103].

Liu et al. (2021) and Klein et al. (2021) examine a generalization of an unmodified YS-style model which permits growth in the money supply over time, which they call the "Growth, Exchange, and Distribution" (GED) model [104,105]. Each time-step, total wealth $W(t)$ is increased by a factor of $1 + \mu$, and the wealth influx $\mu W(t)$ is distributed among agents such that agent i receives $w_i^\lambda / (\sum_j w_j^\lambda)$. For subunitary values of λ, poorer agents disproportionately benefit from the growth in the money supply and a quasi-stationary distribution exists; otherwise, the system exhibits wealth condensation as in the unaltered YS-style model. The mechanism of apportioning the surpluses attained from growth in these models is similar to that employed by Vallejos et al. (2018), in which surpluses from growth apportioned according to a more indirect "wealth power" parameter [106].

3.2 Networks and preferential attachment

It is a notable and well-established result in the random asset exchange literature that the effect of adjacency networks on the equilibrium distribution of wealth depends heavily on the type of model. While, for instance, the specific nature of the network has a decisive effect on the steady state wealth distribution in BM-style models, the opposite tends to be true for KWE-style models. Networks of exchange are an important aspect of real economic systems, and as such there has been a significant effort to study the effect they have on various types of RAE models.

Interestingly, models characterized by unidirectional exchange exhibit greater sensitivity to network structure than bidirectional exchange models. Chatterjee (2009), for example, introduces a toy model in which agents exchange fixed fractions of their wealth on a directed network characterized by a disorder parameter p [107]. Higher values of p produced networks where more agents had similar incoming and outgoing connections; the distributions obtained therefrom were more Gamma-like, as opposed to the Boltzmann-like distributions obtained from lower values of p. Martínez-Martínez and López-Ruiz (2013) study a unidirectional model with random exchange fractions, meant to represent payments on a non-complete graph [108]. This "directed random exchange" (DRE) model thus has the exchange rule:

$$\mathbf{M} = \begin{bmatrix} \varepsilon & 0 \\ 1-\varepsilon & 1 \end{bmatrix} \tag{32}$$

As in Chatterjee (2009), the choice of adjacency network affects the equilibrium distribution of the DRE model. For the fully-connected case, the equilibrium distribution $p(w)$ is again exactly Gamma, with shape parameter $\frac{1}{2}$ [109]. Notably, this implies that the equilibrium distribution possesses a singularity at 0, likely explaining why Martínez-Martínez and López-Ruiz observed a condensation-like phenomenon even on fully-connected networks.

Sánchez et al. (2007) investigate a model in which agents populate a one-dimensional lattice. Each agent's wealth grows in a deterministic fashion as a product of a linear "natural growth" term and an exponential "control" term, which retards growth as the difference between an agent's wealth and the average wealth of its neighbors increases [110]. While this system produces a pure power law distribution, different parametrizations or arrangements of agents' neighborhoods can produce Boltzmann–Gibbs distributions as well [111,112].

A handful of models have included the additional possibility of agents exchanging connections or positions on a lattice as well as units of wealth. Gusman et al. (2005) define an IGAV model on a random network in which the winner of an exchange is rewarded with additional connections on the network, producing a power law regime [113]. Aydiner et al. (2019) examine a CCM-style bidirectional exchange model on a one-dimensional lattice, in which some fraction of agents exchange lattice position each iteration of the simulation [114]. Fernandes and Tempere (2020) likewise consider a variation of the CC model in which agents on a two-dimensional lattice randomly switch positions on the lattice such that the average wealth difference between neighboring nodes is reduced [115]. This ultimately results in perfect wealth segregation and uniformly higher inequality.

The effect of extremal dynamics on the distribution of wealth over agents on a network is thoroughly studied in the "conservative exchange market" (CEM) model [116–118]. This model populates a lattice with agents who possess wealth levels in the range $[0,1]$, and each time step sees the poorest agent's wealth randomly re-randomized at the expense or benefit of its two closest neighbors. The selection rule in this model induces self-organizing behavior such that almost all agents end up with wealth levels above a "poverty line," which proves to be higher in the restricted lattice case than in the fully-connected case. This model has been utilized by a number of follow-up papers over the years: Iglesias et al. (2010) use this model to compare two different redistribution schemes, and Ghosh et al. (2011) consider its mean-field approximation [119,120]. Chakraborty et al. (2012) and Braunstein et al. (2013) study the same dynamics on various other networks [121,122]. Finally, Paul et al. (2022) demonstrate that the level of the "poverty line" increases with saving propensity in the case of a CC-style model coupled with extremal dynamics [123] (Fig. 5).

A concept closely related to adjacency is that of preferential attachment, which defines the likelihood of two agents interacting as a function of endogenous variables. The variable chosen is usually wealth, representing the fact that, in real economies, both the rich and the poor tend to interact more often with people of similar socioeconomic status to themselves. Because of its non-discrete nature, preferential attachment can allow for somewhat more dynamic interactions than adjacency networks can, permitting agents who become wealthy to access the networks of the rich and not totally disallowing chance rich–poor interactions.

Laguna et al. (2005) study the effect of this phenomenon on the IGAV model by imposing the restriction that a given agent is only permitted to interact with another agent if the difference between their two wealth levels is less than a given threshold value u [124]. Large values of u unsurprisingly replicate the IGAV model, while small values freeze the system entirely. Intermediate values of u, however, produce a self-organizing separation within the distribution of wealth, with a gap separating rich agents from poor ones spontaneously arising. This bimodal distribution persists even for high values of the poor-bias parameter f.

Chakraborty and Manna (2010) study a model with simple preferential attachment behavior, such that richer agents engage in exchange more frequently [125]. That is, the probability that agent i is selected as the first trader is proportional w_i^{α}, and the probability that j is selected as the second trader is proportional w_j^{β}. The limit as either exponent goes to infinity yields purely extremal mechanics, while $\alpha = \beta = 0$ is identical to the CCM model. Goswami and Sen (2014) defines a more complicated attachment function, wherein the probability of a given pair of agents (i, j) interacting depends on i's total wealth, the difference in wealth between i and j, and the number of past interactions between agents i and j [126]. The strength of each factor is modulated by a corresponding exponent, and, when applied to the classical BDY model, the choice of mod-

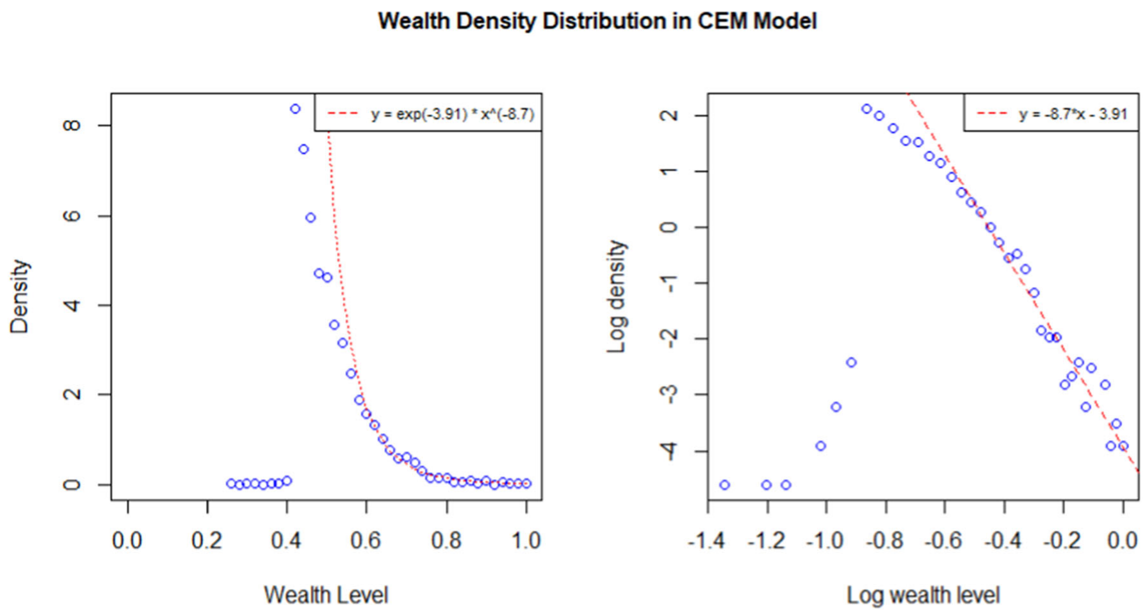

Fig. 5 Stationary distribution produced by the CEM model, with the best log-linear fit of the right tail of the distribution shown. Simulation was performed with $N = 5000$ over 10^7 iterations

ulation has a significant effect on the Pareto index of the steady-state distribution.

3.3 Goods and rationality

Despite their reductive nature, many of the simplifying assumptions discussed above are not uncommon to find in the economics literature as well. Many neoclassical models of simple "exchange economies" study the distribution of endowed and conserved assets absent wealth creation, and comparatively few consider the effect of exchange networks or other kinds of barriers to freely-associating exchange between agents (an important source of imperfect competition and thus market inefficiency). Rather, the main distinction between RAE models and those found in the mainstream economics literature lies in the fact that the former typically study ensembles of agents exchanging money directly in a stochastic fashion, while the latter typically study ensembles of rational (i.e., utility-maximizing) agents exchanging goods, with money exchange being an implicit consequence of goods exchange. A number of attempts have been made to partially bridge this difference by introducing goods and rationality into RAE models.

Chakraborti et al. (2001) study a model with both a fixed commodity supply Q and money supply M distributed among a population of agents [127]. These agents have a simple but reasonable utility function: they first seek to ensure their level of goods q_i exceeds some subsistence level q_0, and then seek to maximize their money holdings m_i; agents with $q_i > q_0$ thus find agents with $q_j < q_0$ to sell their excess goods to at a fixed price of 1. Not surprisingly, the steady-state distribution of this system is found to be sensitive to the global quantities Q and N; if the commodity supply is limited ($Q/N < q_0$), some fraction of agents will necessarily fall below subsistence level, while if the money supply is limited, agents lack the ability to redistribute the commodity supply in an efficient manner. A similar model with stochastic price fluctuations is considered by Chatterjee and Chakrabarti (2006), in which wealth is taken to be the sum of money and commodity holdings [128]. In both models, the money distribution exhibits a Pareto tail with index 1 while the commodity distribution is exponential so long as neither Q nor M is restricted.

Silver et al. (2002) consider a model with a more sophisticated utility function, in which agents possess stochastically time-varying Cobb-Douglas utility functions of the form [129]:

$$u_{i,t}(a_{i,t}, w_{i,t}) = (a_{i,t})^{f_{i,t}}(w_{i,t} - a_{i,t})^{1-f_{i,t}} \tag{33}$$

where $a_{i,t}$ represents agent i's holdings of the money commodity at time t, $w_{i,t} - a_{i,t}$ represents agent i's holdings of non-money commodities at time t, and $f_{i,t}$ is a random variable independently and identically distributed across both indices. Each agent then chooses to re-allocate his wealth between money and non-money commodities in such a way that maximizes $u_{i,t}$ subject to supply constraints. Simulations of this system produce a wealth distribution well-fit by a Gamma distribution with a shape parameter of 1 and a rate parameter of $1/\alpha$, where α represents the global supply of the money commodity.

More recently, a handful of economists have made progress working in the other direction, recovering well-studied dynamics for the time-evolution of wealth from "micro-founded" models (i.e., models in which aggregate dynamics are derived from the directed behavior of an ensemble of utility-maximizing agents). Benhabib et al. (2011) consider an overlapping generations model in which each individual allocates income received from idiosyncratic and stochastic returns in order to maximize utility, which is derived both from consumption and bequest of wealth to the next generation; the resultant stationary distribution is shown to have a Pareto tail [130]. Benhabib et al. (2019) then calibrate this model to U.S. data and find that both the stochastic growth factor and differential (wealth-varying) levels of intended bequest as a fraction of lifetime earnings play a significant role in determining the time-evolution of income inequality [131]. Gabaix et al. (2016) suggest a possible microeconomic formulation—based on an stochastic optimal investment problem—to recover dynamics very similar to the mean-field Bouchaud-Mézard model:

$$dw_{it} = \mu dt + \sigma dB_{it} + g_{it} dN_{it} \tag{34}$$

where g_{it} are i.i.d. random variables drawn from an arbitrary distribution and N_{it} is a Lévy jump process [132]. Gabaix and coauthors also discuss the problem of the too-slow convergence of the model to the steady-state level of inequality when compared to empirical data, and a modification based on "growth types" and "scale dependence" (individuals belong to one of finitely many types which define different growth dynamics) is proposed to solve this problem. In a similar vein, Berman et al. (2020) find that—contrary to the widespread assumptions that (a) empirical wealth and income dynamics have a stationary distribution and (b) the system converges more or less rapidly to said stationary distribution—aggregate transfers from poor to rich (corresponding to a $J < 0$ and precluding the possibility of a stationary distribution) predominate in most recent data [133]. Moreover, Berman and coauthors find that, when a stationary distribution does exist, the time to relaxation is extraordinarily long.

However, not every exchange model with goods is paired with rational agents. Ausloos and Pękalski (2007) consider a model with money, goods, and completely stochastic agent behavior [134]. Each time step, one agent decides via coin toss whether to purchase a nonzero number of goods. If so, he randomly selects a fraction of his money to spend and taps another agent to sell to him. If this second agent has enough goods to sell and has a desire to sell (again decided via coin toss), the exchange takes place. This model produces a distribution of wealth which interpolates between two power laws as time progresses, while the distribution of goods follows a static power-law. In general, those agents that are rich in terms of money are poor in terms of goods, and vice versa.

Another interesting line of research has concerned itself with defining traditional macroeconomic ensembles which produce results equivalent to RAE models. For example, Chakrabarti and Chakrabarti (2009) demonstrate that the dynamics of the CCM model can be replicated in a neoclassical framework with rational agents producing differentiated goods and trading in order to maximize time-varying Cobb-Douglas utility functions for goods and money [135]. In this case, the stochastic nature of exchange in the CCM model is represented by random variations in agents' utility functions in the analog model. Tao (2015) derives the entropy-maximizing exponential distribution as the statistical equilibrium of an Arrow-Debreu market system populated by agents with such time-varying utility functions [136]. More recently, Quevedo and Quimbay (2020) have extended this formulation to permit agents to save a portion s of goods possessed, naturally leading to an equivalent non-conservative RAE model [137].

3.4 Strategic behavior

Another way to model "smarter" agent behavior is to integrate game-theoretic or machine learning dynamics into RAE models. This integration can take various forms, including bilateral agreement, strategic heterogeneity, and behavioral evolution, just to name a few.

In Heinsalu and Patriarca's aforementioned 2014 paper, the authors consider the effect of introducing an acceptance criterion—a probabilistic factor defining the odds a given agent will agree to engage in a proposed transaction. This factor is a function of the wealth levels of each agent (increasing the wealthier an agent's trading partner is). and both agents need to agree to a transaction for it to take place [74]. In both the BDY and IE models, the choice of any symmetrical acceptance criterion (whether linear, exponential, etc.) only impacts the time of relaxation to equilibrium, but not the shape equilibrium itself. Asymmetrical decision criteria cause the equilibrium distribution to lose its universal form and to depend instead on the rule chosen. For the CC model, however, introducing even a symmetric criterion causes the equilibrium to lose its Gamma-like shape.

Sun et al. (2008) investigate a KWE model in which each agent can follow one of four strategies, chosen at random before the simulation begins [138]. The exchange rule between two agents depends on their respective strategies: two of the strategies are passive and tend toward equalizing the wealth of the two agents, while the other two are aggressive and tend toward classical theft-and-fraud exchange. As in Heinsalu and Patriarca (2014), the introduction of heterogeneous trading strategies leads to a steady-state distribution heavily sensitive to the model parameters, specifically those defining the rate of success of the aggressive strategies against the passive strategies.

Heterogeneity in strategies is often studied alongside dynamics for updating agents' strategies, representing a rudimentary form of learning. Hu et al. (2006, 2007, 2008), for example, consider a model in which each agent begins as either a "cooperator" or a "defector" and plays a series of prisoner's dilemma and public goods games with his neighbors [139–141]. After each game, an agent identifies the strategy of his richest neighbor and adopts it with some probability defined by his most recent payout, causing more successful strategies to propagate throughout the network. In a similar vein, da Silva and de Figueirêdo (2014) investigate an adaptive variation of the CCM model in which each agent i has a fixed probability γ_i of being able to update his savings parameter according to a pre-defined rule each time step [142]. Neñer and Laguna (2021b) study a variation on a poor-biased YS-style model with non-zero saving propensity, in which a fraction of agents are subjected to a genetic evolutionary algorithm after each Monte Carlo simulation step to update their exchange parameters, which approach the optimal values determined by Neñer and Laguna (2021a) [49,143]. A BM-style model coupled with game-theoretic dynamics is extensively analyzed by Degond et al. (2014) [144].

3.5 Class division

As mentioned above, one of the key features of empirical income distributions which RAE models attempt to capture is the bifurcation of the overall distribution into distinct exponential and Pareto ("thermal" and "superthermal," following Silva and Yakovenko (2004) [145]) components. While some models attempt to replicate this two-regime behavior while preserving homogeneity of system dynamics (e.g., by distributing a behavioral parameter throughout the population or imposing a specific network structure), a number of authors have instead sought to do so by defining separate system dynamics for agents with large wealth. It is very natural to identify the exponential bulk of the income distribution with labor income and the power law tail with capital gains, seeing as Pareto's original observations came from data for property incomes [145]. In this way, asymmetric system dynamics represent the fact that, in real economies, the rich do indeed have access to income streams not available to the majority of the population [146].

Simple models which have this class division "baked in" are easily able to produce two-regime structures of wealth. Yarlagadda and Das (2005) and Das and Yarlagadda (2005), for instance, introduce a model in which trading dynamics differ for agents on opposite sides of a fixed wealth threshold [147,148]. Poorer agents engage in bilateral exchange exactly as in Chakraborti and Chakrabarti (2000), while richer agents engage in exchange—with a different saving parameter—against the gross system, representing forms of leverage only available to the wealthy. Quevedo and Quimbay (2020) likewise study a trading model in which a fixed fraction of the population acts as "producers," who employ the remainder of the population as "workers" [137]. Producers trade wealth and pay their associated workers a portion of the exchanged quantity, creating two differently-shaped Gamma distributions for producer and worker income which, when combined, create a clear two-regime distribution.

Lim and Min (2020) consider the case in which the CCM model is partitioned into two classes by a wealth percentile threshold and a "solidarity effect" among agents below said threshold is introduced [149]. If two agents belong to the same class, then exchange proceeds according to the usual CCM dynamics. But if the agents belong to different classes, the lower-class agent gathers some fraction of the class into a coalition and wins a portion of the upper-class agent's wealth with a probability equal to the ratio between the coalition's wealth and the total wealth of all agents involved in the exchange. This solidarity factor turns out to be crucial for the generation of a realistic wealth distribution, as without it the middle income stratum collapses and one obtains a bimodal distribution, as in Laguna et al. (2005).

However, imposing a fixed boundary differentiating the upper class from the lower is not necessarily the best approach; analysis has shown that the "superthermal" component of the income distribution is highly volatile, fluctuating in size with the stochastic movements of financial markets [145]. Thus, a more accurate representation of real-world class dynamics would see a porous boundary between classes, with some fraction of upper-class agents going broke and falling into the lower class and some fraction of lower-class agents "making it" and entering the upper class. A number of models attempt to capture this aspect of the distribution's right tail by setting class boundaries dynamically. Russo (2014) investigates a model without exchange in which a new wealth percentile threshold defining the size of the upper class is chosen from the uniform distribution at each time step [150]. Agents above that threshold see their wealth augmented by a multiplicative stochastic process, while agents below it have their wealth augmented by an additive stochastic process. A different approach is forwarded by Smerlak (2016), who constructs a Markov process defining transition probabilities between a finite number of stratified classes [151]. Agents in higher classes derive proportionally greater amounts of income from a multiplicative process subject to shocks, and consequently exhibit much greater fluctuations in wealth compared to the majority of agents, who persist at low levels of wealth indefinitely.

Finally, the authors wish to highlight the unique and striking "social architecture" (SA) model of Wright (2005), which sees agents spontaneously self-organize into three distinct classes [152]. Wright defines an ensemble with three types of agents—employers, employees, and the unemployed—and in each iteration, an agent i is randomly chosen to be "active." The activities agent i engages in depends on its status: if i is an employer, it pays as many of its employees as it can afford; if i is an employee, it receives a wage and spends it on consumption goods produced by an employer; and if agent i is unemployed, a random (wealthy) agent is chosen to hire i, assuming his level of wealth is sufficient to pay i's wages. Although the initial conditions of the simulation posit complete equality of agents (all agents begin unemployed and with equal wealth), the population quickly self-organizes into a three-class regime with a distribution of wealth characterized by an exponential bulk and a Pareto tail (Fig. 6). The exact nature of this distribution becomes clear when

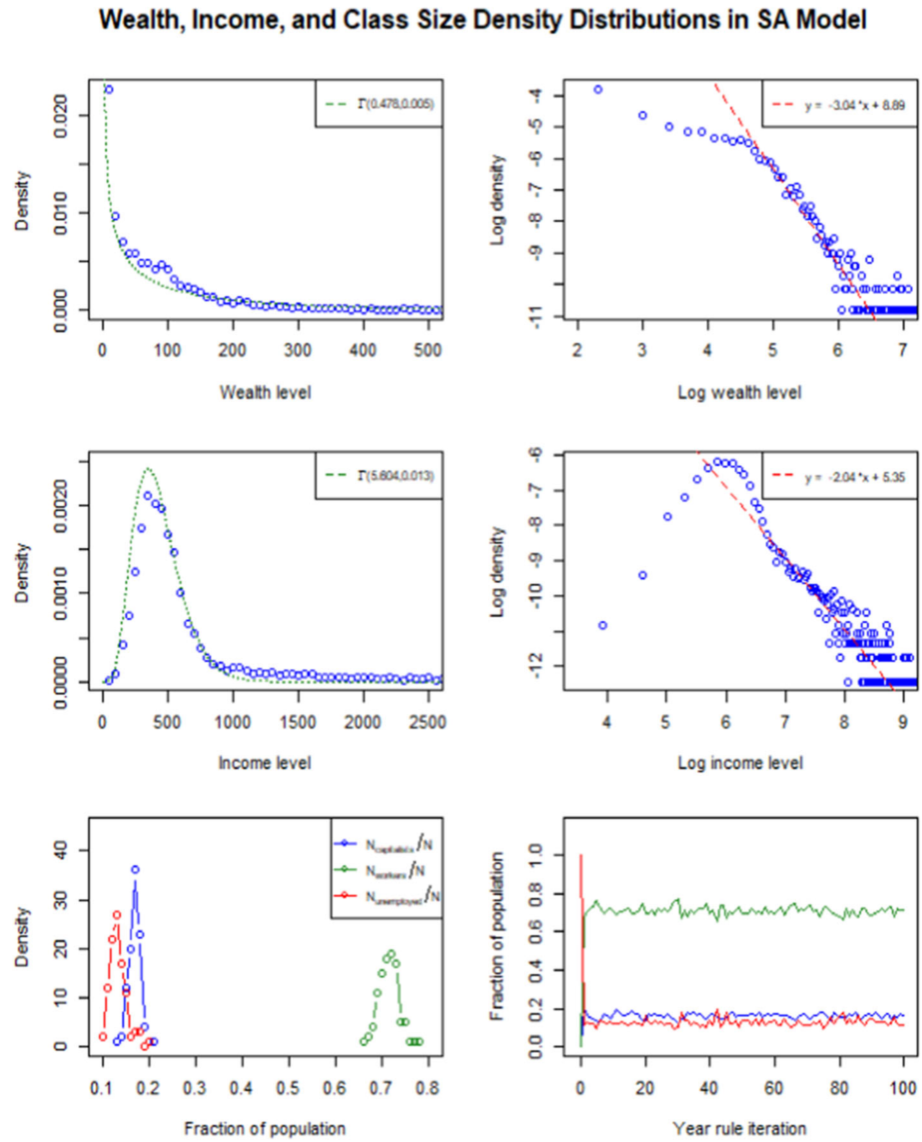

Fig. 6 Stationary distributions of wealth, income, and class size produced by the SA model of Wright (2005). Simulation was performed with the parameterization $N = 5000$, $\bar{w} = 100$, $[w_a, w_b] = [10, 90]$, over 10^2 "year rule" iterations ($6 \cdot 10^6$ time-steps)

disaggregated by class: the wealth of "employee" agents is governed by an exponential distribution, while that of "employer" agents is well-fit by a power law. This result is consonant the argument forwarded by Montroll and Shlesinger (1982) and contrasts with explanations of the two-regime distribution which rely on endogenous differences between agents. Unfortunately, Wright's model has seen few direct extensions, though a similar self-organizing model is studied in Lavička et al. (2010) [153].

3.6 Taxation and redistribution

A good deal of attention has been dedicated to the potential usefulness of RAE models for analyzing the effectiveness of tax policy. Earlier studies such as Guala (2009) and Toscani (2009) consider the effect of introducing a simple "income tax," in which a fixed fraction of each exchange is withdrawn by an external body ("the state") and redistributed uniformly, into mean-conservative KWE models; this alteration does not alter the exponential nature of the steady state distribution [154,155]. Diniz and Mendes (2012) extend this result for multiple different taxation rules in a CC model, representing both income taxes (taxes on transaction amounts) and wealth taxes (taxes on wealth level) [156]. Bouleau and Chorro (2017) contrast the effect of income and wealth taxes on YS-style models, demonstrating analytically that income taxes alone are not sufficient to prevent condensation [157]. Similarly, Burda et al. (2019) investigate the dynamics of a BM-style model in which all wealth flow rates $J_{ij} < 0$—a parametrization which normally causes the system to condense—paired with a redistributive mechanism [158]. A sufficiently strong mechanism succeeds in preventing condensation and recovering a heavy-tailed wealth distribution, with a multimodal critical phase also being observed. A number of non-standard redistribution rules in the context of a YS-style model are examined by Lima et al. (2022) [159].

More recently, interest in the problem of identifying optimal tax rates in RAE models has grown, often borrowing techniques from control theory to do so. Bouchaud (2015) extends the BM model to permit a wealth tax capable of reallocating wealth between the private and public sectors, with different growth rate parameters [160]. By maximizing expected economic growth, an optimal tax rate in the interval $(0, 1)$ is obtained so long as the difference between sectoral growth rates lies within an intermediate range. Düring et al. (2018) develop a finite-horizon model predictive control mechanism for the CPT model to find a feasible tax regime which minimizes a cost function representing some metric of inequality, and consider various objective functions and redistribution schemes [161]. Zhou and Lai (2022) investigate an idiosyncratic model of individual wealth growth and formulate both an additive and a multiplicative control mechanism to modulate the excessive growth of the right tail of the ensemble's wealth distribution [162]. Lastly, Wang et al. (2022) pair a CPT model with an evolutionary description of agents' decision-making competence—which feeds back into their saving propensities—and a model predictive control mechanism to reduce inequality [163].

3.7 Miscellanea

Though the outline above (and Table 2 below) enumerates the most widely-studied variations of RAE models, it should by no means be considered exhaustive. The flexibility of the RAE framework makes it easy to introduce new system dynamics and isolate the effects of a given modification. Just to name a few examples which do not fit neatly into any of the categories above, Pareschi and Toscani (2014) investigate the effect of variable agent knowledge on the CPT model, obtaining the counterintuitive result that the most knowledgeable agents tend not to be the richest ones; Trigaux (2005) examines the effect of introducing altruistic behavior to a subpopulation and finds a very strong equalizing effect when combined with redistribution; Coelho et al. (2005) and Patrício and Araújo (2021) model the propagation of wealth on a generational network to study the stratifying effect of inheritance; and Dimarco et al. (2020) use a class-based framework to characterize the effect of pandemics on wealth inequality [164–168].

The RAE literature has also given rise to a number of wholly new analytical techniques. Ballante et al. (2020) demonstrate that fitting the distribution of saving propensities to real-time economic data in a generalized CCM model via statistical sampling may be useful as a leading indicator of economic stressors which have the potential to increase inequality [169]. Luquini et al. (2020) establish a formal equivalence between KWE models and population-based random search algorithms used in computer science, and speculate that econophysical models could ultimately be used as a benchmark model in cybernetics [170]. Finally, dos Santos et al. (2022) propose a computational technique by which the crossover point between the exponential and Pareto regimes can be identified within data sets of real income distributions, aiding in the empirical study of economic inequality [171].

4 Discussion

From the ambition and breadth of recent contributions to the literature, it is clear that random asset exchange modeling is being increasingly recognized as a highly versatile tool which has the potential to find wide application even beyond its original use as a descriptive model. It is also clear that, in seeking to explain the characteristic features of wealth and income distributions, these models have highlighted the existence of a number of more fundamental economic phenomena underlying those features, such as the inherently diffusive nature of exchange economies and the emergence of apparent power laws from overlapping exponential functions.

Furthermore, it has become apparent that the random asset exchange modeling literature as a whole has a number of non-trivial implications. Namely, all of the models discussed seem to imply that a large proportion of observed economic inequality is the result of luck and the inherently diffusive (entropy-increasing) nature of exchange itself, and not the result of interpersonal differences in industriousness, entrepreneurialism, or intelligence. While some authors have taken this to mean that the "natural," entropy-maximizing level of inequality is by definition fair, such a conclusion is far too strong and veers into the territory of naturalistic fallacies. Instead, the conclusion one ought to draw from this cardinal result of the RAE literature depends on one's own subjective beliefs about the "ideal" level of inequality—however that is determined—as compared to the prevailing level of inequality. For proponents of relatively unrestrained capitalism, who have argued that inequality plays an important stimulative role in the economy by encouraging people to work harder in the hopes of achieving better economic outcomes [172], the implications of said result are quite positive: the laws of statistical mechanics naturally guarantee such inequality without the help of market-distorting conditions such as the formation of monopolies or the institutionalization of economic thievery! On the other hand, for those policy makers who aim to reduce the level of inequality in modern, developed economies, the corresponding implication may be somewhat more dismal. For them, the primary implication of these models is that altering government policies to make market economies operate more "fairly" by, for example, introducing progressive taxation can only do so much. At the end of the day, large scale regimes of wealth redistribution, such as wealth taxes, may be necessary in order to reduce inequality below the level that is endogenous to exchange-based systems.

Table 2 Notable papers in the random asset exchange literature, disaggregated by taxon and theme

Notable papers				
Theme	KWE: Theft & Fraud	KWE: Yard Sale	Bouchaud-Mézard	Other
Canon.	Angle (1986, 1992, 1993) [30–32]	Chakraborti (2002) [58]	Bouchaud and Mézard (2000) [4]	Biham et al. (1998) [91]
	Bennati (1988, 1993) [35,36]	Sinha (2003) [1]	Di Matteo et al. (2003) [82]	Huang and Solomon (2001) [92]
	Ispolatov et al. (1998) [29]	Iglesias et al. (2004) [71]	Scafetta et al. (2004) [83]	Solomon and Richmond
	Drăgulescu and Yakovenko (2000) [2]	Caon et al. (2007) [73]	Huang (2004) [84]	(2001, 2002) [95,96]
	Chakraborti and Chakrabarti (2000) [3]	Moukarzel et al. (2007) [63]	Ichinomiya (2012a, b; 2013) [87–89]	Souma and Nirei (2005) [93]
	Chatterjee et al. (2003, 2004) [47,48]	Boghosian (2014a, 2014b) [59,173]		Basu and Mohanty (2008) [94]
	Cordier et al. (2005) [53]			
	Repetowicz et al. (2005, 2006) [50,51]			
	Düring and Toscani (2008) [54]			
	Bisi and Spiga (2010) [55]			
	Zhou et al. (2021) [56]			
Non-cons.	Slanina (2004) [100]	Liu et al. (2021) [104]	N/A: by definition, all BM models are non-conservative.	Heinsalu and Patriarca (2014) [74]
	Cordier et al. (2005) [53]	Klein et al. (2021) [105]	A BM-style model in which wealth is conserved is studied in Johnston et al. (2005) [86]	
	Coelho et al. (2008) [101]			
	Bisi et al. (2009) [97]			
	Bassetti and Toscani (2010) [98]			
	Chen et al. (2013) [102]			
	Schmitt et al. (2014) [103]			
Net-works and Pref. Attach-ment	Chatterjee (2009) [107]	Gusman et al. (2005) [113]	Souma et al. (2001) [78]	Pianegonda et al. (2003) [116]
	Chakraborty and Manna (2010) [125]	Laguna et al. (2005) [124]	Di Matteo et al. (2003) [82]	Iglesias et al. (2003) [117]
	Martínez-Martínez and López-Ruíz (2013) [108]	Bustos-Guajardo and Moukarzel (2012) [64]	Scafetta et al. (2004) [174]	Pianegonda and Iglesias (2004) [118]
	Goswamy and Sen (2014) [126]		Garlaschelli and Loffredo (2004, 2008) [79,80]	Sánchez et al. (2007) [110]
	Aydiner et al. (2019) [114]		Ma et al. (2013) [81]	Iglesias et al. (2010) [119]
	Fernandes and Tempere (2020) [115]			Ghosh et al. (2011) [120]
				Chakraborty et al. (2012) [121]

Table 2 continued

Notable papers				
Theme	KWE: Theft & Fraud	KWE: Yard Sale	Bouchaud-Mézard	Other
				Braunstein et al. (2013) [122]
				Paul et al. (2022) [123]
Goods	Chakraborti et al. (2001) [127]			Ausloos and Pękalski (2007) [134]
	Chatterjee and Chakrabarti (2006) [128]			
Ration-ality	Chakrabarti and Chakrabarti (2009) [135]		Benhabib et al. (2011) [130]	Silver et al. (2002) [129]
	Tao (2015) [136]		Gabaix et al. (2016) [132]	
	Quevedo and Quimbay (2020) [137]		Benhabib et al. (2019) [131]	
			Berman et al. (2020) [133]	
Strat-egies	Sun et al. (2008) [138]	Neñer and Laguna (2021b) [143]	Degond et al. (2014) [175]	Hu et al. (2006, 2007, 2008) [139–141]
	da Silva and de Figueirêdo (2014) [142]			Heinsalu and Patriarca (2014) [74]
Class div.	Yarlagadda and Das (2005) [147]			Wright (2005) [152]
	Das and Yarlagadda (2005) [148]			Lavička et al. (2010) [153]
	Lim and Min (2020) [149]			Russo (2014) [150]
	Quevedo and Quimbay (2020) [137]			Smerlak (2016) [151]
Redist.	Guala (2009) [154]	Boghosian (2014a, 2014b) [59,173]	Bouchaud (2015) [160]	Zhou and Lai (2022) [162]
	Toscani (2009) [155]	Boghosian et al. (2015) [176]	Burda et al. (2019) [158]	
	Diniz and Mendes (2012) [156]	Boghosian et al. (2017) [65]		
		Bouleau and Chorro (2017) [157]		
		Devitt-Lee et al. (2018) [66]		
		Düring et al. (2018) [161]		
		Li et al. (2019) [67]		
		Polk and Boghosian (2021) [68]		
		Lima et al. (2022) [159]		
		Wang et al. (2022) [163]		
Analytical Results	Patriarca et al. (2004a, 2004b) [44,45]	Boghosian et al. (2015) [176]	Medo (2009) [77]	Katriel (2014) [75]
	Patriarca et al. (2005) [34]	Neñer and Laguna (2021b) [49]	Torregrossa and Toscani (2017) [85]	
	Gupta (2006) [43]			
	Lallouache et al. (2010) [46]			
	Katriel (2015) [109]			
	Lanchier (2017) [42]			

It must also be noted that canonical RAE models are not without their own set of problems. Most are still incapable of replicating all of the characteristic features of wealth and income distributions. For wealth distributions, as has been discussed, these include non-negligible segments of the population with non-positive wealth and possibly a power law right tail; for income distributions, these include an exponential or log-normal bulk, and an at least apparent power law tail with exponent between -2 and -3.

More importantly, while all of the models discussed above serve as excellent demonstrations of the role random chance plays in generating the inequalities observed in market economies, the literature has not yet been able to provide an adequate *explanation* for the emergence of the distributional features which it posits to be universal. That is to say, it has not yet been able to identify and describe the concrete system dynamics, common to all market economies, which generate the characteristic features of inequality.

Models which impose specific distributions on an endogenous parameter throughout the population (thriftiness, size of social network, etc.) clearly have the capability of producing nearly any desired distribution, but such results have far less explanatory power seeing as they merely defer the question. If one observes a given distribution of wealth because there exists an underlying distribution of a certain behavioral parameter, why is this parameter distributed the way it is throughout the population? One ultimately returns to Pareto's own unsatisfying explanation for his law—that economic inequality is purely the result of intrinsic differences between individuals—and finds oneself no closer to actually understanding the crux of the issue.

More promise is shown by models which offer, following the terminology of Reddy (2020), "processual" accounts of inequality [177]. Such accounts introduce production and class relationships into their models as fundamental processes of economic systems. This approach reflects concrete asymmetries in the economy, reduces the degree of unnecessary abstraction present within the models, and permits the identification of different segments of wealth and income distributions with different social positions. Unfortunately, models and analyses which take this approach still constitute a small part of the literature, with Wright (2005) and Lavička et al. (2010) remaining the two most notable examples [152, 153].

Another significant problem pertains to the relationship between the distribution of wealth and the distribution of income, the nature of which the literature has not consistently grasped. Wealth and income are two linked but quite distinct quantities. Wealth can take a wide variety of forms—money, consumption goods, real estate, debts, or even information and skills can all be considered forms of wealth! Income, on the other hand, typically refers more narrowly to the amount of "wealth," however it is enumerated, received by an individual in a given time period, prior to expenses. This quantity of course augments one's existing wealth, but the straightforward relation of income as the time-derivative of wealth only holds under the simplifying assumptions that the individual in question has no expenses and that the individual's wealth is not subject to any endogenous changes of its own: that is, no articles of wealth are consumed, fluctuate in value, are traded for articles of differing value, etc.

For the most part, random asset exchange models are concerned with the distribution of an undifferentiated, non-consumable, exchangeable asset—usually a stand-in for money—among an ensemble of agents. Thus, these distributions are best interpreted as wealth distributions, and it is inapt to compare them to empirical distributions of income. Moreover, many of the desirable features of the wealth distributions obtained by these models do not translate to the implicit, corresponding income distributions: Xu et al. (2010) note that reconstructing time-series of agents' income within canonical KWE-style models actually produces income distributions which are Gaussian, as opposed to exponential, directly contrary to the available data [178]! Once again, a greater focus on developing models in a more processual vein, which explicitly link wealth to assets and income to salaries and wages paid out by firms to employees, could be useful in clearing up this confusion.

But these issues should serve as an impetus to clarify, rather than indict, the RAE modeling project, because further progress toward an econophysical explanation for inequality is sorely needed. Again per Horowitz et al., 61% of all American adults believe that the level of inequality in the United States today is too high [5]. Of that number, 81% believe that this problem will require either major policy interventions or a complete restructuring of the economy to address. There exists a clear political will, at least in the U.S., to reduce the degree of inequality that has been allowed to develop over the past few decades. Despite that fact, there exists no consensus on what the major contributors to economic inequality in the U.S. even are. While there are a number of potential explicators commonly cited in the Pew survey—such as industrial outsourcing, tax structure, and intrinsic differences between individuals—no single one is viewed by a majority of the population as a decisive factor. Needless to say, intelligent decision-making about the policies needed for a fairer economy and society necessitates a clearer understanding of the processes responsible for generating inequality in the first place. Much work remains to be done before such a satisfactory understanding is reached.

Author contributions

MG performed the literature review, drafted the manuscript, and generated all figures. OG provided guidance and conceptual expertise at all stages of the writing process and contributed to the review and editing of the manuscript.

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Declarations

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