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Fluxon dynamics in long Josephson junctions with nontrivial current-phase relation

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Abstract. A novel second-order partial differential equation that describes the phase dynamics in the long Josephson junction and takes into account the flow of unpaired electrons across the junction is proposed. It has the form of the modified sine-Gordon equation with the nontrivial current–phase relation. The spatial behavior of the penetrated magnetic flux quantum (Josephson vortex or fluxon) is analyzed for the different values of the insulating layer transparency. The proposed equation of motion is used to investigate the dependence of the equilibrium fluxon velocity on the constant bias current for the different values of the insulating layer transparency. The self-consistent approach where the dissipation coefficient also depends on the layer transparency is applied. The resulting current–voltage characteristics (CVCs) have demonstrated better fluxon mobility for smaller values of the transparency coefficient.

1 Introduction

The SIS (superconductor/insulator/superconductor) tunnel junction is a superconducting structure containing a thin insulator film placed between two massive superconductors. Existence of the dissipationless current [1–4] flowing through a SIS tunnel junction is one of the most remarkable phenomena in superconductivity. The value of this current depends on the difference $\varphi = \Theta_L - \Theta_R$ between the phases $\Theta_{L,R}$ of the macroscopic wave functions of the left and right superconductors (also known as Josephson phase). In the simplest case, this dependence is considered to be sinusoidal [5]. The sinusoidal law is used in the main textbooks [3,4] and in the majority of scientific papers on the Josephson effect. In reality, the phase difference φ adjusts to an absolute value of a dissipationless current flowing through the SIS type tunnel junction connected to a circuit with a current source. In general, when the Josephson junction is placed in an external magnetic field, the phase difference φ depends on the spatial coordinates. This dependence is governed by the so-called Ferell–Prange equation or stationary sine-Gordon equation [6]. It is a second-order differential equation containing a physical quantity λ_j called a Josephson penetration depth. This characterizes a typical depth of a magnetic field penetration into the Josephson junction.

The critical current I_C of the junction can also be less than the current I_B set by an external source. As a result, a voltage drop across the junction appears and the Josephson phase φ becomes a time-dependent function [6]. In this case, the phase difference φ evolution is a solution of a nonstationary sine-Gordon equation [7,8]. This equation is completely integrable with the help of the inverse scattering transform [7]. In addition to the Josephson penetration depth λ_j, the so-called Swihart velocity $\bar{c}$ appears as a maximal velocity of the electromagnetic waves in the junction. The nonstationary sine-Gordon equation supports topological soliton solutions that describe the penetration of the external magnetic field into the junction in the form of individual vortices (fluxons) carrying a magnetic flux quantum. Fluxons are topologically protected against small perturbation and their remarkable stability leads to their application in various quantum computing applications [11–13]. Although a huge number of publications have been already devoted to the fluxon dynamics in the Josephson junctions, most of those papers consider the supercurrent–phase $I_S(\varphi)$ dependence to be sinusoidal [7–9].

Within the last 2 decades, the non-sinusoidal current–phase relations have been studied for different variants of the superconducting tunnel junctions [14–16]. For example, the biharmonic current–phase relation can appear in the SFS (superconductor/ferro-pg magnet/superconductor) and SIFS (superconductor/insulator/ferromagnet/superconductor) junctions [17,18] and in asymmetric arrays of three-junction SQUIDs

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[19]. Such a relation brings about new interesting phenomena such as relativistic time dilation [19], radiationless fluxon propagation [20,21], or significant change of the escape rate [22]. A more complex current–phase relation appears in [23] for the SIS type junctions and in [24] for the SISIS type junctions that contain a two-gap superconductor.

In this paper, we present a novel equation for the Josephson phase dynamics with the non-sinusoidal current–phase relation. Such a nontrivial current–phase relation can appear in the problems where the so-called depairing effects [25,26,28,29] are important. The influence of the insulating layer transparency on the mathematical structure of the current–phase relation in pure SIS superconducting tunnel junctions was investigated in [26,27]. A Josephson junction with an arbitrary concentration of nonmagnetic impurities was considered in [29]. The aim of this paper is to investigate the properties of Josephson vortices (fluxons) of the nonstationary modified sine-Gordon equation with the nontrivial current–phase relation obtained in [29]. Moreover, we would like to approach in the self-consistent way the fluxon dynamics in the presence of dc bias and normal electron flow. The latter will bring a dissipative term into the sine-Gordon equation that contains a coefficient inversely proportional to the junctions resistance which naturally must depend on the electron transmission ratio. This dependence will be taken into account when the fluxon velocity is computed as a function of the dc bias.

This paper is organized as follows. The next section describes the model and the basic equations of motion. In Sect. 3, the properties of the fluxon solutions are discussed. The dissipative and dc-biased fluxon dynamics is studied in Sect. 4. Conclusions are given in the last section.

2 Model and basic equations

2.1 Derivation of the current–phase relation

In this paper, the non-sinusoidal dependence of the supercurrent I_S on the Josephson phase φ obtained in paper [29] will be used

$$I_S(\varphi) = I_C\sqrt{1-\varepsilon^2}\,\frac{\sin\varphi}{1-\varepsilon\cos\varphi}. \tag{1}$$

Here, the dimensionless parameter ε $(0<\varepsilon<1)$

$$\varepsilon = \frac{1}{\sqrt{1+2\tau^2 q_\infty^2}} \tag{2}$$

measures the deviation of the current–phase relation from the sinusoidal form. We need to define the parameters τ and q_∞ that appear in Eq. (2). It is worthwhile to discuss briefly the origin of this formula. In our investigation, we will consider an SIS type junction without impurities. The dimensionless parameter τ was obtained in papers [26,29] and has the following form:

$$\tau^2 = \left[\frac{\xi_0}{\xi(T)}\right]^2 = \frac{12}{7\zeta(3)}\left(1-\frac{T}{T_c}\right), \tag{3}$$

where the coherence length is given by $\xi_0 = v_F/2\pi T_c$, v_F is Fermi velocity, T_c is superconducting critical temperature, and $\xi(T)$ is the characteristic length of the Ginzburg-Landau theory. The constant $\zeta(3) \cong 1.2$ is defined by the Riemann zeta function $\zeta(s) = \sum_{n=1}^{+\infty} n^{-s}$. It is important to note that the nontrivial current–phase relation (1) is valid for $T \lesssim T_c$.

The constant q_∞ [26,29] is defined by the following expression:

$$q_\infty = \frac{21\zeta(3)}{\pi^2}\left(\int_0^1 y^2\mathcal{R}(y)\,dy\right)^2\left(\int_0^1 y\mathcal{D}(y)\,dy\right)^{-1} + \frac{\pi^4}{28\zeta(3)}\int_0^1 y^3\mathcal{R}(y)\,dy. \tag{4}$$

The insulator film is modelled by the potential $U(z) = U_0\delta(z)$. The function $\mathcal{D}(y) = y^2/\left[y^2+(mU_0/p_F)^2\right]$ defines the transparency of the insulating layer as a function of the electron incidence angle θ on the IS interface $(y=\cos\theta)$. Here, p_F is the Fermi momentum and m is the electron mass. The function $\mathcal{R}(y) = 1-\mathcal{D}(y)$ is the reflection coefficient. We introduce the transmission coefficient $D \equiv \mathcal{D}(1)$ for the electrons incident in the normal direction of the IS interface. Hence, the following identity takes place: $(mU_0/p_F)^2 = 1/D-1$. After computing the three integrals in Eq. (4), we arrive to the final expression for the constant q_∞

$$q_\infty = \frac{\pi^4}{56\zeta(3)}\left(\frac{1}{D}-1\right) \times\left[1+\left(\frac{1}{D}-1\right)\ln(1-D)\right] + \frac{42\zeta(3)}{\pi^2}\left(\frac{1}{D}-1\right)^3 \times\frac{\left\{\sqrt{\frac{D}{1-D}}-\arctan\sqrt{\frac{D}{1-D}}\right\}^2}{1+\left(\frac{1}{D}-1\right)\ln(1-D)}. \tag{5}$$

As a result, we have a functional dependence that links the parameter $\varepsilon = \varepsilon(D)$ used in Eq. (1) and the transparency of the insulating barrier D. As can be seen from Fig. 1, it is a monotonic increasing function defined over the interval $\varepsilon \in [0,1]$ with $\varepsilon(0)=0$ and $\varepsilon(1)=1$. Because of Eq. (3), this function depends also on the ratio T/T_c. If the temperature is close to critical, this dependence is sufficient if $D \lesssim 1/2$. For $D \gtrsim 0.8$, the parameter ε almost reaches its saturation

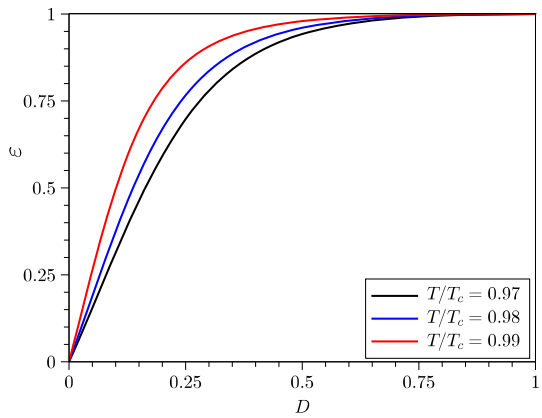

Fig. 1 Parameter ε as a function of the [see Eqs. (1–3) and (5)] barrier transparency D for three different values of temperature

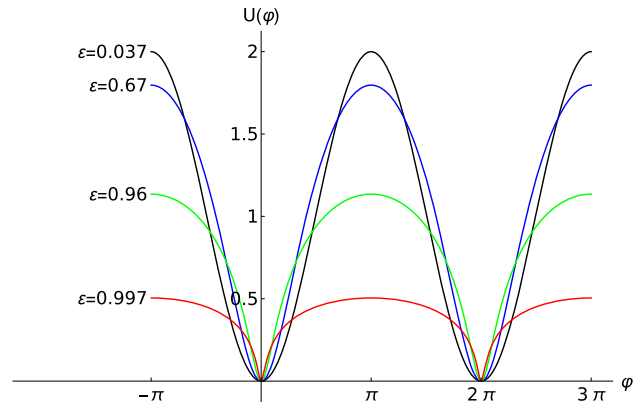

Fig. 2 Potential $U(\varphi)$ [see Eq. (8)] for different values of ε

value $\varepsilon = 1$. We will keep $T = 0.98\ T_c$ throughout the paper.

2.2 Modified sine-Gordon equation

We are interested in the magnetic flux dynamics in the large Josephson junction. In this case, the Josephson phase depends both on space and time. For the nontrivial current–phase relation (1), it is governed by the modified sine-Gordon equation

$$\varphi_{xx} - \frac{1}{\bar{c}^2}\varphi_{tt} - \frac{\sqrt{1-\varepsilon^2}}{\lambda_j^2}\frac{\sin\varphi}{1-\varepsilon\cos\varphi} = 0, \tag{6}$$

where λ_j is the Josephson penetration depth and $\bar{c}$ is the Swihart velocity. The subscripts $_x$ and $_t$ correspond to the space and time differentiation, respectively. It should be noted that various modifications of the sine-Gordon equation are not uncommon in physics beyond the Josephson junctions. They have been used to describe the dynamics of dislocations [30], spin waves in ^{3}He [31], or nonlinear electromagnetic waves in Dirac-like superlattices [32]. It is convenient to introduce the dimensionless variables $x \to x/\lambda_j$ and $t \to \bar{c}t/\lambda_j$. As a result, we obtain the following dimensionless second-order partial differential equation (PDE):

$$\varphi_{tt} - \varphi_{xx} = -\sqrt{1-\varepsilon^2}\frac{\sin\varphi}{1-\varepsilon\cos\varphi} = -U'(\varphi), \tag{7}$$

where the auxiliary potential $U(\varphi)$

$$U(\varphi) = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\ln\left(\frac{1-\varepsilon\cos\varphi}{1-\varepsilon}\right), \tag{8}$$

has been introduced to simplify the analysis of this equation. This potential is presented in Fig. 2. The parameter ε varies within the interval $[0, 1]$. In the limit $\varepsilon \to 0$, the standard sine-Gordon equation is restored: $\lim_{\varepsilon\to 0} U(\varphi) = 2\sin^2(\varphi/2)$. In the limit $\varepsilon \to 1$, the wells of $U(\varphi)$ become more narrow and barrier height $\Delta U = U(\pi) - U(0) = \sqrt{1-\varepsilon^2}\ln[(1+\varepsilon)/(1-\varepsilon)]/\varepsilon$ tends to 0. The Josephson plasmon frequency is given by the following equation $\omega(q) = \{[(1+\varepsilon)/(1-\varepsilon)]^{1/2} + q^2\}^{1/2}$. Here, the plasma frequency in the long wave limit $\omega(0)$ increases when ε increases.

3 Solutions of the modified sine-Gordon equation

We have introduced the modified sine-Gordon equation (7) that describes the fluxon dynamics in the long Josephson junctions with the nontrivial current–phase relation (1). We are interested in the particular case of the traveling wave solutions $\varphi = \varphi(x - Vt) \equiv \varphi(\xi)$ that propagate with the constant dimensionless velocity V. As a result, the second-order PDE turns into the ordinary differential equation (ODE) with respect to the new spatial variable $\xi = x - Vt$

$$\left(1 - V^2\right)\frac{d^2\varphi}{d\xi^2} = U'(\varphi) = \sqrt{1-\varepsilon^2}\frac{\sin\varphi}{1-\varepsilon\cos\varphi}. \tag{9}$$

This equation can be interpreted as a newtonian equation of motion for the particle with mass $1 - V^2 > 0$ moving in the potential $-U(\varphi)$. It is known that the magnetic flux quantum in long Josephson junctions is carried by topological solitons (fluxons). Solitons are spatially localized waves that should satisfy the following boundary conditions:

$$\lim_{\xi\to-\infty}\varphi(\xi) = 0,\ \lim_{\xi\to+\infty}\varphi(\xi) = \pm 2\pi,\ \lim_{\xi\to\pm\infty}\frac{d\varphi}{d\xi} = 0. \tag{10}$$

Here, the signs "+" and "−" corresponds to a fluxon or antifluxon, respectively. Without loss of generality, we will focus only on fluxons. As a result, the modified sine-Gordon equation (9) can be rewritten as a first-

order ODE

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{2}{1-V^2}U(\varphi) = \frac{2}{1-V^2}\frac{\sqrt{1-\varepsilon^2}}{\varepsilon} \times \ln\left(\frac{1-\varepsilon\cos\varphi}{1-\varepsilon}\right). \quad (11)$$

In the limit $\varepsilon \ll 1$, the right side of a first-order differential equation (11) can be expanded in the Taylor series

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{2\sqrt{1-\varepsilon^2}}{1-V^2}\left\{1-\cos\varphi + \sum_{n=1}^{+\infty}\frac{1-(\cos\varphi)^{n+1}}{n+1}\varepsilon^n\right\}. \quad (12)$$

The limiting case of an infinitesimal dielectric layer transparency means that $D \ll 1$. In this particular case, a formula (5) for the constant q_∞ can be replaced by the following asymptotics:

$$q_\infty \cong \frac{28\zeta(3)}{3\pi^2}\frac{1}{D}, \qquad D \ll 1. \quad (13)$$

Thus, the constant q_∞ tends to infinitely large values, and, consequently, the dimensionless parameter ε is infinitesimally small [see Eq. (2)]. As a result, the differential equation (12) can be simplified. If only the lowest order of ε in expansion (12) is taken into account we end up with the standard sine-Gordon equation $\varphi_{tt} - \varphi_{xx} + \sin\varphi = 0$. The soliton solutions of this equation are well known [7]: $\varphi(x,t) = 4\arctan(\exp z)$, $z = \pm\frac{x-x_0-Vt}{\sqrt{1-V^2}}$.

Taking into account the $\mathcal{O}(\varepsilon)$ terms yields the ODE

$$\left(\frac{d\varphi}{d\xi}\right)^2 = \frac{2(1-\cos\varphi)}{1-V^2}\left(1+\varepsilon\frac{1+\cos\varphi}{2}\right). \quad (14)$$

which is equivalent to the double sine-Gordon equation. The solution of this equation is well known ([31])

$$\varphi(x,t) = -2\arctan\left[\frac{\sqrt{1+\varepsilon}}{\sinh\left(\pm\frac{x-x_0-Vt}{\sqrt{1-V^2}}\sqrt{1+\varepsilon}\right)}\right]. \quad (15)$$

For the further expansion, the $\mathcal{O}(\varepsilon^2)$ terms should be taken into account. That will lead to the triple sine-Gordon equation and so on.

Let us now assume that $D \lesssim 1$. In this particular case, a formula (5) for the constant q_∞ can be replaced by the following asymptotics:

$$q_\infty \cong \frac{\pi^4}{56\zeta(3)}(1-D), \qquad D \lesssim 1. \quad (16)$$

Table 1 Values of the barrier transparency D and the dimensionless parameter ε to be used in Figs. 3-4

D	0.01	0.2	0.5	0.8
ε	0.037	0.67	0.96	0.997

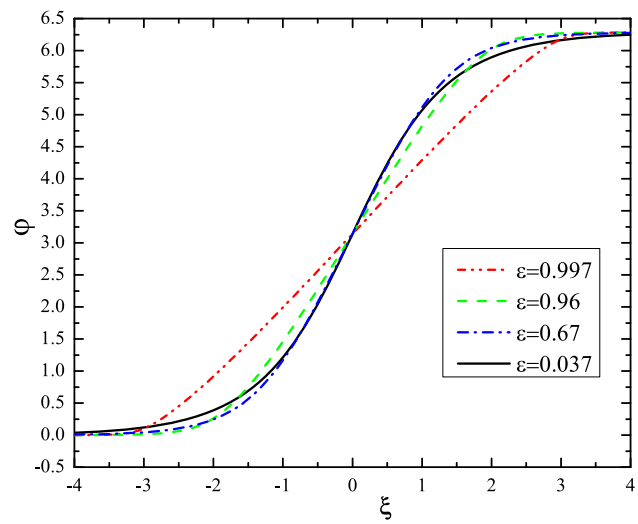

Fig. 3 Spatial distribution of the phase difference φ for one fluxon in the long Josephson junction with the different values of the barrier transparency D. The temperature is $T = 0.98\ T_c$ and the dimensionless fluxon velocity equals $V = 0.5$

Unfortunately, it is impossible to solve Eq. (11) explicitly for an arbitrary ε. Therefore, we have to use numerical methods. This have been done for several values of ε that include the extreme values and the intermediate ones. These values are also given in Table 1.

The respective coordinate dependence of the Josephson phase for the fluxon solutions with different ε is given in Fig. 3. The penetrated magnetic field distribution $H(\xi) = H_0 d\varphi/d\xi$ in the fluxon core is shown in Fig. 4. Here, we have the quantity $H_0 = \hbar/2e\mu_0\lambda_j\Lambda$ that contains the total penetration depth $\Lambda = \lambda_1 + \lambda_2 + d$, where $\lambda_{1,2}$ are the London penetration depth for the superconductors and d is the thickness of the insulating area.

These figures allow us to make several conclusions. (i) As the parameter ε grows, the approximate width of the fluxon increases. This increase is well seen in the limit $\varepsilon \to 1$, but is insignificant otherwise. (ii) Significant departure from the sinusoidal law makes the fluxon borders more sharp. In particular, the phase distribution for $\varepsilon \lesssim 1$ is almost linear in the fluxon core. The magnetic field distribution in the limit $\varepsilon \to 1$ is more box-like. For small and intermediate values of ε, the fluxon borders are rather blurred. At the same time, the maximal value of the penetrated magnetic field $H(\xi = 0)$ decreases.

Considering the asymptotics (16), we can see that the constant q_∞ tends to infinitely small values when

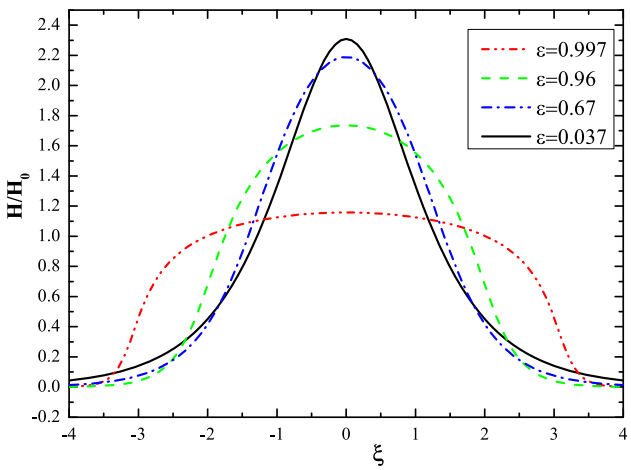

Fig. 4 Spatial distribution of the dimensionless magnetic field $H/H_0 = d\varphi/d\xi$ for one fluxon in the long Josephson junction with the different values of the barrier transparency D. The system parameters are the same as in Fig. 3

$D \lesssim 1$. This means that the values of the dimensionless parameter ε are close to 1. Due to the fact that in the limit $\varepsilon \lesssim 1$, the potential barrier $U(\varphi)$ becomes significantly flat we can approximate it around the value $\varphi = \pi$

$$U(\varphi) \approx U(\pi) + \frac{U''(\pi)}{2}(\varphi - \pi)^2 = \frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right) - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\frac{(\varphi-\pi)^2}{2}. \tag{17}$$

Taking into account the numerical results shown in Fig. 3 where the phase distribution in the fluxon core is almost linear, we restrict ourselves only with the constant term in (17). It should also be mentioned that in the limit $\varepsilon \to 1$, or, alternatively, $\delta \equiv 1 - \varepsilon \to 0$, the constant term in the expansion (17) decreases as $\delta^{1/2}\ln(1/\delta)$, while the quadratic term decreases faster, as $\delta^{1/2}$. The first-order differential equation (11) in the fluxon core can be replaced by the more simplified version

$$\frac{d\varphi}{d\xi} = \sqrt{\frac{2U(\pi)}{1-V^2}} = \sqrt{\frac{2\sqrt{1-\varepsilon^2}}{\varepsilon(1-V^2)}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)} \equiv C(\varepsilon), \tag{18}$$

which yields the following solution: $\varphi(\xi) = C(\varepsilon)\xi + \pi$. The fluxon slope is defined by the constant $C(\varepsilon)$ and it decays as $\delta^{1/4}\sqrt{\ln(1/\delta)}$ as $\delta = 1 - \varepsilon \to 0$. Elsewhere away from the vortex core, we assume the linear approximation of Eq. (9) around the minima of the potential U. Hence, the phase distribution in the fluxon tails will be exponential. Finally, the whole approximate solution can be written as

$$\varphi(\xi) = \begin{cases} A\exp\left[\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{1}{4}}\frac{\xi}{\sqrt{1-V^2}}\right], & \xi < -\xi_i, \\ C(\varepsilon)\xi + \pi, & |\xi| < \xi_i, \\ 2\pi - A\exp\left[-\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{1}{4}}\frac{\xi}{\sqrt{1-V^2}}\right], & \xi > \xi_i, \end{cases} \tag{19}$$

where the parameters A and ξ_i are obtained from the boundary conditions $\varphi(\xi_i + 0) = \varphi(\xi_i - 0)$, $\varphi'(\xi_i + 0) = \varphi'(\xi_i - 0)$. The fluxon core is located in the interval $[-\xi_i, \xi_i]$. From the above boundary conditions, we get the fluxon halfwidth

$$\xi_i = -\sqrt{1-V^2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{1}{4}} + \pi\left[\frac{2}{1-V^2}\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\right]^{-\frac{1}{2}}, \tag{20}$$

and the parameter A

$$A = C(\varepsilon)\sqrt{1-V^2}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{\frac{1}{4}} e^{\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{\frac{1}{4}}\frac{\xi_i}{\sqrt{1-V^2}}} = \chi\exp\left(\frac{\pi}{\chi} - 1\right), \ \chi = \sqrt{\frac{2(1-\varepsilon)}{\varepsilon}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}. \tag{21}$$

One can observe from Eq. (20) that $\xi_i > 0$ if $\varepsilon \to 1$, because the first term in that equation becomes negligibly small, while the second one increases. The value of the fluxon halfwidth obtained from Eq. (20) yields $\xi_i \approx 2.74$ for the parameters of Fig. 3 which is close to the numerical result.

4 Dc-driven fluxon dynamics and the current–voltage characteristics

The physically realistic case should take into account the flow of unpaired electrons across the junction. Also, the junction can be biased by the spatially uniform dc bias I_B. In this case, the evolution equation for the Josephson phase should be rewritten as

$$\varphi_{tt} - \varphi_{xx} + \sqrt{1-\varepsilon^2}\frac{\sin\varphi}{1-\varepsilon\cos\varphi} + \alpha\varphi_t + \gamma = 0. \tag{22}$$

The dimensionless parameter $\alpha = \hbar\omega_j/[2eR(D)I_C]$ is responsible for the normal electron contribution to the total current and $\gamma = I_B/I_C$ is the dimensionless external current. The Josephson plasma frequency is connected to the Josephson penetration depth and Swihart velocity: $\omega_j = \bar{c}/\lambda_j$. The dimensionless parameter $\alpha \propto R^{-1}(D)$ depends on the transmission ratio D. Since D and ε are mutually and uniquely connected through Eqs. (2),(3) and (5) [see also Fig. 1], we may

write $\alpha = \alpha(\varepsilon)$. The junction resistance as a function of D was obtained in [33]

$$R(D) = \frac{1}{e^2 v_F N(0) \int_0^1 z\mathcal{D}(z)\, dz},$$

$$\int_0^1 zD(z)dz = \frac{1}{2}\left[1 + \left(\frac{1}{D} - 1\right)\ln(1-D)\right]. \quad (23)$$

The integral in (23) is a monotonically increasing function that varies from 0 at $D = 0$ when there is no normal electron transmission to 1/2 at $D = 1$ (maximal transmission). Thus, the damping parameter α varies from $\alpha(\varepsilon = 0) = 0$ till some constant $\alpha_1 \equiv \alpha(\varepsilon = 1) = \hbar\omega_j e v_F N(0)/4I_C$. This constant α_1 will be considered as some maximal value which is attained when transmission is maximal. For our computations, we will choose it to be $\alpha_1 = 0.2$ as it corresponds to the most common order of the dissipation parameters in long Josephson junctions [8].

The driven and damped sine-Gordon equation (22) has been studied in detail first in Ref. [7]. If we are interested in the fluxon dynamics on large times $t \to +\infty$, we can use the energy balance approximation discussed in the above paper. As a result, we arrive to the equation for the junction energy E time evolution

$$\frac{dE}{dt} = -\gamma \int_{-\infty}^{+\infty} \varphi_t dx - \alpha(\varepsilon) \int_{-\infty}^{+\infty} \varphi_t^2 dx, \quad (24)$$

$$E = \int_{-\infty}^{+\infty} \left[\frac{\varphi_t^2 + \varphi_x^2}{2} + U(\varphi)\right] dx. \quad (25)$$

We suppose that there is only one fluxon in the junction and it satisfies the boundary conditions (10). Thus, the first integral in (24) reduces to $\pm 2\pi\gamma V$. The sign "+" corresponds to a fluxon and "−" to an antifluxon. Without the loss of generality, we will consider only fluxons. At large times $t \gg \alpha^{-1}(\varepsilon)$, the system will settle on the attractor that corresponds to the fluxon moving with some equilibrium velocity V_∞. The dissipative energy losses due to the normal electron tunneling [second integral in (24)] will be compensated by the energy input due to dc bias. Hence, the total energy must be constant, $dE/dt = 0$. As a result, using Eq. (11) one can obtain from Eq. (24) the following equation for the equilibrium velocity V_∞:

$$2\pi\gamma V_\infty = \alpha(\varepsilon) \int_{-\infty}^{+\infty} \varphi_t^2 dx = \alpha(\varepsilon) V_\infty^2 \int_0^{2\pi} \varphi_\xi d\varphi$$
$$= \alpha(\varepsilon) V_\infty^2 \sqrt{\frac{2\sqrt{1-\varepsilon^2}}{\varepsilon(1-V_\infty^2)}} \int_0^{2\pi} \sqrt{\ln\left(\frac{1-\varepsilon\cos\varphi}{1-\varepsilon}\right)} d\varphi. \quad (26)$$

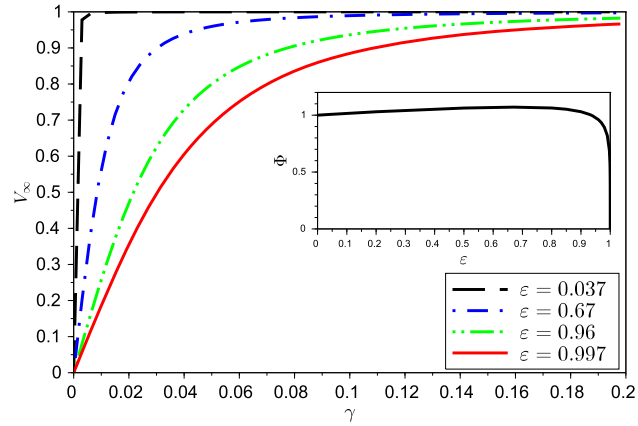

Fig. 5 Dependence of the equilibrium fluxon velocity on the dc bias for $\varepsilon = 0.037$, $\varepsilon = 0.67$, $\varepsilon = 0.97$ and $\varepsilon = 0.997$. The inset illustrates the function $\Phi(\varepsilon)$ [see Eq. (28)]

This equation can be easily solved with respect to V_∞. Thus, the equilibrium velocity expression reads

$$V_\infty = \frac{\mathrm{sign}(\gamma)}{\sqrt{1 + \left[\frac{4\alpha(\varepsilon)}{\pi\gamma}\Phi(\varepsilon)\right]^2}}, \quad (27)$$

where the function $\Phi(\varepsilon)$ is given by the following integral:

$$\Phi(\varepsilon) = \frac{1}{4\sqrt{2}}\sqrt{\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}} \int_0^{2\pi} \sqrt{\ln\left(\frac{1-\varepsilon\cos\varphi}{1-\varepsilon}\right)} d\varphi$$
$$\to \begin{cases} 1 + \frac{\varepsilon}{6} + \mathcal{O}(\varepsilon^2), & \varepsilon \to 0, \\ 0, & \varepsilon \to 1. \end{cases} \quad (28)$$

If $\varepsilon \to 0$ (in the usual model of the driven damped Josephson junction), it can be seen that the result (27) for the equilibrium velocity V_∞ reduces to the well-known McLaughlin–Scott formula [7].

The equilibrium fluxon velocity as a function of the dimensionless bias is presented in Fig. 5.

The auxilliary function $\Phi(\varepsilon)$ defines the behavior of the slope of the $V_\infty(\gamma)$ in the limit $|\gamma| \ll 1$, $V_\infty \approx \pi\gamma/4\alpha(\varepsilon)\Phi(\varepsilon)$. As one can see from the inset of Fig.5, this function Φ has a maximum at $\varepsilon \lesssim 1$. It is slightly larger than 1 for almost all values of ε except some small interval when $\varepsilon \to 1$. Within this interval, the function $\Phi(\varepsilon)$ sharply decreases to zero. In the limit of small transparency, the superconductivity is strong and the dissipative effects due to the normal electron flow are weak. Thus, the product $\alpha(\varepsilon)\Phi(\varepsilon) \to 0$ and the slope of the velocity dependence is large. Hence, the fluxon is highly mobile. This occurs in the standard sine-Gordon limit. As ε increases, the fluxon velocity decreases, because the superconductivity is significantly suppressed. In the high transparency limit ($\varepsilon \to 1$) $\Phi(\varepsilon) \to 0$, we again must observe increase of the fluxon velocity. It should happen for very high transparencies

$D > 0.99$ which are very unlikely to be achieved. This happens, because the fluxon energy ($|\gamma| \ll 1$, $\alpha \ll 1$) which is given by the equation

$$E(V_\infty) \approx \frac{8}{\sqrt{1-V_\infty^2}} = \frac{2\pi\gamma}{\alpha(\varepsilon)\Phi(\varepsilon)}\sqrt{1+\left[\frac{4\alpha(\varepsilon)}{\pi\gamma}\Phi(\varepsilon)\right]^2}, \tag{29}$$

diverges as $\Phi(\varepsilon) \to 0$. Although the maximum of the penetrated magnetic field decreases with $\varepsilon \to 1$ as shown in Fig. 4, the fluxon size drastically increases. Since the real Josephson junction length studied in experiments hardly exceeds $30\lambda_j$, it is very unlikely that such large fluxons can ever be studied. We conclude that the fluxon velocity increase in the limit $\varepsilon \to 1$ is a mathematical artifact irrelevant for real physical setups.

5 Conclusions

The modified sine-Gordon equation that describes the fluxon dynamics in the long Josephson junctions with the nontrivial current–phase relation is derived. This relation originates, because the depairing effects are taken into account. The dimensionless parameter ε measures the deviation from the standard sinusoidal current–phase relation. This parameter is related to the normal electron transmission coefficient through the insulating barrier, D, in such a way that for the zero transparency, we are in the sine-Gordon limit, while for high transparency, the current–phase relation is strongly non-sinusoidal. Since it is not possible to solve explicitly the resulting modified sine-Gordon equation, the numerical methods are used to analyze the fluxon (Josephson vortex) spatial behavior. The analytical results can only be obtained in the case of the small dielectric layer transparency. In that case, the modified sine-Gordon equation is reduced to the standard sine-Gordon or double sine-Gordon equation. In the case of high transparency, the fluxon shape is strongly distorted as compared to the soliton solution of the standard sine-Gordon equation. The penetrated magnetic field distribution has smaller amplitude in the fluxon center, but the fluxon width is larger and its borders are rather sharp.

The dependence of the fluxon velocity on the uniformly applied dc bias is derived, as well. We have developed the self-consisted approach when the dissipation constant in the modified sine-Gordon equation also depends on the transmission coefficient. In the small transparency limit, the superconducting effects are strong, and as a result, the fluxon is highly mobile. As the transparency increases, the role of the normal electron current increases, and, consequently, the superconducting effects are suppressed. This leads to decreasing of the fluxon velocity.

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Author contributions

All authors have contributed equally to the paper.

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