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Thermodynamics of the independent harmonic oscillators with different frequencies in the Tsallis statistics in the high physical temperature approximation

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Abstract. We study the thermodynamic quantities in the system of the N independent harmonic oscillators with different frequencies in the Tsallis statistics of the entropic parameter q ($1 < q < 2$) with escort average. The norm equations are derived, and the physical quantities are calculated with the physical temperature. It is found that the number of oscillators is restricted below $1/(q-1)$. The energy, the Rényi entropy $S_q^{(R)}$, and the Tsallis entropy $S_q^{(T)}$ are obtained by solving the norm equations approximately at high physical temperature and/or for small deviation $q-1$. The energy is q-independent at high physical temperature when the physical temperature is adopted, and the energy is proportional to the number of oscillators and physical temperature at high physical temperature. The form of the Rényi entropy is similar to that of the von-Neumann entropy, and the Tsallis entropy is given through the Rényi entropy. The physical temperature dependence of the Tsallis entropy is different from that of the Rényi entropy. The Tsallis entropy is bounded from the above, while the Rényi entropy increases with the physical temperature. The ratio of the Tsallis entropy to the Rényi entropy is small at high physical temperature. The relation between the physical temperature T_{ph} and the temperature T (the inverse of the Lagrange multiplier) is obtained, and the quantity as a function of T and q can be obtained through T_{ph}. We calculate the free energy $F_q^{(R)}$ which is defined with T_{ph} and $S_q^{(R)}$ and the free energy $F_q^{(T)}$ which is defined with T and $S_q^{(T)}$. The relation between $\partial F_q^{(R)}/\partial T_{\mathrm{ph}}$ and $S_q^{(R)}$ and the relation between $\partial F_q^{(T)}/\partial T$ and $S_q^{(T)}$ are shown.

1 Introduction

The various statistics have been proposed to describe the phenomena which show power-like distributions. For example, the power-like probability distribution appears when the stochastic equation is solved [1–4]. An extension of the Boltzmann–Gibbs statistics is the Tsallis statistics, and the statistics has been applied in various branches of science [5]. The escort average is often adopted to calculate the physical quantities in the Tsallis statistics. The Tsallis statistics has the entropic parameter q, and the statistics approaches the Boltzmann–Gibbs statistic as q approaches one.

The entropic parameter q is often restricted. The normalizability of the probability requires that q is less than two [6]. The parameter q is also restricted because physical quantities are restricted [6–9]. For example, the energy density should be finite and the number of particles should be positive, and these requirements show that the maximum value of q is smaller than two. The

limitation of q was also derived using the conjugate variables theorem [10].

Simple systems have been adopted to study the effects of statistics. The classical gas model was adopted, and it was found that the energy is proportional to the number of particles and the physical temperature [11–20] in the Tsallis statistics. It was also found that the number of the particles are restricted [12]. The thermodynamic quantities for a classical harmonic oscillator were also calculated in the Tsallis statistics with escort average. The partition function was calculated and the energy was obtained [21].

The calculations of the thermodynamic quantities for the harmonic oscillators are required in the Tsallis statistics. A field is decomposed into harmonic oscillators with different frequencies to calculate physical quantities. The results for the harmonic oscillators with different frequencies in the Tsallis statistics will be helpful to calculate physical quantities in various systems.

The system of harmonic oscillators with different frequencies should be studied in the Tsallis quantum statistics by introducing the physical temperature T_{ph} which is a function of the temperature T and the entropic parameter q, because a system of harmonic

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oscillators is the base of calculations and because the physical temperature may be the observed temperature of the system. It is better to represent the physical quantities with T_{ph} and q, though calculations may not be easy for the power-like distribution, because the distribution of N independent harmonic oscillators is not decomposed into the product of the distribution of a single harmonic oscillator even when the oscillators are independent, where N is the number of harmonic oscillators. The physical quantity as a function of T_{ph} and q can be represented with T and q when T_{ph} is represented with T and q. Like the restriction for free particles, it is also important to study the restriction of N for independent harmonic oscillators, because the restriction does not exist in the conventional statistics.

In this paper, we study the thermodynamic quantities in the system of the N independent harmonic oscillators in the Tsallis statistics of the entropic parameter q. The range of q is set between one and two in this study. The escort average is employed to obtain physical values. In Sect. 2, we briefly review the Tsallis statistics. In Sect. 3, we study the N independent harmonic oscillators with different frequencies. The norm equations are derived, and the equations are solved approximately. The expression of the energy is obtained with the physical temperature. The expressions of Tsallis and Rényi entropies are represented with the partition function Z. The physical temperature T_{ph} is represented with the temperature T (the inverse of the Lagrange multiplier) and the entropic parameter q, and free energies are calculated. In Sect. 4, the validity of the results in this study is discussed and the results in this study are compared with those in the previous studies. The last section is assigned for conclusions.

2 Brief review of the Tsallis statistics

The Tsallis statistics [5,21] is based on the Tsallis entropy $S_q^{(T)}$ with the entropic parameter q. The entropy $S_q^{(T)}$ is defined by

$$S_q^{(T)} = \frac{1 - \mathrm{Tr}\left[\hat{\rho}^q\right]}{q - 1}, \tag{1}$$

where $\hat{\rho}$ is the density operator and Tr means trace. The expectation value (escort average) of an operator $\hat{A}$, $\langle\hat{A}\rangle$, is defined by

$$\langle\hat{A}\rangle = \frac{\mathrm{Tr}\left[\hat{\rho}^q\hat{A}\right]}{\mathrm{Tr}\left[\hat{\rho}^q\right]}. \tag{2}$$

We apply the maximum entropy principle to obtain the density operator. The density operator $\hat{\rho}$ is obtained by extremizing $S_q^{(T)}$ under the normalization condition $\mathrm{Tr}\left[\hat{\rho}\right] = 1$ and the energy constraint:

$$U = \frac{\mathrm{Tr}\left[\hat{\rho}^q\hat{H}\right]}{\mathrm{Tr}\left[\hat{\rho}^q\right]}, \tag{3}$$

where U is the energy. The density operator $\hat{\rho}$ in the Tsallis statistics with the escort average is obtained:

$$\hat{\rho} = \frac{1}{Z}\left(1 - (1 - q)\frac{\beta}{c_q}(\hat{H} - U)\right)^{\frac{1}{1-q}}, \tag{4a}$$

$$Z = \mathrm{Tr}\left[\left(1 - (1 - q)\frac{\beta}{c_q}(\hat{H} - U)\right)^{\frac{1}{1-q}}\right], \tag{4b}$$

$$c_q = \mathrm{Tr}\left[\hat{\rho}^q\right], \tag{4c}$$

where β is the inverse temperature. This statistics is often called Tsallis-3 statistics. The partition function Z is related to c_q:

$$c_q = Z^{1-q}. \tag{5}$$

The inverse physical temperature is given by

$$\beta_{\mathrm{ph}} = \beta/c_q. \tag{6}$$

The physical temperature T_{ph} is given as $1/\beta_{\mathrm{ph}}$. It may be worth to mention in relation to Lagrange multipliers that different functionals can be adopted for a given problem in the variational method.

The thermodynamic quantities are calculated with the above density operator for the N independent harmonic oscillators with different frequencies in the following section.

3 The independent harmonic oscillators with different frequencies

3.1 Derivation of norm equations

We attempt to derive a norm equation by calculating c_q in two ways. One way is the method using the relation $c_q = Z^{1-q}$ and the other way is the method by calculating $c_q = \mathrm{Tr}\left[\hat{\rho}^q\right]$ directly. We obtain the norm equation by equating these results.

We treat the N independent harmonic oscillators with different frequencies. The Hamiltonian $\hat{H}$ is

$$\hat{H} = \sum_{j=1}^{N}\hbar\omega_j\left(\hat{n}_j + \frac{1}{2}\right), \tag{7}$$

where $\hat{n}_j$ is the number operator with the subscript j. We treat the above Hamiltonian in the Tsallis statistics of $1 < q < 2$: $(2 - q)/(q - 1)$, $1/(q - 1)$, and $q/(q - 1)$ are positive.

We introduce a parameter $E_{\rm ref}$ and calculate the partition function Z from Eqs. (4b) and (7) by using the number states:

$$Z = \sum_{n_1,\cdots,n_N=0}^{\infty} \left\{ 1 + (q-1)\beta_{\rm ph}\left(\frac{\hbar}{2}(\omega_1 + \cdots + \omega_N) - U\right) \right.$$
$$\left. + (q-1)\beta_{\rm ph}\left(\hbar\omega_1 n_1 + \cdots + \hbar\omega_N n_N\right)\right\}^{\frac{1}{1-q}}$$
$$= ((q-1)\beta_{\rm ph}E_{\rm ref})^{\frac{1}{1-q}}$$
$$\times \sum_{n_1,\cdots,n_N=0}^{\infty} (\lambda_N + a_1 n_1 + \cdots + a_N n_N)^{\frac{1}{1-q}}, \quad (8)$$

where

$$\lambda_N = \frac{1 + (q-1)\beta_{\rm ph}\left(\sum_{i=1}^{N}\frac{1}{2}\hbar\omega_i - U\right)}{(q-1)\beta_{\rm ph}E_{\rm ref}}, \quad (9a)$$

$$a_j = \frac{\hbar\omega_j}{E_{\rm ref}}. \quad (9b)$$

Equation (8) is represented with Barnes zeta function $\zeta_{\rm B}(s,\alpha|\boldsymbol{\omega}_N)$ (see Eq. (B.10)):

$$Z = ((q-1)\beta_{\rm ph}E_{\rm ref})^{\frac{1}{1-q}} \zeta_{\rm B}(1/(q-1), \lambda_N|\mathbf{a}_N)$$
$$\mathbf{a}_N = (a_1, a_2, \cdots, a_N). \quad (10)$$

The condition $s > N$ for the parameters of the Barnes zeta function in the present case is

$$\frac{1}{q-1} > N. \quad (11)$$

This means that the number of the oscillators is restricted. We also calculate c_q directly as

$$c_q = {\rm Tr}\left[\hat{\rho}^q\right] = Z^{-q}((q-1)\beta_{\rm ph}E_{\rm ref})^{\frac{q}{1-q}}\zeta_{\rm B}(q/(q-1), \lambda_N|\mathbf{a}_N). \quad (12)$$

From Eqs. (5), (10), and (12), we have the following norm equation:

$$((q-1)\beta_{\rm ph}E_{\rm ref})\,\zeta_{\rm B}(1/(q-1), \lambda_N|\mathbf{a}_N) = \zeta_{\rm B}(q/(q-1), \lambda_N|\mathbf{a}_N). \quad (13)$$

We obtain another norm equation from Eq. (3) in the similar way:

$$U = \frac{\sum_{j=1}^{N}(1-q)\hbar\omega_j\frac{\partial}{\partial a_j}\zeta_{\rm B}(\frac{1}{q-1}, \lambda_N|\mathbf{a}_N)}{\zeta_{\rm B}(\frac{q}{q-1}, \lambda_N|\mathbf{a}_N)} + \sum_{j=1}^{N}\frac{1}{2}\hbar\omega_j. \quad (14)$$

We attempt to obtain the physical quantities by solving the norm equations in the next subsection.

3.2 Energy and entropies

We attempt to find the expressions of physical quantities in this subsection. For $\lambda_N \gg 1$, we have the following expressions using Eq. (B.14):

$$\zeta_{\rm B}(1/(q-1), \lambda_N|\mathbf{a}_N)$$
$$\sim \frac{(q-1)^N}{\left(\prod_{j=0}^{N-1}((2-q)-j(q-1))\right)\left(\prod_{j=1}^{N}a_j\right)(\lambda_N)^{\frac{1}{(q-1)}-N}}, \quad (15a)$$

$$\zeta_{\rm B}(q/(q-1), \lambda_N|\mathbf{a}_N)$$
$$\sim \frac{(q-1)^N}{\left(\prod_{j=0}^{N-1}(1-j(q-1))\right)\left(\prod_{j=1}^{N}a_j\right)(\lambda_N)^{\frac{q}{(q-1)}-N}}. \quad (15b)$$

In Appendix B, the approximated expression for the Barnes zeta function is given using the approximated expression for the Hurwitz zeta function given in Appendix A. We use these expressions of $\zeta_{\rm B}$ to solve Eq. (13) approximately.

3.2.1 Expression of the energy

We attempt to calculate the energy U by solving Eq. (13). Substituting Eqs. (15a) and (15b) into Eq. (13), we have

$$U = \frac{T_{\rm ph}}{(q-1)}\left(1 - \frac{\prod_{j=0}^{N-1}((2-q)-j(q-1))}{\prod_{j=0}^{N-1}(1-j(q-1))}\right)$$
$$+ \sum_{i=1}^{N}\frac{\hbar\omega_i}{2}, \qquad N < \frac{1}{(q-1)}. \quad (16)$$

It is noted that Eq. (16) does not contain $E_{\rm ref}$. We obtain easily

$$\frac{\prod_{j=0}^{N-1}((2-q)-j(q-1))}{\prod_{j=0}^{N-1}(1-j(q-1))} = 1 - N(q-1). \quad (17)$$

By substituting the above expression into Eq. (16), we have the following expression of U:

$$U = NT_{\rm ph} + \sum_{i=1}^{N}\frac{1}{2}\hbar\omega_i, \qquad N < \frac{1}{(q-1)}. \quad (18)$$

Equation (18) is the well-known form of the energy U in the Boltzmann–Gibbs statistics. It is possible to estimate λ_N using Eq. (18):

$$\lambda_N = \frac{1 - N(q-1)}{(q-1)\beta_{\text{ph}}E_{\text{ref}}}. \tag{19}$$

The numerator of the right-hand side of Eq. (19) is positive, because $N(q-1)$ is less than one. Therefore, the condition $\lambda_N \gg 1$ is satisfied for $(q-1)\beta_{\text{ph}}E_{\text{ref}} \ll 1$: the condition is satisfied at high physical temperature T_{ph} and/or for small deviation $(q-1)$.

The energy U is also calculated directly from Eq. (14). Using the approximated equations (15a) and (15b), the right hand side of Eq. (14) (R. H. S. of Eq. (14)) is

R. H. S. of Eq. (14)

$$= \frac{N}{1-N(q-1)}\left(T_{\text{ph}} + (q-1)\left(\sum_{j=1}^{N}\frac{1}{2}\hbar\omega_j\right)\right.$$

$$\left. -(q-1)U\right) + \sum_{j=1}^{N}\frac{1}{2}\hbar\omega_j. \tag{20}$$

Therefore, we have

$$U = NT_{\text{ph}} + \sum_{j=1}^{N}\frac{1}{2}\hbar\omega_j. \tag{21}$$

3.2.2 Expressions of the entropies

The Tsallis entropy $S_q^{(T)}$ is represented as

$$S_q^{(T)} = \frac{1-c_q}{q-1} = \frac{1-Z^{1-q}}{q-1}. \tag{22}$$

The Rényi entropy $S_q^{(R)}$, which is rarely called Rényi-like auxiliary function, is related to the Tsallis entropy:

$$S_q^{(R)} = \frac{1}{1-q}\ln(1 + (1-q)S_q^{(T)}). \tag{23}$$

It may be worth to mention that the entropies can be defined without applying the maximum entropy principle. This equation is represented with c_q as

$$S_q^{(R)} = \frac{1}{1-q}\ln c_q = \frac{1}{1-q}\ln e^{(1-q)\ln Z} = \ln Z. \tag{24}$$

We calculate Z approximately using Eq. (15a):

Substituting Eq. (16) into Eq. (25), we obtain

$$Z = \frac{\left(\prod_{j=0}^{N-1}(1-j(q-1))\right)^{\frac{1}{q-1}-N}}{\left(\prod_{j=1}^{N}(\beta_{\text{ph}}\hbar\omega_j)\right)\left(\prod_{j=0}^{N-1}((2-q)-j(q-1))\right)^{\frac{q}{q-1}-N}}. \tag{26}$$

We find the relation between dU and $dS_q^{(R)}$. The Rényi entropy is given by $\ln Z$. For the fixed N and q, we have

$$dS_q^{(R)} = d\ln Z = N\frac{dT_{\text{ph}}}{T_{\text{ph}}}. \tag{27}$$

With Eqs. (18) and (27), we have

$$dU = NdT_{\text{ph}} = T_{\text{ph}}dS_q^{(R)}. \tag{28}$$

We also have the following expression of Z using Eqs. (25) and (18):

$$Z = \frac{1}{\left(\prod_{j=1}^{N}(1-j(q-1))\right)\left(\prod_{j=1}^{N}(\beta_{\text{ph}}\hbar\omega_j)\right)(1-N(q-1))^{\frac{1}{q-1}-N}}. \tag{29}$$

The Rényi entropy $S_q^{(R)}$ is obtained by calculating $\ln Z$ with Eq. (29). The Rényi entropy $S_q^{(R)}$ for small $N(q-1)$ is obtained by expanding the logarithm of Eq. (29) with respect to $N(q-1)$:

$$S_q^{(R)} = \ln Z \tag{30a}$$

$$= L_N(T_{\text{ph}}) + N + \frac{1}{2}N(q-1) + N \times O((N(q-1))^2), \tag{30b}$$

where $L_N(T_{\text{ph}})$ is defined by

$$L_N(T_{\text{ph}}) = \sum_{j=1}^{N}\ln\left(\frac{T_{\text{ph}}}{\hbar\omega_j}\right). \tag{31}$$

We remember that $N(q-1)$ is less than one.

$$Z = \frac{1}{\left(\prod_{j=0}^{N-1}((2-q)-j(q-1))\right)\left(\prod_{j=1}^{N}(\beta_{\text{ph}}\hbar\omega_j)\right)\left(1+(q-1)\beta_{\text{ph}}\left(\frac{1}{2}\sum_{i=1}^{N}\hbar\omega_i - U\right)\right)^{\frac{1}{q-1}-N}}. \tag{25}$$

The Tsallis entropy $S_q^{(T)}$ as a function of T_{ph} and q is obtained by calculating Z^{1-q} with Eq. (29). With Eq. (30b), the Tsallis entropy $S_q^{(T)}$ is also given using the following relation:

$$S_q^{(T)} = \frac{1 - e^{-(q-1)S_q^{(R)}}}{q - 1}. \tag{32}$$

It is found from Eqs. (29) and (30a) that the Rényi entropy as a function of T_{ph} is unbounded from the above. It is also found from Eq. (32) that the Tsallis entropy as a function of T_{ph} is bounded.

We obtain the ratio of $S_q^{(T)}$ to $S_q^{(R)}$. Hereafter, we represent $L_N(T_{\mathrm{ph}})$ as L_N^{ph} for simplicity. The ratio $S_q^{(T)}/S_q^{(R)}$ from Eqs. (32) and (30b) is

$$\frac{S_q^{(T)}}{S_q^{(R)}} = \frac{1 - e^{-(q-1)S_q^{(R)}}}{(q-1)S_q^{(R)}}$$

$$\sim \frac{1 - e^{-(q-1)L_N^{\mathrm{ph}}} e^{-N(q-1)} e^{-\frac{1}{2N}(N(q-1))^2} e^{O((N(q-1))^3)}}{(q-1)L_N^{\mathrm{ph}} + N(q-1) + \frac{1}{2N}(N(q-1))^2 + O((N(q-1))^3)}. \tag{33}$$

We note that the quantity $N(q-1)$ is not negative and less than one. The ratio $S_q^{(T)}/S_q^{(R)}$ is approximately $1/((q-1)L_N^{\mathrm{ph}})$ at sufficiently high physical temperature which satisfies $(q-1)L_N^{\mathrm{ph}} \gg 1$. This ratio is $1 - (q-1)L_N^{\mathrm{ph}}/2$ for $(q-1)L_N^{\mathrm{ph}} \ll 1$ at sufficiently high physical temperature: the relation $(q-1)N \ll (q-1)L_N^{\mathrm{ph}}$ is satisfied at high physical temperature because of $L_N^{\mathrm{ph}} \gg N$. The expression of the Tsallis entropy can be obtained from Eq. (33), and the simple expressions of the Tsallis entropy can be obtained for $(q-1)L_N^{\mathrm{ph}} \gg 1$ and $(q-1)L_N^{\mathrm{ph}} \ll 1$.

3.3 The physical temperature represented with the temperature T and the entropic parameter q

It is possible to describe the physical temperature T_{ph} with the temperature T and the entropic parameter q. The physical temperature is given by

$$T_{\mathrm{ph}} = c_q T = Z^{1-q} T. \tag{34}$$

The energy and the entropies can be represented with T and q instead of T_{ph} and q. The physical temperature T_{ph} as a function of T and q is given when Z^{1-q} as a function of T and q is given. Therefore, we give the expression of Z^{1-q}.

To obtain the expression of Z, we rewrite β_{ph} using the relation $\beta_{\mathrm{ph}} = \beta/Z^{1-q}$. With this replacement, we have

$$\prod_{j=1}^{N} \beta_{\mathrm{ph}} \hbar \omega_j = Z^{N(q-1)} \prod_{j=1}^{N} \beta \hbar \omega_j. \tag{35}$$

Using the above relation, from Eq. (29), we have

$$Z^{1+N(q-1)}$$
$$= \frac{1}{\left(\prod_{j=1}^{N}(1-j(q-1))\right)\left(\prod_{j=1}^{N}(\beta\hbar\omega_j)\right)(1-N(q-1))^{\frac{1}{q-1}-N}}. \tag{36}$$

This expression leads to

$$Z^{1-q} = \left(\prod_{j=1}^{N}(1-j(q-1))\right)^{\frac{(q-1)}{1+N(q-1)}}$$
$$\times \left(\prod_{j=1}^{N}(\beta\hbar\omega_j)\right)^{\frac{(q-1)}{1+N(q-1)}} (1-N(q-1))^{\frac{1-N(q-1)}{1+N(q-1)}}. \tag{37}$$

The expression of a physical quantity as a function of T and q can be obtained with Eqs. (34) and (37) from the expression with T_{ph} and q.

It is also possible to obtain the relation between T_{ph} and T by using the expression of $S_q^{(R)}$. The partition function Z is directly related to the Rényi entropy: $Z^{1-q} = \exp\left(-(q-1)S_q^{(R)}\right)$. Therefore, T_{ph} is related to T: $T_{\mathrm{ph}} = \exp\left(-(q-1)S_q^{(R)}\right)T$. With the expression of $S_q^{(R)}$, Eq. (30b), we have

$$T_{\mathrm{ph}} = \exp\left(-(q-1)\left(L_N^{\mathrm{ph}} + N + \frac{1}{2}N(q-1) + N\right.\right.$$
$$\left.\left. \times O((N(q-1))^2)\right)\right)T$$
$$= T_{\mathrm{ph}}^{N(1-q)}\left(\prod_{j=1}^{N}(\hbar\omega_j)^{q-1}\right)$$
$$\exp\left(-N(q-1) - \frac{1}{2N}(N(q-1))^2 + O((N(q-1))^3)\right)T$$
$$\equiv K T_{\mathrm{ph}}^{N(1-q)}\left(\prod_{j=1}^{N}(\hbar\omega_j)^{q-1}\right)T, \tag{38}$$

where K is explicitly given by

$$K = \exp\left(-N(q-1) - \frac{1}{2N}(N(q-1))^2 + O((N(q-1))^3)\right). \tag{39}$$

Therefore, we obtain

$$T_{\mathrm{ph}} = K^{\frac{1}{1+N(q-1)}}\left(\prod_{j=1}^{N}(\hbar\omega_j)^{q-1}\right)^{\frac{1}{1+N(q-1)}} T^{\frac{1}{1+N(q-1)}}$$
$$\equiv e^{-(q-1)S_q^{(R)}}T. \tag{40}$$

Equation (40) gives the relation between T_{ph} and T. The expression of $\exp\left(-(q-1)S_q^{(R)}\right)$ as a function of T and q is explicitly obtained from Eq. (40). The expression of a physical quantity as a function of T and q can also be obtained with Eq. (40). The expressions of $S_q^{(R)}$ and $S_q^{(T)}$ with T and q can be obtained using the following relation:

$$
\begin{aligned}
Z^{1-q} &= e^{-(q-1)S_q^{(R)}} \\
&= K^{\frac{1}{1+N(q-1)}} \left(\prod_{j=1}^{N}(\hbar\omega_j)^{q-1}\right)^{\frac{1}{1+N(q-1)}} T^{\frac{-N(q-1)}{1+N(q-1)}}.
\end{aligned}
$$
(41)

3.4 Free energies

We calculate the free energy approximately. We have the following relation:

$$
T_{\text{ph}}dS_q^{(R)} = TdS_q^{(T)}.
$$
(42)

Therefore, it is possible to define the following functions (free energies):

$$
F_q^{(R)} = U - T_{\text{ph}}S_q^{(R)},
$$
(43a)
$$
F_q^{(T)} = U - TS_q^{(T)}.
$$
(43b)

The function $F_q^{(R)}$ with Eqs. (18) and (30b) is

$$
\begin{aligned}
F_q^{(R)} &= \left(NT_{\text{ph}} + \sum_{j=1}^{N}\frac{1}{2}\hbar\omega_j\right) \\
&\quad - T_{\text{ph}}\left(L_N^{\text{ph}} + N + \frac{1}{2}N(q-1) + N \times O((N(q-1))^2)\right).
\end{aligned}
$$
(44)

Therefore, $F_q^{(R)}$ at high T_{ph} is

$$
F_q^{(R)} \sim -T_{\text{ph}}\sum_{j=1}^{N}\ln\left(\frac{T_{\text{ph}}}{\hbar\omega_j}\right) - \frac{1}{2}T_{\text{ph}}N(q-1).
$$
(45)

From Eq. (45), we calculate $\partial F_q^{(R)}/\partial T_{\text{ph}}$:

$$
\frac{\partial F_q^{(R)}}{\partial T_{\text{ph}}} = -\sum_{j=1}^{N}\ln\left(\frac{T_{\text{ph}}}{\hbar\omega_j}\right) - \frac{1}{2}N(q-1) - N.
$$
(46)

The right-hand side of Eq. (46) equals $-S_q^{(R)}$ from Eq. (30b). Therefore, we have

$$
\frac{\partial F_q^{(R)}}{\partial T_{\text{ph}}} = -S_q^{(R)}.
$$
(47)

The function $F_q^{(T)}$ is calculated using the relation $T_{\text{ph}} = \exp\left(-(q-1)S_q^{(R)}\right)T$. The function $F_q^{(T)}$ with Eq. (18) is

$$
\begin{aligned}
F_q^{(T)} &= \left(NT_{\text{ph}} + \sum_{j=1}^{N}\frac{1}{2}\hbar\omega_j\right) - T\left(\frac{1-e^{-(q-1)S_q^{(R)}}}{q-1}\right) \\
&= \left(N + \frac{1}{(q-1)}\right)e^{-(q-1)S_q^{(R)}}T - \frac{T}{(q-1)} + \sum_{j=1}^{N}\frac{1}{2}\hbar\omega_j.
\end{aligned}
$$
(48)

Therefore, $F_q^{(T)}$ with Eq. (40) is

$$
\begin{aligned}
F_q^{(T)} &= \left(N + \frac{1}{(q-1)}\right)K^{\frac{1}{1+N(q-1)}} \\
&\quad \times \left(\prod_{j=1}^{N}(\hbar\omega_j)^{q-1}\right)^{\frac{1}{1+N(q-1)}} T^{\frac{1}{1+N(q-1)}} - \frac{T}{(q-1)} \\
&\quad + \sum_{j=1}^{N}\frac{1}{2}\hbar\omega_j.
\end{aligned}
$$
(49)

In the same way, we calculate $\partial F_q^{(T)}/\partial T$:

$$
\frac{\partial F_q^{(T)}}{\partial T} = \left(\frac{1}{q-1}\right)e^{-(q-1)S_q^{(R)}} - \frac{1}{q-1} = -S_q^{(T)}.
$$
(50)

We have the relations, $\partial F_q^{(R)}/\partial T_{\text{ph}} = -S_q^{(R)}$ and $\partial F_q^{(T)}/\partial T = -S_q^{(T)}$, with Eqs. (45) and (49).

4 Validity and comparison of results

4.1 Validity of the results

The validity can be checked by taking the Boltzmann–Gibbs limit ($q \to 1$) and by comparing the results in this study with the results given in the previous study.

First, we take the Boltzmann–Gibbs limit. The quantity Z^{1-q} approaches one as q approaches one, and the physical temperature T_{ph} approaches the temperature T as q approaches one. The Boltzmann–Gibbs limit of the partition function Z is

$$
\lim_{q \to 1} Z = \exp(\beta U_{q=1})\text{Tr}\left[\exp(-\beta\hat{H})\right],
$$
(51)

where $U_{q=1}$ is the value of the energy at $q = 1$. Therefore, the Boltzmann–Gibbs limit of the Rényi entropy $S_q^{(R)}$ is

$$
\lim_{q \to 1} S_q^{(R)} = \beta U_{q=1} + \ln\left(\text{Tr}\left[\exp(-\beta\hat{H})\right]\right).
$$
(52)

This equation indicates

$$S_{\text{conv}} = \lim_{q \to 1} S_q^{(R)}, \qquad S_{\text{conv}} = -\text{Tr}\left[\rho_{\text{conv}} \ln \rho_{\text{conv}}\right],$$
(53)

because $U_{q=1}$ is equivalent to U_{conv} which is the energy in the conventional statistics, where ρ_{conv} is the density operator in the conventional statistics. This equivalence is explicitly shown by taking the Boltzmann–Gibbs limit. From Eq. (18), at high temperature, we have

$$\lim_{q \to 1} U = NT.$$
(54)

The right-hand side of Eq. (54) is just the energy in the conventional statistics. The energy, Eq. (54), is the same as the energy for the harmonic oscillators with the same frequencies [22]. From Eq. (30b), at high temperature, we have

$$\lim_{q \to 1} S_q^{(R)} = L_N(T) + N.$$
(55)

The right-hand side of Eq. (55) is just the entropy in the conventional statistics.

Next, we compare the results in this paper with the results for a classical harmonic oscillator in the previous work [21]. The results in Ref. [21] correspond to the results for $N = 1$ in this paper. The energy U and the partition function Z in Ref. [21] are given by

$$U/\varepsilon = (2 - q)^{\frac{1}{q}} t^{\frac{1}{q}},$$
(56a)

$$Z = (2 - q)^{\frac{1}{q(1-q)}} t^{\frac{1}{q}},$$
(56b)

where $t = T/\varepsilon$. The quantity ε corresponds to $\hbar\omega_{j=1}$.

We can calculate the physical temperature T_{ph} to describe the quantities with the physical temperature from Eq. (56b). The physical temperature calculated from Eq. (56b) is given by

$$T_{\text{ph}} = Z^{1-q} T = (2 - q)^{\frac{1}{q}} \varepsilon^{1-\frac{1}{q}} T^{\frac{1}{q}}.$$
(57)

The energy from Eq. (56a) is represented as follows:

$$U = (2 - q)^{\frac{1}{q}} \varepsilon^{1-\frac{1}{q}} T^{\frac{1}{q}}.$$
(58)

Therefore, we have the energy represented with T_{ph}:

$$U = T_{\text{ph}}.$$
(59)

The partition function Z, Eq. (56b), is

$$Z = (2 - q)^{\frac{1}{q(1-q)}} \varepsilon^{-\frac{1}{q}} T^{\frac{1}{q}} = (2 - q)^{\frac{1}{1-q}} \left(\frac{T_{\text{ph}}}{\varepsilon}\right).$$
(60)

We can obtain the results for $N = 1$ from Eqs. (18) and (29) for comparison. The same expression of the energy is obtained by ignoring the zero-point energy from Eq. (18). The same expression of the partition function Z is also obtained from Eq. (29). The results for a harmonic oscillator in the present study is consistent with the results in the previous study.

4.2 Comparison between the results in this study and the general results in the Rényi statistics

The Rényi statistics is based on the extremization of the Rényi entropy with the conventional expectation value of an operator $\hat{A}$, $\text{Tr}\left[\hat{\rho}\hat{A}\right]$, under the normalization condition $\text{Tr}\left[\hat{\rho}\right] = 1$ [23,24]. In Ref. [24], the quantities in the canonical ensemble were discussed with $z = 1/(q-1)$ and the two cases were dealt: one is that z is extensive, and the other is that z is intensive. The requirement $N(q-1) < 1$ exists in the present calculations in the Tsallis statistics, and the entropic parameter q goes to one when N goes to infinity. The Tsallis statistics becomes to be the conventional statistics when q approaches one.

First, we compare the energy in this paper with the general result in Ref. [24] which is given under the conditions $N \to \infty$, $V \to \infty$, $\nu = V/N = const.$, and $\tilde{z} = z/N = const.$, when z is extensive, where V is the volume. The quantity $N(q-1)$ is constant when $\tilde{z}$ is constant, and q approaches one as N approaches infinity. It was shown that a homogeneous function of first degree is $N[a(T, \nu, \tilde{z}) + O(N^{-\alpha})]$ $(\alpha > 0)$ in the limit. In the present calculations, the energy U is given as $NT + \sum_{j=1}^{N} \hbar\omega_j/2$ for $q = 1$, because T_{ph} equals T for $q = 1$. The behavior of U is similar to the result in Ref. [24] with fixed $N(q-1)$ in the limit.

Next, we compare the energy in this paper with the general result in Ref. [24] which is given under the conditions $N \to \infty$, $V \to \infty$, $\nu = V/N = const.$, and $z = const.$, when z is intensive. It was also shown that a homogeneous function of first degree is $N[a(T, \nu, z) + O(N^{-\alpha})]$ $(\alpha > 0)$ in the limit. The behavior of U may also be similar to the result in Ref. [24] for large N and small $q - 1$ when T_{ph} is (approximately) equal to T, though the limit $N \to \infty$ cannot be taken because of the condition $N(q-1) < 1$ in the present calculations. It may be natural that T_{ph} is (approximately) equal to T, because the physical temperature T_{ph} is determined using the Rényi entropy and because q is close to one for large N.

The calculations in the Rényi statistics are required in the case of the harmonic oscillators for comparison, because the probability in the Rényi statistics is different from the probability in the Tsallis statistics.

5 Conclusions

We studied the thermodynamic quantities in the system of the N independent harmonic oscillators with different frequencies ω_j in the Tsallis statistics of the entropic parameter q $(1 < q < 2)$. The number of the oscillators N was fixed and the escort average was adopted in this study. We derived the norm equations, and the expressions of physical quantities with the physical temperature were obtained. We obtained the partition function Z, the energy U, the Rényi entropy $S_q^{(R)}$, and the Tsallis entropy $S_q^{(T)}$ by solving the norm equations approximately at high physical temperature $T_{\rm ph}$ and/or for small deviation $q - 1$.

It was found from the condition for the parameters of the Barnes zeta function that the number of harmonic oscillators N is less than $1/(q-1)$. The restriction of the number of the harmonic oscillators exists, as the restriction was previously given for the classical gas [12]. As expected, the supremum $1/(q-1)$ goes to infinity when q approaches one.

The energy U is q-independent at high physical temperature when the physical temperature is adopted. The energy is proportional to the number of harmonic oscillators N and the physical temperature $T_{\rm ph}$ at high physical temperature when zero-point energy is ignored: the expression of the energy is the well-known expression, $U = N T_{\rm ph} + \sum_j \hbar \omega_j / 2$. The Rényi entropy $S_q^{(R)}$ is given as $\sum_{j=1}^{N} \ln(T_{\rm ph}/(\hbar \omega_j))$ for the independent harmonic oscillators at high physical temperature. The Rényi entropy with the same frequencies, $\omega \equiv \omega_1 = \cdots = \omega_N$, is given by $N \ln(T_{\rm ph}/(\hbar \omega))$ which is well-known expression for the N independent harmonic oscillators with the same frequencies. The Tsallis entropy $S_q^{(T)}$ was obtained through the Rényi entropy. The variation for the Rényi entropy is simply given as $dS_q^{(R)} = N dT_{\rm ph}/T_{\rm ph}$ for the fixed N, and the well-known relation between dU and $dS_q^{(R)}$ is also obtained: $dU = T_{\rm ph} dS_q^{(R)}$.

The physical temperature dependence of the Tsallis entropy is different from that of the Rényi entropy. The Rényi entropy contains the term that is $\sum_{j=1}^{N} \ln(T_{\rm ph}/(\hbar \omega_j))$. Therefore, the Rényi entropy increases with the physical temperature, and is unbounded from the above. In contrast, the Tsallis entropy increases with the physical temperature, and is bounded from the above. The ratio of the Tsallis entropy to the Rényi entropy, $S_q^{(T)}/S_q^{(R)}$, is small at high physical temperature. The difference between the Tsallis and Rényi entropies is large at high physical temperature.

The quantity as a function of $T_{\rm ph}$ and q can be represented with T and q. It may be useful to give the relation between $T_{\rm ph}$ and T. This relation was obtained using the partition function Z, and also obtained using the representation of the Rényi entropy because of $S_q^{(R)} = \ln Z$. The T-dependence of the quantity can be found through the $T_{\rm ph}$-dependence of the quantity.

We calculated the function $F_q^{(R)}$ defined as $U - T_{\rm ph} S_q^{(R)}$ and the function $F_q^{(T)}$ defined as $U - T S_q^{(T)}$. It was shown that these functions have the same relations as shown in the conventional free energy: the relations, $\partial F_q^{(R)}/\partial T_{\rm ph} = -S_q^{(R)}$ and $\partial F_q^{(T)}/\partial T = -S_q^{(T)}$, are satisfied. It is worth to mention that $T_{\rm ph} dS_q^{(R)}$ is equal to $T dS_q^{(T)}$. These quantities are related to the first law of thermodynamics. It was already shown that the Legendre transform structure is robust against the choice of the entropic form and the constraints such as energy mean value [25].

The system of the independent harmonic oscillators with different frequencies is basic, and the results in this study will give the insight on other physical systems. The author believes that the present study will be helpful for the reader to study the system represented with oscillators in unconventional statistics such as the Tsallis statistics.

Appendix A: Approximate expression of Hurwitz zeta function

The Hurwitz zeta function [26–28] is defined by

$$\zeta_{\rm H}(s, \alpha) := \sum_{n=0}^{\infty} \frac{1}{(\alpha + n)^s}. \tag{A.1}$$

We treat the case of $s > 1$ and $\alpha > 0$ in this appendix.

Let $B_n(x)$ be Bernoulli polynomials which are defined by

$$\frac{t e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{A.2}$$

The Bernoulli number B_n in this paper is defined [1] by

$$B_n := B_n(x = 1). \tag{A.3}$$

We use the Euler–Maclaurin formula. Let a and b be integer with $a < b$ and let $f(x)$ be continuously differentiable for M-times. The Euler–Maclaurin formula is

$$\sum_{n=a}^{b} f(n) = \int_a^b dx f(x) + \frac{1}{2}(f(b) + f(a))$$
$$+ \sum_{k=1}^{M-1} \frac{B_{k+1}}{(k+1)!}(f^{(k)}(b) - f^{(k)}(a))$$
$$- \frac{(-1)^M}{M!} \int_a^b dx B_M(x - [x]) f^{(M)}(x), \tag{A.4}$$

---

[1] The Bernoulli number B_n is often defined as $B_n(x = 0)$. It may worth to mention that $B_n(x = 0) = B_n(x = 1)$ for $n \neq 1$.

where $f^{(k)}$ is k-th derivative and $[x]$ is the Gauss symbol (the floor function).

We attempt to find the expression of $\zeta_H(1+z, \alpha)$ for $z > 0$ using the Euler–Maclaurin formula. The right-hand side of Eq. (A.1) is an infinite series. Therefore, we first consider the following finite series:

$$\zeta_{H,m}(s, \alpha) = \sum_{n=0}^{m} \frac{1}{(\alpha + n)^s}. \qquad (A.5)$$

We set $f(x)$ as $1/(\alpha + x)^{1+z}$ and apply the Euler–Maclaurin formula. By taking the limit $m \to \infty$, we have the expression of $\zeta_H(1 + z, \alpha)$. The integral part converges when α is positive. We finally obtain

$$
\begin{aligned}
&\zeta_H(1 + z, \alpha) \\
&= \frac{1}{z\alpha^z} + \frac{1}{2\alpha^{1+z}} \\
&\quad + \sum_{k=1}^{M-1} \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} \frac{\Gamma(z+k+1)}{\Gamma(z+1)} \frac{1}{\alpha^{z+k+1}} \\
&\quad - \frac{(-1)^M}{M!} \int_0^\infty dx\, B_M(x - [x]) f^{(M)}(x) \quad (z > 0, \alpha > 0).
\end{aligned}
\qquad (A.6)
$$

The function $\zeta_H(1+z, \alpha)$ can be rewritten [29]. For example, $\zeta_H(1 + z, \alpha)$ is given by

$$
\begin{aligned}
&\zeta_H(1 + z, \alpha) \\
&= \frac{1}{z\alpha^z} + \frac{1}{2\alpha^{1+z}} + \frac{1}{z} \sum_{k=2}^{M} \frac{B_k}{k!} \frac{\Gamma(z+k)}{\Gamma(z)} \frac{1}{\alpha^{z+k}} \\
&\quad - \frac{(-1)^M}{M!} \int_0^\infty dx\, B_M(x - [x]) f^{(M)}(x) \quad (z > 0, \alpha > 0),
\end{aligned}
\qquad (A.7)
$$

because B_{2n+1} is zero for $n \geq 1$ and $\Gamma(z+1) = z\Gamma(z)$.

It is possible to estimate the integral of Eq. (A.6) by setting M. For example, the upper value of the integral with $M = 2$ is estimated:

$$\left| \frac{1}{2!} \int_0^\infty B_2(x - [x]) f^{(2)}(x) \right| \leq \frac{C_2}{2!} \int_0^\infty \left| f^{(2)}(x) \right|, \qquad (A.8)$$

where C_2 is the maximum value of $|B_2(x)|$ in the range of $0 \leq x \leq 1$.

From Eq. (A.6), we find that the $\zeta_H(1 + z, \alpha)$ for $\alpha \gg 1$ behaves

$$\zeta_H(1 + z, \alpha) \sim \frac{1}{z\alpha^z}. \qquad (A.9)$$

Appendix B: Approximate expression of Barnes zeta function

The Barnes zeta function [30,31] is defined by

$$
\begin{aligned}
&\zeta_B(s, \alpha | \boldsymbol{\omega}_N) \\
&\quad := \sum_{n_1, \cdots, n_N = 0}^{\infty} \frac{1}{(\alpha + \omega_1 n_1 + \cdots + \omega_N n_N)^s}, \\
&\qquad \boldsymbol{\omega}_N = (\omega_1, \omega_2, \cdots, \omega_N),
\end{aligned}
\qquad (B.10)
$$

where $s > N$, $\alpha > 0$, and $\omega_j > 0$.

We define Ω_N as

$$\Omega_N := \alpha + \omega_1 n_1 + \cdots + \omega_N n_N. \qquad (B.11)$$

The function ζ_B is represented as

$$
\begin{aligned}
\zeta_B(s, \alpha | \boldsymbol{\omega}_N) &= \sum_{n_1, \cdots, n_N = 0}^{\infty} \frac{1}{(\Omega_N)^s} \\
&= \frac{1}{(\omega_N)^s} \sum_{n_1, \cdots, n_{N-1} = 0}^{\infty} \sum_{n_N = 0}^{\infty} \frac{1}{((\Omega_{N-1}/\omega_N) + n_N)^s} \\
&= \frac{1}{(\omega_N)^s} \sum_{n_1, \cdots, n_{N-1} = 0}^{\infty} \zeta_H(s, \Omega_{N-1}/\omega_N).
\end{aligned}
\qquad (B.12)
$$

We have $\zeta_H(1 + z, \alpha) \sim 1/(z\alpha^z)$ for $\alpha \gg 1$. Therefore, for sufficiently large α, we have

$$
\begin{aligned}
\zeta_B(1 + z, \alpha | \boldsymbol{\omega}_N) &\sim \frac{1}{(\omega_N)^{1+z}} \sum_{n_1, \cdots, n_{N-1} = 0}^{\infty} \frac{1}{z(\Omega_{N-1}/\omega_N)^z} \\
&= \frac{1}{z\omega_N} \zeta_B(z, \alpha | \boldsymbol{\omega}_{N-1}).
\end{aligned}
\qquad (B.13)
$$

Using the recurrence relation, Eq. (B.13), we have the approximate expression of ζ_B for $\alpha \gg 1$:

$$
\begin{aligned}
&\zeta_B(1 + z, \alpha | \boldsymbol{\omega}_N) \\
&\sim \frac{1}{z(z-1)\cdots(z-(N-1))} \frac{1}{\omega_1 \omega_2 \cdots \omega_N} \frac{1}{\alpha^{z-(N-1)}} \\
&= \frac{1}{\left(\prod_{j=0}^{N-1} (z - j) \right) \left(\prod_{j=1}^{N} \omega_j \right) \alpha^{z-(N-1)}} \quad (z - (N-1) > 0).
\end{aligned}
\qquad (B.14)
$$

The condition $z - (N - 1) > 0$ is rewritten as $1 + z - N > 0$. This condition is equivalent to the condition $s > N$ with $s = 1 + z$ for $\zeta_B(s, \alpha | \boldsymbol{\omega}_N)$.

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