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Non-conservative kinetic model of wealth exchange with saving of production

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Abstract. In this paper we propose a non-conservative kinetic model of wealth exchange with saving of production as an extension of the Chakraborti–Chakrabarti model of money exchange. Using microeconomic arguments, we achieve rules of interaction between economic agents that depend on two exogenous parameters, the exchange aversion of the agents (λ) and the saving of production (s), such that in the limit $s = 0$, these rules can be reduced to the ones of the Chakraborti–Chakrabarti model. The non-conservative dynamics are approached analytically through a mean field approximation and the Boltzmann kinetic equation. Both approximations allow us to compute a theoretical rate of exponential growth (g) and to fit the emergent distributions of wealth to a gamma probability density function, in such a way that g, the fit parameters and the Gini index can be expressed analytically in terms of λ and s. In general, the emergent distributions do not reach a stationary state, however it is possible to study the emergence of self-similar distributions that hold the gamma pattern and maximize the Shannon entropy. With the purpose of addressing labor income, we explore additionally the effect of salary income in the model by defining a two-class structure where population is separated into workers and producers. This assumption leads to an emergent rate of economic growth $\widetilde{g}_e$. The macroeconomic implications of this model are studied by means of the wealth/income ratio, which can be predicted as $s/\widetilde{g}_e$, in accordance with the Solow model of economic growth. The results in this paper allow to tie some of the important facts of the modern economic speech, as well as the microeconomic theory, with some methods and ideas developed in the context of non-conservative exchange models.

1 Introduction

The study and development of agent-based models of economic systems inspired by the statistical mechanics has been an active field of research of econophysics since the early XXIst century [1–4]. This approach has proposed an important set of tools, most of them coming from the kinetic theory of gases and the condensed matter physics, aimed to understand the dynamics behind the economic inequality from a microscopic point of view. Certain class of these models imposes the conservation of total wealth and total income as an analogy to the energy conservation, due to elastic collisions in an ideal gas [5–7]. Those closed economic dynamics constitute a useful simplification, but with a restricted generalizability in economics, that allows to model transaction processes where money is locally conserved and there is no return from wealth, for example, the distribution of salaries or wages [8]. However, a more accurate approach to macroeconomics requires non-closed dynamics, governed by the production of goods and return from wealth, which inherently leads to non-conservative economic scenarios characterized by the presence of economic growth [9].

A non-conservative approach to the study of the distribution of wealth and income is usually introduced in econophysics through the return from wealth, which is obtained as a consequence of stochastic multiplicative processes that model the investment as a wealth injection to the system, in such a way that the evolution in time of the average wealth describes an exponential growth pattern [10–12]. The effects of this general scheme of stochastic processes on the wealth distributions have been extensively investigated by different studies, which include important economic factors as taxation and loan [13–16]. Nevertheless, the macroeconomic phenomenon of economic growth has not been enough explored within the scenario of econophysics, and only until recent years the study of the microeconomic dynamics associated to the emergence of these exponential growth patterns and its relation to the phenomenon of economic inequality has attracted some attention [17–19].

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In this work we propose a kinetic wealth-exchange model driven by the saving of production, one of the key factors governing the economic growth, in neoclassical macroeconomics [20,21]. This model states a non-conservative extension of the gas-like model of money exchange with marginal saving propensity λ developed by Chakraborti and Chakrabarti [7] (henceforth referred as CC model). With this in mind, we start from the microeconomic formalism based on the production and exchange of economic goods constrained to stochastic preferences that has been introduced by Chakrabarti et al. [22]. Thus, each economic agent that is involved in a trading saves a fraction s of the goods that they produce and buy, in such a way that these are not perishable anymore. As a result, we obtain rules of interaction between economic agents, which can be reduced to the ones of the CC model for the particular case $s = 0$. However, we redefine the parameter λ as the exchange aversion of the agents, in the frame of this new non-conservative model, because it does not lead to capitalization and only limits the available amount of wealth of each agent at any transaction. The effect of this new parameter tends to right-skew the emergent distributions, in spite of this, the distributions hold the original gamma pattern of the CC model [23], in most of the cases, and the Gini index can be computed in a similar way of reference [24].

We present two analogous approaches to the main results of the model, that connect with the methodologies of study of non-conservative models. First, we investigate the formation of wealth using a mean field approximation similar to the one proposed by Bouchaud and Mézard in their model of wealth condensation [10], which allows us to understand the exponential behavior of the average wealth, and to compute an analytical rate of growth g in terms of λ and s. As a complementary analysis, we use the Boltzmann equation approach presented in references [25,26] to compute recursively higher moments. In particular, we solve the differential equations for the first and second moments, in order to calculate the parameters of the gamma distribution pattern. As it is expected, the emergent distributions do no reach a stationary state, however, it is possible to study the emergence of a quasi-steady state, where the Shannon entropy is maximized and the distributions hold the gamma pattern, by defining the self-similar wealth $\frac{w}{\langle w\rangle}$.

The exponential increasing of the total wealth and production of the system is related to the phenomenon of economic growth. However, with the purpose of addressing the effect of labor income, we separate the total population of the system into N_l workers and N_p producers. In this frame, every producer has associated N_l/N_w workers that receive an average salary proportional to a fraction $(1-\alpha)$ of the total production, where α is the share of capital income. We study these dynamics within the framework of neoclassical macroeconomics, as an approach to the recent studies on economic inequality made by Piketty [27]. So, we present their direct relation to the Solow Model of economic growth [20], which states as a golden rule that on the long run the ratio between wealth and income $(W(t)/Y(t))$ tends to the ratio between the rate of saving and rate of economic growth $s/\widetilde{g}_e$.

All the results are tested numerically using Monte Carlo simulations. For this purpose, we proceed in the usual way, considering a pool with a fixed number of economic agents N, which interact pairwise according to the rules established in the model. The evolution in time of the system is studied by recording the wealth, production and entropy at every time-step. The first two state variables are computed as the sum over the individual wealth and production of all the economic agents, which allows us to measure the economic growth of the system, in absence of population growth [28].

This paper is presented in the following structure. In Section 2, we present the microeconomics of the model, by extending the formalism of the CC model. The analytical approach to the moments of the distribution, using both the mean field approximation and the Boltzmann equation, is presented in Section 3. Section 4 is devoted to the numerical analysis of the emergent distributions, testing the underlying results from the previous sections. On the other hand, the macroeconomic implications of the model are discussed in Section 5, where we study the wealth/income ratio. Conclusions and remarked ideas are presented in Section 6.

2 Microeconomics of the non-conservative gas-like model

We consider an economic system with a fixed number of agents N that interact in a market by exchanging a fraction of their wealth. The transactions between pairs of economic agents occur following the dynamic of a pairwise collision between molecules of a gas. At every time t, two economic agents i and j are randomly selected to trade in the market following the equations: $w_i^* = w_i + \Delta w_i$ and $w_j^* = w_j + \Delta w_j$, where $w_{i,j} \equiv w_{i,j}(t)$ and $w_{i,j}^* \equiv w_{i,j}(t+1)$ are, respectively, the wealth of each agent before and after trading, and therefore these interaction rules are well defined if Δw_i and Δw_j are known.

The economic transactions obey the production of goods, in the same spirit of the CC model [7]. Thus, the agent i produces an amount X of certain good, and the agent j produces an amount Z of a different good. Each agent tries to buy an amount of the good produced by the other agent selling a fraction of their own good and using a portion of their wealth, in such a way that at the end of a transaction the agent i holds an amount x_i of their own good and an amount z_i of the other good, and the agent j ends with x_j and z_j. This market dynamic is governed by the preferences of the agents for each good, which are defined through the Cobb-Douglas utility functions:

$$\mathscr{U}_i(x_i, z_i, w_i^*) = [Ax_i]^{\theta}\,[Az_i]^{\phi}\,[w_i^*]^{\lambda}, \tag{1}$$

$$\mathscr{U}_j(x_j, z_j, w_j^*) = [Ax_j]^{\theta}\,[Az_j]^{\phi}\,[w_j^*]^{\lambda}. \tag{2}$$

The terms $[Ax_i]^{\theta}$, $[Ax_j]^{\theta}$, $[Az_i]^{\phi}$, $[Az_j]^{\phi}$ establish the preferences for consuming, such that the amounts exchanged of the goods satisfy the market conditions

$x_i + x_j = X$ and $z_i + z_j = Z$, and the powers are normalized to 1 as $\theta + \phi + \lambda = 1$. The last condition implies that the utility functions are constant return to scale, which is assume for simplicity as in [22,29]. Note that the factor A that multiplies the exchanged amounts decreases the proclivity of the agents to consume, in such a way that if $A = 1$, then the goods are completely consumed after trading, and if $A = 1 - s$, the agents save a fraction s of the production, such that $s\epsilon[0, 1]$.

In general, the powers in the utility functions are assumed to be equal for both agents. This fact allows to simplify the dynamics of the model as in the CC model [7], and to include the stochastic nature of the transactions by means of θ and ϕ, as we show below. On the other hand, the parameters λ and s are not determined within the model, conversely both are set as exogenous parameters describing the overall exchange aversion of the agents and capacity of the entities that compose the economy to save a fraction of the production.

The constraints over consumption are introduced using the inequalities shown below, which establish that the agents' consumption, added to their remaining wealth after trading (left-hand side of the inequality) cannot exceed their total wealth before trading. This last is defined as the individual capital w_i and w_j at t, added to the value of their total production $p_x X$ and $p_z Z$ (right-hand side of the inequality), where p_x and p_z are the selling prices of the goods.

$$Ap_x x_i + Ap_z z_i + w_i^* \leq w_i + p_x X, \tag{3}$$

$$Ap_x x_j + Ap_z z_j + w_j^* \leq w_j + p_z Z. \tag{4}$$

At every time, the agents pursuit to maximize their utility subjected to constraints (3) and (4). Thus, there are defined the Lagrangian functions:

$$\mathscr{L}_i(x_i, z_i, w_i^*, \mu_i) = [Ax_i]^\theta [Az_i]^\phi [w_i^*]^\lambda - \mu_i \left[Ap_x x_i + Ap_z z_i + w_i^* - w_i - p_x X\right], \tag{5}$$

$$\mathscr{L}_j(x_j, z_j, w_j^*, \mu_j) = [Ax_j]^\theta [Az_j]^\phi [w_j^*]^\lambda - \mu_j \left[Ap_x x_j + Ap_z z_j + w_j^* - w_j - p_z Z\right], \tag{6}$$

where μ_i and μ_j are Lagrange multipliers.

This maximization problem is solved using the Lagrange multipliers method as in the "Microeconomic foundation of the kinetic exchange models" introduced by Chakraborti et al. [29]. Hence, taking the derivatives equal to zero: $\frac{\partial \mathscr{L}_{i,j}}{\partial x_{i,j}} = \frac{\partial \mathscr{L}_{i,j}}{\partial z_{i,j}} = \frac{\partial \mathscr{L}_{i,j}}{\partial w^*_{i,j}} = \frac{\partial \mathscr{L}_{i,j}}{\partial \mu_{i,j}} = 0$, and solving the equation system, there are obtained the demand functions:

$$x_i = \frac{\theta}{\lambda A p_x} w_i^*, \quad z_i = \frac{\phi}{\lambda A p_z} w_i^*, \quad w_i^* = \lambda[w_i + p_x X], \tag{7}$$

$$x_j = \frac{\theta}{\lambda A p_x} w_j^*, \quad z_j = \frac{\phi}{\lambda A p_z} w_j^*, \quad w_j^* = \lambda[w_j + p_z Z]. \tag{8}$$

It is clear from equations (7) and (8) that the demand for goods decreases as their prices increase. Additionally, the individual wealth at $t + 1$ is proportional to the total production of each agent. Using the market conditions $x_i + x_j = X$ and $z_i + z_j = Z$, we obtain the following expressions for the clearing prices of goods:

$$p_x = \frac{\varepsilon(1 - \lambda)[w_i + w_j]}{X[\lambda - s]}, \tag{9}$$

$$p_z = \frac{(1 - \varepsilon)(1 - \lambda)[w_i + w_j]}{Z[\lambda - s]}, \tag{10}$$

where the variable $\varepsilon = \frac{\theta}{\theta + \phi}$ was introduced as a random factor, uniformly distributed over the domain $[0, 1]$.

Note that the evolution of the individual wealth can be directly computed now by replacing the clearing prices in the expressions for w_i^* and w_j^*. However, it is useful to express it in the fashion $w_i^* = w_i + \Delta w_i$, $w_j^* = w_j + \Delta w_j$. Adding and subtracting w_i and w_j in the corresponding expression, then the interaction rules are now known because we obtain:

$$\Delta w_i = \frac{1 - \lambda}{\lambda - s} \lambda \left[\varepsilon(w_i + w_j) - w_i\right] + \frac{1 - \lambda}{\lambda - s} s w_i, \tag{11}$$

$$\Delta w_j = \frac{1 - \lambda}{\lambda - s} \lambda \left[-\varepsilon(w_i + w_j) + w_i\right] + \frac{1 - \lambda}{\lambda - s} s w_j. \tag{12}$$

For the case $s = 0$, it is easy to check that $\Delta w = \Delta w_i = -\Delta w_j$, which implies that the evolution of individual wealth reduces to mere monetary exchange, in accordance with the CC model [7]. In this case, the expressions for Δw_i and Δw_j allow to the same interaction rules of the CC model given by $w_i^* = w_i + \Delta w$ and $w_j^* = w_j - \Delta w$, where $\Delta w = (1 - \lambda)\left[\varepsilon(w_i + w_j) - w_i\right]$ [7], being w now only related to money because of the lack of production saving. On the other hand, if $s \neq 0$, then $\Delta w_i \neq \Delta w_j$ and total wealth $W(t)$ is not conserved in time, in contrast, $W(t)$ grows as:

$$W(t + 1) = W(t) + \Delta w_i + \Delta w_j = W(t) + \frac{1 - \lambda}{\lambda - s} s(w_i + w_j). \tag{13}$$

In order to guarantee the convergence of the model and avoid the presence of debt, the exchange aversion λ is set as an upper limit for s. Thus, we restrict the analysis to the cases that satisfy $\lambda > s$. Note that the expressions for Δw_i and Δw_j contain exchange terms similar to the CC model $\frac{1-\lambda}{\lambda-s}\lambda\left[\varepsilon(w_i + w_j) - w_i\right]$, and non-conservative terms $\frac{1-\lambda}{\lambda-s} s w_{i,j}$, analogous to the models of references [10–12]. However, the wealth growth induced by this last term is completely determined by exogenous parameters, in contrast to what happens in the models of wealth exchange with stochastic growth as the proposed in references [10–12].

3 Analytical approach to the moments of the distribution

An important fact of the extension introduced in the previous section is the possibility of connecting some the methods developed in the context of non-conservative models in references [10,12,25] with the CC model. In this order of ideas, we present in this section an analytical study of the moments of the distribution using a mean field approximation to the pairwise dynamics between agents, and the Boltzmann-type approach of references [25,26]. In particular, we calculate explicitly the expression for the first and second moments, which are used, together with a numerical analysis in the next section, for computing the parameters of the emergent distributions.

All the results presented are tested using numerical simulations. With this in mind, we define a set of agents $N = 1000$ with one unit of initial individual wealth, such that the total wealth of the system starts at $W(t_0) = 1000$. The economic system evolves according to the dynamics described above. Thus, at every time-step, two economic agents are randomly selected, using a generator of random uniform numbers, to interact according to equations (11) and (12). In addition, we compute the state variables of the system: total wealth $W(t)$ and total production $Y(t)$, at every time-step. The first one is obtained by summing over the wealth of every economic agent, and the second is obtained computing and summing equations (9) and (10), which represent the production of each couple of agents involved in a transaction at any time-step. For all the cases studied, we take an average over 10^5 ensembles.

3.1 Mean field approximation to formation of wealth

According to the dynamics of the model, the individual wealth of an agent k at any time $t+1$ has a probability $1/N$ of increasing their initial value w_k due to the interaction rule defined by equation (11) and a probability $1/N$ of increasing due to the interaction rule (12). Hence, assuming that any agent k feels an average influence from their environment, given by the average wealth over all agents $\langle w\rangle = \frac{1}{N}\sum_{i=1}^{N} w_i$, we redefine both equations as follows:

$$\Delta w_k' = \frac{1-\lambda}{\lambda - s}\lambda\left[\varepsilon(w_k + \langle w\rangle) - w_k\right] + \frac{1-\lambda}{\lambda - s} s w_k, \quad (14)$$

$$\Delta w_k'' = \frac{1-\lambda}{\lambda - s}\lambda\left[-\varepsilon(\langle w\rangle + w_k) + \langle w\rangle\right] + \frac{1-\lambda}{\lambda - s} s w_k. \quad (15)$$

Note that equations (14) and (15) constitute an approximate form of equations (11) and (12) where we assumed that the interaction with the other agent can be reduced to an average over all the economic agents. Hence, the behavior in time of the individual wealth is given by:

$$\begin{aligned} w_k^* &= w_k + \frac{1}{N}\left\{\Delta w_k' + \Delta w_k''\right\} \\ &= w_k + \frac{2}{N}\frac{(1-\lambda)}{(\lambda - s)} s w_k + \frac{1}{N}\frac{(1-\lambda)}{(\lambda - s)}\lambda\left[\langle w\rangle - w_k\right]. \end{aligned} \quad (16)$$

Using the fact that $w_k^* - w_k = \frac{w_k(t+1) - w_k(t)}{t+1-t}$, we get the following continuous form of equation (16), under the limit $\Delta t \to 0$:

$$\frac{dw_k}{d\tau} = g w_k + J\left[\langle w\rangle - w_k\right], \quad (17)$$

where $g = \frac{(1-\lambda)}{(\lambda-s)} s$ is the rate of growth of the average wealth, $J = \frac{(1-\lambda)}{(\lambda-s)}\frac{\lambda}{2}$ is the rate of exchange and $\tau = \frac{2t}{N}$ is the normalized time, which obeys the fact that at every time-step, only one couple, from the set of $\frac{N}{2}$, trades in the market.

Equation (17) is analogous to the process of evolution of the individual wealth in the Bouchaud and Mézard model [10]. Nevertheless, the factor g which gives account of the growth of wealth does not have a stochastic behavior related with a stochastic process, inducing wealth condensation. Taking and average over k and integrating equation (17), we find that the average wealth evolves in time as $\langle w\rangle = \langle w_0\rangle \exp(g\tau)$, where the constant $\langle w_0\rangle$ is the average wealth per agent at $t = 0$ that depends on the initial conditions of the system. In Figure 1a, we show the behavior of the average wealth simulated for different cases of λ and s and fitted using the previous result.

3.2 The Boltzmann equation approach

In general, the dynamics of the kinetic exchange models of markets can be studied as a one-dimensional collision between Maxwell molecules [25]. When a trading occurs in the market, the individual wealth of a pair of economic agents is transformed as $[w, v] \to [w^*, v^*]$, in analogy to the change of the velocities due to a binary collision. This process is described in terms of the following equations:

$$w^* = p_1 w + q_1 v, \quad (18)$$

$$v^* = p_2 w + q_2 v. \quad (19)$$

Note that the variables $w \equiv w_i$, $v \equiv w_j$ have been introduced replacing the index notation, with the purpose of simplifying the equations along this section, and $p_{1,2}$, $q_{1,2}$ give the detail of the transaction. In particular, for the model considered in this paper, relations (11) and (12) lead to:

$$p_1 = \frac{\lambda}{\lambda - s}\left[\lambda + \varepsilon(1-\lambda) - s\right], \quad (20)$$

$$q_1 = \frac{\lambda}{\lambda - s}\left[\varepsilon(1-\lambda)\right], \quad (21)$$

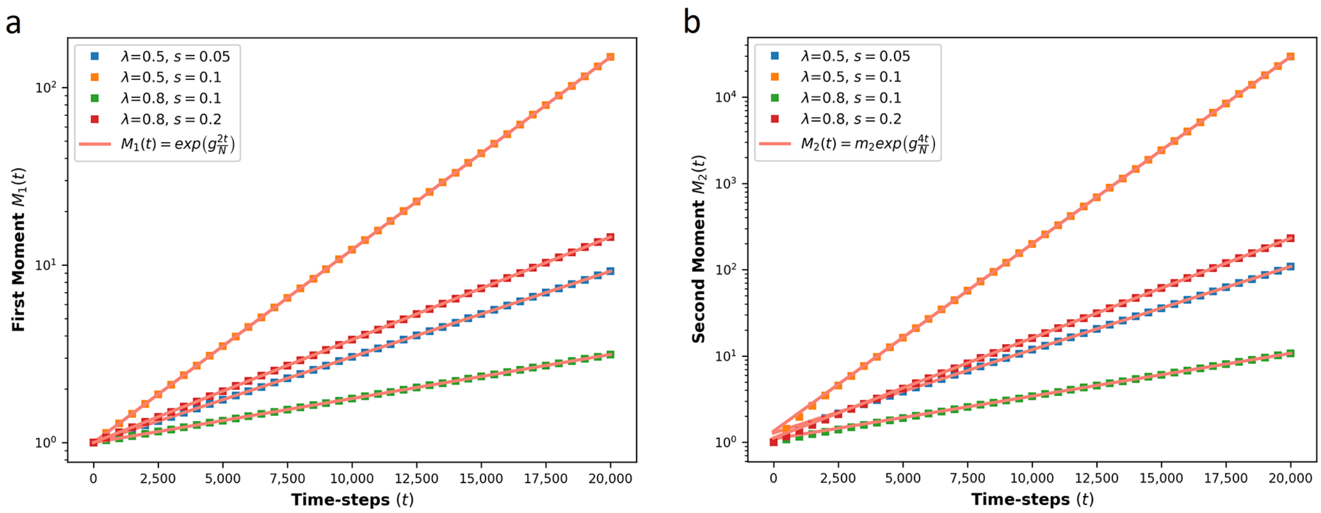

Fig. 1. Evolution in time of the first and second moments of the distribution of wealth. The moments computed using simulated data follow the analytical expressions obtained through the mean field approximation and the Boltzmann kinetic equation. In particular, for the second moment, the coefficient multiplying the exponential function reads as $m_2 = \frac{\lambda^2(\lambda+2-3s)}{3(\lambda-s)(\lambda+s-\lambda s)-2\lambda^2(1-\lambda)}$.

$$p_2 = \frac{\lambda}{\lambda - s}[(1-\varepsilon)(1-\lambda)], \tag{22}$$

$$q_2 = \frac{\lambda}{\lambda - s}[\lambda + (1-\varepsilon)(1-\lambda) - s]. \tag{23}$$

In the kinetic theory of gases, the probabilistic description of such transformation is made through the Boltzmann kinetic equation:

$$\frac{\partial f}{\partial t} = Q(f, f), \tag{24}$$

where the term $Q(f, f)$ is known as the collision operator, which determines the evolution of the distribution function due to the collision process. In general, the mathematical definition of $Q(f, f)$ obeys the physical characteristics of the collision, which are related with a cross section, however, in the case of an economic transaction, its formulation is simpler. A useful approach to this binary dynamics shown in references [25,26] is achieved through the following weak formulation of the Boltzmann kinetic equation:

$$\frac{d}{d\tau}\int f(w,\tau)\Phi(w)dw = \frac{1}{2}\left\langle \int f(w)f(v)\left[\Phi(w^*) + \Phi(v^*) - \Phi(w) - \Phi(v)\right] dwdv\, dw^* dv^* \right\rangle. \tag{25}$$

This equation studies the effect of the collision operator in smooth test functions $\Phi(w)$. Thus, assuming $\Phi(w) = w^r$ we can compute the evolution in time of the rth moment as:

$$\frac{d}{d\tau}\int f(w,\tau)w^r dw = \frac{1}{2}\left\langle \int f(w)f(v)\left[(w^*)^r + (v^*)^r - w^r - v^r\right] dwdv\, dw^* dv^* \right\rangle. \tag{26}$$

Using the binomial expansion, the last expression leads to the following recursive relation:

$$\frac{d}{d\tau}M_r(\tau) = \langle p_i^r + q_i^r - 1\rangle_+ M_r(\tau) + \sum_{k=1}^{n-1}\binom{r}{k}\langle p_i^k q_i^{r-k}\rangle_+ M_k(\tau)M_{r-k}(\tau), \tag{27}$$

where the notation $\langle\psi(p_i, q_i)\rangle_+ := \frac{1}{2}\langle\psi(p_1, q_1) + \psi(p_2, q_2)\rangle$ has been introduced, as in reference [26], in order to abbreviate the expected values with respect to variables defining the factors $p_{1,2}$, $q_{1,2}$.

By standard methods of ordinary differential equations, we find through recursive integration the following solutions of equation (27) for $r = 1, 2$:

$$M_1(\tau) = M_1(0)\exp[\langle p_i + q_i - 1\rangle_+ \tau], \tag{28}$$

$$M_2(\tau) = \frac{2\langle p_i q_i\rangle_+ \exp[2\langle p_i + q_i - 1\rangle_+ \tau]}{\langle 1 - p_i^2 - q_i^2\rangle_+ + 2\langle p_i + q_i - 1\rangle_+} + \frac{c}{\exp[\langle 1 - p_i^2 - q_i^2\rangle_+ \tau]}, \tag{29}$$

where c is a constant of integration that can be neglected in the limit $\tau \to \infty$ only if $\langle 1 - p_i^2 - q_i^2\rangle_+ > 0$. In particular, for the factors (20), (22), (21) and (23), this condition is satisfied at every time, due to the fact that the order of magnitude of the exogenous parameters is $\lambda, s \sim \mathscr{O}\left(10^{-1}\right)$ and $\lambda > s$. Therefore, (29) reduces to:

$$M_2(\tau) = \frac{2\langle p_i q_i\rangle_+ \exp[2\langle p_i + q_i - 1\rangle_+ \tau]}{\langle 1 - p_i^2 - q_i^2\rangle_+ + 2\langle p_i + q_i - 1\rangle_+}, \tag{30}$$

and the moments computed explicitly in terms of $p_{1,2}$ and $q_{1,2}$ read as:

$$M_1(\tau) = \exp\left(\frac{(1-\lambda)}{(\lambda - s)} s\tau\right), \quad (31)$$

$$M_2(\tau) = m_2 \exp\left(2\frac{(1-\lambda)}{(\lambda - s)} s\tau\right), \quad (32)$$

where

$$m_2 = \frac{\lambda^2(\lambda + 2 - 3s)}{3(\lambda - s)(\lambda + s - \lambda s) - 2\lambda^2(1-\lambda)}. \quad (33)$$

Note that the result obtained for the first moment is exactly the same achieved by means of the mean field approximation. In Figure 1, we show the results for the first and second moment computed using simulated data for different values of λ and s. In both cases the theoretical relations reproduce accurately the behavior in time of data, however for the second moment, the level of accuracy increases as t becomes higher, as consequence of the approximation made to neglect the integration constant.

4 Emergent wealth distributions

In the context of the CC model, it is generally acknowledged that the best fitting of simulated data is achieved by means of the gamma probability density function (PDF):

$$f(w) = \frac{1}{a\Gamma(b)}\left(\frac{w}{a}\right)^{b-1} \exp\left(-\frac{w}{a}\right) dw, \quad (34)$$

where a is the scale parameter and b the shape.

According to Patriarca et al. [23], the gamma distribution pattern constitutes the analytical emergent distribution of the CC model, such that $b = 1 + \frac{3\lambda}{1-\lambda}$ and $a = 1/b$. However, it has been shown in later studies, by direct comparison of the moments of the distribution computed using both a fixed-point distribution approach and the Boltzmann kinetic equation, that this conjecture is not right [30,31]. In spite of this conclusion, it is clear that this pattern reproduces correctly the main macroscopic properties of the model, including the behavior of the wealth inequality captured by the Gini index [24]. In this order of ideas, we study in this section the validity of the gamma pattern for fitting the emergent distributions of the non-conservative extension proposed in previous sections.

An underlying property of the gamma PDF is that the rth moment of the distribution can be computed using the scale and shape parameters as $M_r = \frac{a\Gamma(r+b)}{\Gamma(b)}$ [32]. In particular, for the first and second moments, the previous expression reads as $M_1 = ab$ and $M_2 = a^2 b(b+1)$. Therefore, using the results (31) and (32), a and b can be computed as:

$$a = \frac{M_1(\tau)}{b} = (m_2 - 1)\exp(g\tau) = \frac{(1-\lambda)(\lambda^2 + 3s)}{3(\lambda - s)(\lambda + s - \lambda s) - 2\lambda^2(1-\lambda)} \exp(g\tau), \quad (35)$$

$$b = \frac{1}{m_2 - 1} = \frac{3(\lambda - s)(\lambda + s - \lambda s) - 2\lambda^2(1-\lambda)}{(1-\lambda)(\lambda^2 + 3s)}. \quad (36)$$

In the case $s = 0$, the rate of growth is $g = 0$, which implies that the exponential growth disappears and the parameters a and b match the expression predicted by Patriarca et al. [23]. On the other hand, for $s \neq 0$, the scale of the distribution grows exponentially in time, while the shape remains constant. In Figures 2a–2d, we present the numerical analysis of the emergent distributions obtained at different time-steps for different values of λ and s; all the cases are fitted using Maximum Likelihood Estimation (MLE). On the whole, the gamma distribution is accepted for every case by the Kolmogorov–Smirnov test, with a level of significance $\alpha = 0.5$. In addition, the theoretical relations (35) and (36) match, with a high level of accuracy, the value of the parameters estimated using MLE. The comparison between both results is shown in subplots (e and f).

4.1 Quasi-steady state and wealth inequality

In general, we do not expect to reach a steady state for the emergent distributions, as consequence of the non-conservative dynamics of the model, inducing the exponential increasing of the moments. This behavior is well captured by the scale parameter a of the gamma pattern as it is shown in equation (35). However, one can study the emergence of a quasi-steady state by defining the normalized wealth $\widetilde{w} = \frac{w}{\langle w \rangle}$. Under this condition, the PDF (34) reads as:

$$f(\widetilde{w}) = \frac{1}{\widetilde{a}\Gamma(b)}\left(\frac{\widetilde{w}}{\widetilde{a}}\right)^{b-1} \exp\left(-\frac{\widetilde{w}}{\widetilde{a}}\right) d\widetilde{w}, \quad (37)$$

where the scale parameter becomes the time constant variable: $\widetilde{a} = a_0 = \frac{a}{\langle w \rangle}$, due to the fact that $a(t) = a_0 \exp(g\tau)$; while the shape parameter is not altered by the effect of the normalization.

The results obtained for $\lambda = 0.8$ and $s = \{0, 0.1, 0.2, 0.3\}$ are shown in Figure 3. On the whole, the distribution tends to decrease its peakedness for higher values of s, which implies that it becomes less egalitarian as s increases. This effect is reflected by the Gini index, which increases proportionally to s. The computed values of this metric for fixed $l = \{0.2, 0.5, 0.8\}$ are shown in Figure 4 and fitted using the theoretical expression for the Gini index of the gamma PDF [33]:

$$G = \frac{1}{\sqrt{\pi}} \frac{\Gamma(b + \frac{1}{2})}{\Gamma(b+1)}, \quad (38)$$

where b is given by equation (36).

The emergent distributions are successfully characterized by means of the gamma PDF, however, in the limit $s \to \lambda$ the goodness of fit is lost for the majority of the cases. Thus, the accuracy of the prediction made using the theoretical relation (38) decreases in that limit. It is important to remark that the Gini index of the distribution only depends on the shape parameter and is

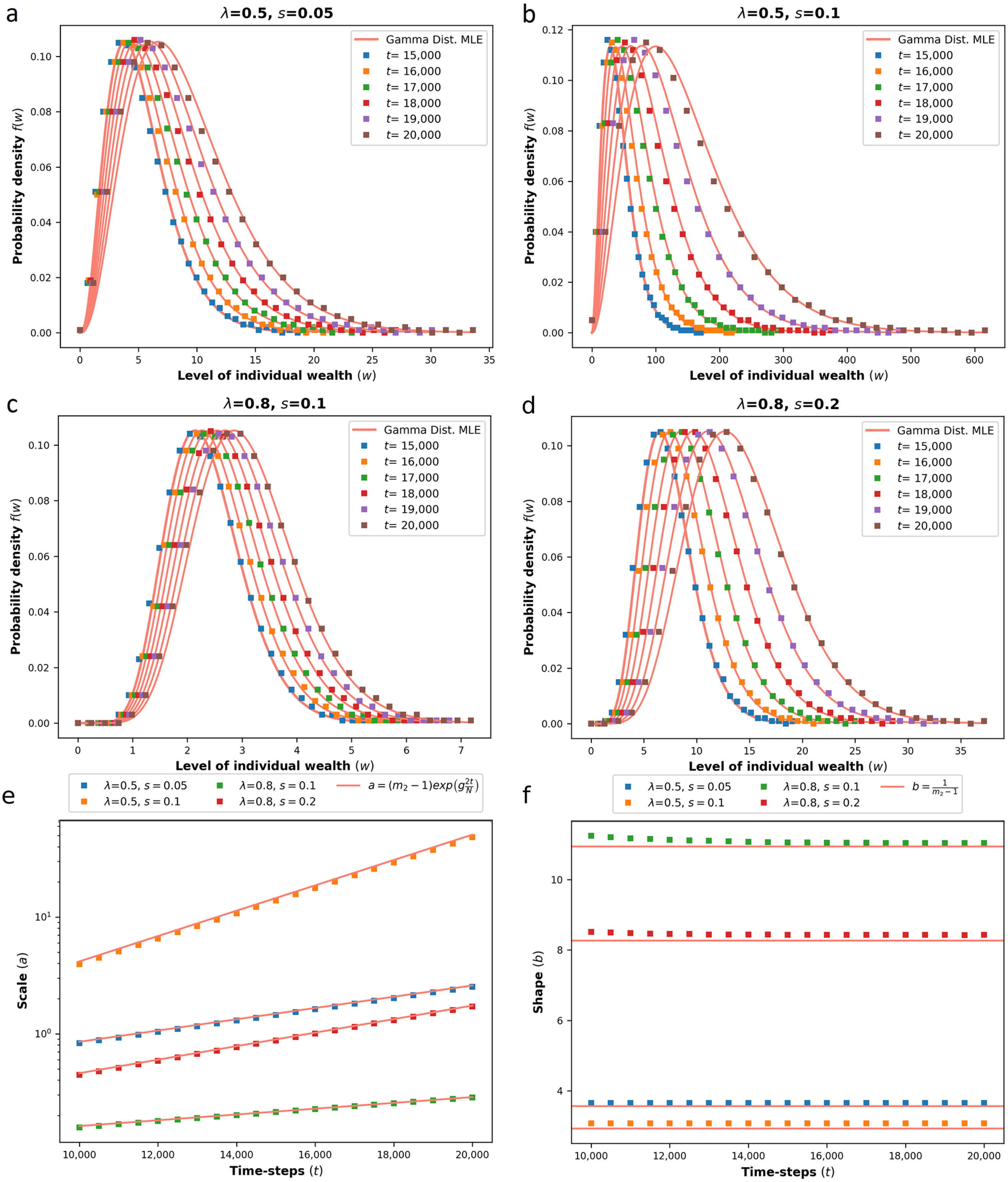

Fig. 2. Evolution in time of the emergent wealth distributions. (a–d) Simulated data is well fitted by the gamma PDF. The parameters were computed, in all the cases, using MLE. (e and f) The exponential behavior of the first moments is captured by the scale parameter a, which follows the function $a = (m_2 - 1)\exp(g\frac{2t}{N})$, while the shape parameter remains constant in time. In all the cases studied, their values are well fitted by the theoretical functions obtained using the Boltzmann equation.

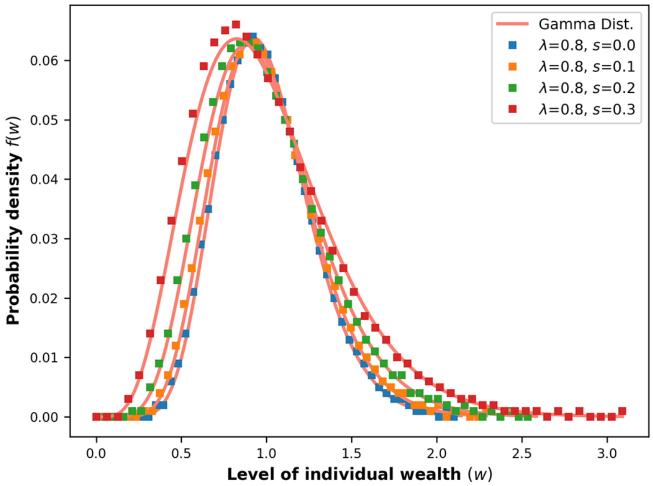

Fig. 3. Simulated data is well fitted using the gamma distribution at quasi-steady state of normalized wealth distributions. In this case, the parameters are time independent and can be directly computed as $a = m_2 - 1$, $b = \frac{1}{a} = \frac{1}{m_2 - 1}$.

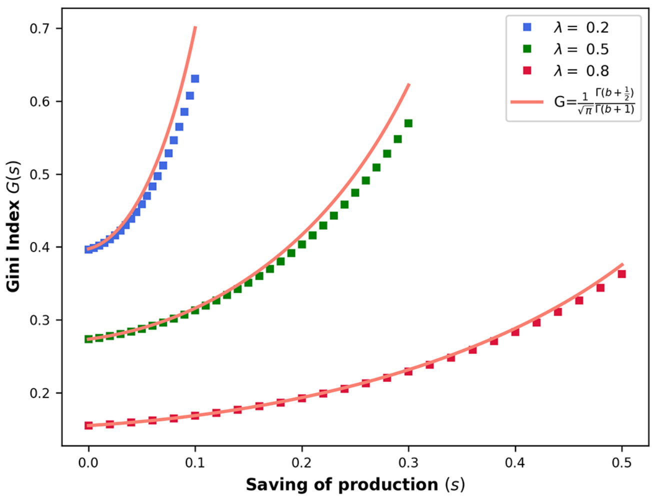

Fig. 4. The spectrum of values of the Gini index is wider than in the CC model. Simulated data shows that $G > 0.5$ as $s \to \lambda$. However, in that case the predictive power of the gamma distribution decreases.

independent of the time. Additionally, the spectrum of values for the Gini index is greater than 0.5, which removes the restriction of the CC model imposed by the case $\lambda = 0$.

The effect of dividing by $\langle w \rangle$ induces a quasi-stationary state, in which the entropy reaches a maximum, as we show in Figure 5. This state variable is computed from simulated data using the Shannon's entropy definition $H = \sum_{l=1}^{m} p_l \log p_l$, [34] defining a fixed wide $d = \frac{\max[w_k(t)] - \min[w_k(t)]}{m}$, where $k = 1, 2, 3, \ldots, N$ and the terms $\max/\min[w_k(t)]$ are, respectively, the money of the richest and the poorest agent during all the time interval studied. Note that the m levels of money used for every λ and s must be the same at every time, thus, the maximum and minimum of w_k represent an absolute value, which is defined over the whole time interval considered. In general, we found that for higher values of λ, the relaxation process takes more time. Conversely, the system reaches faster the quasi-stationary state as s increases.

5 Evolution of income

Until know, we have focused on the behavior of wealth. However, the non-conservative dynamics of the model were introduced as a direct consequence of production of goods, which has important effects on the income and the long-term behavior of the model, as we discuss below.

According to the kinetic-like dynamics defined in Section 2, at every time-step, the total production can be computed as $Y(t) = p_x X + p_z Z$. Using equations (9) and (10), the previous expression becomes:

$$Y(t) = \frac{1 - \lambda}{\lambda - s}[w_i + w_j]. \tag{39}$$

5.1 Labor income

Note that the dynamics described so far assume that all the agents are producers, which implies that the only source of income is the return from individual wealth by means of total production, that is, the share of capital income is $\alpha = 100\%$. However, according to empirical data from modern economies, this value constitutes only 30% of total income [27,35]. Therefore, a more realistic approach must take into account the income due to labor.

In order to introduce the effect of income salary into the model, we suppose a two-class structure where total population is composed of workers and producers. The total amount of workers is set to be $N_l = 0.9N$, while producers are $N_p = 0.1N$. For simplicity, we suppose that every producer has associated N_l/N_p workers, that is, the size of firms is homogeneous, and all the employees receive the same average salary. In this frame, the wealth of two producers i and j, that are randomly selected to trade in the market, changes as:

$$w_i^* = w_i + \alpha \Delta w_i, \tag{40}$$

$$w_j^* = w_j + \alpha \Delta w_j, \tag{41}$$

where Δw_i and Δw_j are defined by (11) and (12). On the other hand, the average salary of an agent l from the set of $2\frac{N_l}{N_p}$ workers associated to both producers is defined as:

$$\langle Y_l \rangle = \frac{N_p}{2N_l}(1 - \alpha)(\Delta w_i + \Delta w_j), \tag{42}$$

and their wealth at t increases as $w_l^* = w_l + \langle Y_l \rangle$.

Recalling the mean field approximation introduced in Section 3, it is clear that under this new dynamics, the average wealth of producers increases as:

$$\begin{aligned} w_k^* &= w_k + \frac{\alpha}{N_p}\left\{\Delta w_k' + \Delta w_k''\right\} \\ &= w_k + \frac{2\alpha}{N_p}\frac{(1-\lambda)}{(\lambda - s)} s w_k + \frac{\alpha}{N_p}\frac{(1-\lambda)}{(\lambda - s)}\lambda\left[\langle w \rangle - w_k\right]. \end{aligned} \tag{43}$$

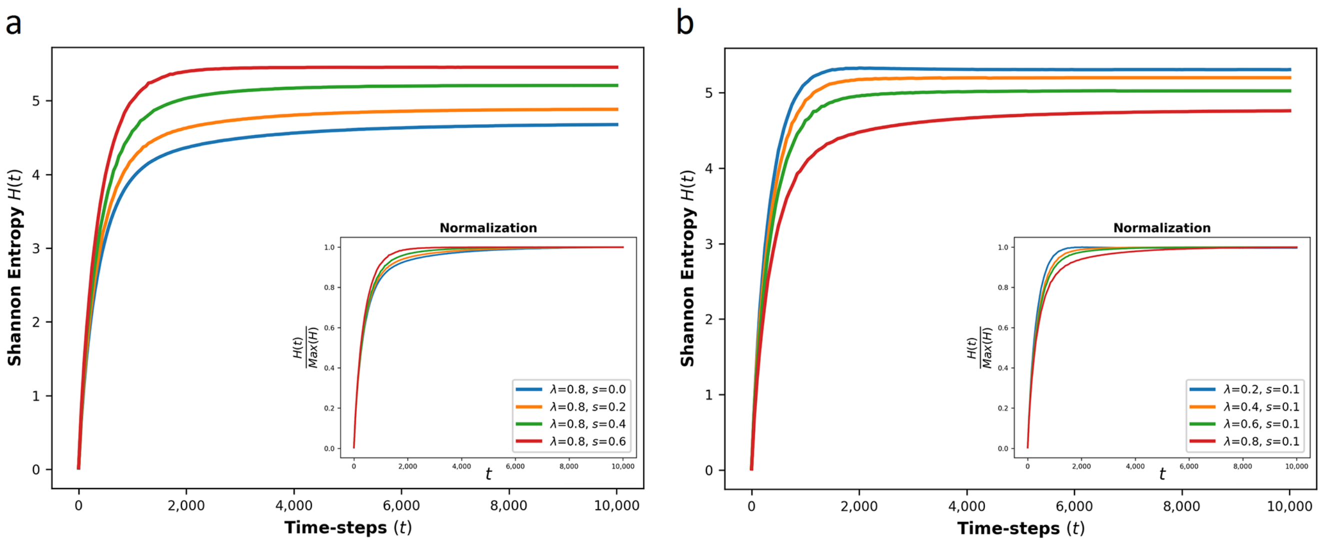

Fig. 5. The quasi-stationary state of self-similar distributions for $\widetilde{w} = \frac{w}{\langle w \rangle}$ is studied using the Shannon entropy. (a) The relaxation process is faster for higher values of s. (b) Conversely, it is slower for higher λ. All the curves were normalized to one dividing by their maximum value of entropy, in order to make easier the comparison between curves.

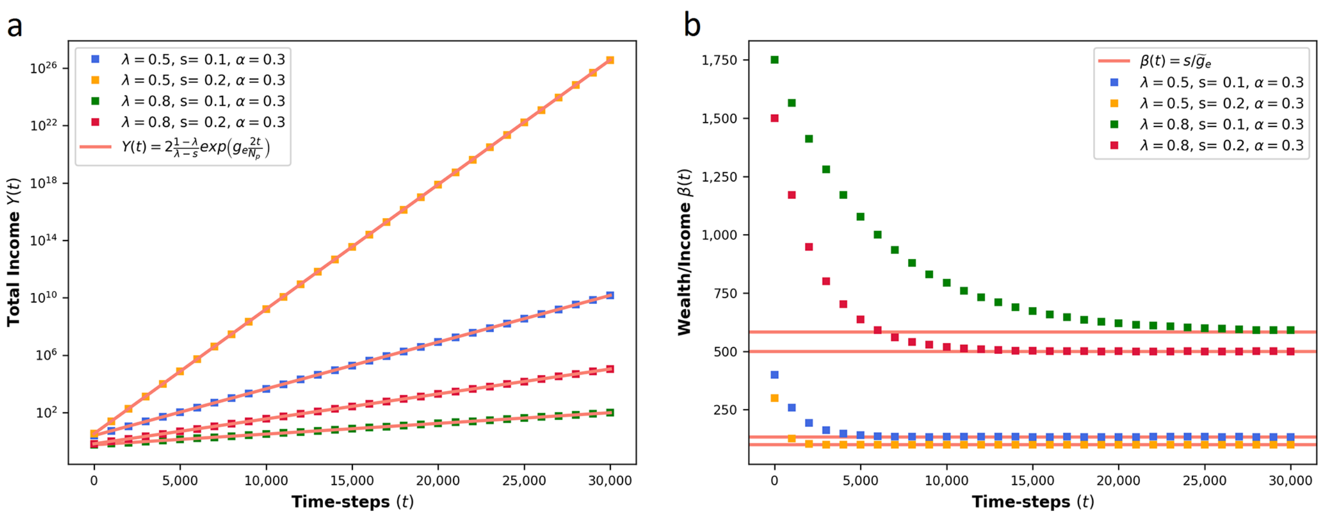

Fig. 6. (a) Simulated data of production, for different cases of s and λ, is well fitted by the exponential function $Y(t) = 2\frac{(1-\lambda)}{(\lambda-s)}\exp(g_e \frac{2t}{N_p})$. (b) The asymptotic behavior of the wealth/income ratio can be predicted by means of theoretical value $s/\widetilde{g}_e$, where $\widetilde{g}_e$ is the rate of growth divided by the rate of interaction between producers $\frac{N_p}{2}$.

The solution of this equation, in the limit $\Delta t \to \infty$, is analogous to the one obtained in previous sections: $\langle w_p \rangle = \langle w_0 \rangle \exp\left(g_e \frac{2t}{N_p}\right)$, however, in this case the rate of growth is $g_e = \frac{(1-\lambda)}{(\lambda-s)}\alpha s$.

Note that the size of population remains constant in time, as well as the number of workers associated to each producer. Therefore, the production only increases as consequence of wealth hold by producers. In this line, equation (39) can be approached analytically as: $Y(t) = 2\frac{(1-\lambda)}{(\lambda-s)}\exp(g_e \frac{2t}{N_p})$. In Figure 6a, we present the evolution of the total income computed using simulated data and fitted to this exponential function, setting $\alpha = 0.3$.

5.2 Wealth/income ratio

Despite the modification in the population structure, the total wealth of the system still increases according to equation (13), due to the fact that the proportion α goes to producers and $(1-\alpha)$ goes to workers, but on the whole both fractions sum 1. Thus, comparing (13) and (39) we can rewrite the evolution of the total wealth in time as:

$$W(t+1) = W(t) + sY. \tag{44}$$

In a continuum time horizon, the previous equation reads as $\frac{dW}{dt} = sY$. This result constitutes one of the main macroeconomic hypothesis about the capitalization in neoclassical economics [36]. Replacing $Y(t) =$

$Y_0 \exp\left(g_e \frac{2t}{N_p}\right)$, where $Y_0 = 2\frac{(1-\lambda)}{(\lambda-s)}$, and integrating both sides of equation we obtain:

$$W(t) = \frac{s}{\widetilde{g}_e} Y(t) + \left[W_0 - \frac{s}{\widetilde{g}_e} Y_0\right], \tag{45}$$

where $\widetilde{g}_e = \frac{2g_e}{N}$ is the rate of economic growth normalized by the rate of trading between producers defined in the kinetic-like process. Now, dividing by $Y(t)$, the last expression becomes:

$$\beta(t) = \frac{s}{\widetilde{g}_e} + \left[W_0 - \frac{s}{\widetilde{g}_e} Y_0\right] \frac{1}{Y(t)}. \tag{46}$$

In the limit $t \to \infty$, the factor $\frac{1}{Y(t)}$ vanishes the second term in the right-hand side of the equation, thus we obtain:

$$\lim_{t\to\infty} \beta(t) = \frac{s}{\widetilde{g}_e}. \tag{47}$$

The concept of infinity in the previous result describes a very large, but reasonable, period of time in which the economy reaches a steady state. This result is coherent with the prediction made by Solow [20], which establishes that in the long-term, the wealth/income ratio tends to the ratio between savings and the rate of economic growth. However, we obtained it as an emergent property of the microscopic dynamics of the model, in contrast with the macroeconomic analysis made by Solow, where the micro and the macroeconomics are connected by means of the aggregation property of the production function [37]. In Figure 6b, we present the behavior of the wealth/income ratio computed using simulated data and predicted by the relation (47).

The behavior of wealth/income ratio $\beta(t)$ is a well-known fact in the context of macroeconomics since the middle XXth century. Its implications in the analysis of economic inequality has been revisited in recent years by Thomas Piketty, who explored the influence of the inherited wealth in the divergence of wealth distributions, using the asymptotic behavior of $\beta(t)$ for approaching the empirical data of the 10 richest economies in the world [38]. In that context, limit (47) is known as the second fundamental law of capitalism.

5.3 Wealth distributions for workers and producers

In the context of two-structure population, we observe the emergence of two different distributions of wealth. The first one given by the salaries paid to workers at every time-step, and the second one analogous to the distributions discussed in previous sections, related to producers. In general, both distributions are well fitted using the gamma PDF, however, their parameters are different in both cases. In all the cases studied, the emergent distribution for workers is more egalitarian than distributions for producers. In Figure 7, we present the results of simulations for $\lambda = 0.5$ and $s = 0.1$, using the normalized wealth $\frac{w}{\langle w\rangle}$ defined in the previous section. The level of inequality is measured, as in the previous results, using the Gini index, which is higher for the distribution of wealth related to producers.

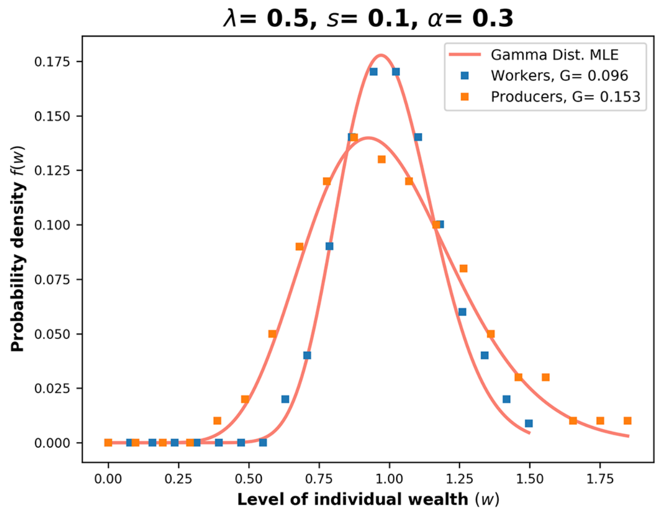

Fig. 7. Distributions of wealth for workers and producers setting $\lambda = 0.5$, $s = 0.1$ and $\alpha = 0.3$. The distribution obtained for workers is generated by salaries paid at every time-step. On the other hand, the process leading to distributions for producers is given by the kinetic-like interactions described in previous sections, multiplied by the share of capital income $\alpha = 0.3$. In general, the value of the Gini index (G) is higher in the distributions for producers.

It is important to remark that the origin of the wealth distributions for workers are the salaries paid at every time-step. Therefore, this distribution is actually an income distribution, due to the fact that workers do not receive any kind of rent. In general, the income distributions obtained from the model are more egalitarian than the wealth distributions for producers, this fact satisfies empirical observations made in the context of economics [35,39].

The assumption of a two-class structure constitutes a strong simplification of the society structure in modern economies, that we made in order to propose an approach to heterogeneous economies with a microeconomic perspective. However, a more realistic approach to modern industrialized economies should take into account the fact that population on the top decile of capital income are also on the top decile of labor income. As a matter of fact, empirical evidence for countries as Sweden and Norway shows that the share of wage on total income for the top decile of the population constitutes approximately the 25% of income since the late XXth century [40,41].

6 Discussion and conclusions

The kinetic wealth-exchange model introduced in this paper extends the CC model, by imposing the saving of production in the formalism of the utility function. The maximization of the utility leads to a generalized dynamic, depending on two exogenous parameters: the saving of production s and the original parameter of the CC model λ, which is redefined in this context as the exchange aversion of the agents, due to the fact that it represents the utility of protecting wealth from risky tradings [22].

In the limit case $s = 0$, the dynamics and results are the same of the CC model, where the agents interact by means of money transactions. On the other hand, in the non-conservative regime $s \neq 0$, the model is extended to wealth exchange, and the economic agents are able to capitalize, by saving a fraction of the production. Despite the differences between the nature of both interactions, the results of the non-conservative model hold the gamma distribution pattern for s sufficiently far from λ. In addition, considering higher values of s allows to reproduce economic scenarios characterized by a tougher level of inequality, such that the Gini index is greater than 0.5. However, the accuracy of the predictions made by the gamma pattern is reduced in these cases, due to the divergence in the interaction rules induced as $s \to \lambda$.

As a first approach to the non-conservative dynamics, we considered in this paper a constant rate of saving of production. According to analysis made by Piketty, this reduction can be assumed as a value representing the average rate of saving in a society [27]. However, a more accurate approach can be achieved considering a distribution pattern for s. We expect this modification eventually leads to the emergence of power laws, as in references [10,25], due to the stochastic multiplicative term that appears for the evolution of wealth.

An important fact of introducing saving of production is the possibility of modeling two-class economies. In this line, we extend the kinetic formalism to a model where population is divided into workers and producers. Thus, the income is separated into capital income and labor income, where the last is paid to a fixed number of agents associated to each producer, every time that they are involved in a transaction. In this context, we obtain that the emergent distributions can be separated into distributions of wealth, related to producers, and distributions of wealth related to income paid to workers. And moreover, the long-term behavior of the wealth/income ratio can be predicted as the ratio between savings and economic growth.

In line with the previous argument, the non-conservative extension allows to tie important problems in the context of modern economics and agent-based modeling, with the microeconomic formalism proposed in the CC model. First, the exponential dynamics of the average wealth are obtained with a clear microeconomic perspective, leading to the emergence of well-known distribution patterns and macroscopic properties. And second, the explicit emergence of the economic growth, governing the exponential increasing of income in time, leads to the second fundamental law of capitalism, proposed by Piketty as one of the key factors governing economic inequality [27].

Author contribution statement

Both authors have contributed equally to the statement of the model and the theoretical results presented in this paper. The numerical analysis and simulations have been performed by D.S. Quevedo.

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References

1. D. Stauffer, Physica A **336**, 1 (2004)
2. V.M. Yakovenko, J.B. Rosser Jr., Rev. Mod. Phys. **81**, 1703 (2009)
3. C. Schinckus, Physica A **389**, 3814 (2010)
4. T. Lux, Eur. Phys. J. Special Topics **225**, 3255 (2016)
5. S. Ispolatov, P.L. Krapivsky, S. Redner, Eur. Phys. J. B **2**, 267 (1998)
6. A. Dragulescu, V.M. Yakovenko, Eur. Phys. J. B **17**, 723 (2000)
7. A. Chakraborti, B.K. Chakrabarti, Eur. Phys. J. B **17**, 167 (2000)
8. J. Tempere, Eur. Phys. J. B **91**, 38 (2018)
9. M. Gallegati, S. Keen, T. Lux, Physica A **370**, 1 (2006)
10. J.P. Bouchaud, M. Mézard, Physica A **282**, 536 (2000)
11. F. Slanina, Phys. Rev. E **69**, 046102 (2004)
12. S. Cordier, L. Pareschi, G. Toscani, J. Stat. Phys. **120**, 253 (2005)
13. R. Coelho, P. Richmond, P. Barry, S. Hutzler, Physica A **387**, 3847 (2008)
14. B.M. Boghosian, Phys. Rev. E **89**, 042804 (2014)
15. B. Düring, L. Pareschi, G. Toscani, Eur. Phys. J. B **91**, 265 (2018)
16. J. Li, B.M. Boghosian, C. Li, Physica A **516**, 423 (2019)
17. K. Liu, N. Lubbers, W. Klein, J. Tobochnik, B.M. Boghosian, H. Gould, arXiv:1305.0794v1 [q-fin.GN]
18. Y. Berman, Y. Shapira, Physica A **467**, 562 (2017)
19. Y. Berman, Y. Shapira, M. Schwartz, Europhys. Lett. **118**, 38004 (2017)
20. R.M. Solow, Q. J. Econ. **70**, 65 (1956)
21. T.W. Swan, Econ. Rev. **32**, 334 (1956)
22. A.S. Chakrabarti, B.K. Chakrabarti, Physica A **388**, 4151 (2009)
23. M. Patriarca, A. Chakraborti, K. Kaski, Phys. Rev. E **70**, 016104 (2004)
24. A. Ghosh, A. Chatterjee, I. Jun-ichi, B.K. Chakrabarti, Physica A **450**, 465 (2016)
25. L. Pareschi, G. Toscani, J. Stat. Phys. **124**, 740 (2006)
26. D. Matthes, G. Toscani, J. Stat. Phys. **130**, 1087 (2008)
27. T. Piketty, *Capital in the Twenty-First Century* (Harvard University Press, 2014)
28. T. Piketty, E. Saez, Science **344**, 838 (2014)
29. B.K. Chakrabarti, A. Chakraborti, S.R. Chakravarty, A. Chatterjee, *Econophysics of Income and Wealth Distributions* (Cambridge University Press, 2013)
30. P. Repetowicz, S. Hutzler, P. Richmond, Physica A **356**, 641 (2005)
31. M. Lallouache, A. Jedidi, A. Chakraborti, Sci. Cult. (India) **76**, 478 (2010)
32. V.P. Singh, *Entropy-Based Parameter Estimation in Hydrology* (Springer, 1998)
33. J.B. McDonald, B.C. Jensen, J. Am. Stat. Assoc. **74**, 856 (1979)
34. C.E. Shannon, Bell Sys. Tech. J. **27**, 379 (1948)
35. E. Saez, G. Zucman, Q. J. Econ. **131**, 519 (2016)

36. D. Romer, *Advanced Macroeconomics* (McGraw-Hill, 2006)
37. D. Acemoglu, *Introduction to Modern Economic Growth* (Princeton University Press, 2009)
38. T. Piketty, G. Zucman, Q. J. Econ. **129**, 1255 (2014)
39. M. Kuhn, M. Schularick, U.I. Steins, J. Political Econ. **128**, 00 (2020)
40. J. Roine, D. Waldenström, J. Public Econ. **92**, 366 (2008)
41. R. Aaberge, A.B. Atkinson, S. Königs, J. Econ. Inequal. **16**, 295 (2018)