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Painlevé analysis, group classification and exact solutions to the nonlinear wave equations

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Abstract. This paper is concerned with the general regular long-wave (RLW) types of equations. By the combination of Painlevé analysis and Lie group classification method, the conditional Painlevé property (PP) and Bäcklund transformations (BTs) of the nonlinear wave equations are provided under some conditions. Then, all of the point symmetries of the nonlinear RLW types of equations are obtained, the exact solutions to the equations are investigated. Particularly, some explicit solutions are provided by the special function and Φ-expansion method.

1 Introduction

Biswas [1] investigated the regularized long-wave (RLW) equation with power-law nonlinearity using the solitary-wave ansatz. Such celebrated regularized long-wave equation plays a significant role in fluid mechanics, physical applications and nonlinear science. The classical models of the equation, such as $u_t + u_x + uu_x + au_{xxt} = 0$ (where $a \neq 0$ is a constant, compare with equation (3) as follows) have been studied extensively (see, e.g. [1–6] and the references cited therein). Particularly, this classical equation is also called the Benjamin-Bona-Mahony (BBM) equation, which arises as an alternative model to the KdV equation for small-amplitude, long wavelength surface water waves. These types of RLW equations are of importance also in fluid mechanics, nonlinear theory and integrable system. In [2], the solitary wave solutions to the equation are obtained by the decomposition method. In [3], some solitary wave solutions to the BBM equation are obtained by the so called direct method. In fact, these solitary wave solutions are traveling wave solutions actually. Moreover, for the classical RLW equation as above, the symmetries are given in [4,5], and the relationship between partial differential equations (PDEs) soluble by inverse scattering and ordinary differential equations (ODEs) of Painlevé type are presented [6].

In the present paper, we investigate the general nonlinear regular long-wave (RLW) types of equations as follows:

$$u_t + \alpha(1 + u^p)u_x + \beta u_{xxt} = 0, \tag{1}$$

and

$$u_t + \alpha u^p u_x + \beta u_{xxt} = 0, \tag{2}$$

where $u = u(x, t)$ denotes the unknown function with respect to the space variable x and time t, while α, β and p are arbitrary nonzero constants. In physical applications, $p > 0$ is assumed generally.

We note that the general RLW equations (1) and (2) include a lot of important nonlinear wave equations as its special cases. For example, if $p = 1, 2$, then equation (1) reduces to the following usual forms of RLW equations:

$$u_t + \alpha(1 + u)u_x + \beta u_{xxt} = 0, \tag{3}$$

$$u_t + \alpha(1 + u^2)u_x + \beta u_{xxt} = 0. \tag{4}$$

If $p = 1, 2$, then equation (2) reduces to the following classical and modified RLW equations (c.f., KdV and mKdV equations):

$$u_t + \alpha uu_x + \beta u_{xxt} = 0, \tag{5}$$

$$u_t + \alpha u^2 u_x + \beta u_{xxt} = 0. \tag{6}$$

Furthermore, in view of p is an any constant, let $p = 1/2$, then equations (1) and (2) become the following nonlinear wave equations with fractional exponents, respectively

$$u_t + \alpha(1 + u^{\frac{1}{2}})u_x + \beta u_{xxt} = 0, \tag{7}$$

$$u_t + \alpha u^{\frac{1}{2}} u_x + \beta u_{xxt} = 0, \tag{8}$$

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and so on. In recent years, such fractional exponential nonlinear equations are playing an increasingly important part in physics and applications.

In general, it is very difficult to cope with exact solutions to the nonlinear partial differential equations (NLPDEs), and there are few systematic and effective methods so far. However, we know that the Lie symmetry analysis is a powerful method for dealing with symmetry, integrability and exact solutions to NLPDEs [4–18], e.g., Ma [17,18] studied the symmetries and conservation laws (CLs) systematically for some nonlinear systems, and presented the inverse scattering transforms and soliton solutions to nonlocal reverse-time nonlinear Schröinger equations in [19], these are very meaningful results in symmetry analysis and integrable system. On the other hand, the Painlevé test could to provide useful information for integrability of NLPDEs and rational solutions sometimes based on the truncated expansion technique [20–22]. However, the combination of Painlevé test and Lie symmetry analysis method for dealing with NLPDEs is a new idea. In [16], we considered the integrable properties, symmetries and exact solutions to the variable-coefficient PDEs by the Painlevé test and Lie symmetry analysis method. In the current paper, we shall develop the combination of Painlevé analysis and Lie group classification method for dealing with integrable conditions, complete group classifications and exact solutions to the nonlinear RLW types of equations. In fact, this combination is very efficient to deal with nonlinear PDEs. We summarize the contribution and novelty of the current paper as follows:

– we develop the combination of Painlevé test and Lie symmetry analysis method for tackling symmetries, exact solutions and integrable properties of NLPDEs;
– we employ the Lambert W function function and Φ-expansion method to investigate explicit solutions to the nonlinear RLW types of equations.

The rest of this paper is structured as follows: In Section 2, the Painlevé test is employed to tackle the general nonlinear RLW equations, the conditional Painlevé property (PP) and Bäcklund transformations of the NLPDEs are obtained. In Section 3, the Lie group classification is performed, all of the point symmetries of the nonlinear RLW types of equations are presented, and the integrability of the general RLW equation is considered. In Sections 4 and 5, the symmetry reductions and exact explicit solutions to the nonlinear RLW equations are provided by the dynamical system method. Particularly, some exact explicit solutions to the nonlinear PDEs are obtained by the Lambert W function and Φ-expansion procedure. Finally, some new findings and remarks are given in Section 6.

2 Painlevé analysis for the generalized RLW equations

In this section, we assume that the parameter p is a positive integer only, i.e., $p = 1, 2, \ldots$ Thus, the Painlevé analysis can be performed for the generalized RLW equations (1) and (2) through the WTC procedure [16,20–22].

Firstly, we assume

$$u = \phi^{-\rho} \sum_{j=0}^{\infty} u_j \phi^j, \tag{9}$$

where $\phi = \phi(x,t)$, $u_j = u_j(x,t)$ are analytic functions in a neighborhood of the noncharacteristic singular manifold, $u_0 \neq 0$ and ρ is a positive integer.

For the generalized RLW equations (1) and (2), by the leading order analysis, we get

$$-(p+1)\rho - 1 = -\rho - 3, \tag{10a}$$

$$-\rho\alpha u_0^{p+1}\phi_x - \rho(\rho+1)(\rho+2)\beta u_0 \phi_x^2 \phi_t = 0. \tag{10b}$$

From (10a), we have

$$\rho = \frac{2}{p}. \tag{11}$$

Substituting it into (10b), we get

$$u_0^p = -\frac{2(1+p)(2+p)}{p^2\alpha}\beta\phi_x\phi_t, \tag{12}$$

where the condition $u_0 \neq 0$ is satisfied for $p > 0$. Then, in view of (11), we have the following result:

Theorem 1. *Let p be a positive integer. If equations (1) and (2) are integrable, then $p = 1, 2$.*

This theorem gives the necessary condition for RLW types of equations (1) and (2) are integrable. Under this condition, equation (1) becomes equations (3) and (4), equation (2) becomes equations (5) and (6), respectively. Furthermore, in view of (12), when $p = 1$, we have $u_0 = -\frac{12\beta}{\alpha}\phi_x\phi_t$. When $p = 2$, we have $|u_0| = \sqrt{-\frac{6\beta}{\alpha}\phi_x\phi_t}$.

Now, taking $p = 1$ for an example, substituting (9) into equation (3), we have

$$j = 0, \quad u_0 = -\frac{12\beta}{\alpha}\phi_x\phi_t, \tag{13}$$

$$j = 1, \quad u_1 = \frac{36\beta}{5\alpha}\phi_{xt} + \frac{12\beta}{5\alpha}\left(\frac{\phi_x\phi_{tt}}{\phi_t} + \frac{\phi_{xx}\phi_t}{\phi_x}\right), \tag{14}$$

$$\begin{aligned} j = 2, \quad & -2u_0\phi_t - 2\alpha u_0\phi_x + \alpha u_1 u_{0,x} + \alpha u_0 u_{1,x} \\ & - \alpha u_1^2\phi_x - 2\alpha u_0 u_2 \phi_x - 2\beta(u_{0,xx}\phi_t + 2u_{0,xt}\phi_x \\ & + 2u_{0,x}\phi_{xt} + u_{0,t}\phi_{xx} + u_{0,xxt}) + 2\beta(2u_{1,x}\phi_x\phi_t \\ & + u_{1,t}\phi_x^2 + 2u_1\phi_x\phi_{xt} + u_1\phi_{xx}\phi_t) = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} j = 3, \quad & u_{0,t} - u_1\phi_t + \alpha u_{0,x} - \alpha u_1\phi_x \\ & + \alpha(u_2 u_{0,x} + u_1 u_{1,x} + u_0 u_{2,x}) - \alpha u_1 u_2 \phi_x - \alpha u_0 u_3 \phi_x \\ & + \beta u_{0,xxt} - \beta(u_{1,xx}\phi_t + 2u_{1,xt}\phi_x \\ & + 2u_{1,x}\phi_{xt} + u_{1,t}\phi_{xx} + u_{1,xxt}) = 0, \end{aligned} \tag{16}$$

$$j=4,\quad u_{1,t}+\alpha u_{1,x}+\alpha(u_3u_{0,x}+u_2u_{1,x}+u_1u_{2,x}+u_0u_{3,x})+0\cdot u_4=0. \tag{17}$$

By (13)–(16), we can get u_0, u_1, u_2 and u_3 in a unique manner. But from (17), we cannot get u_4 uniquely, so $j=4$ is a resonance. In fact, we have the recursion relations for equation (3) as follows:

$$\begin{aligned}\beta(j+1)(j-4)(j-6)\phi_x^2\phi_t u_j &= -u_{j-3,t}\\ &-(j-4)u_{j-2}\phi_t-\alpha u_{j-3,x}-\alpha(j-4)u_{j-2}\phi_x\\ &-\alpha\sum_{k=0}^{j-1}u_{k,x}u_{j-k-1}-\alpha\sum_{k=1}^{j-1}(j-k-2)u_k u_{j-k}\\ &-\beta u_{j-3,xxt}-\beta(j-4)(u_{j-2,xx}\phi_t+2u_{j-2,xt}\phi_x\\ &+2u_{j-2,x}\phi_{xt}+u_{j-2,t}\phi_{xx}+u_{j-2,xxt})\\ &-\beta(j-3)(j-4)(2u_{j-1,x}\phi_x\phi_t+u_{j-1,t}\phi_x^2\\ &+2u_{j-1}\phi_x\phi_{xt}+u_{j-1}\phi_{xx}\phi_t).\end{aligned} \tag{18}$$

Clearly, we can see that the resonances occur at $j=-1,4,6$. The compatibility conditions at $j=4,6$ are satisfied identically for arbitrary chosen u_4 and u_6. We now specialize (9) by setting the resonance functions $u_4=u_6=0$. Moreover, in view of (18), we have

$$\begin{aligned}j=5,\quad -6\beta\phi_x^2\phi_t u_5 &= -u_{2,t}-u_3\phi_t-\alpha u_{2,x}-\alpha u_3\phi_x\\ &-\alpha(u_4u_{0,x}+u_3u_{1,x}+u_2u_{2,x}+u_1u_{3,x}+u_0u_{4,x})\\ &-\alpha(u_1u_4+u_2u_3)-\beta u_{2,xxt}\\ &-\beta(u_{3,xx}\phi_t+2u_{3,xt}\phi_x+2u_{3,x}\phi_{xt}+u_{3,t}\phi_{xx}+u_{3,xxt})\\ &-2\beta(2u_{4,x}\phi_x\phi_t+u_{4,t}\phi_x^2+2u_4\phi_x\phi_{xt}+u_4\phi_{xx}\phi_t),\end{aligned} \tag{19}$$

$$\begin{aligned}j=6,\quad 0 &= -u_{3,t}-2u_4\phi_t-\alpha u_{3,x}-2\alpha u_4\phi_x\\ &-\alpha(u_5u_{0,x}+u_4u_{1,x}+u_3u_{2,x}+u_2u_{3,x}\\ &+u_1u_{4,x}+u_0u_{5,x})-\alpha(2u_1u_5+2u_2u_4+u_3^2)-\beta u_{3,xxt}\\ &-2\beta(u_{4,xx}\phi_t+2u_{4,xt}\phi_x+2u_{4,x}\phi_{xt}+u_{4,t}\phi_{xx}+u_{4,xxt})\\ &-6\beta(2u_{5,x}\phi_x\phi_t+u_{5,t}\phi_x^2+2u_5\phi_x\phi_{xt}+u_5\phi_{xx}\phi_t).\end{aligned} \tag{20}$$

Furthermore, let $u_3=0$, from (16), we get

$$\begin{aligned}&u_{0,t}-u_1\phi_t+\alpha u_{0,x}-\alpha u_1\phi_x+\alpha(u_2u_{0,x}+u_1u_{1,x}\\ &+u_0u_{2,x})-\alpha u_1u_2\phi_x+\beta u_{0,xxt}-\beta(u_{1,xx}\phi_t\\ &+2u_{1,xt}\phi_x+2u_{1,x}\phi_{xt}+u_{1,t}\phi_{xx}+u_{1,xxt})=0.\end{aligned} \tag{21}$$

On the other hand, in view of (19), we get $u_5=0$. Generally, from $u_3=u_4=u_6=0$, by the induction method, we can prove $u_j=0$, for all $j\geq 2$.

In this case, we can say that equation (3) possesses the Painlevé property (PP) under the conditions (17), (20) and (21), more specifically, it is called the conditional Painlevé property [21]. Furthermore, we have

$$u=u_0\phi^{-2}+u_1\phi^{-1}+u_2, \tag{22}$$

where $u_{2,t}+\alpha u_{2,x}+\alpha u_2u_{2,x}+\beta u_{2,xxt}=0$, that is, $u_2=u_2(x,t)$ satisfies equation (3), while u_0 and u_1 are given by (13) and (14). Thus, we obtain the following result:

Theorem 2. *Under the conditions (17), (20) and (21), equation (3) possesses the Painlevé property, and it has a Bäcklund transformation (22).*

In addition, we can derive the other integrable properties of the RLW equation, the details are omitted.

In view of (11), we can see that these generalized regular long-wave equations (1) and (2) are non-integrable unless $p=1,2$.

3 Lie symmetry analysis for the generalized RLW equations

In this section, we assume that the exponent p is an any real constant. Now, let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned}x&\to x+\epsilon\xi(x,t,u),\\ t&\to t+\epsilon\tau(x,t,u),\\ u&\to u+\epsilon\phi(x,t,u),\end{aligned}$$

with a small parameter $|\epsilon|\ll 1$. The geometric vector field associated with the group of transformations above can be written as

$$V=\xi(x,t,u)\partial_x+\tau(x,t,u)\partial_t+\phi(x,t,u)\partial_u, \tag{23}$$

where $\xi(x,t,u)$, $\tau(x,t,u)$ and $\phi(x,t,u)$ are coefficient functions to be determined.

If the vector field (VF) (23) generates a symmetry of equations (1) and (2), then V must satisfy the following Lie's symmetry condition:

$$\mathrm{pr}^{(3)}V(\Delta)|_{\Delta=0}=0, \tag{24}$$

where $\Delta=u_t+\alpha(1+u^p)u_x+\beta u_{xxt}$ for equation (1) and $\Delta=u_t+\alpha u^p u_x+\beta u_{xxt}$ for equation (2), respectively.

Then, by using the Lie group classification method, we have

Theorem 3. *If $p>0$, α and β are arbitrary nonzero constants, then all of the geometric vector fields of the generalized regular long-wave equations (1) and (2) are given by the following cases:*

Case 1. For equation (1), there are two subcases as follows:

(I-1) If $p=1$, then the VF of equation (1) is

$$V_1=\partial_x,\quad V_2=\partial_t,\quad V_3=t\partial_t-(u+1)\partial_u. \tag{25}$$

(I-2) If $p \neq 1$ and $p > 0$, then the VF of equation (1) is

$$V_1 = \partial_x, \quad V_2 = \partial_t. \tag{26}$$

Case 2. For equation (2), the VF of this equation is

$$V_1 = \partial_x, \quad V_2 = \partial_t, \quad V_3 = pt\partial_t - u\partial_u, \tag{27}$$

where $p > 0$ is an arbitrary real constant.

Proof. Firstly, in view of $p > 0$, we have two cases to be considered only:

(1) $p = 1$;
(2) $p > 0$ and $p \neq 1$.

Then, for the coefficient functions ξ, τ and ϕ, and in view of α and β are arbitrary nonzero constants, through the symmetry condition (24), we can get the following determining equations:

$$\xi = \xi(x), \quad \tau = \tau(t), \quad \phi = a(t)u + b(t). \tag{28}$$

Thirdly, in terms of the above symmetry condition, we discuss the cases (1) and (2) respectively, where the repeat and trivial cases are omitted, then all of the vector fields (25)–(27) of equations (1) and (2) are obtained.

Furthermore, we have to check that the vector fields are closed under the Lie bracket, respectively. Taking the vector field (25) for an example, we have

$$[V_1, V_1] = [V_2, V_2] = [V_3, V_3] = 0, \quad [V_1, V_2] = -[V_2, V_1] = 0$$
$$[V_1, V_3] = -[V_3, V_1] = 0, \quad [V_2, V_3] = -[V_3, V_2] = -pV_2.$$

Obviously, the group classifications are complete with respect to the parameters of equations (1) and (2). The proof is completed.

Based on the complete group classifications of equations (1) and (2), we can give the VFs of the specific RLW equations straightforwardly. Now, as examples, we consider equations (3)–(8).

If $p = 1$, then equation (1) reduces to equation (3). Thus, the VF of this equation is (25).

If $p = 2$, then equation (1) reduces to equation (4). Thus, the VF of this equation is (26).

On the other hand, if $p = 1$, then equation (2) reduces to equation (5). Thus, we get the VF of equation (5) as follows

$$V_1 = \partial_x, \quad V_2 = \partial_t, \quad V_3 = t\partial_t - u\partial_u. \tag{29}$$

If $p = 2$, then equation (2) reduces to equation (6). Thus, we get the VF of equation (6) as follows:

$$V_1 = \partial_x, \quad V_2 = \partial_t, \quad V_3 = 2t\partial_t - u\partial_u. \tag{30}$$

If $p = 1/2$, then the VF of equation (7) is (26) as well. In view of (27), we get the VF of equation (8) as follows:

$$V_1 = \partial_x, \quad V_2 = \partial_t, \quad V_3 = t\partial_t - 2u\partial_u. \tag{31}$$

We note that (29) is the same as the result given in [4,5] (see [4], p. 194 and [5], p. 804, respectively). Similarly, the VFs of the other RLW types of equations can be provided in terms of Theorem 3.

Now, we consider the symmetry-based integrability of the nonlinear wave equations. In view of Theorem 3, we have

Corollary 1. *Suppose that $p > 0$, α and β are arbitrary nonzero constants, if equation (1) is integrable, then $p = 1$.*

In other words, if $p \neq 1$, then the general nonlinear RLW type of equation (1) is non-integrable. From the above discussion, we can see that equations (4) and (7) are non-integrable generally.

4 Symmetry reductions and exact solutions

Firstly, we note that these equations have constant solution $u(x,t) = c$, for c is a constant. In what follows, we will consider the non-constant solutions to these nonlinear equations.

4.1 The traveling wave solutions to the GRLW equations (1) and (2)

For $V = vV_1 + V_2$ ($v \neq 0$ denotes the wave speed), we have

$$u = f(\xi), \tag{32}$$

where $\xi = x - vt$. Substituting (32) into equation (1), we reduce the equation to the following ordinary differential equation (ODE):

$$\beta v f''' - \alpha f^p f' + (v - \alpha) f' = 0, \tag{33}$$

where $f' = df/d\xi$, $p > 0$.

Now, we solve this equation by the dynamical system method. In view of (33), we have

$$\beta v f'' - \frac{\alpha}{p+1} f^{p+1} + (v - \alpha) f + k = 0,$$

where k is an integration constant.

Let $f' = y$, then we get the following planar dynamical system

$$\frac{df}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\alpha}{\beta v(p+1)} f^{p+1} + \frac{\alpha - v}{\beta v} f + k.$$

Solving this system, we have $y = \sqrt{\frac{2\alpha}{\beta v(p+1)(p+2)} f^{p+2} + \frac{\alpha - v}{\beta v} f^2 + kf + c_1}$. Thus, we get that $\xi = \int \frac{1}{y(f)} df + c_2$, for k and c_i $(i = 1, 2)$ are constants. From this we can get that the general solution to equation (33) is $f = f(\xi, k, c_1, c_2)$. Thus, we obtain the exact solution to equation (1) as follows

$$u(x,t) = f(x - vt, k, c_1, c_2), \tag{34}$$

where $v \neq 0$ denotes the wave speed, k and c_i $(i = 1, 2)$ are arbitrary constants. This is the traveling wave solution to equation (1).

Similarly, for equation (2), by the traveling wave transformation (32), we have the following ODE:

$$\beta v f''' - \alpha f^p f' + v f' = 0, \tag{35}$$

where $f' = df/d\xi$, $p > 0$, and $v \neq 0$ is the wave speed. Similar to equation (33), we can solve this equation by using the dynamical system method, and obtain the exact traveling wave solutions to equation (2). The details are omitted here.

In particular, if $p = 1, 2$, then we can get the traveling wave solutions to equations (3)–(6), respectively. For the concrete parameters, the exact explicit traveling wave solutions to the equations are obtained in terms of the elliptic functions [23,24].

4.2 Similarity reductions and exact solutions to the RLW equations (2), (5) and (6)

(i) For V_3, we have

$$u = t^{-\frac{1}{p}} f(\xi), \tag{36}$$

where $\xi = x$. Substituting (36) into equation (2), we reduce this equation to the following ODE:

$$\beta f'' - \alpha p f^p f' + f = 0, \tag{37}$$

where $f' = df/d\xi$, $p > 0$. This is a nonlinear ODE, we cannot get the exact explicit solutions generally for the arbitrary parameter p. In particular, if $p = 1$, then we reduce equation (5) to the following ODE:

$$\beta f'' - \alpha f f' + f = 0. \tag{38}$$

If $p = 2$, then we reduce equation (6) to the following ODE:

$$\beta f'' - 2\alpha f^2 f' + f = 0. \tag{39}$$

Let $f' = y$, then we transform equation (38) into the following planar dynamical system

$$\frac{df}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\alpha}{\beta} f y - \frac{1}{\beta} f. \tag{40}$$

This system has a first integral

$$H(f, y) = 2\alpha\beta y + 2\beta \log|\alpha y - 1| - \alpha^2 f^2 = h, \tag{41}$$

where h is an arbitrary constant. Then, we can analyze the dynamical behavior and get the exact solutions by using the dynamical system method [14,15,25], the details are omitted. On the other hand, we can consider the general solution to equation (38) directly (see Subsection 4.3).

(ii) For $V = cV_1 + V_3$, we have

$$u = t^{-\frac{1}{p}} f(\xi), \tag{42}$$

where $\xi = x - \frac{c}{p} \log t$. Substituting (42) into equation (2), we reduce this equation to the following ODE:

$$\beta c f''' + \beta f'' - \alpha p f^p f' + c f' + f = 0, \tag{43}$$

where $f' = df/d\xi, c \neq 0$ is an arbitrary constant. In particular, if $p = 1$, then we reduce equation (5) to the following ODE:

$$\beta c f''' + \beta f'' - \alpha f f' + c f' + f = 0. \tag{44}$$

If $p = 2$, then we reduce equation (6) to the following ODE:

$$\beta c f''' + \beta f'' - 2\alpha f^2 f' + c f' + f = 0. \tag{45}$$

4.3 Similarity reductions and exact solutions to the RLW equation (3)

(i) For V_3, we have

$$u = t^{-1} f(\xi) - 1, \tag{46}$$

where $\xi = x$. Substituting (46) into equation (3), we reduce this equation to the following ODE:

$$\beta f'' - \alpha f f' + f = 0, \tag{47}$$

where $f' = df/d\xi$. Clearly, this second-order nonlinear ODE is equivalent to the following planar dynamical system

$$\frac{df}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{\alpha}{\beta} f y - \frac{1}{\beta} f. \tag{48}$$

Solving the system, we get the following general solution to equation (47)

$$\int \frac{df}{W\left(c_1 e^{\frac{\alpha^2}{2\beta} f^2}\right) + 1} = \frac{1}{\alpha}\xi + c_2, \tag{49}$$

where c_1 and c_2 are arbitrary constants, $W = W(x)$ is the Lambert W function given by $W(x)e^{W(x)} = x$ (see [26] and the references therein for detail).

In view of (49), we have $f = f(\xi, c_1, c_2)$. Thus, we obtain the exact solution to equation (3) as follows

$$u(x, t) = t^{-1} f(x, c_1, c_2) - 1, \tag{50}$$

where c_1 and c_2 are arbitrary constants, $f = f(\xi, c_1, c_2)$ is given by (49).

Similarly, we can get the exact solutions to the other equations by the procedure.

(ii) For $V = cV_1 + V_3$, we have

$$u = t^{-1} f(\xi) - 1, \tag{51}$$

where $\xi = x - c\log t$. Substituting (51) into equation (3), we reduce this equation to the following ODE:

$$c\beta f''' + \beta f'' - \alpha f f' + cf' + f = 0, \tag{52}$$

where $f' = df/d\xi$, $c \neq 0$ is an arbitrary constant.

From the above discussions, we have the following result:

Proposition 1. *From the Painlevé test and Lie symmetry analysis points of view, the general nonlinear RLW equation (1) has only the traveling wave solutions, in addition to the constant solution $u(x,t) = c$, for c is a constant.*

However, for the other RLW types of equations, by the Painlevé test and symmetry analysis method, we can get the other forms of exact solutions even for the non-integrable cases.

5 Exact explicit solutions to the RLW types of equations

In this section, we deal with some particular exact explicit solutions to the general RLW equations by the Φ-expansion method [27,28] based on the homogeneous balance principle (HBP).

Similar to the Painlevé test in Section 2, we suppose that $p > 0$ is an integer. In view of equations (33) and (35), integrating them with respect to ξ yields

$$\beta v f'' - \frac{\alpha}{p+1} f^{p+1} + (v - \alpha) f + k = 0, \tag{53}$$

and

$$\beta v f'' - \frac{\alpha}{p+1} f^{p+1} + v f + l = 0, \tag{54}$$

respectively, where k and l are integration constants to be determined.

Suppose that the solutions to equations (53) and (54) can be expressed by a polynomial with respect to Φ as follows:

$$f(\xi) = \sum_{j=0}^{m} a_j \Phi^j, \quad \Phi = \frac{G'}{G}, \tag{55}$$

where $G = G(\xi)$ satisfies the second-order linear ODE in the following form

$$G'' + \lambda G' + \mu G = 0, \tag{56}$$

here λ and μ are arbitrary real numbers.

Then, by the homogeneous balance method, we have

$$m = \frac{2}{p}. \tag{57}$$

It is the same as (11) given by the Painlevé test in Section 2. So, we have the two cases as follows:

(i) If $p = 1$, then $m = 2$.

(ii) If $p = 2$, then $m = 1$.

Furthermore, the two cases are discussed one by one, we can get the explicit traveling wave solutions to the RLW equations through the Φ-expansion procedure, respectively. Taking equation (53) for example, we have the following results:

Case (i) In this case, equation (53) becomes the following ODE:

$$\beta v f'' - \frac{\alpha}{2} f^2 + (v - \alpha) f + k = 0, \tag{58}$$

where $f' = df/d\xi$.

Clearly, this equation has a solution

$$f(\xi) = a_2\Phi^2 + a_1\Phi + a_0, \quad a_2 \neq 0, \tag{59}$$

where Φ is given by (55) and (56), the constants v, k and a_i $(i = 0, 1, 2)$ are to be determined later.

Substituting (59) into equation (58), and comparing the coefficients, we have

$$\begin{aligned} a_1 &= \frac{12\beta\lambda(a_0+1)}{\beta\lambda^2 + 8\beta\mu + 1}, \quad a_2 = \frac{12\beta(a_0+1)}{\beta\lambda^2 + 8\beta\mu + 1}, \\ v &= \frac{\alpha(a_0+1)}{\beta\lambda^2 + 8\beta\mu + 1}, \\ k &= \frac{-12\alpha\beta^2\mu(\lambda^2+2\mu)^2(a_0+1)^2}{(\beta\lambda^2 + 8\beta\mu + 1)^2} - \frac{\alpha a_0(a_0+1)}{\beta\lambda^2 + 8\beta\mu + 1} \\ &\quad + \frac{1}{2}\alpha a_0^2 + \alpha a_0, \end{aligned} \tag{60}$$

where λ, μ and a_0 are arbitrary constants, such that $\beta\lambda^2 + 8\beta\mu + 1 \neq 0$.

By using (60), the expression (59) can be written as

$$f(\xi) = a_2\left(\frac{G'}{G}\right)^2 + a_1\left(\frac{G'}{G}\right) + a_0. \tag{61}$$

Equation (61) gives the formula of the solutions to equation (58) provided that the integration constant k in equation (58) is taken as that in (60).

Then, in view of equation (56), we have three types of exact explicit traveling wave solutions to equation (3) as follows:

When $\lambda^2 - 4\mu > 0$, the traveling wave solution to equation (3) is

$$\begin{aligned} u(x,t) &= \frac{3\beta(\lambda^2-4\mu)(a_0+1)}{\beta\lambda^2 + 8\beta\mu + 1} \\ &\quad \times \left[\frac{c_1 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt) + c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)}{c_2 \sinh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt) + c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)}\right]^2 \\ &\quad - \frac{3\beta\lambda^2(a_0+1)}{\beta\lambda^2 + 8\beta\mu + 1} + a_0, \end{aligned} \tag{62}$$

where v is given in (60), a_0, c_1 and c_2 are arbitrary constants.

When $\lambda^2-4\mu<0$, the traveling wave solution to equation (3) is

$$u(x,t)=\frac{3\beta(4\mu-\lambda^2)(a_0+1)}{\beta\lambda^2+8\beta\mu+1}\times\left[\frac{c_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)-c_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)}{c_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)+c_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)}\right]^2-\frac{3\beta\lambda^2(a_0+1)}{\beta\lambda^2+8\beta\mu+1}+a_0, \tag{63}$$

where v is given in (60), a_0, c_1 and c_2 are arbitrary constants.

When $\lambda^2-4\mu=0$, the traveling wave solution to equation (3) is

$$u(x,t)=\frac{12\beta(a_0+1)c_2^2}{(\beta\lambda^2+8\beta\mu+1)[c_1+c_2(x-vt)]^2}-\frac{3\beta\lambda^2(a_0+1)}{\beta\lambda^2+8\beta\mu+1}+a_0, \tag{64}$$

where v is given in (60), a_0, c_1 and c_2 are arbitrary constants.

Case (ii) In this case, equation (53) becomes the following ODE:

$$\beta v f''-\frac{\alpha}{3}f^3+(v-\alpha)f+k=0, \tag{65}$$

where $f'=df/d\xi$.

In view of (ii), this equation has a solution

$$f(\xi)=a_1\Phi+a_0,\quad a_1\neq 0, \tag{66}$$

where Φ is given by (55) and (56), the constants v, k and a_i $(i=0,1,2)$ are to be determined later.

Substituting (66) into equation (65), and comparing the coefficients, we have

$$a_0=\pm\sqrt{\frac{3\beta\lambda^2}{-\beta\lambda^2+4\beta\mu+2}},\quad a_1=\pm\frac{2}{\lambda}\sqrt{\frac{3\beta\lambda^2}{-\beta\lambda^2+4\beta\mu+2}},$$

$$v=\frac{2\alpha}{-\beta\lambda^2+4\beta\mu+2},\quad k=0, \tag{67}$$

where α, β, λ and μ are arbitrary constants, such that $\beta\lambda^2-4\beta\mu-2\neq 0$.

By using (67), the expression (66) can be written as

$$f(\xi)=a_1\left(\frac{G'}{G}\right)+a_0. \tag{68}$$

This gives the formula of the solutions to equation (65) provided that the integration constant k in equation (65) is taken as that in (67).

Then, in view of equation (56), we have three types of exact explicit traveling wave solutions to equation (4) as follows:

When $\lambda^2-4\mu>0$, the traveling wave solution to equation (4) is

$$u_{1,2}(x,t)=\pm\sqrt{\frac{3\beta(\lambda^2-4\mu)}{-\beta\lambda^2+4\beta\mu+2}}\times\frac{c_1\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)+c_2\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)}{c_2\sinh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)+c_1\cosh\frac{\sqrt{\lambda^2-4\mu}}{2}(x-vt)}, \tag{69}$$

where the wave speed v is given in (67), a_0, c_1 and c_2 are arbitrary constants.

When $\lambda^2-4\mu<0$, the traveling wave solution to equation (4) is

$$u_{3,4}(x,t)=\pm\sqrt{\frac{3\beta(4\mu-\lambda^2)}{-\beta\lambda^2+4\beta\mu+2}}\times\frac{-c_1\sin\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)+c_2\cos\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)}{c_1\cos\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)+c_2\sin\frac{\sqrt{4\mu-\lambda^2}}{2}(x-vt)}, \tag{70}$$

where the wave speed v is given in (67), a_0, c_1 and c_2 are arbitrary constants.

When $\lambda^2-4\mu=0$, the traveling wave solution to equation (4) is

$$u_{5,6}(x,t)=\pm 2\sqrt{\frac{3\beta}{-\beta\lambda^2+4\beta\mu+2}}\frac{c_2}{c_1+c_2(x-vt)}, \tag{71}$$

where the wave speed v is given in (67), a_0, c_1 and c_2 are arbitrary constants.

Similarly, we can consider equation (54), then the explicit traveling wave solutions to the RLW equation (6) are provided successively.

Remark 1. From our previous discussion, we can see that the necessary condition of the existence of the Φ-expansion solution coincides with the necessary condition of the Painlevé integrable given in Section 2. Generally, if a nonlinear PDE has the Φ-expansion solution, then it is called Φ-integrable [27]. So, we can say that the above equations are Φ-integrable.

6 Conclusion and further discussion

In the current paper, the general nonlinear RLW types of equations are investigated by the combination of Painlevé analysis and Lie group classification method. The conditional PP, BTs, symmetries and exact solutions to the equations are obtained, and the integrability of the nonlinear PDEs is investigated based on the Lie group classification and Φ-expansion method for the first time.

Then, some explicit solutions to the nonlinear PDEs are provided through the so called Φ-expansion procedure. Furthermore, we find that the combination of Painlevé test, Lie group classification and the Φ-expansion method is compatible for dealing with integrability and exact solutions to the nonlinear PDEs.

On the other hand, by the symmetry integration of ODEs, we can reduce the ODEs to the more lower-order ODEs if its symmetries are given. Now, we consider equation (37) as an example.

Clearly, equation (37) has a symmetry $V = \partial/\partial\xi$, such that $V\xi = 1$ and $Vf = 0$. Setting $f = y$, $\xi = w$, and substituting it into (37), we have

$$-\beta w_{yy} - \alpha p y^p w_y^2 + y w_y^3 = 0, \tag{72}$$

where $w_y = dw/dy$. Let $w_y = z$, then we have

$$\frac{dz}{dy} = -\frac{\alpha p}{\beta} y^p z^2 + \frac{1}{\beta} y z^3, \tag{73}$$

i.e.,

$$\frac{d\eta}{dy} = \frac{\alpha p}{\beta} y^p - \frac{1}{\beta\eta} y, \tag{74}$$

where $\eta = z^{-1}$. So, we reduce the nonlinear second-order ODE (37) to the first-order ODE (73) by the symmetry integration method.

Through the same procedure, we can reduce equations (33), (35), (43) and (52) to second-order ODEs, respectively. The details are omitted in the present paper.

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Author contribution statement

All the authors were involved in the preparation of the manuscript. All the authors have read and approved the final manuscript.

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