Eur. Phys. J. B (2018) 91: 26
https://doi.org/10.1140/epjb/e2017-80165-9

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Stochastic resonance in a harmonic oscillator subject to random mass and periodically modulated noise

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Received 21 March 2017 / Received in final form 25 July 2017
Published online 7 February 2018 –

Abstract. In many practical systems, the periodic driven force and noise are introduced multiplicatively. However, the corresponding researches only focus on the first order moment of the system and its stochastic resonance phenomena. This paper investigates a harmonic oscillator subject to random mass and periodically modulated noise. Using Shapiro-Loginov formula and the Laplace transformation technique, the analytic expressions of the first-order and second-order moment are obtained. According to the analytic expressions, we find that although the first-order moment is always zero but second-order moment is periodic which is different from other harmonic oscillators investigated. Furthermore, we find the amplitude and average of second-order moment have a non-monotonic behavior on the frequency of the input signal, noise parameters and other system parameters. Finally, the numerical simulations are presented to verify the analytical results.

1 Introduction

The concept of stochastic resonance (SR) which an intermediate noise intensity can lead to the maximum response of a stochastic system was introduced by Benzi et al. in 1981 [1]. Due to its wide applications in physics, biology, chemistry and engineering [2–7], it has attracted enormous attention and has been studied extensively in recent decades. And SR phenomenon has been found in various systems. Gitterman used the term "stochastic resonance" in the wide sense, meaning the nonmonotonic (resonance) dependence of the output signal or some function of it (moments, autocorrelation functions, power spectrum or signal-to-noise ratio) on the characteristics of the noise (the noise amplitude or correlation time) [8] and this phenomenon is often called generalized stochastic resonance [9].

The early researches focused on non-linear systems with additive white noises [10,11]. However, colored noises widely exist in various systems, especially biological system. Recently researches show that SR can occur in linear systems with multiplicative colored noise [12–23].

The harmonic oscillator is a simple system and widely used in physics. A particle of mass $m > 0$ moving in a parabolic potential $U(x,t) = \omega^2 x(t)/2$ driven by a periodic force can be described by following equation:

$$m\frac{\mathrm{d}^2 x\left(t\right)}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x\left(t\right)}{\mathrm{d}t} + \omega^2 x\left(t\right) = A\sin\Omega t, \tag{1}$$

where $x(t)$ is the displacement of the particle, $\gamma > 0$ is the friction constant and $A\sin\Omega t$ is the periodic driven force. For non-zero temperatures, the particle is collided by the surrounding molecules, so equation (1) needs to be supplemented by thermal noise $\eta(t)$.

$$m\frac{\mathrm{d}^2 x\left(t\right)}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x\left(t\right)}{\mathrm{d}t} + \omega^2 x\left(t\right) = A\sin\Omega t + \eta\left(t\right). \tag{2}$$

However, in many systems, the surrounding molecules not only collide with the particle but also adhere to the particle randomly. This phenomenon has been found in various systems, especially biological and chemistry systems [24–30]. In these systems, the mass of particle is random. Taking the fluctuation of mass into account, we arrive at the following equation

$$\left(m + \xi\left(t\right)\right)\frac{\mathrm{d}^2 x\left(t\right)}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}x\left(t\right)}{\mathrm{d}t} + \omega^2 x(t) = A\sin\Omega t + \eta(t). \tag{3}$$

Gitterman considered this system and analyzed the first moment and found the stochastic resonance phenomenon occurred [31–33]. Particularly, an RLC electrical circuit subject to a voltage $V(t)$ with a fluctuation inductance L can be regarded as a random mass system [12]

$$\left[L + \xi(t)\right]\frac{\mathrm{d}^2 J}{\mathrm{d}t^2} + R\frac{\mathrm{d}J}{\mathrm{d}t} + \frac{1}{C}J = \frac{\mathrm{d}V}{\mathrm{d}t}. \tag{4}$$

Considering the input periodic signal (driven force), the signal and noise may act on each other multiplicatively. It is introduced by Dykman et al. when they studied SR

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in an asymmetric bistable potential [34]. This noise which is called periodically modulated noise, is not uncommon, for example, at the output of any amplifier whose amplification factor varies periodically with time. Periodically modulated noise exists in many practical system and the SR in these systems have been studied [13,16,35].

Considering an RLC electrical circuit given by equation (4), if the input signal $V(t)$ is multiplied by a noise, the system is a harmonic oscillator subject to a random mass and periodically modulated noise. The previous works on SR in harmonic oscillator subject to random mass and periodically modulated noise only considered the first-order moment. However, the output of the system is a stochastic process, to study the behavior of a stochastic process, only the first-order moment is not enough. In this paper, we focus on studying the SR phenomenon of the second-order moment in a harmonic oscillator subject to random mass, periodically modulated noise and an additive unmodulated noise. The structure of this paper is organized as follows. In Section 2, the model of the system is introduced and we provide the analytic expressions of the first-order and second-order moments. In Section 3, the SR phenomenon for the second-order moment is discussed. In Section 4, we verify the analytical result by numerical simulations. Finally, conclusions are drawn in Section 5.

2 Analytic solution

2.1 Model

We consider a harmonic oscillator subject to random mass, periodically modulated noise and an additive unmodulated noise as follows:

$$(m+\xi(t))\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\gamma\frac{\mathrm{d}x}{\mathrm{d}t}+\omega^2x=A\cos\Omega t\cdot\psi(t)+\eta(t),\quad (5)$$

where $A=0$ and $\Omega=0$ are the amplitude and frequency of the periodic signal, $m>0$ is the mass of the oscillator, $\gamma>0$ is the friction constant. $\xi(t)$, $\psi(t)$, $\eta(t)$ are noises and they are independent. In this paper, all noises are modeled as asymmetric dichotomous noise (random telegraph noise) which is Markov processes and can be reduced to Gaussian white noise and white shot noise [36]. $\xi(t)$ is the fluctuation of the mass and takes two values $A_1>0$ and $-B_1<0$. To make sure the particle mass is positive, we assume $m>B_1$. p_1 is the transition rate from A_1 to $-B_1$ and q_1 is the transition rate from $-B_1$ to A_1. $\psi(t)$ is the periodically modulated noise which jumps between two values $A_2>0$ and $-B_2<0$. p_2 is the transition rate from A_2 to $-B_2$ and q_2 is the transition rate from $-B_2$ to A_2. $\eta(t)$ is the additive noise which takes two value $A_3>0$ and $-B_3<0$. The transition rate from A_3 to $-B_3$ is p_3 and the reverse rate is q_3. We denote

$$\begin{aligned}
&D_1=A_1B_1,\lambda_1=p_1+q_1,\Delta_1=A_1-B_1\\
&D_2=A_2B_2,\lambda_2=p_2+q_2,\Delta_2=A_2-B_2\\
&D_3=A_3B_3,\lambda_3=p_3+q_3,\Delta_3=A_3-B_3,
\end{aligned}$$

the statistical properties of the noises are

$$\begin{aligned}
\langle\xi(t)\rangle=0,\langle\psi(t)\rangle&=0,\langle\eta(t)\rangle=0,\\
\langle\xi(t)\xi(s)\rangle&=D_1\exp(-\lambda_1|t-s|)\\
\langle\psi(t)\psi(s)\rangle&=D_2\exp(-\lambda_2|t-s|)\\
\langle\eta(t)\eta(s)\rangle&=D_3\exp(-\lambda_3|t-s|).
\end{aligned}$$

To make $\langle\xi(t)\rangle$=0, $\langle\psi(t)\rangle$=0, $\langle\eta(t)\rangle$=0, the parameters of noise satisfy the following equation

$$A_iq_i=B_ip_i,\quad i=1,2,3.$$

Noticing that $\xi(t)$, $\psi(t)$, $\eta(t)$ are independent, we have

$$\begin{aligned}
\langle\xi(t)\psi(t)\xi(s)\psi(s)\rangle&=D_1D_2\exp(-\lambda_{12}|t-s|)\\
\langle\xi(t)\eta(t)\xi(s)\eta(s)\rangle&=D_1D_3\exp(-\lambda_{13}|t-s|)\\
\lambda_{12}&=\lambda_1+\lambda_2;\lambda_{13}=\lambda_1+\lambda_3.
\end{aligned}$$

2.2 First-order moment

First, we introduce two properties of dichotomous noises:

– (Shapiro-Loginov formula) [37]. Assume $\xi(t)$ and $x(t)$ are two stochastic processes. $\xi(t)$ is a stochastic process which satisfies

$$\langle\xi(t)\rangle=0;\langle\xi(t)\xi(s)\rangle=D_\xi\exp(-\lambda|t-s|),$$

and $x(t)$ is some function of $\xi(t)$, then we have the following Shapiro-Loginov formula

$$\left\langle\xi(t)\frac{\mathrm{d}^nx(t)}{\mathrm{d}t^n}\right\rangle=\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda\right)^n\langle\xi(t)x(t)\rangle;$$

– assume $\xi(t)$ is a asymmetric dichotomous noise which takes two value $A>0$ and $-B<0$. The transition rate from A to $-B$ is p and the inverse rate is q. Then we have the following equation;

$$\left\langle\xi^2(t)x(t)\right\rangle=\Delta\langle\xi(t)x(t)\rangle+D\langle x(t)\rangle,$$

where

$$D=AB;\Delta=A-B.$$

Now according to the properties introduced above, we obtain the following equation by averaging over all realizations of the trajectory of the stochastic equation (5) and using the Shapiro-Loginov formula

$$\begin{aligned}
&m\frac{\mathrm{d}^2\langle x(t)\rangle}{\mathrm{d}t^2}+\frac{\mathrm{d}^2\langle\xi(t)x(t)\rangle}{\mathrm{d}t^2}+2\lambda_1\frac{\mathrm{d}\langle\xi(t)x(t)\rangle}{\mathrm{d}t}\\
&+\lambda_1^2\langle\xi(t)x(t)\rangle+\gamma\frac{\mathrm{d}\langle x(t)\rangle}{\mathrm{d}t}+\omega^2\langle x(t)\rangle=0.\quad (6)
\end{aligned}$$

To find $\langle\xi(t)x(t)\rangle$, we multiply equation (5) by $\xi(t)$ and average it. Then we obtain

$$\begin{aligned}(m+\Delta_1)&\frac{\mathrm{d}^2\langle\xi(t)x(t)\rangle}{\mathrm{d}t^2}\\&+(2\Delta_1\lambda_1+2m\lambda_1+\gamma)\frac{\mathrm{d}\langle\xi(t)x(t)\rangle}{\mathrm{d}t}\\&+\left[\lambda_1^2(m+\Delta_1)+\gamma\lambda_1+\omega^2\right]\langle\xi(t)x(t)\rangle\\&+D_1\frac{\mathrm{d}^2\langle x(t)\rangle}{\mathrm{d}t^2}=0.\end{aligned}\tag{7}$$

Equations (6) and (7) are the homogeneous linear differential equations of $\langle x(t)\rangle$ and $\langle\xi(t)x(t)\rangle$ with constant coefficients. By applying results in Appendix A, we obtain the steady-state solution of equations (6) and (7) as follows:

$$\langle x(t)\rangle_{st}=\langle x(t)\rangle|_{t\to\infty}=0;$$
$$\langle\xi(t)x(t)\rangle_{st}=\langle\xi(t)x(t)\rangle|_{t\to\infty}=0;$$

and the characteristic equation

$$\sum_{i=0}^{4}s_{1i}s^i=0,$$

where

$$\begin{aligned}s_{14}&=m^2+m\Delta_1-D_1;\\s_{13}&=\gamma\Delta_1+2m\gamma-2D_1\lambda_1+2m\Delta_1\lambda_1+2m^2\lambda_1;\\s_{12}&=\omega^2\Delta_1+2m\omega^2+m\Delta_1\lambda_1^2+m^2\lambda_1^2+3m\gamma\lambda_1+\gamma^2\\&\quad+2\gamma\Delta_1\lambda_1-D_1\lambda_1^2;\\s_{11}&=\gamma\omega^2+2\omega^2\Delta_1\lambda_1+2\omega^2\lambda_1+\gamma\Delta_1\lambda_1^2+\gamma m\lambda_1^2+\gamma^2\lambda_1;\\s_{10}&=\omega^2\Delta_1\lambda_1^2+\omega^2m\lambda_1^2+\omega^2\gamma\lambda_1+\omega^4.\end{aligned}$$

According to Routh-Hurwitz stability criterion, we obtain the stability condition

$$s_{13}s_{12}>s_{14}s_{11};s_{11}s_{12}s_{13}>s_{14}s_{11}^2+s_{10}s_{13}^2.$$

2.3 Second-order moment

In this part, we try to find the analytic expression of the second-order moment of $x(t)$. Firstly, we find the stochastic differential equations of $x^2(t)$; secondly, we obtain the ordinary differential equations of $\langle x^2(t)\rangle$ by using Shapiro-Loginov formula; finally, we obtain the analytic expression of $\langle x^2(t)\rangle$ by solving the differential equations. Noticing that $\langle x(t)\rangle=0$, the second-order moment $\langle x^2(t)\rangle$ is also the variance of $x(t)$.

Denote $y_1=x^2, y_2=x(\mathrm{d}x/\mathrm{d}t), y_3=(\mathrm{d}x/\mathrm{d}t)^2$, then differentiate them, we can obtain the following stochastic differential equations

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}t}&=2y_2\\(m+\xi)\frac{\mathrm{d}y_2}{\mathrm{d}t}&=(m+\xi)y_3-\gamma y_2-\omega^2y_1\\&\quad+A\cos\Omega t\cdot\psi x+\eta x\\(m+\xi)\frac{\mathrm{d}y_3}{\mathrm{d}t}&=-2\gamma y_3-2\omega^2y_2+2A\cos\Omega t\cdot\psi\frac{\mathrm{d}x}{\mathrm{d}t}+2\eta\frac{\mathrm{d}x}{\mathrm{d}t}.\end{aligned}\tag{8}$$

In a similar way as Section 2.2, we average equation (8) and equation (8) multiplied by $\xi(t)$, then denotes

$$g_1=\langle y_1\rangle, g_2=\langle y_2\rangle, g_3=\langle y_3\rangle$$
$$g_4=\langle\xi y_1\rangle, g_5=\langle\xi y_2\rangle, g_6=\langle\xi y_3\rangle,$$

the linear differential equations of $g_1\sim g_6$ can be written as

$$\begin{aligned}&\frac{\mathrm{d}g_1}{\mathrm{d}t}-2g_2=0\\&m\frac{\mathrm{d}g_2}{\mathrm{d}t}+\frac{\mathrm{d}g_5}{\mathrm{d}t}+\omega^2g_1+\gamma g_2-mg_3+\lambda_1g_5-g_6\\&\quad=A\cos\Omega t\cdot\langle\psi x\rangle+\langle\eta x\rangle\\&m\frac{\mathrm{d}g_3}{\mathrm{d}t}+\frac{\mathrm{d}g_6}{\mathrm{d}t}+2\omega^2g_2+2\gamma g_3+\lambda_1g_6\\&\quad=2A\cos\Omega t\cdot\left\langle\psi\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle+2\left\langle\eta\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle\\&\frac{\mathrm{d}g_4}{\mathrm{d}t}+\lambda_1g_4-2g_5=0,\\&D_1\frac{\mathrm{d}g_2}{\mathrm{d}t}+(m+\Delta_1)\frac{\mathrm{d}g_5}{\mathrm{d}t}-D_1g_3+\omega^2g_4+(m\lambda_1+\Delta_1\lambda_1\\&\quad+\gamma)g_5-(m+\Delta_1)g_6=A\cos\Omega t\cdot\langle\xi\psi x\rangle+\langle\xi\eta x\rangle\\&D_1\frac{\mathrm{d}g_3}{\mathrm{d}t}+(m+\Delta_1)\frac{\mathrm{d}xg_6}{\mathrm{d}t}+2\omega^2g_5+(m\lambda_1+\Delta_1\lambda_1\\&\quad+2\gamma)g_6=2A\cos\Omega t\cdot\left\langle\xi\psi\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle+2\left\langle\xi\eta\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle.\end{aligned}\tag{9}$$

To solve equation (9), we need to find $\langle\psi x\rangle$, $\langle\eta x\rangle$, $\langle\psi(dx/dt)\rangle$, $\langle\eta(dx/dt)\rangle$, $\langle\xi\psi x\rangle$, $\langle\xi\eta x\rangle$, $\langle\xi\psi(dx/dt)\rangle$ and $\langle\xi\eta(dx/dt)\rangle$. In order to calculate them, we multiply equation (5) by ψ, $\xi\psi$, η, $\xi\eta$ and average them. Then the equations of $\langle\psi x\rangle$,$\langle\xi\psi x\rangle$ and $\langle\eta x\rangle$,$\langle\xi\eta x\rangle$ are obtained by using properties of dichotomous noise.

$$\begin{aligned}&m\frac{\mathrm{d}^2\langle\psi x\rangle}{\mathrm{d}t^2}+(2m\lambda_2+\gamma)\frac{\mathrm{d}\langle\psi x\rangle}{\mathrm{d}t}\\&\quad+\left(m\lambda_2^2+\gamma\lambda_2+\omega^2\right)\langle\psi x\rangle+\frac{\mathrm{d}^2\langle\xi\psi x\rangle}{\mathrm{d}t^2}+2\lambda_{12}\frac{\mathrm{d}\langle\xi\psi x\rangle}{\mathrm{d}t}\\&\quad+\lambda_{12}^2\langle\xi\psi x\rangle=AD_2\cos\Omega t\\&D_1\frac{\mathrm{d}^2\langle\psi x\rangle}{\mathrm{d}t^2}+2D_1\lambda_2\frac{\mathrm{d}\langle\psi x\rangle}{\mathrm{d}t}+D_1\lambda_2^2\langle\psi x\rangle+(m+\Delta_1)\\&\quad\times\frac{\mathrm{d}^2\langle\xi\psi x\rangle}{\mathrm{d}t^2}+\left[2\lambda_{12}(m+\Delta_1)+\gamma\right]\frac{\mathrm{d}\langle\xi\psi x\rangle}{\mathrm{d}t}\\&\quad+\left[\lambda_{12}^2(m+\Delta_1)+\gamma\lambda_{12}+\omega^2\right]\langle\xi\psi x\rangle=0,\end{aligned}\tag{10}$$

and

$$\begin{aligned}
&m\frac{\mathrm{d}^2\langle\eta x\rangle}{\mathrm{d}t^2}+(2m\lambda_3+\gamma)\frac{\mathrm{d}\langle\eta x\rangle}{\mathrm{d}t}\\
&+\left(m\lambda_3^2+\gamma\lambda_3+\omega^2\right)\langle\eta x\rangle+\frac{\mathrm{d}^2\langle\xi\eta x\rangle}{\mathrm{d}t^2}+2\lambda_{13}\frac{\mathrm{d}\langle\xi\eta x\rangle}{\mathrm{d}t}\\
&\quad+\lambda_{13}^2\langle\xi\eta x\rangle=D_3\\
&D_1\frac{\mathrm{d}^2\langle\eta x\rangle}{\mathrm{d}t^2}+2D_1\lambda_3\frac{\mathrm{d}\langle\eta x\rangle}{\mathrm{d}t}+D_1\lambda_3^2\langle\eta x\rangle\\
&\quad+(m+\Delta_1)\frac{\mathrm{d}^2\langle\xi\eta x\rangle}{\mathrm{d}t^2}+\left[2\lambda_{13}(m+\Delta_1)+\gamma\right]\frac{\mathrm{d}\langle\xi\eta x\rangle}{\mathrm{d}t}\\
&\quad+\left[\lambda_{13}^2(m+\Delta_1)+\gamma\lambda_{13}+\omega^2\right]\langle\xi\eta x\rangle=0.
\end{aligned}\tag{11}$$

By applying the result in Appendix A, the analytic expressions of the steady-state solutions of equations (10) and (11) can be written as following

$$\begin{aligned}
&\langle\psi x\rangle_{st}=R_1\cos(\Omega t+\phi_1);\langle\xi\psi x\rangle_{st}=R_2\cos(\Omega t+\phi_2)\\
&\langle\eta x\rangle_{st}=\frac{b_{13}b_{22}-b_{12}b_{23}}{b_{11}b_{22}-b_{12}b_{21}};\langle\xi\eta x\rangle_{st}=\frac{b_{11}b_{23}-b_{13}b_{21}}{b_{11}b_{22}-b_{12}b_{21}},
\end{aligned}$$

where

$$\begin{aligned}
R_1&=|H_{\psi1}(j\Omega)|,\phi_1=\arg(H_{\psi1}(j\Omega))\\
R_2&=|H_{\psi2}(j\Omega)|,\phi_2=\arg(H_{\psi2}(j\Omega)),
\end{aligned}$$

$$H_{\psi1}(s)=\frac{a_{13}a_{22}-a_{12}a_{23}}{a_{11}a_{22}-a_{12}a_{21}},H_{\psi2}(s)=\frac{a_{11}a_{23}-a_{13}a_{21}}{a_{11}a_{22}-a_{12}a_{21}},$$

$$\begin{aligned}
a_{11}&=ms^2+(2m\lambda_2+\gamma)s+\left(m\lambda_2^2+\gamma\lambda_2+\omega^2\right)\\
a_{12}&=s^2+2\lambda_{12}s+\lambda_{12}^2\\
a_{13}&=AD_2\\
a_{21}&=D_1s^2+2D_1\lambda_2s+D_1\lambda_2^2\\
a_{22}&=(m+\Delta_1)s^2+\left[2\lambda_{12}(m+\Delta_1)+\gamma\right]s\\
&\quad+\left[\lambda_{12}^2(m+\Delta_1)+\gamma\lambda_{12}+\omega^2\right]\\
a_{23}&=0\\
b_{11}&=m\lambda_3^2+\gamma\lambda_3+\omega^2,b_{12}=\lambda_{13}^2,b_{13}=D_3\\
b_{21}&=D_1,b_{22}=\lambda_{13}^2(m+\Delta_1)+\gamma\lambda_{13}+\omega^2,b_{23}=0,
\end{aligned}$$

and the stability condition

$$s_{i3}s_{i2}>s_{i4}s_{i1};s_{i1}s_{i2}s_{i3}>s_{i4}s_{i1}^2+s_{i0}s_{i3}^2;i=2,3,$$

where

$$\begin{aligned}
s_{i4}&=m^2+m\Delta_1-D_1;\\
s_{i3}&=\gamma\Delta_1+2m\gamma+2m\Delta_1\lambda_i+2m^2\lambda_i-2D_1\lambda_{1i}-2D_1\lambda_i\\
&\quad+2m\Delta_1\lambda_{1i}+2m^2\lambda_{1i};\\
s_{i2}&=(\gamma+2(\Delta_1+m)\lambda_{1i})(\gamma+2\lambda_im)\\
&\quad+(\Delta_1+m)\left(m\lambda_i^2+\gamma\lambda_i+\omega^2\right)\\
&\quad+m\left((\Delta_1+m)\lambda_{1i}^2+\gamma\lambda_{1i}+\omega^2\right)\\
&\quad-D_1\lambda_i^2-D_1\lambda_{1i}^2-4D_1\lambda_i\lambda_{1i};\\
s_{i1}&=(\gamma+2\lambda_{1i}(\Delta_1+m))\left(m\lambda_i^2+\gamma\lambda_i+\omega^2\right)\\
&\quad+(\gamma+2\lambda_im)\left((\Delta_1+m)\lambda_{1i}^2+\gamma\lambda_{1i}+\omega^2\right)\\
&\quad-2D_1\lambda_i\lambda_{1i}^2-2D_1\lambda_i^2\lambda_{1i};\\
s_{i0}&=\left((\Delta_1+m)\lambda_{1i}^2+\gamma\lambda_{1i}+\omega^2\right)\left(m\lambda_i^2+\gamma\lambda_i+\omega^2\right)\\
&\quad-D_1\lambda_i^2\lambda_{1i}^2.
\end{aligned}$$

Furthermore, by using Shapiro-Loginov formula, we obtain

$$\begin{aligned}
\left\langle\psi\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle_{st}&=R_3\sin(\Omega t+\phi_3);\\
\left\langle\xi\psi\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle_{st}&=R_4\sin(\Omega t+\phi_4);\\
\left\langle\eta\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle_{st}&=\lambda_3\frac{b_{13}b_{22}-b_{12}b_{23}}{b_{11}b_{22}-b_{12}b_{21}};\\
\left\langle\xi\eta\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle_{st}&=\lambda_{13}\frac{b_{11}b_{23}-b_{13}b_{21}}{b_{11}b_{22}-b_{12}b_{21}},
\end{aligned}$$

where

$$\begin{aligned}
&R_3=R_1\sqrt{\Omega^2+\lambda_2^2},R_4=R_2\sqrt{\Omega^2+\lambda_{12}^2}\\
&\phi_3=\phi_1+\arctan\left(-\tfrac{\lambda_2}{\Omega}\right)+\pi\\
&\phi_4=\phi_2+\arctan\left(-\tfrac{\lambda_{12}}{\Omega}\right)+\pi.
\end{aligned}$$

Now all of the right of equation (9) are calculated, and can be written as

$$\begin{pmatrix}u_1\\ \vdots\\ u_6\end{pmatrix}\cos2\Omega t+\begin{pmatrix}v_1\\ \vdots\\ v_6\end{pmatrix}\sin2\Omega t+\begin{pmatrix}w_{11}\\ \vdots\\ w_{61}\end{pmatrix}+\begin{pmatrix}w_{12}\\ \vdots\\ w_{62}\end{pmatrix},$$

where

$$\begin{aligned}
u_1&=0;v_1=0;w_{11}=0;w_{12}=0;\\
u_2&=\frac{1}{2}AR_1\cos\phi_1;v_2=-\frac{1}{2}AR_1\sin\phi_1;w_{21}=u_2;\\
w_{22}&=\langle\eta x\rangle;\\
u_3&=AR_3\sin\phi_3;v_3=AR_3\cos\phi_3;w_{31}=u_3;\\
w_{32}&=2\left\langle\eta\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle;\\
u_4&=0;v_4=0;w_{41}=0;w_{42}=0;u_5=\frac{1}{2}AR_2\cos\phi_2;\\
v_5&=-\frac{1}{2}AR_2\sin\phi_2;\\
w_{51}&=u_5;w_{52}=\langle\xi\eta x\rangle;\\
u_6&=AR_4\sin\phi_4;v_6=AR_4\cos\phi_4;w_{61}=u_6;\\
w_{62}&=2\left\langle\xi\eta\frac{\mathrm{d}x}{\mathrm{d}t}\right\rangle.
\end{aligned}$$

Now solve equation (9) by applying the method given in Appendix A, we obtain the analytic steady-state expression of the second-order moment of x.

$$\left\langle x^2\right\rangle_{st}=\left\langle x^2\right\rangle|_{t\to\infty}=R\sin(2\Omega t+\theta)+\alpha_3,\tag{12}$$

where

$$R = \sqrt{(\alpha_2 \cos\beta_2 - \alpha_1 \sin\beta_1)^2 + (\alpha_1 \cos\beta_1 + \alpha_2 \sin\beta_2)^2}$$
$$\theta = \begin{cases} \arctan\left(\frac{(\alpha_1 \cos\beta_1 + \alpha_2 \sin\beta_2)}{(\alpha_2 \cos\beta_2 - \alpha_1 \sin\beta_1)}\right) & \alpha_2 \cos\beta_2 - \alpha_1 \sin\beta_1 \geq 0 \\ \arctan\left(\frac{(\alpha_1 \cos\beta_1 + \alpha_2 \sin\beta_2)}{(\alpha_2 \cos\beta_2 - \alpha_1 \sin\beta_1)}\right) + \pi & \alpha_2 \cos\beta_2 - \alpha_1 \sin\beta_1 < 0 \end{cases}$$

$$\alpha_1 = |H_{x1}(2j\Omega)|; \beta_1 = \arg(H_{x1}(2j\Omega))$$
$$\alpha_2 = |H_{x2}(2j\Omega)|; \beta_2 = \arg(H_{x2}(2j\Omega))$$
$$\alpha_3 = \det(D_{1w}) / \det(D),$$

$$H_{x1}(s) = \det(C_{1u}) / \det(C)$$
$$H_{x2}(s) = \det(C_{1v}) / \det(C),$$

where C_{iz} and D_{iz}, $z = u, v$ are the matrix formed by replacing the ith column of C and D by the column vector z. And matrix C and D are

$$\begin{aligned}
&c_{11} = s; c_{12} = -2; c_{13} = c_{14} = c_{15} = c_{16} = 0;\\
&c_{21} = \omega^2; c_{22} = ms + \gamma; c_{23} = -m; c_{24} = 0;\\
&c_{25} = \lambda_1 + s; c_{26} = -1;\\
&c_{31} = 0; c_{32} = 2\omega^2; c_{33} = ms + 2\gamma; c_{34} = c_{35} = 0;\\
&c_{36} = s + \lambda_1;\\
&c_{41} = c_{42} = c_{43} = 0; c_{44} = s + \lambda_1; c_{45} = -2; c_{46} = 0;\\
&c_{51} = 0; c_{52} = D_1 s; c_{53} = -D_1; c_{54} = \omega^2;\\
&c_{55} = ms + \Delta_1 s + m\lambda_1 + \Delta_1\lambda_1 + \gamma;\\
&c_{56} = -(m + \Delta_1);\\
&c_{61} = c_{62} = 0; c_{63} = D_1 s; c_{64} = 0; c_{65} = 2\omega^2;\\
&c_{66} = ms + \Delta_1 s + m\lambda_1 + \Delta_1\lambda_1 + 2\gamma;
\end{aligned}$$

and

$$\begin{aligned}
&d_{11} = 0; d_{12} = -2; d_{13} = d_{14} = d_{15} = d_{16} = 0;\\
&d_{21} = \omega^2; d_{22} = \gamma; d_{23} = -m; d_{24} = 0;\\
&d_{25} = \lambda_1; d_{26} = -1;\\
&d_{31} = 0; d_{32} = 2\omega^2; d_{33} = 2\gamma; d_{34} = d_{35} = 0; d_{36} = \lambda_1;\\
&d_{41} = d_{42} = d_{43} = 0; d_{44} = \lambda_1; d_{45} = -2; d_{46} = 0;\\
&d_{51} = d_{52} = 0; d_{53} = -D_1; d_{54} = \omega^2;\\
&d_{55} = m\lambda_1 + \Delta_1\lambda_1 + \gamma; d_{56} = -m - \Delta_1;\\
&d_{61} = d_{62} = d_{63} = d_{64} = 0; d_{65} = 2\omega^2;\\
&d_{66} = m\lambda_1 + \Delta_1\lambda_1 + 2\gamma.
\end{aligned}$$

We have known that the average of the output is zero in Section 2.2, therefore the second-order moment is the variance of the response. Equation (12) shows that the variance of x varies periodically with time. It repeats in a sinusoidal fashion with amplitude R, frequency 2Ω and phase θ. It indicates that although the mean of oscillator displacement is zero at any time, but the mean of the distance between the harmonic oscillator and its equilibrium position varies periodically with time. And the frequency is twice than the input signal. The amplitude R is determined by m, γ, ω, Ω, A, λ_1, D_1, Δ_1, λ_2, D_2 and average α_3 is determined by m, γ, ω, Ω, A, λ_1, D_1, Δ_1, λ_2, D_2, λ_3, D_3.

Because first-order moment is always zero, the second-order at time t is the variance of the displacement of oscillator at time t. The analytic solution indicates that the variance of the displacement is a sinusoid with two parameters α_3 and R. α_3 is the average of the variance within a period which means the uncertainty of the hole trajectory.

3 Stochastic resonance results

According to the results in Section 2, we find this system is different from the harmonic oscillator subject to only one multiplicative noise. The average of first-order moment of this system is always zeros but the system with only one multiplicative noise is a sinusoid and the amplitude has SR phenomenon. By multiplying driven force by a colored noise with zero mean, the mean of driven force becomes zero at any time. It leads to the first-order is not periodic and the SR disappear. By further studying, we find the second-order moment of this system is a sinusoid. This indicates that although the average of oscillator displacement is zero at any time but the average of $|x|$ hold the periodicity by adding a new independent multiplicative noise.

In this section, we will discuss the SR phenomena in this system. Because the average of first-order moment is zero, we focus on the SR of second-order moment. For the convenience of analysis, we divide the huge number of parameters into three classes, driven force parameters Ω, A; noise parameters λ_1, D_1, Δ_1, λ_2, D_2, λ_3, D_3; system parameters m, γ, ω.

3.1 The bona fide stochastic resonance

In this part, we discuss the relationship between second-order moment and driven force parameters. Obviously, the second-order moment is multiplied by A, therefore we only discuss the parameter Ω.

In Figures 1a and 1b, we plot the curves of the dependence of the amplitude R and average α_3 of the variance on the Ω which is the frequency of the modulated noise under different system frequency ω. The other parameters are $m = 5$, $\gamma = 1$, $A = 5$, $\lambda_1 = 1$, $D_1 = 3$, $\Delta_1 = 1.5$, $\lambda_2 = 1$, $D_2 = 3$, $\lambda_3 = 1$ and $D_3 = 8$.

As shown in Figures 1a and 1b, all the curves of the amplitude and average have a non-monotonic behavior by increasing the modulated noise frequency Ω. The bona fide stochastic resonance [38] appears. The SR phenomena of average shown in Figure 1a is one-peak SR, the value of the resonant peak decrease and the position of the resonant peak turn to right as increasing the system frequency ω. The SR phenomena of amplitude shown in Figure 1b is one-valley and one-peak SR. By increasing the ω, the value of both peak and valley decrease, meanwhile the position of both peak and valley turn to right.

3.2 The conventional stochastic resonance

In this part, we discuss the relationship between second-order moment and noise intensity. Intensity of $\eta(t)$ will not

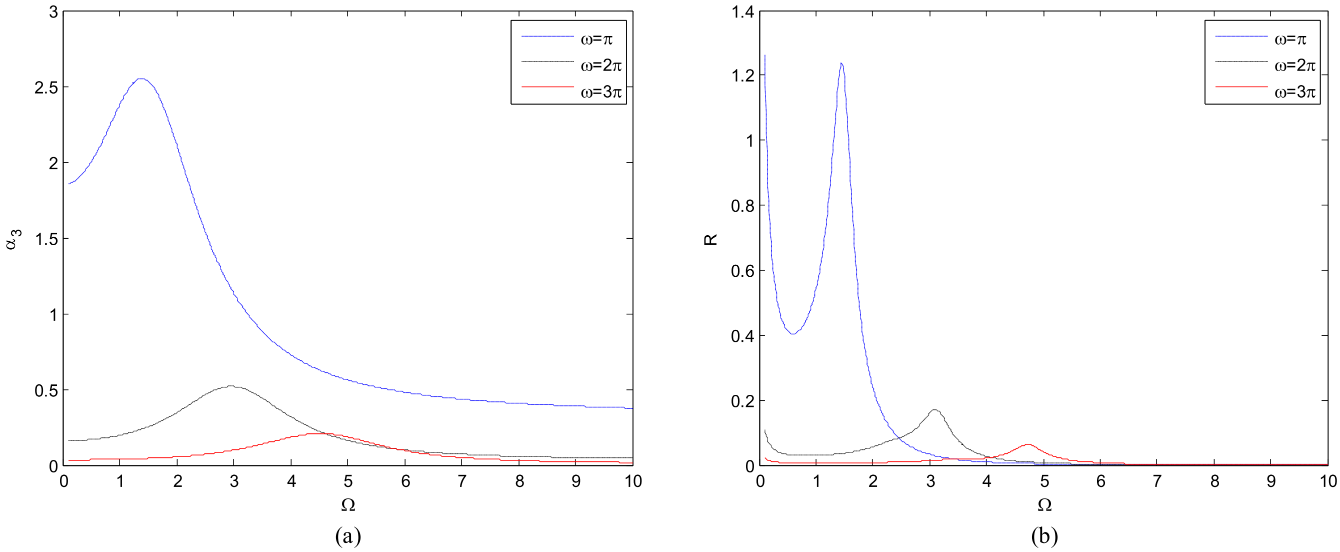


Fig. 1. The amplitude R and average α_3 versus the modulated noise frequency Ω with various system frequency ω.

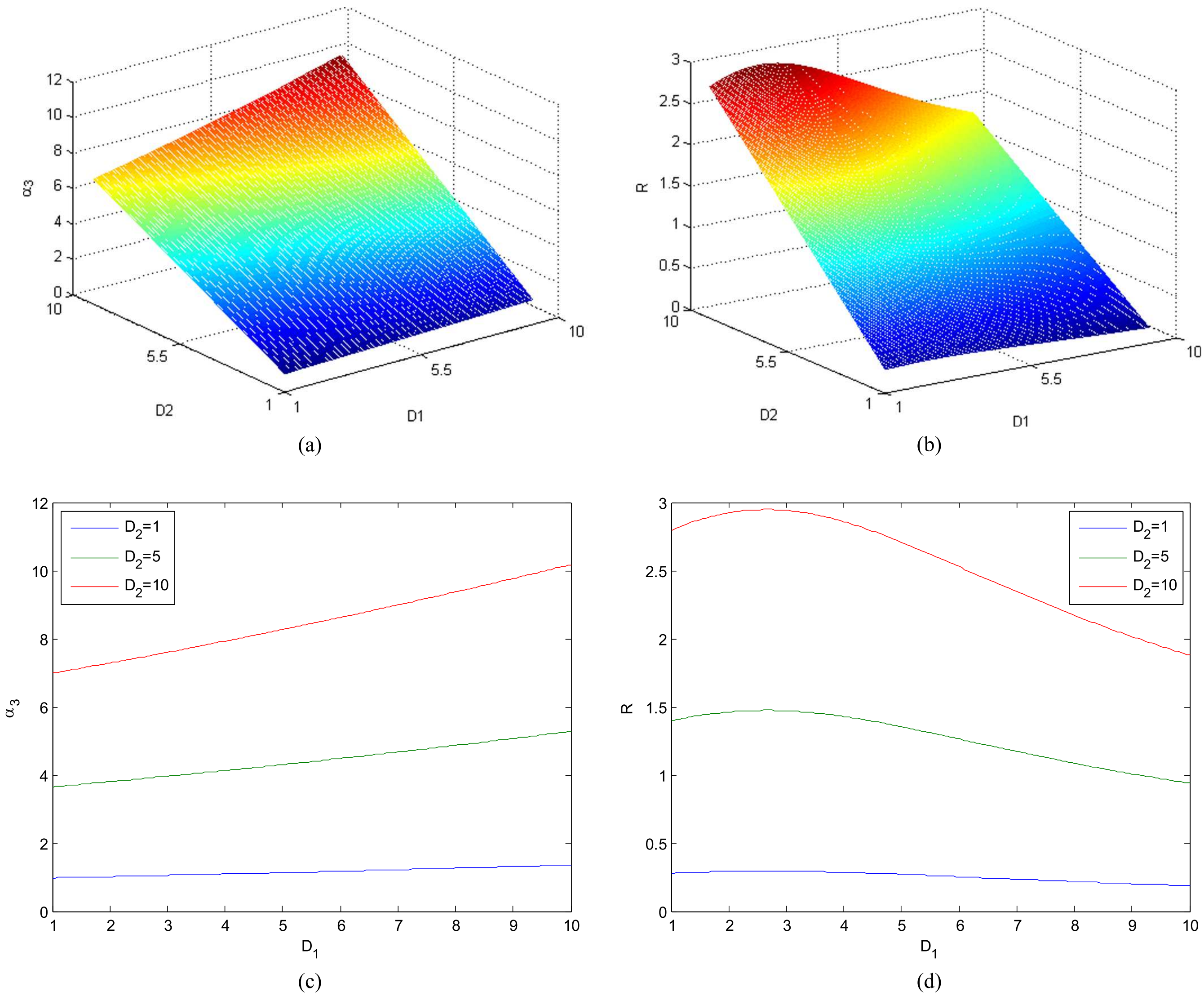


Fig. 2. The amplitude R and average α_3 versus the noise intensity D_1 and D_2.

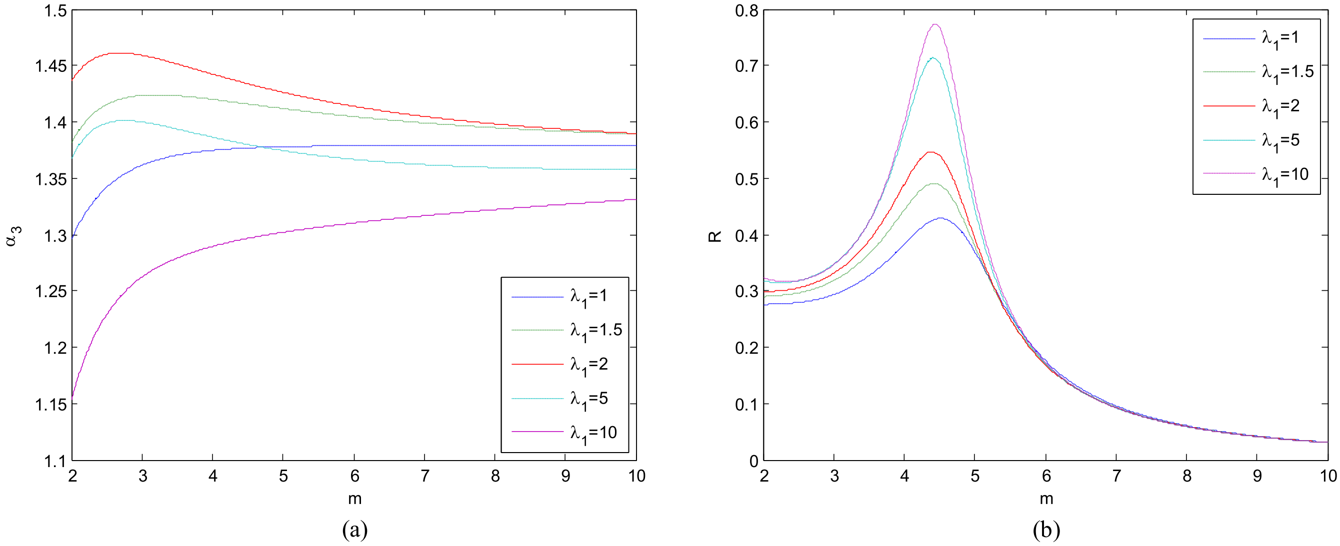


Fig. 3. The amplitude R and average α_3 versus the mass of harmonic oscillator with different λ_1. The other parameters is $\omega = \pi, \gamma = 1$, $A = 5$, $\Omega = \pi/2$, $D_1 = 3$, $D_1 = 1.5$, $\lambda_2 = 3$, $D_2 = 3$, $\lambda_3 = 2$, $D_3 = 8$.

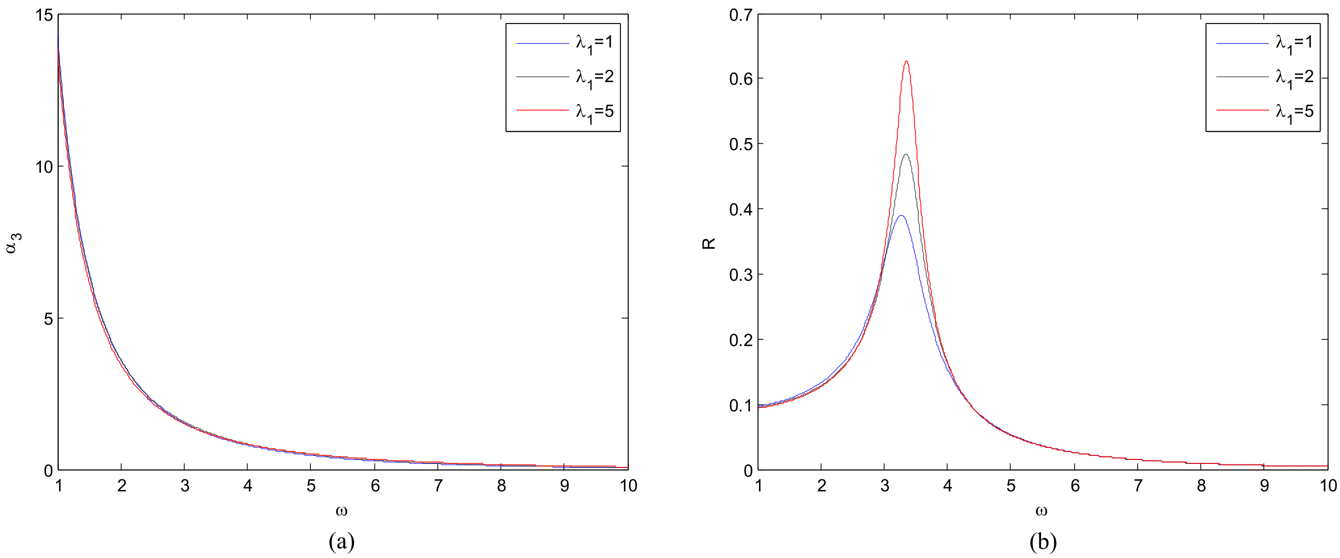


Fig. 4. The amplitude R and average α_3 versus the ω with various λ_1. The other parameters is $m = 5$, $\gamma = 1$, $A = 5$, $\Omega = \pi/2$, $D_1 = 3$, $D_1 = 1.5$, $\lambda_2 = 3$, $D_2 = 3$, $\lambda_3 = 2$, $D_3 = 8$.

lead to SR phenomenon because it is an additive noise in a linear system. We focus on the noise intensity of the two multiplicative noises $\xi(t)$ and $\psi(t)$.

In Figure 2, we plot the curves of the dependence of the amplitude R and average of variance α_3 on the mass fluctuation noise intensity D_1 and modulated noise intensity D_2. The other parameters are $m = 5$, $\gamma = 1$, $\omega = \pi$, $\Omega = \pi/2$, $A = 5$, $\lambda_1 = 1$, $D_1 = 1.5$, $\lambda_2 = 1$, $\lambda_3 = 1$, $D_3 = 8$.

In Figure 2 the average α_3 of the variance is always monotonically increasing at the noise intensities. But the amplitude R have a non-monotonic behavior by increasing the mass fluctuation noise intensity. Conventional stochastic resonance occurs. But by increasing the modulated noise intensity, R is monotonically increasing.

It indicates that in this system, the mass fluctuation is still the main causation of conventional stochastic resonance. By adding the modulated noise, the driven force can choose two values at a fixed time, and the mean is zero. Therefore, different from the system only has one multiplicative noise, the average of first-order is zero in this system. But we can find the periodicity of trajectories by calculating second-order moment and the conventional SR still occurs.

3.3 The generalized stochastic resonance

Gitterman used the term "stochastic resonance" in the wide sense, meaning the nonmonotonic (resonance) dependence of the output signal or some function of it

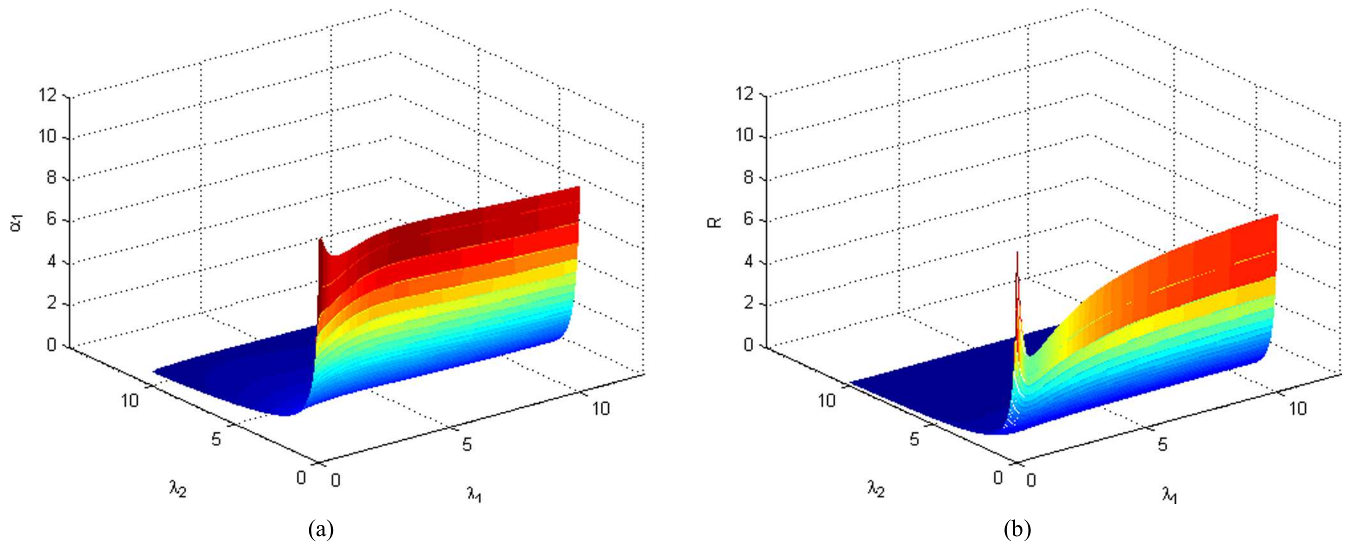


Fig. 5. The amplitude R and average α_3 versus λ_1 and λ_1. The other parameters are $m = 5$, $\gamma = 1$, $A = 5$, $\Omega = \pi/2$, $\omega = \pi$, $D_1 = 3$, $D_1 = 1.5$, $\lambda_2 = 3$, $D_2 = 3$, $D_3 = 8$.

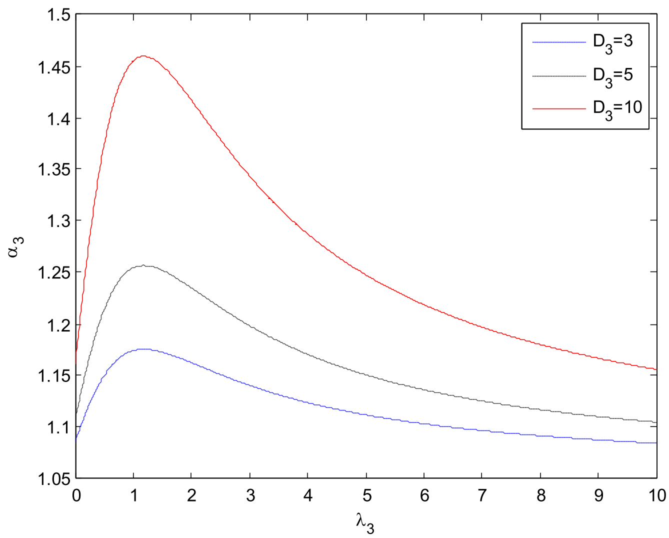


Fig. 6. The amplitude R and average α_3 versus λ_3 with various D_3. The other parameters are $m = 5$, $\gamma = 1$, $A = 5$, $\Omega = \pi/2$, $\omega = \pi$, $\lambda_1 = 1$, $D_1 = 3$, $D_1 = 3$, $\lambda_2 = 3$, $D_2 = 3$.

(moments, autocorrelation functions, power spectrum or signal-to-noise ratio) on the characteristics of the noise (the noise amplitude or correlation time) [8] and this phenomenon is often called generalized stochastic resonance [9]. In this part, we discuss the relationship between second-order moment and other parameters including system parameters and correlation rate of noises.

Figure 3 which shows the dependence of the amplitude and average on the mass of oscillator confirms the existence of SR phenomenon. Figure 3a, when $\lambda_1 = 1$, $\lambda_1 = 10$, the average has no SR phenomenon, and when $\lambda_1 = 1.5$, $\lambda_1 = 2$, $\lambda_1 = 5$, the SR phenomenon occurs. The SR phenomena of average appears and then disappears by increasing λ_1. The SR phenomenon of amplitude always exists with different λ_1, and the resonant peak increases by increasing λ_1.

Indeed, we can multiply equation (5) by $1/m$ to normalize the mass. The variety of mass can convert to noise intensity, system frequency and friction constant. Furthermore, the SR of mass can convert to the SR of noise intensity and system frequency which has been discussed in Sections 3.1 and 3.2.

As Figure 4 shows, only curves of amplitude versus ω have non-monotonic behavior and the peak increases with the increase of λ_1. The curves of average monotonically decrease at ω. Indeed, system frequency ω represents as the restoring force, the displacement and uncertainty of the oscillator should decrease by increasing ω with fixed noise intensity and driven force. But according to Section 3.1, there is a resonance between oscillator internal frequency and driving, so there is a resonant peak for a fixed driven frequency. When $\omega \to \infty$, both R and α_3 decrease to zero, it is obviously that oscillator displaces from its equilibrium position hardly with the increase of restoring force.

In Figure 5, we plot the curves of the dependence of α_3 and R on the noise correlation rates λ_1 and λ_2. The average and amplitude attain a minimum value by increasing λ_1. It means that reverse-resonance takes place. Furthermore, the value of valley decreases by increasing the correlation rate of modulated noise λ_2. By increasing λ_2, both α_3 and R decrease. Specifically, the periodically modulated noise approaching white noise when $\lambda_2 \to \infty$, the periodicity of driven force will disappear and lead to R decrease to zero which means that periodicity of output disappears.

According to the analytic expression, we have known that the properties of $\eta(t)$ does not influence R, we only plot the curves of average α_3 as a function λ_3 in Figure 6 It is shown that average of $\langle x^2 \rangle$ presents a resonance structure with the increasing of λ_3. And the increasing of D_3 enhances the SR phenomenon.

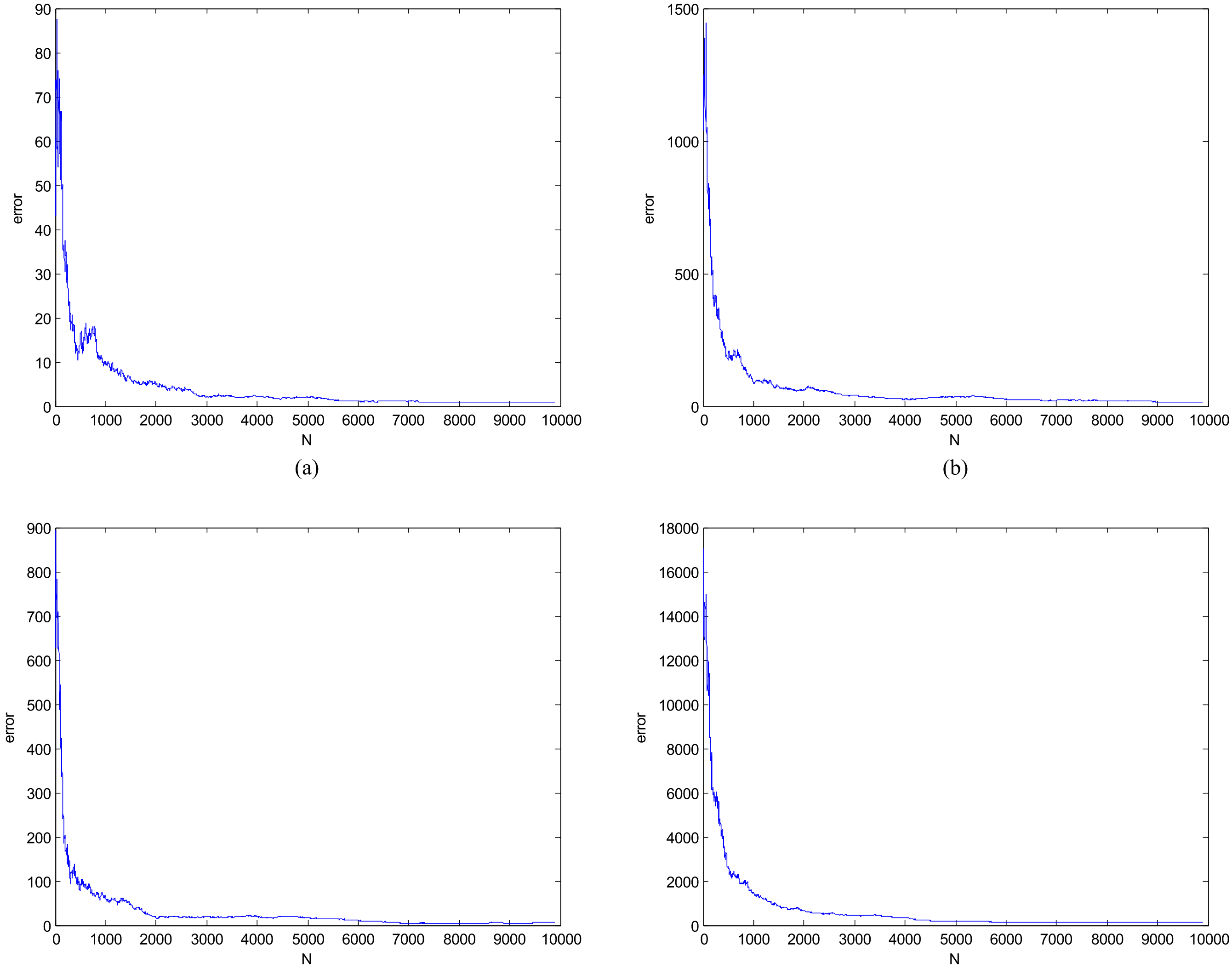


Fig. 7. The error of analytic solutions and average numerical solutions with simulation times N. The error in (a) and (c) is defined by equation (13); the error in (b) and (d) is defined by equation (14). The parameters of (a) and (b) are $m = 5$, $\gamma = 1$, $A = 5$, $\Omega = \pi/2$, $\omega = \pi$, $\lambda_1 = 1$, $D_1 = 3$, $D_1 = 1.5$, $\lambda_2 = 1$, $D_2 = 3$, $\lambda_3 = 1$, $D_3 = 8$; the parameters of (c) and (d) are $m = 5$, $\gamma = 1$, $A = 5$, $\Omega = \pi/5$, $\omega = \pi$, $\lambda_1 = 1$, $D_1 = 3$, $D_1 = 1.5$, $\lambda_2 = 3$, $D_2 = 30$, $\lambda_3 = 2$, $D_3 = 8$.

The discussion in this part indicates that the properties of this system are like the system which has only mass fluctuation. By multiplying driven force by a colored noise with zero mean, the periodic first-order moment becomes zero and does not exhibit these properties. But through calculating the second-order moment, we can find the periodicity of $|x|$ and the SR phenomena in this system.

4 Numerical simulations

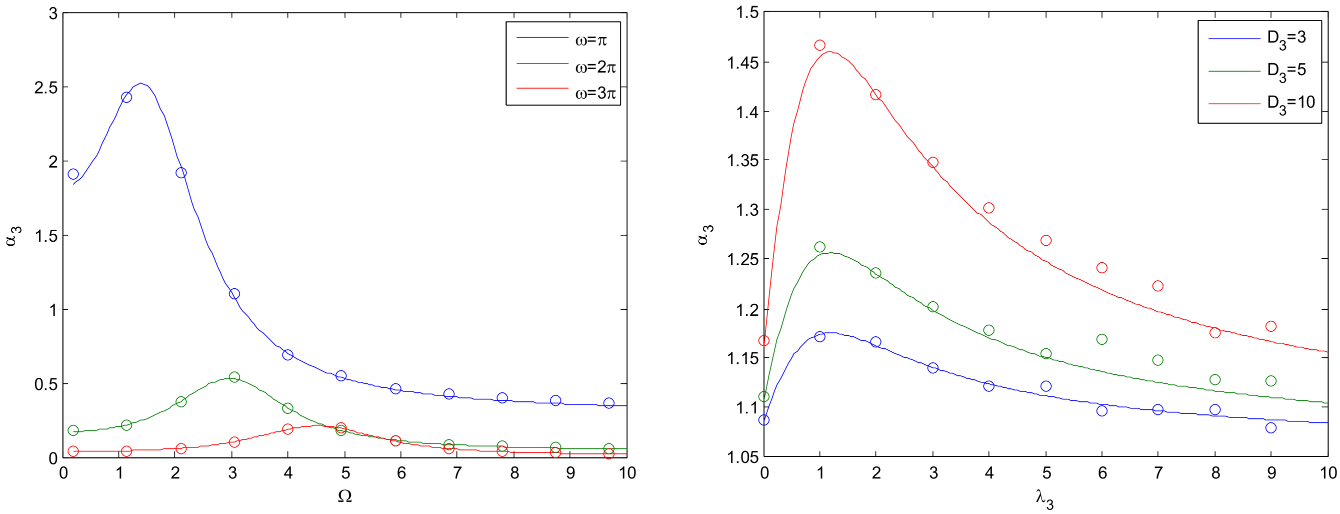
In this section, we chosen two groups of parameters to verify the analytical results by using numerical simulations. The parameters are

- $m = 5, \gamma = 1, A = 5, \Omega = \pi/2, \omega = \pi, \lambda_1 = 1, D_1 = 3, D_1 = 1.5, \lambda_2 = 1, D_2 = 3, \lambda_3 = 1, D_3 = 8$;
- $m = 5, \gamma = 1, A = 5, \Omega = \pi/5, \omega = \pi, \lambda_1 = 1, D_1 = 3, D_1 = 1.5, \lambda_2 = 3, D_2 = 30, \lambda_3 = 2, D_3 = 8$.

We use Runge-Kutta 4th order method to calculate the numerical solutions of equation (5) and we take the time $T = 100$, time step $T_s = 0.01$, number of simulations $N = 10\,000$.

In Figure 7, we plot the error of analytic solutions and average numerical solutions with simulation times N. We only consider the output at $T > 50$ to provide the numerical solution approach the steady-state solution. The error of first-order and second-order are

$$\sum_{i=i_0}^{i_T} \left({}_s x_i - {}_e x_i\right)^2, \tag{13}$$



(a) (b)

Fig. 8. The curves are analytical results and the circles are the numerical results. The other parameters are $m = 5$, $\gamma = 1$, $A = 5$, $\lambda_1 = 1$, $D_1 = 3$, $D_1 = 1.5$, $\lambda_2 = 1$, $D_2 = 3$, $\lambda_3 = 1$, $D_3 = 8$ in (a) and $m = 5$, $\gamma = 1$, $A = 5$, $\Omega = \pi/2$, $\omega = \pi$, $\lambda_1 = 1$, $D_1 = 3$, $D_1 = 3$, $\lambda_2 = 3$, $D_2 = 3$ in (b).

$$\sum_{i=i_0}^{i_T} \left({}_s x_i^2 -{}_e x_i^2\right)^2, \tag{14}$$

where ${}_s x_i$ is the averaged numerical solutions, ${}_e x_i$ is the analytic solutions, ${}_s x_i^2$ is the averaged of square of numerical solutions and ${}_e x_i^2$ is the square of analytic solutions.

According to the results, in Figure 8, we add the numerical results to Figures 1a and 6 with $N = 2500$ to verify the SR phenomena by presenting agreement of the simulation and analytics. As shown in Figure 8, the numerical results present SR phenomena and agree well with the analytical results.

5 Conclusions

In this paper, we study a harmonic oscillator subject to random mass and periodically modulated noise. Many practical systems can be modeled as this harmonic oscillator, for example, an RLC electrical circuit subject to a voltage $V(t)$ multiplied by a noise with a fluctuation inductance L. We obtain the analytic expression of the first-order and second-order moments. In this system, the second-order moment equal to the variance. According to the analytic expressions, we found different results from the other driven harmonic oscillator investigated. By multiplying driven force by a colored noise with zero mean, the mean of driven force becomes zero at any time. It leads to the first-order is not periodic and the SR disappear. The second-order moment of this system is a sinusoid and the frequency is twice than the input signal frequency. It indicates that although the average of oscillator displacement is always zero, the distance between oscillator and its equilibrium position varies periodically. The average and amplitude are determined by the parameters of the system and noise. Furthermore, we found various SR phenomenon occurs in this system: (1) bona fide SR with period of the modulated noise; (2) conventional SR with intensity of the mass fluctuation noise; and (3) generalized SR with other system parameters and correlation rate of the noise.

This work was supported by the National Natural Science Foundation of China (grant number 11301361).

Author contribution statement

This paper is equally contributed by all of the authors.

Appendix A

Here we give some results of second-order nonhomogeneous linear differential equations with constant coefficients used in this paper. For any n-order differential equations, we can denote new variables $y_1 = x, y_i = \frac{\mathrm{d}^{i-1}x}{\mathrm{d}t^{i-1}}, i = 2, \ldots, n$, then the n-order equations can be replaced by a one-order equation. Therefore, without loss of generality, we consider the following equations

$$A_1 \begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} \end{pmatrix} + A_2 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = f, \boldsymbol{x}(0) = \boldsymbol{x}_0. \tag{A.1}$$

If $f = \sum_{m=1}^{M} f_m$, the solutions of equation (A.1) is the sum of the solutions of following equations

$$A_1 \begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} \end{pmatrix} + A_2 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0, \boldsymbol{x}(0) = x_0 - \sum_{m=1}^{M} x_{0m}, \tag{A.2}$$

and

$$A_1 \begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}x_n}{\mathrm{d}t} \end{pmatrix} + A_2 \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = f_m, \boldsymbol{x}(0) = x_{0m}, m = 1, \ldots, M. \tag{A.3}$$

The characteristic equation of equations (A.1)–(A.3) is

$$\det(sA_1 + A_2) = 0.$$

According to the theory of linear differential equations, if all roots of the characteristic equation of equation (A.1) have negative real part, then for all $x(0)$ the solutions of equation (A.2) satisfies

$$\lim_{t\to\infty} x(t) = 0.$$

Furthermore, let $t \to +\infty$, the influence of the initial conditions will vanish. Instead of solving the characteristic equation, we can obtain that equation (A.1) is stability or not by using Routh-Hurwitz stability criterion.

In this paper, f_m is either constant or sinusoid, we will give the solution of equation (A.3) in the following when f_m is constant or sinusoid.

Firstly, we solve the equation (A.3) when f_m is constant. Assume $f_m = (c_1, \ldots, c_n)^T$ and $x_i = c_{i1}t + c_{i0}, i = 1, \ldots, n$, and insert into equation (A.3), we have

$$\begin{pmatrix} A_2 & 0 \\ A_1 & A_2 \end{pmatrix} \begin{pmatrix} c_{11} \\ \vdots \\ c_{n1} \\ c_{10} \\ \vdots \\ c_{n0} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}. \tag{A.4}$$

We can obtain the solution of equation (A.3) by solving the linear algebraic equation (A.4), particularly if $\det(A_2) \neq 0$, we have

$$x = A_2^{-1} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix},$$

with initial condition

$$x(0) = A_2^{-1} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Secondly, consider $f_m = C \sin \varOmega t$, where C is a column vector. By using Laplace transformation and denoting $x_0 = 0$, we obtain

$$A \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = C \frac{\varOmega}{s^2 + \varOmega^2}, \tag{A.5}$$

where $A = (sA_1 + A_2)$ and X_i is the Laplace transformation of x_i. Solve equation (A.5), we have

$$X_i(s) = H_i(s) \frac{\varOmega}{s^2 + \varOmega^2} = \frac{\det(A_{ic})}{\det(A)} \frac{\varOmega}{s^2 + \varOmega^2},$$

where A_{ic} is the matrix A formed by replacing the ith column of A by the column vector C. Use inverse Laplace transformation, the solution for equation (A.3) can be written as the following

$$x_i = \int_0^t h_i(t - \tau) \sin(\varOmega t)\, \mathrm{d}\tau, \tag{A.6}$$

where $h_i(t)$ is the inverse Laplace transformation. From the points of signal and system, x_i can be regarded as the response of system $H_i(s)$ to the sine signal input. Thus, we can calculate (A.6) without using inverse Laplace transformation, x_i can be written as

$$x_i = R \sin(\varOmega t + \phi),$$

where

$$R = |H_i(j\varOmega)|; \phi = \arg(H_i(j\varOmega)).$$

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