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Continuous time persistent random walk: a review and some generalizations*

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Abstract. We review some extensions of the continuous time random walk first introduced by Elliott Montroll and George Weiss more than 50 years ago [E.W. Montroll, G.H. Weiss, J. Math. Phys. **6**, 167 (1965)], extensions that embrace multistate walks and, in particular, the persistent random walk. We generalize these extensions to include fractional random walks and derive the associated master equation, namely, the fractional telegrapher's equation. We dedicate this review to our joint work with George H. Weiss (1930–2017). It saddens us greatly to report the recent death of George Weiss, a scientific giant and at the same time a lovely and humble man.

1 Introduction

Among the many aspects of the theory of random walks and their countless applications developed by George H. Weiss, he dedicated some amount of his vast work to multistate random walks [1,2]. These are random walks with internal states. In the standard and simplest formulation of walks with internal states, the walker randomly changes its internal state while the substrate (lattice or continuum) remains translationally invariant [2], that is, these states are associated with a fixed location.

According to Weiss, one of the earliest works containing the notion of randomly changing internal states is that of Lennard-Jones, who in the 1930s proposed a model for diffusing particles on surfaces that allowed the particles to be either immobile on the substrate (attached to a vibrating string tied to a substrate location) or in a state of diffusive motion (changing location on the substrate) [3,4]. This model is an example of a two-state random walk, a particular case of multistate walks, which has many notable applications in chromatography and electrophoretic systems. In the latter, particles in a medium are assumed to be either mobile in response to an external field, or otherwise immobile by being, for instance, entangled in a gel [5,6]. Disentanglement of the trapped particle is usually accomplished by thermal fluctuations and is therefore random.

A persistent random walk (PRW) allows one to incorporate a property analogous to momentum within the framework of diffusion theory. In the original one-dimensional formulation by Fürth [7] and Taylor [8], the walk takes place on a lattice in discrete time, and at each node the probability that the walker continues walking in the direction it took in the previous step differs from the probability of reversing direction. When the former probability is greater than the latter, the walker "persists" in moving along one direction. When both probabilities are equal, the PRW reduces to the ordinary random walk with uncorrelated steps. One salient characteristic of the PRW is that in the diffusive limit (well known, but also introduced later in this paper) the probability density function for the displacement at time t satisfies the telegrapher's equation rather than the more common diffusion equation associated with the ordinary random walk.

In 1989 the two of us (Masoliver and Lindenberg) together with George Weiss [9] developed the continuous time version of the persistent random walk, also allowing for steps of arbitrary length on a line. Their generalization of the formulation of the problem included Markovian and non-Markovian walks within the same formalism. In the diffusion limit such an extension leads to a variety of evolution equations more general than the telegrapher's equation. The continuous time persistent random walk (CTPRW) has recently been reformulated by Masoliver [10] to include a fractional version of the persistent walk which leads to the fractional telegrapher's equation in its standard form.

Here we review these developments, complementing the direct method of reference [9] by deriving the persistent walk as a particular case of multistate walks along the line followed in Weiss's book [2]. This has the advantage of introducing new readers to the topic of

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multistate random walks. These walks seem not to be well known despite their great potential in many theoretical developments and applications.

We limit our discussion to walks in one dimension. Some of the discussion can easily be extended to higher dimensions, while other aspects are more difficult to generalize in this way. In Section 2 we present a short review of multistate random walks, the basic walks on which were built the subsequent developments presented in this review. In Section 3 we add the notion of persistent walks, which can be seen as a type of multistate walk. We consider two states, namely, one in which the walker walks in one direction and the other in the opposite direction. Persistence occurs when a step in one direction is more likely to be followed by one in the same direction than in the opposite direction. A persistent random walk can under appropriate conditions be associated with the famous telegrapher's equation which encompasses the persistence via a second time derivative of the probability distribution. Other conditions on the persistent continuous time random walk lead to other partial differential equations. For instance, one can allow non-Markovian walks. These equations are always more complex than the telegrapher's equation. When there is no persistence, that is, when the probability of subsequent steps does not depend on the past, the telegrapher's equation reduces to the ordinary diffusion equation. The telegrapher's equation also reduces to the diffusion equation at long times. At short times, the telegrapher's equation exhibits wave-like behavior. In Section 5 we present some of the most recent advances in this area, namely, a discussion of fractional persistent random walks. This is an interesting development in the description of so-called "anomalous" dynamics. Here we again observe the dual short time (wave-like) and long time (diffusive) behavior. Section 6 focuses on the fractional telegrapher's equation. This equation has been the subject of ad hoc assumptions, but we focus on its systematic derivation. This specific equation also exhibits the dual wave-like and diffusive behavior observed above. Finally, in Section 7 we offer some closing remarks.

Because there are a number of repeated abbreviations in this paper, we assume that a glossary that can easily be referred to would be useful. We list the abbreviations here in their order of appearance in the following text:

- Persistent random walk (PRW)
- Continuous time persistent random walk (CTPRW)
- Continuous time random walk (CTRW)
- Probability density function (PDF)
- Telegrapher's equation (TE)
- Fractional telegrapher's equation (FTE)

2 Multistate random walks

We next present a short review of the formalism of multistate random walks. We suppose that the substrate is continuous and that time is measured in continuous units. In other words, we will outline the continuous time version of the multistate random walk. The case of discrete substrates (i.e., lattices) and discrete times requires only minor modifications.

Before addressing multistate walks, we pause for a moment and briefly sketch the (single-state) continuous time random walk (CTRW) of Montroll and Weiss [11]. In its original and simplest formulation, the walker starting at x_0 initially jumps to $x_0 + \Delta x_1$, then waits there for a random time interval τ_1 (also called a *sojourn*) and randomly jumps to a new position $x_0 + \Delta x_1 + \Delta x_2$, waits there another random interval of time τ_2 and makes another random jump, and so on. The jumps are instantaneous, that is, they take no time. The time instants at which a given sojourn ends and another one begins are called *regeneration points* because the walk continues from there as though that were the starting time and position of the walk. In what follows, and for the sake of clarity, we will assume that the random walker is initially at $x_0 = 0$ and that $t = 0$ is a regeneration point. We assume that the sojourn times and the jumping distances are mutually independent random variables (in the original theory the walk occurs on a lattice and the jumping distance is the distance between nearest neighbor sites on the lattice). The random walk is then determined by two probability density functions (PDFs): the distribution of sojourn times $\psi(\tau)$, and the distribution of jumping distances $f(\Delta x)$.

The principal quantity of interest is the PDF $p(x,t)$ that the random walker is at position x at time t. The celebrated Montroll-Weiss equation is the Fourier-Laplace transform $\widehat{\widetilde{p}}(\omega,s)$ of $p(x,t)$ in terms of the Laplace transform $\widehat{\psi}(s)$ of the waiting-time density, and the Fourier transform $\widetilde{f}(\omega)$ of the jump density. Assuming $p(x,0) = \delta(x)$, the Montroll-Weiss equation reads [11]

$$\widehat{\widetilde{p}}(\omega,s) = \frac{[1-\widehat{\psi}(s)]/s}{1-\widehat{\psi}(s)\widetilde{f}(\omega)}. \tag{1}$$

It is possible to generalize the formalism by allowing the walker some freedom of motion between regeneration points. This type of generalization was suggested by Montroll in 1950 [12] and more formally addressed by Weiss in 1976 [13]. It was reformulated some years later by Shlesinger et al. [14] (see also Ref. [15])[1]. Here we basically review the approach of the latter. In one such generalization the "jump" from one regeneration point to the next need not be instantaneous, and the walker need not be static during a sojourn. In other words, the walker can move during each sojourn either deterministically or randomly. A sojourn is still the time between two regeneration points, but the jump density $f(x,t)$ is now also a function of position x and time t between the two regeneration points. In fact, $f(x,t)$ is the PDF for the displacement of the walker at time t *inside a given sojourn*. However, $f(x,t)$ is not the probability density of the displacement in a single complete sojourn. The latter, denoted by $f(x)$, is given by

$$f(x) = \int_0^\infty f(x,t)\psi(t)dt. \tag{2}$$

[1] We thank an anonymous referee for pointing us to references [12,15].

The statistical properties of the walk are determined by the product

$$h(x,t) = f(x,t)\psi(t), \tag{3}$$

which is the joint PDF for the displacement during a complete sojourn to be given by x and the time spent in that sojourn to be equal to t. Note that in writing equation (3) we are assuming that inside any sojourn, duration of the sojourn and displacement are independent of each other.

A first step in the calculation of $p(x,t)$ relies on knowledge of an auxiliary function, $\rho(x,t)$, which represents the joint probability density for a regeneration point to occur at time t while the walker is at position x at that time. This density obeys the renewal equation

$$\rho(x,t) = h(x,t) + \int_0^t dt' \int_{-\infty}^{\infty} h(x-x', t-t')\rho(x',t')dx'. \tag{4}$$

This equation simply says that if a regeneration point occurs at time t, it must either be the first (after the initial one at $t=0$), described by the term $h(x,t)$ on the right hand side, or that an earlier regeneration point occurred at time $t' < t$ when the walker was at x' and no further regeneration points occurred during the time interval $t-t'$, all of this integrated over all possible intermediate times t' and intermediate positions x'.

In order to obtain $p(x,t)$ in terms of the auxiliary density $\rho(x,t)$, we also need the function

$$H(x,t) = f(x,t)\Psi(t), \tag{5}$$

where $\Psi(t)$ is the cumulative probability that a given time interval between consecutive regeneration points is greater than t,

$$\Psi(t) = \int_t^{\infty} \psi(t')dt'. \tag{6}$$

The new function $H(x,t)$ is the joint PDF for the displacement during an incomplete sojourn to be given by x with the complete sojourn taking a time greater than t. With this function we can write

$$p(x,t) = H(x,t) + \int_0^t dt' \int_{-\infty}^{\infty} H(x-x', t-t')\rho(x',t')dx'. \tag{7}$$

The interpretation of this equation is analogous to that of equation (4). Thus, the displacement x at time t is either within the first sojourn, described by $H(x,t)$ on the right hand side, or else an earlier sojourn occurred at time $t' < t$ with the walker at x' and the time interval to the next regeneration point exceeded $t-t'$.

Solving the integral equation (4) for the function $\rho(x,t)$ is easily carried out by implementing a joint Fourier-Laplace transform,

$$\widehat{\widetilde{\rho}}(\omega,s) = \int_0^{\infty} e^{-st}dt \int_{-\infty}^{\infty} e^{i\omega x}\rho(x,t)dx, \tag{8}$$

which turns (4) into a simple algebraic equation whose solution reads

$$\widehat{\widetilde{\rho}}(\omega,s) = \frac{\widehat{\widetilde{h}}(\omega,s)}{1-\widehat{\widetilde{h}}(\omega,s)}. \tag{9}$$

Substituting this equation into the Fourier-Laplace transform of equation (7), i.e.,

$$\widehat{\widetilde{p}}(\omega,s) = \widehat{\widetilde{H}}(\omega,s) + \widehat{\widetilde{H}}(\omega,s)\widehat{\widetilde{\rho}}(\omega,s),$$

immediately yields

$$\widehat{\widetilde{p}}(\omega,s) = \frac{\widehat{\widetilde{H}}(\omega,s)}{1-\widehat{\widetilde{h}}(\omega,s)}, \tag{10}$$

which is the generalization of the Montroll-Weiss equation (1).

The generalization of this derivation to multistate random walks in which the walker can be in any one of a discrete number n of states is formally straightforward [13,16]. The joint probability density for displacement and time during a complete sojourn (cf. Eq. (3)) now depends on the particular state, that is,

$$h_k(x,t) = f_k(x,t)\psi_k(t), \tag{11}$$

where $\psi_k(t)$ and $f_k(x,t)$ are, respectively, the sojourn density and the displacement density within a single sojourn of the walker while in state k. We also assume that transitions between different states are generated by a Markov chain at regeneration points in which $\alpha_{jk} \geq 0$ is the probability of the transition $j \to k$. That is to say, α_{jk} is the probability for a sojourn in state j to be followed by a sojourn in state k. Obviously

$$\sum_{k=1}^{n} \alpha_{jk} = 1 \quad (j = 1, 2, \ldots, n). \tag{12}$$

We suppose that $t=0$ initiates a sojourn in one of the n states. The probability that the walker is in state k at time $t=0$ is denoted by $\beta_k \geq 0$, and

$$\sum_{k=1}^{n} \beta_k = 1. \tag{13}$$

We denote the PDF of the walker while in state k by $p_k(x,t)$. Since being in one state or another are mutually exclusive events, the probability density of the random walker at time t regardless of state is therefore

$$p(x,t) = \sum_{k=1}^{n} p_k(x,t). \tag{14}$$

In order to obtain the set of densities $p_k(x,t)$ $(k = 1, 2, \ldots, n)$ we follow the procedure outlined above for the single-state CTRW. We define $\rho_k(x,t)$ to be the joint density that a sojourn in state k ends at time t with the walker at position x. These probability densities satisfy the following set of integral equations:

$$\rho_k(x,t) = \beta_k h_k(x,t) + \sum_{j=1}^{n} \alpha_{jk} \int_0^\infty dt' \int_{-\infty}^{\infty} h_k(x-x',t-t')\rho_j(x',t')dx' \tag{15}$$

($k = 1, 2, \ldots, n$), which arises from the same renewal argument as the one following equation (4), but with the additional consideration of transitions between different states governed by the Markov matrix α_{jk}. The first term on the right-hand side of equation (15) accounts for the possibility that the sojourn in state k that ends at time t is the first sojourn, which began at $t = 0$. The remaining terms account for possible earlier transitions to state k from other states. We next define the functions (see Eq. (5))

$$H_k(x,t) = f_k(x,t)\Psi_k(t), \tag{16}$$

which represent the probability densities for the displacement of the walker during a sojourn in state k, a sojourn that lasts longer than t. The densities $p_k(x,t)$ ($k = 1, 2, \ldots, n$) are related to the ρ's by

$$p_k(x,t) = \beta_k H_k(x,t) + \sum_{j=1}^{n} \alpha_{jk} \int_0^\infty dt' \int_{-\infty}^{\infty} H_k(x-x',t-t')\rho_j(x',t')dx'. \tag{17}$$

The reasoning behind this equation is similar to that following equation (7), extended to accommodate changes of state.

As in the single-state case, the set of integral equations (15) for the unknown ρ's can be converted into a system of algebraic equations by taking the joint Fourier-Laplace transform as defined in equation (8). This yields

$$\hat{\tilde{\rho}}_k(\omega,s) = \beta_k \hat{\tilde{h}}_k(\omega,s) + \sum_{j=1}^{n} \alpha_{jk} \hat{\tilde{h}}_k(\omega,s)\hat{\tilde{\rho}}_j(\omega,s) \tag{18}$$

($k = 1, 2, \ldots, n$). This set of linear equations can always be solved for $\hat{\tilde{\rho}}_k(\omega,s)$. This has been proved by Weiss (using a rather unknown theorem by Gerschgorin). We refer the interested reader to reference [2] (pp. 237–238) for details.

The Fourier-Laplace transform of equation (17) yields the following linear relation between p's and ρ's in the Fourier-Laplace domain:

$$\hat{\tilde{p}}_k(\omega,s) = \beta_k \hat{\tilde{H}}_k(\omega,s) + \sum_{j=1}^{n} \alpha_{jk} \hat{\tilde{H}}_k(\omega,s)\hat{\tilde{\rho}}_j(\omega,s). \tag{19}$$

After solving the linear system (18), this allows us to calculate the set of transformed probability densities $\hat{\tilde{p}}_k(\omega,s)$. The complete transformed PDF $\hat{\tilde{p}}(\omega,s)$ is then given by the sum (cf. Eq. (14))

$$\hat{\tilde{p}}(\omega,s) = \sum_{k=1}^{n} \hat{\tilde{p}}_k(\omega,s). \tag{20}$$

As an example we consider the two-state walk. Thus, if the walker must change states at each regeneration point then the matrix elements of the transition matrix α_{jk} are $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{12} = \alpha_{21} = 1$, indicating that the two possible states can occur only in alternating order. In many two-state cases this choice is possible with appropriate adjustment of the waiting time densities $\psi_1(t)$ and $\psi_2(t)$. The two equations satisfied by the $\hat{\tilde{\rho}}_k$ are easily solved and inserted into the two equations for the $\hat{\tilde{p}}_k$. We finally obtain the solutions

$$\hat{\tilde{p}}_1 = \frac{\beta_1 + \beta_2 \hat{\tilde{h}}_2}{1 - \hat{\tilde{h}}_1 \hat{\tilde{h}}_2} \hat{\tilde{H}}_1, \quad \hat{\tilde{p}}_2 = \frac{\beta_2 + \beta_1 \hat{\tilde{h}}_1}{1 - \hat{\tilde{h}}_1 \hat{\tilde{h}}_2} \hat{\tilde{H}}_2. \tag{21}$$

The total transformed PDF, $\hat{\tilde{p}} = \hat{\tilde{p}}_1 + \hat{\tilde{p}}_2$, finally reads

$$\hat{\tilde{p}} = \frac{\left(\beta_1 + \beta_2 \hat{\tilde{h}}_2\right)\hat{\tilde{H}}_1 + \left(\beta_2 + \beta_1 \hat{\tilde{h}}_1\right)\hat{\tilde{H}}_2}{1 - \hat{\tilde{h}}_1 \hat{\tilde{h}}_2}. \tag{22}$$

Moments of the two-state random walk can be obtained by realizing that $\hat{\tilde{p}}(\omega,s)$ is the Laplace transform of the characteristic function. Following this route, the Laplace transform of the nth moment is found to be:

$$\mathcal{L}\left\{\left\langle x^n(t)\right\rangle\right\} = i^{-n} \left.\frac{\partial^n \hat{\tilde{p}}(\omega,s)}{\partial \omega^n}\right|_{\omega=0}. \tag{23}$$

Using equation (22) one can in principle obtain moments of any order in terms of the derivatives of $\hat{\tilde{h}}_k(\omega,s)$ and $\hat{\tilde{H}}_k(\omega,s)$ with respect to ω evaluated at $\omega = 0$. However, this procedure becomes rather cumbersome and basically impractical even for low order moments. We refer the interested reader to reference [2] for additional information.

3 Persistent random walks

The persistent random walk (PRW) is an example of a random walk with internal states, and is a particular case of a two-state random walk allowing one to incorporate the notion of momentum. The standard analysis starts from a random walk on a one-dimensional lattice in discrete time and then assumes that at each node of the lattice the probability that the random walker continues to move in the same direction as in the immediately preceding step is p and that the probability that it reverses direction is $q = 1 - p$. When $p = q = 1/2$, the PRW reduces to the ordinary random walk in which the direction of a step is chosen without reference to that of the previous step.

Some years ago we developed a continuous time generalization of the PRW (hereafter referred to as CTPRW) in one dimension [9]. We next summarize the main features of this generalization.

In the continuous time version of the PRW in one dimension, the notion of persistence is incorporated by assuming that the random walker can be in one of two states, meaning that the walker moves to the right (state $k = 1$

or plus state) or to the left (state $k = 2$ or minus state). The duration and length of each sojourn are random variables. We denote the PDF for the time interval of each sojourn in the plus or minus state by $\psi_\pm(t)$, and the probability that the duration of a given sojourn is greater than t by $\Psi_\pm(t)$. The relationship between each pair of functions is given by equation (6). The composite functions $h_k(x,t)$ and $H_k(x,t)$ $(k = 1, 2)$, now denoted by $h_\pm$ and $H_\pm$, are given by equations (11) and (16).

In the lattice picture the probability that the random walker makes a transition to a neighboring node in the same direction as the previous step was denoted by p. Hence, the probability of n consecutive steps in the same direction is p^n whereas the probability of n consecutive changes of direction is $(1-p)^n$. The continuous analog of this result is found by choosing $\psi_\pm(t)$ to be isotropic, i.e., $\psi_+(t) = \psi_-(t)$, and of exponential form,

$$\psi(t) = \lambda e^{-\lambda t}, \tag{24}$$

where $\lambda^{-1} > 0$ is the mean duration of a sojourn. In this case the regeneration points, that is, the instants of time when changes of direction take place, are distributed according to a Poisson distribution [17]. The cumulative probability $\Psi(t)$ now is (cf. Eq. (6))

$$\Psi(t) = e^{-\lambda t}. \tag{25}$$

The lattice picture concept of "consecutive steps in the same direction" is here contained in the exponential forms, and only points where direction changes are now regeneration points.

Furthermore, in the lattice picture the displacement with each step is always fixed and equal to the distance between lattice nodes. The continuum analog of this requirement implies that the displacement in a single sojourn is proportional to the time spent in that sojourn. In other words, the functions $f_\pm(x,t)$ correspond to deterministic and uniform motion,

$$f_\pm(x,t) = \delta(x \mp ct), \tag{26}$$

where c is the (constant) speed of the walker. We note that any field driving the random walker would result in a nonlinear argument of the δ-function, which would correspond to nonuniform motion. We could choose the motion of the walker in a given sojourn to be random. In such a case, $f_\pm(x,t)$ would be given by an appropriate probability density other than the δ-distribution given in equation (26).

We conclude that the continuous time generalization of the persistent random walk is given by a two-state walk with sojourn displacement densities, $h_\pm$ and $H_\pm$, given by

$$h_\pm(x,t) = \lambda e^{-\lambda t}\delta(x \mp ct), \quad H_\pm(x,t) = \frac{1}{\lambda} h_\pm(x,t). \tag{27}$$

The Fourier-Laplace transforms of these functions are readily found to be

$$\hat{\tilde{h}}_\pm(\omega,s) = \frac{\lambda}{\lambda + s \pm ic\omega} \quad \hat{\tilde{H}}_\pm(\omega,s) = \frac{1}{\lambda + s \pm ic\omega}. \tag{28}$$

Inserting equation (28) into equation (22), recalling isotropy (which, in turn, implies $\beta_1 = \beta_2 = 1/2$), and performing some elementary manipulations, we get

$$\hat{\tilde{p}}(\omega,s) = \frac{2\lambda + s}{s^2 + 2\lambda s + c^2\omega^2}. \tag{29}$$

The Fourier inverse followed by the Laplace transform of this expression is readily obtained, yielding the well-known result [19]

$$p(x,t) = \frac{1}{2} e^{-\lambda t}\Big\{\delta(x-ct) + \delta(x+ct) + \frac{\lambda}{2c}\Theta(ct-|x|)\Big[I_0\big(z(t)\big) + \frac{\lambda}{z(t)} I_1\big(z(t)\big)\Big]\Big\}, \tag{30}$$

where $\delta(x \pm ct)$ are two δ-pulses moving in opposite directions, $\Theta(\cdot)$ is the Heaviside step function, the function $z(t)$ is

$$z(t) = \frac{\lambda}{c}\sqrt{c^2t^2 - x^2}, \tag{31}$$

and $I_0(z)$ and $I_1(z)$ are modified Bessel functions.

Some notable characteristics of the continuous-time generalization of the persistent random walk are apparent from the solution (30). First, there are two δ-functions which decay exponentially in time. These peaks correspond to walkers who at time t have not changed their direction of motion. They are located at $x \pm ct$, and the probability of observing such walkers decreases exponentially as time increases.

A second feature is the presence of the Heaviside function, which makes explicit the exclusion of walkers outside of the interval defined by $|x| = ct$. This is a manifestation of the property of finite speed of signal propagation, since $p(x,t) = 0$ if $|x| > ct$.

A third aspect of equation (30) is that as time increases, $p(x,t)$ relaxes to a Gaussian density. This can be seen by the asymptotic behavior of $I_0(z)$ and $I_1(z)$ as $z \to \infty$, but also more easily by using Tauberian theorems which relate the behavior of a function $f(t)$ as $t \to \infty$ to the small s behavior of its Laplace transform [18]. Taking the limit $s \to 0$ in equation (29), the small s approximation to $\hat{\tilde{p}}(\omega,s)$ is found to be

$$\hat{\tilde{p}}(\omega,s) \simeq \frac{1}{s + (c^2/2\lambda)\omega^2}.$$

Fourier-Laplace inversion yields the Gaussian density

$$p(x,t) \simeq \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt} \quad (t \to \infty), \tag{32}$$

where $D = c^2/2\lambda$.

Let us finally remark that for any value of t, we can also obtain the Gaussian form given in equation (32) from equation (30). Indeed, letting $\lambda \to \infty$ (which means that the duration of each sojourn goes to 0) and also letting $c \to \infty$ such that

$$\frac{c^2}{2\lambda} \to D \text{ (finite)}, \tag{33}$$

we get

$$\widehat{\widetilde{p}}(\omega, s) = \frac{1}{s + D\omega^2}, \tag{34}$$

which, upon inversion, yields the Gaussian density (32). Bearing in mind that the Gaussian density is the PDF for the ordinary random walk in the diffusion limit, we conclude that the PRW reduces to the ordinary random walk in the limit (33).

4 Telegrapher's equation and some generalizations

The telegrapher's equation (TE) was first proposed in the nineteenth century in the context of electrodynamics in the work of Kelvin and of Heaviside [19,20]. It is also useful in thermodynamics [21], population dynamics [22], and random walk theory [9] where the TE is the master equation for the one-dimensional persistent random walk as first presented by Goldstein in the early nineteen fifties for the PRW on a lattice [23]. It also governs the PDF of one-dimensional processes driven by the random telegraph signal, that is, a signal with step-wise transitions between or among different values occurring at random times. In the context of transport theory, the three-dimensional TE is the so-called P_1 approximation to the full transport equation [20,24,25].

In its isotropic form and in the absence of an external field, the TE reads

$$\frac{\partial^2 p}{\partial t^2} + \frac{1}{T}\frac{\partial p}{\partial t} = c^2\frac{\partial^2 p}{\partial x^2}, \tag{35}$$

where T is a characteristic time and c is a characteristic speed. This is a hyperbolic equation which becomes the wave equation as $T \to \infty$ with c fixed, and reduces to the diffusion equation as $T \to 0$ with $c^2T \to D$ finite. The TE equation thus enjoys both wave and diffusion characteristics. From this point of view we can say that the TE describes not only "diffusion with finite propagation velocity" but also "waves with damping" [19]. This duality becomes even more apparent in the mean square deviation from the mean. Scaling time with T it is easy to establish that[2]

$$\frac{\partial^2 p}{\partial t^2} \simeq c^2\frac{\partial^2 p}{\partial x^2} \quad (t \to 0), \qquad \frac{\partial p}{\partial t} \simeq D\frac{\partial^2 p}{\partial x^2} \quad (t \to \infty).$$

This leads to

$$\langle x^2(t)\rangle \sim t^2 \quad (t \to 0), \qquad \langle x^2(t)\rangle \sim t \quad (t \to \infty),$$

showing the transition from ballistic motion to diffusive motion as time progresses. Here and in all subsequent discussions we assume that $\langle x(t)\rangle = 0$.

In reference [9] we generalized Goldstein's approach from the lattice picture to the continuum description. In addition to the TE, this generalization also allowed us to treat non-Markovian walks, which leads to a large variety of partial differential equations of forms much more complicated than the standard TE. We briefly review this generalization.

We first derive the TE from the continuous-time picture. To this end we begin with the exact solution of the CTPRW given in equation (29) and try to find an associated partial differential equation satisfied by $p(x,t)$. We thus multiply both sides of equation (29) by the denominator, rewrite the results as

$$s^2\widehat{\widetilde{p}}(\omega, s) - s + 2\lambda\left[s\widehat{\widetilde{p}}(\omega, s) - 1\right] = -c^2\omega^2\widehat{\widetilde{p}}(\omega, s),$$

and then proceed to Fourier-Laplace inversion. With the initial conditions

$$p(x,0) = \delta(x), \qquad \left.\frac{\partial p(x,t)}{\partial t}\right|_{t=0} = 0, \tag{36}$$

and the inversion formulas [27][3]

$$\mathcal{L}^{-1}\mathcal{F}^{-1}\left\{s^2\widehat{\widetilde{p}}(\omega, s) - s\right\} = \frac{\partial^2 p(x,t)}{\partial t^2},$$

$$\mathcal{L}^{-1}\mathcal{F}^{-1}\left\{s\widehat{\widetilde{p}}(\omega, s) - 1\right\} = \frac{\partial p(x,t)}{\partial t},$$

and

$$\mathcal{L}^{-1}\mathcal{F}^{-1}\left\{\omega^2\widehat{\widetilde{p}}(\omega, s)\right\} = -\frac{\partial^2 p(x,t)}{\partial x^2},$$

we see that $p(x,t)$ satisfies the one-dimensional TE

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda\frac{\partial p}{\partial t} = c^2\frac{\partial^2 p}{\partial x^2}. \tag{37}$$

Further generalizations can be obtained by relaxing the δ form for $f_\pm(x,t)$ and/or the exponential form for $\psi(t)$. We first assume that the motion within each sojourn is now random rather than deterministic, but we retain the exponential form of the waiting time distribution so that switching times are isotropically distributed according to Poisson and $\psi_\pm(t) = \psi(t)$ is given by the exponential form (24). The evolution within a given sojourn is not ballistic. Instead, the walker executes Brownian motion with a drift alternately switching between $+c$ and $-c$ (c constant) and diffusion coefficient D. In this case the densities $f_\pm(x,t)$ assume the Gaussian form

$$f_\pm(x,t) = \frac{1}{2\sqrt{\pi D t}}\exp\left[-\frac{(x \pm ct)^2}{4Dt}\right]. \tag{38}$$

The Fourier-Laplace transform of the composite functions $h_\pm(x,t)$ and $H_\pm(x,t)$ defined in equations (11) and (16) read (compare with Eq. (28))

$$\widehat{\widetilde{h}}_\pm(\omega, s) = \frac{\lambda}{\lambda + s \pm ic\omega + D\omega^2}, \tag{39}$$

and

$$\widehat{\widetilde{H}}_\pm(\omega, s) = \frac{1}{\lambda + s \pm ic\omega + D\omega^2}. \tag{40}$$

[2] Define the dimensionless time scale $t' = t/T$, then drop the prime and use the well known fact that $|\partial^2 p/\partial t^2| \gg |\partial p/\partial t|$ for $t \to 0$ and $|\partial^2 p/\partial t^2| \ll |\partial p/\partial t|$ for $t \to \infty$ [26] (now $D = c^2T^2$).

[3] Note also that $\mathcal{F}^{-1}\{1\} = \delta(x)$.

The total PDF of the walker, which in Fourier-Laplace space is given by equation (22) with $\beta_1 = \beta_2 = 1/2$,

$$\widehat{\widetilde{p}} = \frac{(1+\widehat{\widetilde{h}}_-)\widehat{\widetilde{H}}_+ + (1+\widehat{\widetilde{h}}_+)\widehat{\widetilde{H}}_-}{2(1-\widehat{\widetilde{h}}_+\widehat{\widetilde{h}}_-)}, \tag{41}$$

now reads

$$\widehat{\widetilde{p}}(\omega,s) = \frac{\lambda^2(2\lambda+s+D\omega^2)}{s^2+2\lambda s+2sD\omega^2+(2\lambda D+c^2)\omega^2+D^2\omega^4}. \tag{42}$$

In order to write a partial differential equation for $p(x,t)$ we proceed as in the derivation of the TE outlined above. Fourier-Laplace inversion yields

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda\frac{\partial p}{\partial t} = (c^2+2\lambda D)\frac{\partial^2 p}{\partial x^2} + 2D\frac{\partial^3 p}{\partial x^2 \partial t} - D^2\frac{\partial^4 p}{\partial x^4}, \tag{43}$$

which reduces to the TE when D is set equal to 0.

We next relax the assumption that the distribution of regeneration times of the PRW is Poissonian. We thus assume a general form for $\psi(t)$ other than the exponential density given in equation (24)[4]. The walk is now no longer Markovian [2,9]. However, we return to the requirement that the random walker moves within each sojourn in ballistic uniform manner rather than randomly. That is to say, the densities $f_\pm(x,t)$ retain the ballistic form given in equation (26). Therefore, the Fourier-Laplace transforms of the sojourn displacement densities $h_\pm(x,t)$ are given by

$$\widehat{\widetilde{h}}_\pm(\omega,s) = \int_{-\infty}^{\infty} dx \int_0^\infty e^{i\omega x - st}\delta(x\mp ct)\psi(t)dt = \int_0^\infty e^{-(s\mp i\omega c)t}\psi(t)dt.$$

That is,

$$\widehat{\widetilde{h}}_+(\omega,s) = \widehat{\psi}(z), \quad \widehat{\widetilde{h}}_-(\omega,s) = \widehat{\psi}(z^*), \tag{44}$$

where

$$z = s - i\omega c,$$

and $z^* = s + i\omega c$ is the complex conjugate.

Since the Laplace transform of the cumulative probability $\Psi(t)$ defined in equation (6) is given in terms of the Laplace transform of $\psi(t)$ by

$$\widehat{\Psi}(s) = \frac{1}{s}\left[1-\widehat{\psi}(s)\right], \tag{45}$$

we easily see that

$$\widehat{\widetilde{H}}_+(\omega,s) = \frac{1-\widehat{\psi}(z)}{z}, \quad \widehat{\widetilde{H}}_-(\omega,s) = \frac{1-\widehat{\psi}(z^*)}{z^*}. \tag{46}$$

[4] We still assume isotropy in the sense that $\psi_+(t) = \psi_-(t)$ and $\beta_+ = \beta_- = 1/2$.

Substituting equations (44) and (46) into equation (22) we obtain

$$\widehat{\widetilde{p}}(\omega,s) = \frac{1}{2[1-\widehat{\psi}(z)\widehat{\psi}(z^*)]}\left\{\frac{1}{z}\left[1-\widehat{\psi}(z)\right]\left[1+\widehat{\psi}(z^*)\right] + \frac{1}{z^*}\left[1-\widehat{\psi}(z^*)\right]\left[1+\widehat{\psi}(z)\right]\right\}. \tag{47}$$

This generalization reduces to the solution (29) of the TE when $\psi(t) = \lambda e^{-\lambda t}$, in which case $\widehat{\psi}(s) = \lambda/(\lambda+s)$.

Consider the slower than exponential decay

$$\psi(t) = \lambda^2 t e^{-\lambda t},$$

for which

$$\widehat{\psi}(s) = \left(\frac{\lambda}{\lambda+s}\right)^2.$$

After substituting into equation (47) and following the procedure outlined above, the PDF is found to be the solution of the rather complicated equation

$$4\lambda\frac{\partial^3 p}{\partial t^3} + 6\lambda^2\frac{\partial^2 p}{\partial t^2} + 4\lambda^3\frac{\partial p}{\partial t} = 2\lambda^2 c^2\frac{\partial^2 p}{\partial x^2} + 4\lambda c^2\frac{\partial^3 p}{\partial x^2\partial t} - c^4\frac{\partial^4 p}{\partial x^4}. \tag{48}$$

Note that if we divide both sides of this equation by $4\lambda^3$ and go to the diffusive limit ($\lambda \to \infty$ and $c \to \infty$ with $c^2/2\lambda = D$ finite), equation (48) reduces to the diffusion equation, as shown earlier for the TE.

We mention that in the case of uniform ballistic motion, the Laplace transform of the nth moment is given by equation (23) with the expression for $\widehat{\widetilde{p}}(\omega,s)$ given in equation (47). For the mean square displacement we can write

$$\mathcal{L}\left\{\left\langle x^2(t)\right\rangle\right\} = \frac{2c^2}{s^2}\left[\frac{1}{s} + \frac{2\widehat{\psi}^1(s)}{1-\widehat{\psi}^2(s)}\right]. \tag{49}$$

Using $\widehat{\psi}(s)$, the moments of the sojourn time are given by

$$T_n \equiv \int_0^\infty t^n\psi(t)dt = (-1)^n\widehat{\psi}^{(n)}(0).$$

Here and in the equation above the superscripts (n) denote the nth derivative with respect to s. When the first and second moment are finite, that is, if

$$\widehat{\psi}(s) \sim 1 - T_1 s + \frac{1}{2}T_2 s^2 + \cdots \quad \text{as } s \to 0,$$

one finds in this limit that

$$\mathcal{L}\left\{\left\langle x^2(t)\right\rangle\right\} \sim \left(\frac{c^2\sigma^2}{T_1}\right)\frac{1}{s^2},$$

where $\sigma^2 = T_2 - T_1^2$. Using Tauberian theorems we conclude that

$$\left\langle x^2(t)\right\rangle \sim t \quad (t\to\infty).$$

This is the result expected for an ideal diffusion process.

When the mean sojourn time is finite but the second moment is not, it is possible for the mean square displacement to be asymptotic to a power of t other than the first. For example, suppose that

$$\widehat{\psi}(s) \sim 1 - T_1 s + (Ts)^{1+\alpha}, \quad s \to 0,$$

where $0 < \alpha < 1$ and T is a positive parameter with dimension of time. Note that this expression implies that $\psi(t) \sim t^{-(2+\alpha)}$. For this case the behavior of $\langle x^2(t)\rangle$ goes as

$$\left\langle x^2(t)\right\rangle \sim t^{2-\alpha} \quad (t \to \infty).$$

This is so-called superdiffusive transport where the mean square displacement increases faster than the first power of time.

Finally, when $\psi(t)$ is asymptotically a stable law (e.g., the Levy distribution), then

$$\widehat{\psi}(s) \sim 1 - (Ts)^{\alpha}, \quad s \to 0,$$

$(0 < \alpha < 1)$ and the mean square displacement has the asymptotic form

$$\left\langle x^2(t)\right\rangle \sim t^{\alpha} \quad (t \to \infty).$$

We have thus arrived at subdiffusive transport in which the mean square displacement increases more slowly than t.

5 Fractional persistent random walks

In this section and the next we review some implementations of persistent random walks in the context of anomalous diffusion. The link between PRWs and anomalous diffusion was previewed at the end of the previous section; some fractional aspects of the PRW, particularly those concerning asymptotic results, were treated by George Weiss in Chapter 6 of his book [2]. Chiefly based on reference [10], we will discuss this link in more detail.

For more than two decades, so-called "anomalous transport" and "anomalous diffusion" have been the object of intense research in many branches of physics. There is an immense literature on the subject; for rather complete reports of the field see, for instance [28–37] and references therein. Especially recommended to the newcomer is the less technical but excellent introduction by Klafter and Sokolov [38].

Anomalous diffusion arises in extremely disordered systems such as random media and fractal structures [39], and its most distinctive characteristic is that the mean square deviation follows the asymptotic law [29,30]

$$\langle x^2(t)\rangle \sim t^{\alpha} \tag{50}$$

$(t \to \infty)$, where $\alpha > 0$ is any positive real number. The range $0 < \alpha < 1$ describes subdiffusion, $\alpha = 1$ corresponds to the (normal) diffusive regime, and $\alpha > 1$ describes superdiffusion.

The concept of anomalous diffusion first emerged from the theory of random processes and, specifically, from continuous time random walks. It was first applied to diffusion of charge carriers in organic semiconductors by Scher and Montroll in the 1970s [40,41]. As we have seen, in the original formulation of the CTRW the Fourier-Laplace transform of the PDF is given by the Montroll-Weiss equation (1). A fractional version of the CTRW (which results in anomalous diffusion) is obtained from equation (1) in the so-called "fluid limit" [42], that is, for large times and distances [43] upon Laplace and Fourier inversion of the assumed forms [32,42,44]

$$\widehat{\psi}(s) = 1 - (Ts)^{\alpha}\dots \quad (s \to 0), \tag{51}$$

$$\widetilde{f}(\omega) = 1 - (L\omega)^{2\gamma}\cdots \quad (\omega \to 0), \tag{52}$$

using Tauberian theorems. Here $0 < \alpha \leq 1$, $0 < \gamma \leq 1$ and T and L are positive constant parameters measured in units of time and length, respectively. Introducing equations (51) and (52) into equation (1) we obtain

$$\widehat{\widetilde{p}}(\omega, s) = \frac{s^{\alpha-1}}{s^{\alpha} + D\omega^{2\gamma}}, \tag{53}$$

where $D = L^{2\gamma}/T^{\alpha}$. The mathematical properties of the corresponding PDF, $p(x,t)$, have been thoroughly studied and very clearly presented by Mainardi and collaborators [44–46]. One of these properties is the scaling relation [29,42,46,47]

$$p(\mathbf{r}, t) = t^{-\alpha/2\gamma} g\left(\frac{x}{t^{\alpha/2\gamma}}\right), \tag{54}$$

which results in the scaling of the mean square displacement [42]

$$\langle x^2(t)\rangle = Mt^{\alpha/\gamma}, \tag{55}$$

showing that subdiffusion appears when $\alpha < \gamma$ and superdiffusion when $\alpha > \gamma$. Although very appealing, equation (55) has limited utility since M exists only when $\gamma = 1$. When $\gamma \neq 1$, M turns out to be infinite and the mean square displacement is no longer finite [42].

When $\gamma = 1$ but α is not an integer, we have so-called "time-fractional diffusion", the case $0 < \alpha < 1$ corresponding to subdiffusion and $\alpha > 1$ to superdiffusion. When $\alpha = 1$ but γ is not an integer, equation (53) describes a Levy walk. This case is always associated with superdiffusion and is called "space-fractional diffusion" [32,42].

We next address a generalization of the CTPRW to include fractional motion. Recall that the CTPRW is described by the sojourn displacement densities $h_{\pm}(x,t)$ and $H_{\pm}(x,t)$ given in equations (11) and (16) with $k = 1, 2$ denoted by $h_{\pm}$ and $H_{\pm}$. The Fourier-Laplace transforms of these densities are given in equation (28). We observe that in the fluid limit $(s, \omega \to 0)$ the functions $\widehat{\widetilde{h}}_{\pm}(\omega, s)$ behave as

$$\widehat{\widetilde{h}}_{\pm}(\omega, s) = 1 - (1/\lambda)s \mp i(c/\lambda)\omega\cdots, \tag{56}$$

with similar expansions for $\widehat{\widetilde{H}}_{\pm}(\omega, s)$.

In order to determine a fractional generalization of the CTPRW we will follow the steps of the derivation of the fractional generalization of the CTRW as sketched above in equations (51) and (52). In place of equation (56) we write the Fourier-Laplace transform of $\widehat{\widetilde{h}}_\pm(\omega, s)$ in the fluid limit as

$$\widehat{\widetilde{h}}_\pm(\omega, s) = 1 - (Ts)^\alpha \mp i(L\omega)^{2\gamma} \cdots, \tag{57}$$

where $0 < \alpha \le 1$, $0 < \gamma \le 1$ and, as before, T and L are arbitrary constants setting a characteristic time and a characteristic length, respectively. Note that to the same degree of approximation in which s and ω are small, the expansion (57) is equivalent to

$$\widehat{\widetilde{h}}_\pm(\omega, s) = \frac{1}{1 + (Ts)^\alpha \pm i(L\omega)^{2\gamma}} \cdots. \tag{58}$$

This equation is the cornerstone for building a fractional generalization of the persistent random walk. Before proceeding further we note that other generalizations based on equation (57) in combination with equation (41) are possible. However, as we will see next, the generalization based on equation (58) (instead of Eq. (57)) seems to be the only one leading to the fractional telegrapher's equation in the standard form [10].

We assume a fluid limit-approximation for $\hat{\tilde{H}}_\pm(\omega, s)$ consistent with approximation (58). To this end we recall that the $h_\pm(x, t)$ are the joint densities for the length and duration of sojourns in the plus or minus state. Therefore, their time marginal densities are

$$\int_{-\infty}^{\infty} h_\pm(x, t)dx = \psi_\pm(t),$$

where $\psi_\pm(t)$ are the pdf's for sojourn duration in each state. Correspondingly,

$$\int_{-\infty}^{\infty} H_\pm(x, t)dx = \Psi_\pm(t),$$

where $\Psi_\pm(t)$ is given in equation (6). In Fourier-Laplace space these conditions read

$$\widehat{\widetilde{h}}_\pm(\omega = 0, s) = \widehat{\psi}_\pm(s), \quad \widehat{\widetilde{H}}_\pm(\omega = 0, s) = \widehat{\Psi}_\pm(s),$$

but from equation (6) we see that $\widehat{\Psi}_\pm(s) = [1 - \widehat{\psi}_\pm(s)]/s$, hence

$$\widehat{\widetilde{H}}_\pm(\omega = 0, s) = \frac{1}{s}[1 - \widehat{\widetilde{h}}_\pm(\omega = 0, s)].$$

Inserting equation (58) into this expression yields

$$\widehat{\widetilde{H}}_\pm(\omega = 0, s) = \frac{T(Ts)^{\alpha-1}}{1 + (Ts)^\alpha} \cdots,$$

which leads us to assume that

$$\widehat{\widetilde{H}}_\pm(\omega, s) = \frac{T(Ts)^{\alpha-1}}{1 + (Ts)^\alpha \pm i(L\omega)^\gamma} \cdots, \tag{59}$$

as $s \to 0$ and $\omega \to 0$. We stress that this is a conjecture because the numerator of equation (59) might have depended on ω as well.

Substitution of equations (58) and (59) into equation (41) along with simple algebra yields

$$\widehat{\widetilde{p}}(\omega, s) = \frac{s^{\alpha-1}(s^\alpha + 2\lambda)}{s^{2\alpha} + 2\lambda s^\alpha + c^2\omega^{2\gamma}}, \tag{60}$$

where

$$\lambda \equiv 1/T^\alpha, \quad c \equiv L^\gamma/T^\alpha. \tag{61}$$

Equation (60) determines the probability distribution of the fractional persistent random walk, and it constitutes the generalization of the CTPRW to include fractional motion.

We can invert the Laplace transform in equation (60) and obtain an analytic expression for the characteristic function $\widetilde{p}(\omega, t)$ of the fractional CTPRW. We will not present this expression here and refer the interested reader to reference [10] for more information.

The fractional CTPRW reduces to the fractional CTRW in two cases: (i) when $\lambda, c \to \infty$ such that c^2/λ is finite and (ii) for large values of time. Case (i) is readily obtained from equation (60) by dividing the numerator and denominator by λ and then letting $\lambda \to \infty$. This is equivalent to assuming that $T \to 0$. Then also letting $c \to \infty$ such that

$$\frac{c^2}{2\lambda} \simeq \frac{L^{2\lambda}}{2T^\alpha} \longrightarrow D \text{ (finite)},$$

we arrive at equation (53), which is the result for the CTRW. For case (ii) we use Tauberian arguments [18]. Passing to the limit $s \to 0$ in equation (60) we see that the small s approximation to the transformed PDF also coincides with equation (53),

$$\hat{\tilde{p}}(\omega, s) \simeq \frac{s^{\alpha-1}}{s^\alpha + D\omega^{2\gamma}} \quad (s \to 0), \tag{62}$$

(with $D = c^2/2\lambda$), which shows that as $t \to \infty$ the fractional CTPRW reduces to the fractional CTRW.

The persistent random walk possesses another limit which is not related to the ordinary random walk. Focusing on fractional walks, we will see that this limit can also be obtained from two different approaches: (iii) when $\lambda \to 0$ (i.e. $T \to \infty$) and at the same time $L \to \infty$ such that $c = L^\gamma/T^\alpha$ remains finite, and (iv) for small values of time. For case (iii), the limit $\lambda \to 0$ and c finite in equation (60) yields

$$\widehat{\widetilde{p}}(\omega, s) = \frac{s^{2\alpha-1}}{s^{2\alpha} + c^2\omega^{2\gamma}}, \tag{63}$$

which is unrelated to the CTRW. For case (iv) we readily see that as $s \to \infty$ (i.e., $|s| \gg \lambda$) we have $s^{2\alpha} + 2\lambda s^\alpha \simeq s^{2\alpha}$, and from equation (60) we write

$$\widehat{\widetilde{p}}(\omega, s) \simeq \frac{s^{2\alpha-1}}{s^{2\alpha} + c^2\omega^{2\gamma}} \quad (s \to \infty), \tag{64}$$

which coincides with equation (63). Since $s \to \infty$ corresponds to $t \to 0$ (i.e., $t \ll \lambda^{-1}$), the comparison of this equation with equation (62) clearly shows the completely different behavior of the PDF $p(x,t)$ from that of cases (i) and (ii) as $t \to 0$.

The duality between the limit expressions (62) and (63) is the fractional version of the duality shown earlier by the non-fractional PRW (and also shown by the telegrapher's equation) between wavelike behavior at short times and diffusion behavior at long times. We now look at this duality for the mean square displacement and, in the next section, for the fractional telegrapher's equation.

If we consider the time-fractional PRW (i.e., when $\gamma = 1$) we can obtain moments of any order for the random walker (note that this is not possible in the space fractional case in which $0 < \gamma < 1$). Recall that the Laplace transform of the nth moment is given in terms of the Fourier-Laplace transform of the PDF given in equation (23) and duplicated here for convenience:

$$\mathcal{L}\Big\{\big\langle x^n(t)\big\rangle\Big\} = i^{-n}\left.\frac{\partial^n \widehat{\widetilde{p}}(\omega,s)}{\partial\omega^n}\right|_{\omega=0}. \tag{65}$$

For the time-fractional walk, $\hat{\tilde{p}}(\omega,s)$ is given by equation (60) with $\gamma = 1$,

$$\widehat{\widetilde{p}}(\omega,s) = \frac{s^{\alpha-1}(s^\alpha + 2\lambda)}{s^{2\alpha} + 2\lambda s^\alpha + c^2\omega^2}. \tag{66}$$

By combining equations (65) and (66) we readily see that the first moment is zero, while the Laplace transform of the second moment reads

$$\mathcal{L}\Big\{\big\langle x^2(t)\big\rangle\Big\} = \frac{2c^2}{s^{\alpha+1}(s^\alpha + 2\lambda)}. \tag{67}$$

This expression can be inverted to obtain the exact expression for $\langle x^2(t)\rangle$ in terms of a Mittag-Leffler function [56]. We will not present this solution here (see Ref. [10] for details). Instead, we restrict ourselves to the asymptotic expressions of the mean square displacement for short and long times.

Using Tauberian theorems, we see from equation (67) that as $s \to \infty$, the Laplace transform of the mean square displacement goes as $1/s^{2\alpha+1}$, which implies that

$$\big\langle x^2(t)\big\rangle \sim t^{2\alpha} \quad (t \to 0). \tag{68}$$

For the non-fractional case $\alpha = 1$, we recover the well known result that $\langle x^2(t)\rangle \sim t^2$, that is, the motion is ballistic at short times. We also observe that for small times we have subdiffusion if $0 < \alpha < 1/2$, normal diffusion if $\alpha = 1/2$ and superdiffusion if $1/2 < \alpha \le 1$.

In an analogous way, when $s \to 0$ the transformed mean-square displacement goes as $1/s^{\alpha+1}$. Hence,

$$\big\langle x^2(t)\big\rangle \sim t^{\alpha} \quad (t \to \infty), \tag{69}$$

and the mean square displacement is always subdiffusive at long times.

6 The fractional telegrapher's equation

We saw earlier that the master equation for the Markovian non-fractional CTPRW is the telegrapher's equation (37). Analogously, the master equation for the fractional CTPRW is the fractional telegrapher's equation (FTE). This has recently been shown by one of us [10]. Here we follow and review portions of that work[5].

As far as we know, there have been few efforts in the literature directed toward deriving (or, at least, justifying on physical grounds) the FTE. In the past decade some work has appeared in the mathematical literature, especially the work of Orshinger and collaborators [55–57], who analyzed mathematical and other formal properties of the FTE but without probing physical considerations. In that work the fractional equation is assumed in an *ad hoc* manner by simply replacing ordinary derivatives in the ordinary TE (37) by fractional derivatives. In this way the standard form of the FTE in one dimension is assumed to be given by

$$\frac{\partial^{2\alpha} p}{\partial t^{2\alpha}} + 2\lambda\frac{\partial^{\alpha} p}{\partial t^{\alpha}} = c^2\frac{\partial^{2\gamma} p}{\partial x^{2\gamma}}, \tag{70}$$

where $0 < \alpha \le 1$, $0 < \gamma \le 1$, and $\lambda > 0$ and c are given parameters. Equation (70) is the space-time FTE. The particular case $\gamma = 1$ is called *time-fractional TE*, while $\alpha = 1$ corresponds to the *space-fractional TE*.

One of the few attempts to provide physical grounds for the time-fractional TE was the work of Compte and Metzler [58,59] (see also Ref. [60]) who, starting from Cattaneo's equation (a modification of Fick's law accounting for non instantaneous diffusion [21]) proposed three different candidates for the one-dimensional time-fractional TE. One of them (the only one addressed here), having the standard form given in equation (70), is derived from the CTRW formalism applied to the probability flux followed by the assumption of a Gaussian distribution for jump lengths $f(x)$ [58]. We here summarize the work of reference [10], which consists of a more general approach based on the persistent random walk and resulting in a space-time FTE agreeing with the standard form given above.

To derive the FTE in the fractional CTPRW picture, we first need to introduce some mathematical formalism concerning fractional derivatives. The Caputo fractional derivative of order $\beta > 0$ of a function $\phi(t)$ is defined by the functional [42,45,46,61,62]

$$\frac{\partial^\beta \phi(t)}{\partial t^\beta} = \frac{1}{\Gamma(n-\beta)}\int_0^t \frac{\phi^{(n)}(t')dt'}{(t-t')^{1+\beta-n}} \tag{71}$$

($n-1 < \beta < n$, $n = 1,2,3,\ldots$). When $\beta = n$ is a positive integer, this derivative coincides with the ordinary derivative $\phi^{(n)}(t)$.

Using this definition we can readily obtain the Laplace transform of the Caputo derivative. Laplace transforming

[5] Generalizing the concept of persistence to dimensions greater than one is inherently difficult which, in turn, hinders obtaining higher-dimensional TEs [48–54].

equation (71) and using the convolution theorem we obtain

$$\mathcal{L}\left\{\frac{\partial^\beta \phi(t)}{\partial t^\beta}\right\} = \frac{1}{\Gamma(n-\beta)}\mathcal{L}\left\{\phi^{(n)}(t)\right\}\mathcal{L}\left\{t^{n-\beta-1}\right\}.$$

With the explicit forms [27]

$$\mathcal{L}\left\{\phi^{(n)}(t)\right\} = s^n\widehat{\phi}(s) - \sum_{k=0}^{n-1} s^{n-1-k}\phi^{(k)}(0),$$

and

$$\mathcal{L}\left\{t^{n-\beta-1}\right\} = \Gamma(n-\beta)s^{\beta-n},$$

the Laplace transform of the Caputo derivative is found to be

$$\mathcal{L}\left\{\frac{\partial^\beta \phi(t)}{\partial t^\beta}\right\} = s^\beta\widehat{\phi}(s) - s^{\beta-1}\phi(0) - \sum_{k=1}^{n-1} s^{\beta-1-k}\phi^{(k)}(0). \qquad (72)$$

In order to derive the space-time FTE (70), another operator is needed: the Riesz-Feller fractional derivative of order β $(0 < \beta \le 2)$ of a function $f(x)$ vanishing at $x \to \pm\infty$. The simplest way to define this operator is via the inverse Fourier transform [42]:

$$\frac{\partial^\beta f(x)}{\partial |x|^\beta} = \mathcal{F}^{-1}\Big\{|\omega|^\beta\widetilde{f}(\omega)\Big\}. \qquad (73)$$

We are now equipped to derive the FTE. We begin with equation (60), which we can rewrite as

$$\left(s^{2\alpha} + 2\lambda s^\alpha + c^2|\omega|^{2\gamma}\right)\widehat{\widetilde{p}}(\omega, s) = s^{2\alpha-1} + 2\lambda s^{\alpha-1}.$$

Taking into account the definition (73) of the Riesz-Feller derivative and recalling that $\mathcal{F}^{-1}\{1\} = \delta(x)$, the Fourier inversion yields

$$\left(s^{2\alpha} + 2\lambda s^\alpha - c^2\frac{\partial^{2\gamma}}{\partial x^{2\gamma}}\right)\widehat{p}(x, s) = \left(s^{2\alpha-1} + 2\lambda s^{\alpha-1}\right)\delta(x).$$

We rewrite this last equation as

$$s^{2\alpha}\widehat{p}(x, s) - s^{2\alpha-1}\delta(x) + 2\lambda\left[s^\alpha\widehat{p}(x, s) - s^{\alpha-1}\delta(x)\right] = c^2\frac{\partial^{2\gamma}\hat{p}}{\partial x^{2\gamma}}. \qquad (74)$$

The next step, to Laplace invert equation (74), and thus to obtain an equation for $p(x,t)$, consists in evaluating the Laplace transform of the fractional derivatives $\partial^\alpha p/\partial t^\alpha$ and $\partial^{2\alpha}p/\partial t^{2\alpha}$ using equation (72). We must separate the cases $\beta = \alpha$ and $\beta = 2\alpha$.

(i) Set $\beta = \alpha$ in equation (72). Since $0 < \alpha \le 1$, we see that $n = 1$. Hence

$$\mathcal{L}\left\{\frac{\partial^\alpha p(x,t)}{\partial t^\alpha}\right\} = s^\alpha\widehat{p}(x, s) - s^{\alpha-1}p(x, 0).$$

Recall that $p(x, 0) = \delta(x)$ [cf. Eq. (36)]. Therefore

$$\frac{\partial^\alpha p(x,t)}{\partial t^\alpha} = \mathcal{L}^{-1}\Big\{s^\alpha\widehat{p}(x, s) - s^{\alpha-1}\delta(x)\Big\}. \qquad (75)$$

(ii) When $\beta = 2\alpha$ $(0 < \alpha \le 1)$ we need to distinguish the cases (a) $0 < \alpha \le 1/2$ and (b) $1/2 < \alpha \le 1$. For case (a) we have $0 < \alpha \le 1/2$, which reproduces the conditions leading to equation (75),

$$\mathcal{L}\left\{\frac{\partial^{2\alpha} p(x,t)}{\partial t^\alpha}\right\} = s^{2\alpha}\widehat{p}(x, s) - s^{2\alpha-1}\delta(x).$$

In case (b) we have $1/2 < \alpha \le 1$ and from equation (72) with $n = 2$ we write

$$\mathcal{L}\left\{\frac{\partial^{2\alpha} p(x,t)}{\partial t^\alpha}\right\} = s^{2\alpha}\widehat{p}(x, s) - s^{2\alpha-1}\delta(x) - s^{2(\alpha-1)}\left.\frac{\partial p(x,t)}{\partial t}\right|_{t=0}.$$

Since $\partial p/\partial t|_{t=0} = 0$ [cf. Eq. (36)] we see that this case coincides with case (a) above. Therefore,

$$\frac{\partial^{2\alpha}p}{\partial t^{2\alpha}} = \mathcal{L}^{-1}\Big\{s^{2\alpha}\widehat{p}(x, s) - s^{2\alpha-1}\delta(x)\Big\}, \qquad (76)$$

$(0 < \alpha \le 1)$. Returning to equation (74) and taking the inverse transform we find

$$\mathcal{L}^{-1}\Big\{s^{2\alpha}\widehat{p}(x, s) - s^{2\alpha-1}\delta(x)\Big\} + 2\lambda\mathcal{L}^{-1}\Big\{s^\alpha\widehat{p}(x, s) - s^{\alpha-1}\delta(x)\Big\} = c^2\frac{\partial^{2\gamma}p}{\partial x^{2\gamma}}.$$

Using equations (75) and (76), we immediately recover the space-time FTE equation (70).

In Section 4 we showed that the ordinary telegrapher's equation (35) enjoys both wave and diffusion characteristics (see also remarks at the end of Sect. 5 involving Eqs. (53) and (64) and also Eqs. (68) and (69)). We will now extend this duality to the FTE.

We start by taking the limit $\lambda \to \infty$ in equation (70) and also letting $c \to \infty$ such that $c^2/2\lambda \to D$ finite. In this way we immediately obtain the "fractional diffusion equation":

$$\frac{\partial^\alpha p}{\partial t^\alpha} = D\frac{\partial^{2\gamma}p}{\partial x^{2\gamma}}. \qquad (77)$$

We also recall that for any value of the parameters λ and c, the fractional diffusion equation (77) is the asymptotic limit of the FTE as $t \to \infty$. Indeed from the solution of equation (70), whose Laplace-Fourier transform is given by equation (60), we have shown in Section 5 that the small s approximation for $\widehat{\widetilde{p}}(\omega, s)$ [corresponding to the large t approximation for $\widetilde{p}(\omega, t)$] is given by equation (53). Using the same procedure as that used in going from equation (60) to the FTE (70), one can go from equation (62) to the fractional diffusion equation thus proving, by virtue of Tauberian theorems, that equation (77) is the asymptotic (long t) approximation to equation (70).

The FTE (70) also contains the so-called "fractional wave equation" as a limiting case. Letting $\lambda \to 0$ (i.e., $T \to \infty$) and at the same time $L \to \infty$ such that $c = L^\gamma/T^\alpha$ remains finite, equation (70) reduces to a wave-like equation,

$$\frac{\partial^{2\alpha}p}{\partial t^{2\alpha}} = c^2\frac{\partial^{2\gamma}p}{\partial x^{2\gamma}}. \qquad (78)$$

Note that when $\alpha = 1/2$ and $\gamma = 1$ this equation reduces to the ordinary diffusion equation. In this sense Mainardi's terminology [45], "fractional diffusion-wave equation", is more precise than fractional wave equation. Finally, we observe that the fractional diffusion-wave equation (78) is the small t limit of the FTE regardless of the value of λ. Indeed, the limit $s \to \infty$ in equation (60) as we have seen yields equation (64) which, after Fourier-Laplace inversion, results in equation (78).

All of this reflects the fact that the FTE embraces two different dynamics: one, at small times, represents fractional wavelike behavior, and another one more clearly apparent at long times is a fractional diffusion-like behavior. This constitutes a generalization of the dual character of the ordinary telegrapher's equation between wavelike dynamics and diffusion-like dynamics as discussed in Section 4.

7 Closing words

The history of continuous time random walks started by Montroll and Weiss over five decades ago has certainly spawned a huge amount of fascinating work in applied mathematics, statistical mechanics, and a large variety of applications, and the work continues. This has been one of those examples whose beginnings are so easily pinpointed and yet whose end is nowhere in sight. We, the authors, have been extremely fortunate to have had the opportunity to work with George Weiss over the years (and one of us also with Elliott Montroll), and this has been a truly great experience.

In this review we have touched on only some of the developments in this subject – there are many others that we have not had the room to include here (see, for instance, the extended and updated bibliography of the preface to this commemorative volume [65]). But the ones we have included have been some of the ones that have kept each of us engaged for so many years. The sequence of topics we have covered is laid in the last paragraph of our Introduction – it is a particular train that has led to some of the most interesting developments in so-called fractional walks. There is no doubt that there is yet a great deal more to come.

There is not much more to say other than thank you, George.

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Author contribution statement

Both authors contributed equally to this paper.

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