Eur. Phys. J. B (2015) 88: 125
DOI: 10.1140/epjb/e2015-50865-3

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Parameter dependence of stochastic resonance in the FitzHugh-Nagumo neuron model driven by trichotomous noise

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Received 17 December 2014 / Received in final form 10 March 2015

Abstract. We investigate the stochastic resonance in a FitzHugh-Nagumo neuron model driven by trichotomous noise and periodic signal, focusing on the dependence of properties of stochastic resonance (SR) on system parameters. The stochastic resonance is shown through several different measures: system response, power spectrum and signal-to-noise ratio. Firstly, it is found that whether the neuron can fire regularly depends on the cooperative effect of the signal frequency and the signal amplitude for the deterministic FHN neuron. When the forcing amplitude alone is insufficient to cause the neuron firing, the neuron can fire with the addition of trichotomous noise. Secondly, we show that power spectrum is maximized for an optimal value of the noise correlation time, which is the signature of SR. Finally, from studying SNR, the specific system parameters are found to optimize the SR phenomenon.

1 Introduction

It has been proved that stochastic noise can enhance the response of nonlinear systems to a weak signal. This phenomenon is called "stochastic resonance" (SR) [1–3]. Originally, SR was proposed to explain why the ice age occurred periodically [4]. The first realization of stochastic resonance in a laboratory experiment was provided by Fauve and Heslot [5], in which the SR is measured by the power spectrum. Then McNamara et al. pointed out that signal-to-noise ratio (SNR) which is obtained from the power spectrum of the interwell motion [6] is a more suggestive signature for SR. The signature of SR is that the SNR passes through a maximum at an optimal value of the input noise parameter. Now, the SR phenomenon has been widely observed for various systems experimentally and theoretically [7,8].

Motivated by the biological applications to neuronal dynamics [9], we employ the FitzHugh-Nagumo (FHN) construct as the model system for investigating SR. FHN model is a simplification of the Hodgkin-Huxley (HH) model [10] of spike generation in squid giant axons. In 1961, FitzHugh sought to reduce the HH model to a simpler set of equations in two state variables while retaining its essential excitation characteristics [11]. The reduced version was experimentally demonstrated by Nagumo et al. using electrical circuits and it has since been called the FHN model [12]. It is a model of reduced complexity providing an insight into the dynamical aspects of neuronal activity, such as the complex responses of neurons to sinusoidal stimuli [13]. This model has been widely used in the literature. For example, Nozaki and Yamamoto investigated the stochastic resonance in a FitzHugh-Nagumo neuronal model driven by colored noise [14]. Heneghan et al. demonstrated that sensory neurons could harness aperiodic stochastic resonance to optimize the detection and transmission of weak stimuli [15].

Although this system has been used in a number of physiologically motivated SR investigations [16–18], few results have been obtained on SR driven by trichotomous noise which is a kind of three-level Markovian noise characterized by three parameters: amplitude, correlation time and flatness [19]. Studying the noise has an important practical significance. Firstly, the trichotomous noise is an important type of non-Gaussian colored noise and is also a particular case of the kangaroo process [20,21] that has applications in various fields of science. For example Tammelo et al. have studied transport of Brownian particles in a simple sawtooth potential subjected to both unbiased thermal and nonequilibrium symmetric three-level Markovian noise [22]. Mankin et al. have explored current reversals in ratchets driven by trichotomous noise [23] and the problem of multiple noise-enhanced stability versus temperature [24]. SR which is induced by trichotomous noise for fractional oscillator has been investigated [25–28]. The anomalous transport phenomena induced by trichotomous noise in periodic system has been discussed and the necessary conditions for various anomalous transport properties have been found [29,30]. In addition to mimicking the effects of the finite correlation time of the real noise, the trichotomous noise may directly provide a

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realistic representation of a real physical situation such as thermal transitions between three configurations or states. Moreover, although both trichotomous noise and dichotomous noise are stationary telegraph processes, the former is more flexible and includes all cases of dichotomous noise [29–32]. When the dynamical system is very complex and cannot be studied analytically, it is essential to solve this problem of a numerical scheme. In this paper, trichotomous noise is obtained numerically and we present the phenomenon of stochastic resonance in a stochastic nonlinear dynamical system subject to trichotomous noise.

This paper is organized as follows. Section 2 presents the FitzHugh-Nagumo neuron model (FHN model). In Section 3, in order to study SR phenomenon, we calculate system response, power spectrum and signal-to-noise ratio numerically. Finally, some discussions and conclusions are given in Section 4.

2 Model system

In this paper, we consider a FHN model [33] with an aperiodic input signal driven by trichotomous noise:

$$\varepsilon\dot{x} = -x(x^2 - 1/4) - \omega + A_0 + S(t) + \xi(t), \qquad (1)$$
$$\dot{\omega} = x - \omega\,, \qquad (2)$$

where $x(t)$ is a "fast" variable representing the membrane voltage of the neuron, $\omega(t)$ is a "slow" (recovery) variable which determines the refractory time, A_0 is a critical value which makes the neuron fire periodically. Throughout the paper we fix $A_0 = -0.31056$. As the state variables $x(t)$ and $\omega(t)$ exhibit dynamics on different time scales, the parameter ε is chosen such that $\varepsilon \ll 1$. We fix $\varepsilon = 0.005$. $S(t) = A\sin 2\pi ft$ is the sinewave input, where A is the amplitude, and f is the frequency of the periodic signal. $\xi(t)$ is assumed to be a zero-mean symmetric trichotomous noise which is a random stationary Markovian process that consists of jumps between three values: a, 0 and $-a$. The jumps follow in time according to a Poisson process, while the values occur with the stationary probabilities: $P_s(a) = P_s(-a) = q$, $P_s(0) = 1-2q$ with $0 < q \leqslant 1/2$. We can obtain the transition probabilities between the states $\pm a$ and 0:

$$P\left(-a, t+t^{'}|a, t\right) = P\left(a, t+t^{'}|-a, t\right) = P\left(\pm a, t+t^{'}|0, t\right) = q\left(1 - e^{-vt^{'}}\right), \qquad (3)$$

$$P\left(0, t+t^{'}|a, t\right) = P\left(0, t+t^{'}|-a, t\right) = (1-2q)\left(1 - e^{-vt^{'}}\right),\ \ t^{'} > 0, v > 0. \qquad (4)$$

The switching rate v is the reciprocal of the noise correlation time: $v = 1/\tau$. In the stationary case the fluctuation process satisfies the following conditions:

$$\langle\xi(t)\rangle = 0, \quad \left\langle\xi(t)\xi(t^{'})\right\rangle = 2qa^2 e^{-v\left|t-t^{'}\right|}. \qquad (5)$$

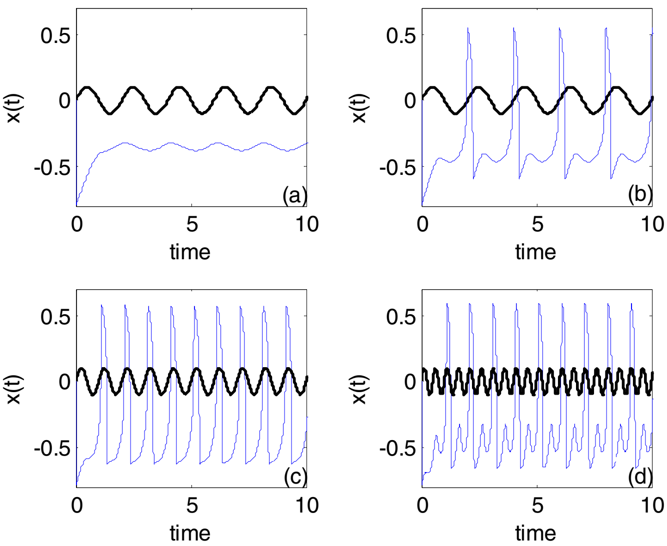

Fig. 1. A time series of membrane potentials of the noiseless FHN model. (a) $A = 0.01$, $f = 0.5$, (b) $A = 0.05$, $f = 0.5$, (c) $A = 0.05$, $f = 1$, (d) $A = 0.05$, $f = 2$. The amplitude for the input signal is enlarged ten times for better viewing (heavy line).

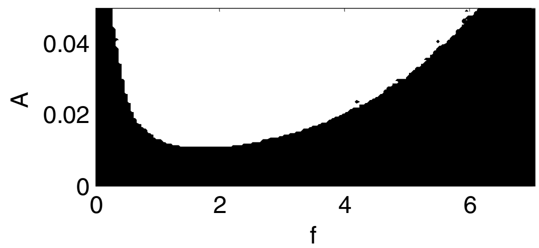

Fig. 2. Phase diagram of the FHN neuron without noise in the parameter space of the forcing frequency f and the forcing amplitude A. The firing state is denoted in white, and the nonfiring state and bistable state in black.

The flatness parameter φ is [19]

$$\varphi = \left\langle\xi^4(t)\right\rangle / \left\langle\xi^2(t)\right\rangle^2 = 1/2q. \qquad (6)$$

It can be seen that the flatness parameter of trichotomous noise can range from 1 to ∞.

3 Stochastic resonance

Figure 1 presents dynamical responses of the noiseless FHN neuron under different sinusoidal current. Apparently, for $A = 0.01$, $f = 0.5$, the signal is too weak to excite a neuron. As the amplitude increases, the neuron is excited to output the spike train. Especially, the forcing amplitude requires of an optimal frequency to bring out the output of the variable x and signal is very well synchronized in time. However, with the frequency further increasing, the neuron does not fire in each period cycle of the external signal, with several cycles being skipped.

Figure 2 depicts the phase diagram of the deterministic FHN neuron in the parameter space of the forcing

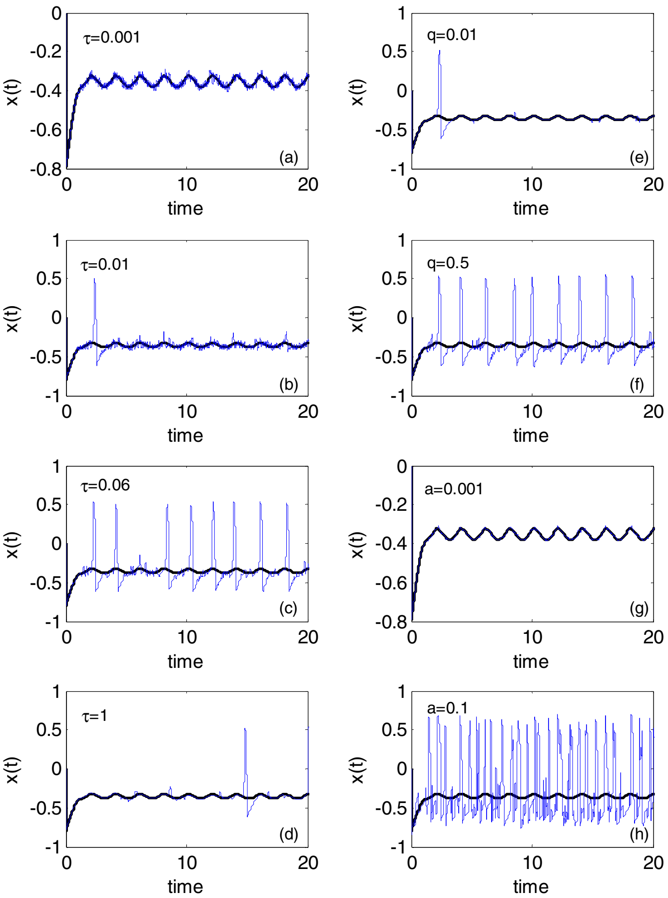

Fig. 3. Membrane potentials for various values of the noise correlation time τ with fixed $A = 0.01$, $f = 0.5$. The profiles of the dynamical response without noise are also superimposed with heavy line. (a)–(d) $a = 0.01$, $q = 0.3$, (e) and (f) $a = 0.01$, $\tau = 0.06$, (g) and (h) $\tau = 0.06$, $q = 0.3$.

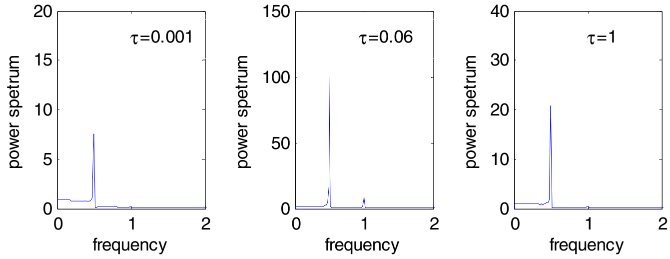

Fig. 4. The power spectrum of x with three different τ value: 0.001, 0.06, 1. Other parameters of the system are $a = 0.01$, $q = 0.3$, $A = 0.01$, $f = 0.5$.

frequency f and the forcing amplitude A, in which the firing state is denoted in white, and the nonfiring state and bistable state in black. It is shown that when the amplitude is too small, the firing cannot appear. Note that the increasing of the forcing amplitude tends to expand the effective frequency domain in which the FHN neuron is excited to fire.

Next, some realizations of the voltage variable are presented to show the effect of trichotomous noise on the dynamic response of the FHN model in Figure 3. Dynamical response without noise is shown in heavy line. We consider a subthreshold signal amplitude $A = 0.01$ that does not allow firing in the absence of noise. Figures 3a–3d display the trajectories for various noise correlation time with fixed $a = 0.01$, $q = 0.3$. For a very small τ, the neuron cannot be excited to fire as the noise would hardly reach the threshold for firing. As the noise correlation time is increased, the neuron shows its basic oscillatory character with bursts. What is more, for small or large noise correlation time, the system spends most of its time fluctuating around the rest potential and displays trains of few short periodic oscillations. It is worth noting that the phenomenon of synchronization is qualitatively visible for moderate noise and the noise-excited oscillations becomes more irregular for larger or smaller noise correlation time. This is a significant of stochastic resonance. With fixed $a = 0.01$, $\tau = 0.06$, increasing q causes less stimulus cycles to be skipped and the phenomenon of synchronization is more easily to be seen as q increases. Remarkably, for $q = 1/2$, the neuron is capable of firing at every circle. With fixed $\tau = 0.06$, $q = 0.3$, we can observe that increasing a causes increase in firing rate of the neuron. However, when a is very large, the noise-excited oscillations becomes more irregular.

To identify the stochastic resonance behavior in the FHN model the power spectral density is calculated with different values of τ in Figure 4. The peak in the power spectrum can be very well pronounced at the frequency of the periodic signal. It is evident that the height of the noise-induced peak in the power spectrum is very small for a weak noise. With the increase of noise correlation time, the height increases. However, with further increase of τ, the height of the peak starts to decrease. This demonstrates that the neuron responds more coherently in the case of $\tau = 0.06$, which agrees with the result presented in Figure 3. This significant observation indicates the possibility of an occurrence of stochastic resonance.

Another dynamical feature of the SR phenomenon can be measured from the SNR. The SNR is obtained from the power spectrum as

$$SNR = 10 \log \frac{S}{N} = 10 \log \frac{S(\omega_0)}{N(\omega_0)}, \quad (7)$$

where the signal power $S = S(\omega_0) = |Y(\omega_0)|^2$ is the magnitude of the output power spectrum $Y(\omega)$ at the input frequency ω_0, and the noise power $N = N(\omega_0)$ at input frequency ω_0 is some average of $|Y(\omega)|^2$ at nearby frequencies [33].

Firstly, we investigate the effect of noise parameter on SNR. The results of calculation of the SNR as the function of noise correlation time are shown in Figure 5 for different values of a. It can be seen that when the appropriate noise is applied, the SNR increases monotonously as a function of τ indicating the occurrence of SR phenomenon. When a is small and large, the SNR increases at first and then decreases as τ increases. Within a certain range of a, the increase of τ induces the value of the SNR to decrease at

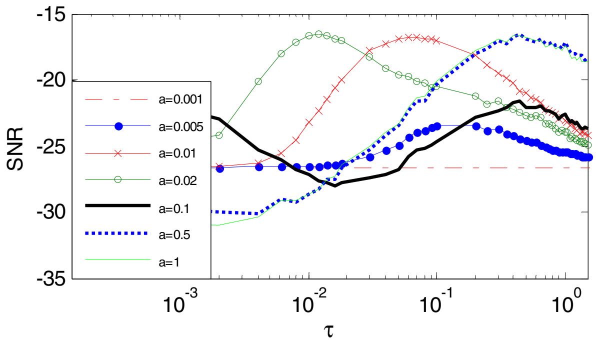

Fig. 5. The performance of SNR as the functions of τ with various values of a. The parameters are $A = 0.01$, $f = 0.5$, $q = 0.3$.

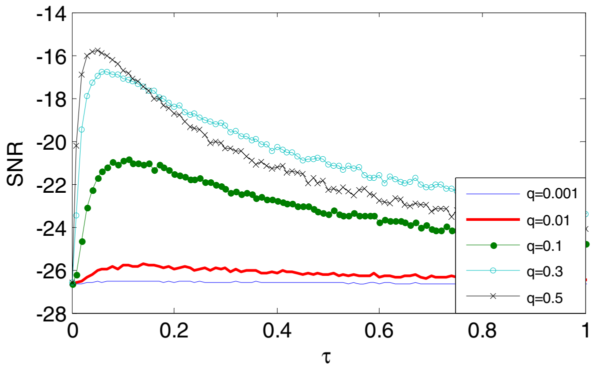

Fig. 6. Plot of SNR as a function of τ for different values of q with $a = 0.01$. The other parameters are identical with Figure 5.

first. It is quite interesting to find that the position of the peak shifts to the left at first and then shifts to the right as the noise amplitude increases. In addition, the peak value of SNR increases initially, then decreases and increases finally with the increasing of a. Note that when a is large enough, the SNR curve changes little with further increase of a. In Figure 6, the various SNR curves with different values of q versus noise correlation time are plotted. It is obvious that the SR phenomenon cannot occur for very small values of q and it can be easily observed by increasing the q value. For an optimal noise correlation time, the SNR possesses a maximum. What is more, the increasing of q enhances the SR phenomenon.

Figure 7 displays SNR as a function of a with different τ. We can observe that as a increases, the SNR first increases reaching to the maximum, then decreases and finally increases again at fixed τ value. However, SNR changes little with further increasing of a. The curves of SNR as a function of q are plotted in Figure 8. At certain noise correlation time the figure displays that with increasing q the SNR first increases and then changes slowly. In conclusion, SR can be enhanced by adjusting noise parameter, which can verify the conclusions showed in Figures 5 and 6.

Now we fix the noise parameter with $a = 0.01$, $q = 0.3$ and study the effect of the input signal on SNR curve in Figure 9. Apparently, with the increase of signal amplitude

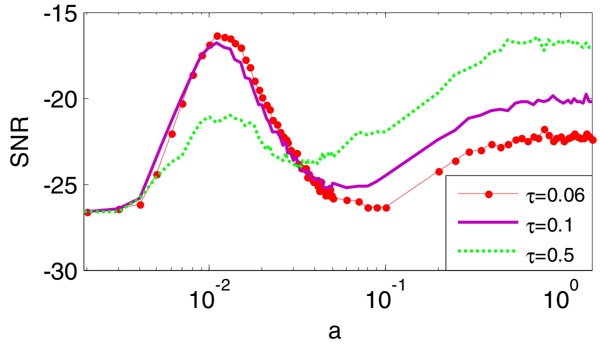

Fig. 7. The curves of SNR versus a for various values of τ with fixed $A = 0.01$, $f = 0.5$, $q = 0.3$.

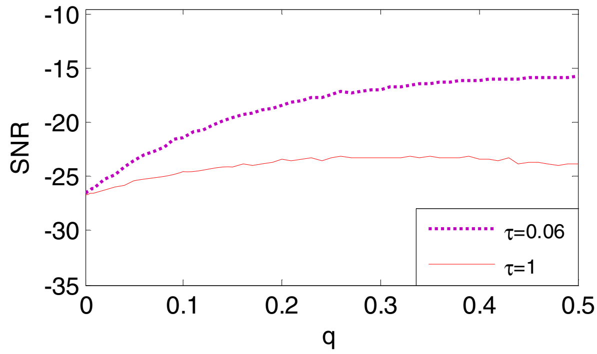

Fig. 8. Plot of SNR as a function of q for different values of τ with $a = 0.01$. The other parameters are identical with Figure 7.

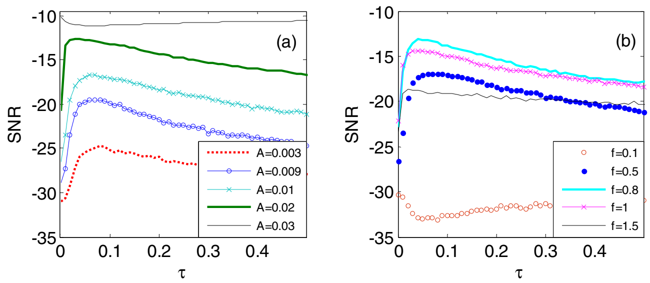

Fig. 9. The SNR curves of a stochastic FH neuron for various forcing amplitudes and frequency with fixed $a = 0.01$, $q = 0.3$. (a) $f = 0.5$, (b) $A = 0.01$.

the height of the SNR peak increases. Note that the SNR value increases as A increases because the dynamical response of the FH neuron becomes more synchronized with stronger periodic stimulus. In addition, we find that the stochastic FH neuron has a critical forcing amplitude A_c, above which the resonance with a maximal SNR disappears. Moreover, we can see that SR occurs only in an optimal range of frequency values. The effect of increasing f is to shift the SNR peak towards smaller vales of τ. Furthermore, we can find an optimal value of f which yields the largest enhancement of the system response. This is due to the fact that the forcing amplitude requires an optimal frequency to make the input signal and the system output to be synergistic which can be seen in Figure 1.

4 Conclusions

In this paper, we mainly consider the phenomenon of SR in a FitzHugh-Nagumo neuron model with trichotomous noise.

Firstly, we demonstrate the effect of signal and noise on the system response. The result shows that under certain conditions the neuron can be excited. Then we discuss the influences of the noise parameter on the system output power spectrum by numerical calculation. The power spectrum exhibits a sharp peak and the peak value presents nonmonotonous behavior as functions of the noise correlation time, which is a typical feature of SR. Finally, the signal-to-noise ratio is used to measure the SR phenomenon. It turned out that SR occurs only in a domain parameter. In addition, the following conclusions can be obtained.

(1) Noise amplitude a first enhances SR, then weakens it and enhances it again. Moreover, the SNR peak shifts towards smaller noise correlation time at first and then shifts towards larger τ value.
(2) Larger q promotes the occurrence of SR.
(3) Larger signal amplitude A promotes the occurrence of SR. However, there is a critical forcing amplitude A_c, above which the resonance with a maximal SNR disappears.
(4) The peak value shifts to smaller noise level with increasing input signal frequency and there is an optimal value of f which yields the largest enhancement of the system response.

Given that FitzHugh-Nagumo model has been widely used as a prototypic model for spiking neurons as well as for cardiac cells, we believe that our results are useful to analyse the excitable cells in living organisms.

This work was supported by the NSF of China (Grant Nos. 11172233, 11372247, 11102157) and Shaanxi province, Program for NCET, NPU Foundation for Fundamental Research, and Graduate Starting Seed Fund. All authors contributed equally to all aspects of this work.

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