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The Immediate Exchange model: an analytical investigation

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Abstract. We study the Immediate Exchange model, recently introduced by Heinsalu and Patriarca [Eur. Phys. J. B **87**, 170 (2014)], who showed by simulations that the wealth distribution in this model converges to a Gamma distribution with shape parameter 2. Here we justify this conclusion analytically, in the infinite-population limit. An infinite-population version of the model is derived, describing the evolution of the wealth distribution in terms of iterations of a nonlinear operator on the space of probability densities. It is proved that the Gamma distributions with shape parameter 2 are fixed points of this operator, and that, starting with an arbitrary wealth distribution, the process converges to one of these fixed points. We also discuss the mixed model introduced in the same paper, in which exchanges are either bidirectional or unidirectional with fixed probability. We prove that, although, as found by Heinsalu and Patriarca, the equilibrium distribution can be closely fit by Gamma distributions, the equilibrium distribution for this model is *not* a Gamma distribution.

1 Introduction

Kinetic exchange models have been widely investigated in recent years within the field of Econophysics, whose aim is to apply ideas and methods from statistical physics to economic questions (see reviews [1–6]). In these models, a large population of agents, each possessing a certain wealth, undergo random pairwise interactions involving transfers of wealth from one agent to another. The precise nature of the interactions differs from one model to another. It is often the case that, as time progresses, the distribution of wealth in the population evolves towards an equilibrium distribution, which is independent of the initial distribution of wealth among the agents. Characterizing this equilibrium wealth distribution for different kinetic exchange models is thus a central question to which many numerical and analytical studies have been devoted. Researchers have also used empirical data on wealth and income distributions in order to examine which theoretical distributions can provide a good fit to the data [2] (Chap. 2), [6–8].

In a recent paper [9], Heinsalu and Patriarca proposed a new kinetic exchange model which they called the Immediate Exchange model. In this model pairs of agents randomly interact, an interaction consisting of each of the agents transferring a random fraction of its wealth to the other agent, where these fractions are independent and uniformly distributed in $[0,1]$. Thus, if agents i,j have wealths x_i, x_j prior to the interaction, their wealths following the interaction are

$$x_i' = (1-\epsilon_i)x_i + \epsilon_j x_j, \quad x_j' = (1-\epsilon_j)x_j + \epsilon_i x_i, \qquad (1)$$

where ϵ_i, ϵ_j are independent and $\epsilon_i, \epsilon_j \sim Uniform([0,1])$. Based on simulation of this process, Heinsalu and Patriarca have concluded that the wealth distribution converges to a Gamma distribution with shape parameter 2. Here we rigorously justify this conclusion, by deriving an infinite population version of the Immediate Exchange model, in which the time-evolution of the wealth distribution is described by iteration of a nonlinear operator on a space of probability distributions (Sect. 2), and showing that Gamma distributions with shape parameter 2 are the fixed points of this operator (Sect. 3). Furthermore, we prove that, starting from a general wealth distribution, iterations of the operator converge to one of these equilibrium distributions, determined by the mean wealth of the initial distribution, which is conserved (Sect. 4).

It is instructive to compare the Immediate Exchange model with some other kinetic exchange models and observe how differences in the microscopic rules of wealth exchange are reflected in the equilibrium distribution that emerges on the global level. In the well-known Drăgulescu-Yakovenko model [10], the wealths of two interacting agents are pooled, and then randomly re-divided among the pair. In contrast to the Immediate Exchange model which leads to a "humped" equilibrium wealth distribution (the Gamma distribution), the Drăgulescu-Yakovenko model leads to an exponential wealth distribution, which thus has a monotone decreasing density, so that the poorest class is the most numerous. The exponential distribution has been found to provide a remarkably good fit to the bulk of some empirical income distributions [2,8] (Chap. 2). Other studies have supported a Gamma distribution as a statistical model for income distributions [7,11]. It is thus of special interest

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to examine which mechanisms in kinetic exchange models can lead to Gamma-type equilibrium distributions. One such mechanism has been proposed in the model of Chakraborti-Chakrabarti [12], namely modifying the Drăgulescu-Yakovenko model by introducing a saving propensity, so that interacting agents share only a certain fraction of their wealths. Simulation of this model yielded equilibrium distributions which are very well fitted by Gamma distributions. Interestingly, later analytical work [13] (see also [2], Chap. 5) has revealed that the equilibrium distribution for this model (in the infinite population limit) is in fact not a Gamma distribution. The Immediate Exchange model offers a different mechanism for generating a Gamma-type distribution, and in this case it turns out, as shown below, that it is indeed exactly a Gamma distribution.

Another class of kinetic exchange models involves unidirectional interactions in which only one of the agents, randomly chosen to be the "loser", transfers a portion of its wealths to the "winner". In Angle's Inequality Process [14,15] the loser transfers a fixed fraction of its wealth to the winner. In Martínez-Martínez and López-Ruiz's Directed Random Market model [16], the fraction of the loser's wealth transferred is random and uniformly distributed in $[0, 1]$. For the Inequality Process an analytic form of the equilibrium distribution is unknown. For the Directed Random Market model it was proved in reference [17] that the equilibrium distribution is a Gamma function with shape parameter $\frac{1}{2}$, which means that its density is monotone decreasing, with a singularity at 0, and thus represents an even more extreme case of the phenomena noted above for the Drăgulescu-Yakovenko model, whereby the poorest group is the largest.

In their paper [9], Heinsalu and Patriarca also presented a "mixed" model in which interactions are either bidirectional as in the Immediate Exchange model, or unidirectional as in the Directed Market model [16], each case occuring with a certain fixed probability. This model is also studied here (Sect. 5) in the infinite-population limit, and we prove that, despite the fact that the equilibrium distribution can be closely fitted by a Gamma distribution, as shown by numerical simulations in [9], the equilibrium distribution is in fact *not* a Gamma distribution.

It should be noted that all the models discussed above and studied below have exponentially decaying tails, and thus do not display the Pareto effect observed in empirical wealth distibutions, wherby the distribution of wealth among the wealthiest 5–10 percent follows a power-law distribution [2] (Chap. 2). Obtaining power-law tails requires introduction of additional features into the models, such as heterogeneity of the agents [18] or dependence of the fraction of wealth exchanged on agents' current wealth [19].

2 Infinite-population version of the Immediate Exchange model

We now formulate the infinite-population discrete-time version of the Immediate Exchange model in the framework of López and co-workers [20,21]. The distribution of wealth is described by a probability density $p_t(x)$ so that $p_t(x)dx$ is the fraction of the population whose wealth is in the interval $[x, x + dx]$ at time $t = 0, 1, 2, \ldots$ It is assumed that at each time step ("day") all agents are randomly paired and exchange wealth according to the rule (1). Assuming the wealth distribution $p_t(x)$ before the interactions of day t take place is given, we derive the probability density $p_{t+1}(x)$ following these interactions, and thus the time-evolution of the distribution of wealth.

The language of probability theory is convenient in deriving the evolution equation. Let us choose a random agent and let U be a random variable representing this agent's wealth before the interaction on day t takes place, and X its wealth following the interaction. Thus the distributions of U and of X are given by the probability densities $p_t(x)$ and $p_{t+1}(x)$, respectively. Let V represent the wealth of the agent with whom our focal agent interacted, which is a random variable whose distribution is also $p_t(x)$. Then we have

$$X = \epsilon_1 U + \epsilon_2 V \tag{2}$$

where ϵ_1, ϵ_2 are independent of each other and of U, V, and uniformly distributed on $[0, 1]$. The probability density $p_{t+1}(x)$ will thus be found by computing the distribution of X given by (2). We use the following simple result.

Lemma 1. Assume W is a non-negative random variable with probability density $p(x)$, and ϵ is a random variable with $\epsilon \sim Uniform([0, 1])$, W, ϵ independent. Then the probability density of the product ϵW is given by:

$$S[p](x) = \int_x^\infty \frac{p(u)}{u} du. \tag{3}$$

Proof.

$$\begin{aligned} P(\epsilon W \le x) &= \int_0^1 \int_0^{\frac{x}{\epsilon}} p(u) du d\epsilon \\ &= \int_0^x p(u) \int_0^1 d\epsilon du + \int_x^\infty p(u) \int_0^{\frac{x}{u}} d\epsilon du \\ &= \int_0^x p(u) du + x \int_x^\infty \frac{p(u)}{u} du \\ \Rightarrow \frac{d}{dx} P(\epsilon W \le x) &= \int_x^\infty \frac{p(u)}{u} du. \end{aligned}$$

Denoting the set of all probability densities on $[0, \infty)$ by $\mathcal{P}$ we can consider S, defined by (3), as an operator $S : \mathcal{P} \to \mathcal{P}$. The above lemma implies that the density of both $\epsilon_1 U$ and $\epsilon_2 V$ is $S[p_t]$. Therefore the density of p_{t+1} of X, which by (2) is the density of the sum of two independent and identically distributed random variables $\epsilon_1 U, \epsilon_2 V$ is given by the convolution:

$$p_{t+1}(x) = (S[p_t] * S[p_t])(x) = \int_0^x S[p_t](x - v) S[p_t](v) dv, \tag{4}$$

or more explicitly

$$p_{t+1}(x) = \int_0^x \left(\int_{x-v}^{\infty} \frac{p_t(u)}{u} du \right) \left(\int_v^{\infty} \frac{p_t(u')}{u'} du' \right) dv$$
$$= \int_0^x \int_y^{\infty} \int_{x-y}^{\infty} \frac{p_t(u)}{u} \frac{p_t(v)}{v} dv du dy. \quad (5)$$

In other words, defining the nonlinear operator $T : \mathcal{P} \to \mathcal{P}$ by:

$$T[p] \doteq S[p] * S[p], \quad (6)$$

we have that the evolution of the wealth distribution for the Immediate Exchange model is given by:

$$p_{t+1} = T[p_t], \quad t = 0, 1, 2, \ldots \quad (7)$$

This is the infinite-population formulation of the Immediate Exchange model.

3 The equilibrium distribution

By (7) the equilibrium distributions are thus the solutions of the functional equation $T[p] = p$, that is:

$$p = S[p] * S[p]. \quad (8)$$

To solve this equation, we apply the Laplace transform

$$\mathcal{L}[p](s) = \int_0^{\infty} e^{-sx} p(x) dx$$

to both sides of (8), and set $\hat{p} = \mathcal{L}[p]$, obtaining

$$\hat{p}(s) = (\mathcal{L}[S[p]](s))^2 .$$

Noting that

$$\mathcal{L}[S[p]](s) = \frac{1}{s} \int_0^s \hat{p}(s') ds', \quad (9)$$

we conclude that the Laplace-transformed version of (8) is:

$$\hat{p}(s) = \left(\frac{1}{s} \int_0^s \hat{p}(s') ds' \right)^2 . \quad (10)$$

To solve this equation, we set

$$g(s) = \sqrt{\hat{p}(s)}$$

and obtain that (10) is equivalent to:

$$g(s) = \frac{1}{s} \int_0^s (g(s'))^2 ds'.$$

Multiplying both sides by s and then differentiating, we obtain the differential equation

$$[sg(s)]' = g(s)^2,$$

that is

$$g'(s) = \frac{1}{s} g(s)[g(s) - 1],$$

a separable equation which is solved to yield:

$$g(s) = \frac{1}{1 + Cs},$$

hence

$$\hat{p}(s) = (g(s))^2 = \frac{1}{(1 + Cs)^2}.$$

The inverse Laplace transform now gives:

$$p(x) = \frac{1}{C^2} x e^{-\frac{x}{C}}.$$

Denoting by w the mean wealth $w = \int_0^{\infty} x p(x) dx$, we have $C = \frac{w}{2}$, which yields

Theorem 1. For each $w > 0$, there exists a unique equilibrium distribution for the Immediate Exchange process satisfying $\int_0^{\infty} x p(x) dx = w$, given by:

$$p_w(x) = \frac{4}{w^2} x e^{-\frac{2}{w} x}. \quad (11)$$

This is the Gamma distribution with shape parameter 2, as found in the simulations of [9].

4 Convergence to the equilibrium distribution

To fully explain the simulation results in reference [9], we need to prove that the iterations (7) converge to an equilibrium distribution (11), starting from an arbitrary initial probability density p_0. Since the process is wealth-preserving (see Lem. 2 below), the value of w will be determined by the mean wealth of the initial density:

$$w = \int_0^{\infty} x p_0(x) dx. \quad (12)$$

Theorem 2. Let $p_0(x)$ be a probability density on $[0, \infty)$ satisfying (12), and such that, for some $\alpha > 1$,

$$M_{\alpha}(p) = \int_0^{\infty} p(x) x^{\alpha} dx < \infty. \quad (13)$$

Then the cumulative probability functions of the iterations (7) converge to that of $p_w(x)$ given by (11), that is for all $x \geq 0$,

$$\lim_{t \to \infty} \int_0^x p_t(u) du = \int_0^x p_w(u) du.$$

Since the proof of Theorem 2 follows the same technique as that used for analogous results for the Drăgulescu-Yakovenko model and the Directed Random Market model [17,22], we will be brief, and refer to those papers for details, indicating only the general argument and some points where calculations somewhat different from those in the above papers are required.

For $\alpha \geq 1$ and $w > 0$, we define $\mathcal{P}_{\alpha,w}$ as the set of all probability densities satisfying (12) and (13). We first show that the operator T defined by (6) maps the space $\mathcal{P}_{\alpha,w}$ into itself.

Lemma 2. If $\alpha \geq 1$, $w > 0$, and $p \in \mathcal{P}_{\alpha,w}$ then $T[p] \in \mathcal{P}_{\alpha,w}$.

Proof. Assume $p \in \mathcal{P}_{\alpha,w}$. Exchanging order of integration, and using the inequality $(x+u)^\alpha \leq 2^{\alpha-1}(x^\alpha + u^\alpha)$, we have:

$$\begin{aligned}
M_\alpha(p) &= \int_0^\infty x^\alpha T[p](x)dx \\
&= \int_0^\infty x^\alpha \int_0^x S[p](x-v)S[p](v)dvdx \\
&= \int_0^\infty x^\alpha \int_0^x \left(\int_{x-v}^\infty \frac{p(u)}{u}du\right)\left(\int_v^\infty \frac{p(u')}{u'}du'\right)dvdx \\
&= \int_0^\infty \left(\int_v^\infty \frac{p(u')}{u'}du'\right)\int_v^\infty x^\alpha \left(\int_{x-v}^\infty \frac{p(u)}{u}du\right)dxdv \\
&= \int_0^\infty \left(\int_v^\infty \frac{p(u')}{u'}du'\right)\int_0^\infty (x+v)^\alpha \left(\int_x^\infty \frac{p(u)}{u}du\right)dxdv \\
&\leq 2^{\alpha-1}\int_0^\infty \left(\int_v^\infty \frac{p(u')}{u'}du'\right)\int_0^\infty (x^\alpha + v^\alpha) \\
&\quad \times \left(\int_x^\infty \frac{p(u)}{u}du\right)dxdv \\
&= 2^{\alpha-1}\int_0^\infty \left(\int_v^\infty \frac{p(u')}{u'}du'\right)\int_0^\infty \frac{p(u)}{u}\int_0^u x^\alpha dxdudv \\
&\quad + 2^{\alpha-1}\int_0^\infty v^\alpha \left(\int_v^\infty \frac{p(u')}{u'}du'\right)\int_0^\infty \frac{p(u)}{u}\int_0^u dxdudv \\
&= \frac{2^{\alpha-1}}{\alpha+1}M_\alpha(p)\int_0^\infty \int_v^\infty \frac{p(u')}{u'}du'dv \\
&\quad + 2^{\alpha-1}\int_0^\infty v^\alpha \int_v^\infty \frac{p(u')}{u'}du'dv \\
&= \frac{2^{\alpha-1}}{\alpha+1}M_\alpha(p) + \frac{2^{\alpha-1}}{\alpha+1}M_\alpha(p) = \frac{2^\alpha}{\alpha+1}M_\alpha(p),
\end{aligned}$$

so that $T[p]$ satisfies (13).

Setting $\alpha = 1$, the above inequality becomes an equality, and we obtain that $M_1(T[p]) = M_1(p)$, so that $T[p]$ satisfies (12).

We define the following metric on the set $\mathcal{P}_{\alpha,w}$, where we now assume $\alpha \in (1,2)$.

$$p, q \in \mathcal{P}_{\alpha,w} \quad \Rightarrow \quad d_\alpha(p,q) \doteq \sup_{s>0} \frac{|\mathcal{L}[p](s) - \mathcal{L}[q](s)|}{s^\alpha}.$$

The finiteness of $d_{\alpha,w}(p,q)$ is ensured whenever $1 < \alpha < 2$, see [22], Lemma 2.3.

We use the following key estimate:

Lemma 3. If $1 < \alpha < 2$, $w > 0$, $p, q \in \mathcal{P}_{\alpha,w}$, then

$$d_\alpha(T[p], T[q]) \leq \frac{2}{\alpha+1} \cdot d_\alpha(p,q).$$

Proof. Recalling (9), we have:

$$\begin{aligned}
\mathcal{L}[T[p]](s) &= (\mathcal{L}[S[p]])^2 \\
&= \left(\frac{1}{s}\int_0^s \hat{p}(s')ds'\right)^2 = \left(\int_0^1 \hat{p}(su)du\right)^2,
\end{aligned}$$

hence, since $|\hat{p}(s)|, |\hat{q}(s)| \leq 1$,

$$\begin{aligned}
&\frac{|\mathcal{L}[T[p]](s) - \mathcal{L}[T[q]](s)|}{s^\alpha} \\
&= \frac{1}{s^\alpha}\left|\left(\int_0^1 [\hat{p}(su) - \hat{q}(su)]du\right)\left(\int_0^1 [\hat{p}(su) + \hat{q}(su)]du\right)\right| \\
&\leq 2\left(\int_0^1 u^\alpha \frac{|\hat{p}(su) - \hat{q}(su)|}{(su)^\alpha}du\right) \\
&\leq 2d_\alpha(p,q)\int_0^1 u^\alpha du = \frac{2}{\alpha+1}d_\alpha(p,q),
\end{aligned}$$

and taking the supremum over $s > 0$ we obtain the result.

Since $\alpha > 1$ implies $\frac{2}{\alpha+1} < 1$ the above lemma implies that T is contracting with respect to the metric d_α, and, by the argument given in reference [17,22], this implies that that the iterates p_t converge to p_w in the metric d_α, and hence in the cumulative probability sense of Theorem 2, concluding the proof of the theorem.

5 The mixed model

We now discuss another model proposed in [9], which is a "mixture" of the Immediate Exchange model discussed above with a model of unidirectional wealth transfers.

In the unidirectional model, when two agents interact, one agent is randomly assigned to be the "loser" and the other the "winner". The loser gives the winner a random fraction ϵ of its wealth. Thus if j is the winner then

$$x_i' = (1-\epsilon)x_i, \quad x_j' = x_j + \epsilon x_i,$$

where $\epsilon \sim Uniform([0,1])$. This model has recently been studied by Martínez-Martínez and López-Ruiz [16] who called it the Directed Random Market. They showed that in the infinite population limit the evolution of the wealth distribution is given by:

$$p_{t+1} = T_D[p_t], \tag{14}$$

where

$$\begin{aligned}
T_D[p](x) &= \frac{1}{2}\int_0^x p_t(x-u)\int_u^\infty \frac{1}{v}p_t(v)dvdu \\
&\quad + \frac{1}{2}\int_x^\infty \frac{1}{u}p_t(u)du.
\end{aligned} \tag{15}$$

In reference [17] it was shown that the corresponding equilibrium distribution is the Gamma distribution with shape parameter $\frac{1}{2}$:

$$p_w(x) = \frac{1}{\sqrt{2w\pi x}}e^{-\frac{x}{2w}},$$

and convergence of the iterations (14) to the equilibrium distribution was proved.

The mixed model proposed in [9] combines the Immediate Exchange model and the Directed Random Market model as follows: for a fixed parameter $\mu \in [0,1]$, when two agents interact, with probability μ a unidirectional money transfer (as in the Directed Random Market model) is carried out, and with probability $1-\mu$ a bidirectional exchange (as in the Immediate Exchange model) is carried out.

In reference [9] the mixed model was investigated by simulations, and it was observed that the resulting wealth distribution is very well fitted by a Gamma distribution with shape parameter $\alpha = 2^{1-2\mu}$. However the authors did note some deviations from the Gamma distribution. In the extreme cases $\mu = 0, \mu = 1$, where the model reduces to the Immediate Exchange and to the directed random market models, respectively, we indeed have the equilibirium distributions with shape parameter $\alpha = 2^{1-2\mu}$, as proved above and in reference [17]. However, as we will show below, for $\mu \in (0,1)$ the equilibrium distribution is *not* a Gamma distribution.

The evolution of the wealth distribution for the mixed model will be given by $p_{t+1} = T_M[p_t]$, with

$$T_M[p] \doteq \mu T_D[p] + (1-\mu)T[p],$$

where T is defined by (6) and T_D by (15). To find the equilibrium distributions we need to solve $T_M[p] = p$, that is:

$$p = \mu T_D[p] + (1-\mu)T[p]. \tag{16}$$

In reference [17] it was shown that, setting $\hat{p}(s) = \mathcal{L}[p](s)$, we have:

$$\mathcal{L}[T_D[p]](s) = \frac{1}{2s}\left[\hat{p}(s)+1\right]\int_0^s \hat{p}(s')ds'$$

and in Section 3 we showed that

$$\mathcal{L}[T[p]](s) = \left(\frac{1}{s}\cdot\int_0^s \hat{p}(s')ds'\right)^2,$$

hence applying the Laplace transform to both sides of (16) gives

$$\hat{p}(s) = \mu\frac{1}{2s}\left[\hat{p}(s)+1\right]\int_0^s \hat{p}(s')ds' + (1-\mu)\left(\frac{1}{s}\int_0^s \hat{p}(s')ds'\right)^2. \tag{17}$$

To solve the functional equation (17), we define

$$h(s) = \frac{1}{s}\int_0^s \hat{p}(s')ds',$$

so that

$$\hat{p}(s) = [sh(s)]' = sh'(s) + h(s), \tag{18}$$

and (17) becomes

$$sh'(s) + h(s) = \mu\frac{1}{2}\left[sh'(s)+h(s)+1\right]h(s) + (1-\mu)(h(s))^2,$$

or, after rearrangement,

$$h'(s) = \frac{2-\mu}{s}\frac{\left[h(s)-1\right]h(s)}{2-\mu h(s)}. \tag{19}$$

This separable differential equation can be solved, but only in implicit form:

$$(1-h(s))^{2-\mu} = Cs^{2-\mu}(h(s))^2. \tag{20}$$

(20) and (18) define $\hat{p}(s)$, from which the equilibrium densities $p(x)$ are obtained by Laplace inversion. However, except in the cases $\mu = 0,1$, one cannot solve (20) for $h(s)$ in a reasonably explicit form.

To verify that the equilibrium distribution is not a Gamma distribution when $\mu \neq 0,1$, we show that the moments of the equilibrium distribution cannot be equal to those of a Gamma distribution. The same idea was used in [13] with regard to the Chakraborti-Chakrabarti model [12]. We compute the moments of integer order of the equilibrium distribution p, $M_k(p) = \int_0^\infty p(x)x^k dx$. By (18) we have:

$$M_k(p) = (-1)^k\hat{p}^{(k)}(0) = (-1)^k(k+1)h^{(k)}(0). \tag{21}$$

Thus $h(0)=1$, $h'(0) = -\frac{1}{2}M_1(p) = -\frac{w}{2}$. By successively differentiating (19) and sending $s \to 0$, we recursively compute the derivatives $h^{(k)}(0)$, $2 \le k \le 4$. By (21) these computations give:

$$M_1(p) = w, \qquad M_2(p) = \frac{3}{2-\mu}w^2,$$
$$M_3(p) = 3\frac{4+\mu}{(2-\mu)^2}w^3, \quad M_4(p) = 5\frac{\mu^2+8\mu+12}{(2-\mu)^3}w^4.$$

Denoting by $q(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\beta}}$ the density of a Gamma distribution, its moments are given by:

$$M_k(q) = \beta^k\prod_{j=0}^{k-1}(\alpha+j). \tag{22}$$

If we wish that $M_1(q) = M_1(p)$, $M_2(q) = M_2(p)$ we need to take $\alpha = \frac{2-\mu}{1+\mu}$, $\beta = \frac{1+\mu}{2-\mu}w$. This gives

$$M_3(q) = 3\frac{4+\mu}{(2-\mu)^2}w^3, \quad M_4(q) = 3\frac{2\mu^2+13\mu+20}{(2-\mu)^3}w^4.$$

While we have $M_k(p) = M_k(q)$, $1 \le k \le 3$ (for the first two this is true by design, while for the third moment it is an interesting "coincidence"), the fourth moment already differs (unless $\mu = 0,1$), proving the equilibrium distribution is *not* a Gamma distribution.

Let us note that the Gamma distribution we fitted above by equating the first two moments gave us shape parameter $\alpha = \frac{2-\mu}{1+\mu}$, while in [9] the fit $\alpha = 2^{1-2\mu}$ was given. If one looks at these two expression in the range $\mu \in [0,1]$, one sees they have very close values. Of course neither of these expressions yields the true equilibrium distribution for the mixed model, since, as shown above, this equilibrium distribution is not Gamma.

6 Conclusion

The Immediate Exchange model, recently put forward by Heinsalu and Patriarca [9], is a natural addition to the existing collection of kinetic exchange models. In this work we have developed an analytical approach to this model, by formulating its infinite-population version. This has enabled us to rigorously prove the fact that the equilibrium distribution of the Immediate Exchange model is a Gamma distribution with shape parameter 2, a result obtained in reference [9] by means of numerical simulation. We have also proved the convergence of the wealth distribution to this equilibrium distribution, starting from an arbitrary initial wealth distribution.

As noted in the introduction, there exist kinetic exchange models, such as the Chakraborti-Chakrabarti model [12], which produce equilibrium distributions which are very well fitted by a Gamma distribution, but for which subsequent analytical study has revealed that the equilibrium distribution is *not* precisely Gamma [13], and indeed no closed form for this distribution is known. By contrast, for the Immediate Exchange model the fact that the equilibrium distribution is exactly Gamma is now established.

We have also studied a more general "mixed" model proposed in reference [9], in which either bi-directional or uni-directional wealth exchanges occur, each with a certain probability. For this model we have proved that the equilibrium distribution is *not* Gamma, despite the fact that it can be closely fitted by a Gamma distribution.

In closing, we make some general remarks on the analytical study of kinetic exchange models. Such study has two distinct stages. The first stage involves the formulation of the infinite-population version of the model as an iterative process on a space of probability distributions, hence obtaining a functional equation for the equilibrium distributions corresponding to the model (the fixed points of the iterative process). This stage can always be carried out. The second stage is investigating the resulting functional equation in order to characterize the equilibrium distributions. Here there is no guarantee that an explicit solution of the functional equation can be found: there is no a priori reason that the equilibrium distribution will be expressible in terms of familiar functions. Indeed it appears that an explicit expression is possible only for a handful of models (such as the Immediate Exchange model, as shown here). However the fact that an explicit solution may not exist does not mean that the equilibrium distribution is not amenable to mathematical analysis: it may be possible to use the functional equation directly to derive qualitative and quantitative properties of the equilibrium distribution's probability density. For example in [23] one can find results regarding the decay of the tail of equilibrium distributions, which apply to general classes of models. It would be of great interest to prove mathematically, for particular models for which an explicit expression for the equilibrium disribution is unavaillable, that the density of the equilibrium distribution is smooth, or monotone decreasing, or unimodal. The mathematical study of equilibrium distributions of kinetic exchange models in cases when it *cannot* be represented by an explicit expression thus seems to offer many challenges and is a potentially rich field for further developments.

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