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Statistical description of the system electrons on the liquid helium surface

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Abstract. System of electrons on the liquid helium surface is considered. General methods for obtaining free energy functional for the systems in mean field approximation are developed. These methods applied for treating systems with particles arranged in a lattice. Thus obtained functional of free energy is analyzed. The localization distance for electron and conditions for existing square or triangular lattices as well as phase transition between them are obtained.

1 Introduction

Particle systems with Coulomb interaction (Coulomb-like systems), such as plasmas, colloidal particles, electrolyte solutions, electron on the helium surface, etc., are widely presented both in nature and under laboratory conditions. An interest to this systems is generated by its applications to the studies of a variety of peculiar phenomena in various fields of science [1–3].

One of the challenges here is a statistical description of Coulomb-like systems with high concentrations of interacting particles [4]. Such effects as formation of various crystal structures, transitions between different phases are observed when concentration increases.

Considerable attention is attracted to the special case of electrons on the surface of a dielectric substrate [5,6]. A possibility of creating a two-dimensional system on the surface of a dielectric medium was predicted by [5–7] and a year later [8] first experiments were carried out. Moreover, the first experimental realization of the Wigner solid, predicted in well-known article [9], was made in an electron system on liquid helium [10]. Studies of these systems are not only of academic interest but can also have some practical applications. For example, it is proposed to use the electrons on a dielectric surface for quantum computations [11].

Electrons located on a dielectric surface have two degrees of freedom only [3,12] and can exist in forms of fluid or Wigner crystal [9,10,13]. It was theoretically and experimentally shown, that electrons on the liquid helium surface can undergo a phase transition, which appears in their ordering stage [10,14] (liquid-to-solid transition). Besides, it is known, that electrons are "floating" above helium surface at quite large distance about 76 Å due to quantum effects [3]. Thus in electron "layer" self-organization studies presence of helium is treated just as "correction" of interelectron potential without consideration of separate helium atoms [3,15–17], which makes this system attractive for theoretical studies. In reference [15] the structural transition of a Wigner lattice from triangle to square was investigated. Moreover, in reference [16] it was shown and later experimentally confirmed [17] that homogeneous distribution of electron density is not always stable, and there are critical parameters when spatial structures, especially periodic deformations and multi-electron dimples, are formed. But this structures can be treated separately from Wigner crystallization, because characteristic length of such structures is much larger than the period of the Wigner lattice.

Modern research in the field of low-dimensional electron systems is based mostly on the quantum field theory [18] and the scaling theory [19]. For example, electron transport properties in heterostructures and electron structures on the liquid-helium surface can be studied using quantum field theory methods [20]. As regards the scaling theory, it was worked out in reference [21]. Nevertheless, these models are complicated for the analysis and require a lot of calculations. Therefore, it would be highly desirable to introduce simpler quasi-classical models of the type [4,22], which could be efficient for the description of the properties of the low-dimensional electron systems.

Another interesting side of studying electrons on liquid helium is that this system is a representative of class of systems with long-range interaction resembling Coulomb one. Dusty plasmas, systems of colloidal particles, electrolyte solutions significantly differ from each other by physical properties, but due to their interparticle interaction formation of similar structures is observed, concerning the formation of stable periodical structures [4,23–29]. Thus, we can expect that methods applicable to two-dimensional electron systems also can be useful for description of other systems of mentioned type.

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In the present contribution we study the influence of new details of the electron interaction related to such effects. We take into account both direct Coulomb repulsion and polarization interaction as well as the interaction due to the deformation of the helium surface [7,15]. Proposed simplified approach is suggested by the fact that the forces, governing self-organization, act on a length scale which is larger than the molecular size; as a consequence many specific details of the molecules of interest are not necessary for studying general features of phases.

We look at the electrons on the liquid helium surface in a way typical for the systems with long range interaction developing necessary formalism of the statistical description of such systems in terms of the mean field approximation [4,30]. The main goal of the present contribution is to work out a model for the description of electrons on a helium surface in the presence of the external electric field applied perpendicularly to the surface. We use it to explore square to triangular lattice transition and Wigner crystallization. In order to do this we employ the concept of the effective interaction energy. For the case of electrons on a cryogenic liquid substrate, distortion of the surface introduces additional physical effects [7,15].

Article is organized as follows. We always adhere principle "from general to concrete". So we start with general description of the interacting particle system in mean-field approximation Section 2. Obtained result can be applied to the wide variety of systems. So we apply obtained equations to the system of Fermi-particles in two-dimensional space Section 2.1. In Section 3 we add assumption about particles arrangement in a lattice. Some general Bravais lattice is analyzed. It can be seen from Section 3 that it is convenient to work in inverse lattice space with probability distribution function for single particle ρ_{SP}. Later in Section 4 we introduce effective interelectron potential and do some assumptions about ρ_{SP}. And in Section 5 we finish our research by obtaining explicit expression for free energy of triangular and square lattices. This expression is approximated and analytically analyzed.

2 Statistical description of interacting particles system

We begin with a brief calculation for inhomogeneous system of interacting particles [22]. In this section we would not specify our system and interaction to keep generosity and extendability of presented theory. As far as we know, stable states of the system minimize its free energy, so with following calculations we aim to get some general expression for it.

In our model macroscopic states of the system are described by a set of occupation numbers. For a wide number of systems Hamiltonian can be written in form:

$$H(n) = \sum_s \epsilon_s n_s + \frac{1}{2}\sum_{s,s'} V_{ss'} n_s n_{s'}.$$

Here ϵ_s is the additive part of particle energy (usually it is kinetic energy, but it can be particle's energy in external field as well), s indicates particle state, $V_{ss'}$ is interaction energy between particles in states s and s', n_s is the occupation number of state s. Most systems mentioned in introduction, including electrons on liquid helium surface, are essentially classical so we neglect any quantum correlations.

Partition function for this kind of system will be:

$$Z = \sum_{\{n_s\}} \exp(-\beta H) = \sum_{\{n_s\}} \exp\left[-\beta\left(\sum_s \epsilon_s n_s + \frac{1}{2}\sum_{s,s'} V_{ss'} n_s n_{s'}\right)\right].$$

In order to perform formal summation in this equation, we use the well-known properties of Gaussian integrals over auxiliary fields, i.e. Hubbard-Stratonovich transformation [31,32]:

$$\exp\left(\frac{\nu^2}{2\vartheta}\sum_{s,s'}\omega_{ss'} n_s n_{s'}\right) = \int_{-\infty}^{\infty} D\varphi \exp\left(\nu\sum_s n_s\varphi_s - \frac{\vartheta}{2}\sum_{s,s'}\omega_{ss'}^{-1}\varphi_s\varphi_{s'}\right),$$

with $D\varphi = \prod_s d\varphi_s / \sqrt{\det(2\pi\beta\omega_{ss'})}$. We can avoid second-order dependence on occupation numbers, carrying it to introduced field. Now partition function can be written as follows[1]:

$$Z = \int D\varphi \exp\left[\sum_s (i\varphi_s - \beta\epsilon_s) n_s - \frac{1}{2\beta}\sum_{s,s'}\left(V_{ss'}^{-1}\varphi_s\varphi_{s'}\right)\right].$$

Let us consider canonical ensemble. We will fix number of particles using Cauchy equation

$$\frac{1}{2\pi i}\oint \xi^{\sum_s n_s - N - 1} d\xi = 1.$$

We get partition function for N-particle system:

$$Z_N = \frac{1}{2\pi i}\oint d\xi \int D\varphi \exp\left[-\frac{1}{2\beta}\sum_{s,s'} V_{ss'}^{-1}\varphi_s\varphi_{s'} - (N+1)\ln\xi\right]\prod_s\sum_{\{n_s\}}\left[\xi\exp(i\varphi_s - \beta\epsilon_s)\right]^{n_s}.$$

Now summation over occupation numbers can be performed according to the type of statistics. As a result we will get

$$Z_N = \frac{1}{2\pi i}\oint d\xi \int D\varphi \exp[-\beta F(\varphi, \xi)],$$

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[1] In future infinite limits of integration will be omitted.

with effective free energy:

$$\beta F(\varphi,\xi)=\frac{1}{2}\sum_{s,s'}V^{-1}_{ss'}\varphi_s\varphi_{s'} + \delta\sum_s \ln\left(1-\delta\xi e^{-\beta\epsilon_s+i\varphi_s}\right)+(N+1)\ln\xi. \tag{1}$$

Variable δ indicates type of statistics under consideration. It equals to $+1$ for Bose-Einstein statistics, 0 for Maxwell-Boltzmann statistics and -1 for Fermi-Dirac statistics.

Last equation we have obtained is an expression for the free energy in auxiliary field representation. Chemical activity of particles can be presented as follows $\xi=\exp(\beta\mu)$. Equation (1) contains the same information as the original partition function with summation over the occupation numbers, i.e. all information about probable states of the system. The partition function represented in terms of the functional integral over auxiliary field corresponds to the sequence of probable equilibrium states with regard to their weights. Extension to the complex plane makes possible to apply saddle-point method and find an asymptotic value of partition function. This helps us avoid using perturbation theory.

Dominant contribution is made by states satisfying extrema condition:

$$\frac{\delta\beta F}{\delta\varphi}=\frac{\delta\beta F}{\delta\xi}=0.$$

Varying (1) we obtain equations for saddle-point states:

$$\frac{1}{\beta}\sum_{s'}V^{-1}_{ss'}\varphi_{s'}-\frac{i\xi e^{-\beta\epsilon_s+i\varphi_s}}{1-\delta\xi e^{-\beta\epsilon_s+i\varphi_s}}=0; \tag{2a}$$

$$\sum_s\frac{\xi e^{-\beta\epsilon_s+i\varphi_s}}{1-\delta\xi e^{-\beta\epsilon_s+i\varphi_s}}=N+1. \tag{2b}$$

We can see from (2b), that expression

$$f_s=\frac{\xi e^{-\beta\epsilon_s+i\varphi_s}}{1-\delta\xi e^{-\beta\epsilon_s+i\varphi_s}} \tag{3}$$

can be treated as an average occupation number for a certain state. Using presented system we can obtain saddle-point states. They can be interpreted as thermodynamically stable distributions.

As is seen, equation (2a) contains inverse matrix. We should either find explicit expression for this matrix as was done for certain interaction potentials in references [22,33,34], or somehow rewrite equations without inverse operators. First approach appears as quite challenging mathematical problem. Besides, the latter is preferable since we don't want to specify potential at this point. Thus we will use (3) and perform inverse transformation to obtain equations without inverse matrix

$$\varphi_s=i\beta\sum_{s'}V_{ss'}f_{s'},$$

$$\frac{1}{2\beta}\sum_{s,s'}V^{-1}_{ss'}\varphi_s\varphi_{s'}=-\frac{\beta}{2}\sum_{s,s'}V_{ss'}f_sf_{s'}.$$

Then free energy can be rewritten

$$\beta F[f,\xi]=-\frac{\beta}{2}\sum_{s,s'}V_{ss'}f_sf_{s'}-\delta\sum_s\ln(1+\delta f_s) + (N+1)\ln\xi(f). \tag{4}$$

Since we are working in terms of canonical ensemble it would be convenient to rewrite equations without chemical potential. We can obtain it from (3)

$$\ln\xi(f)\equiv\beta\mu=\beta(\epsilon_s+E_s)+\ln f_s-\ln(1+\delta f_s),$$

with

$$E_s=\sum_s V_{ss'}f_{s'}. \tag{5}$$

Nevertheless we consider chemical potential to be a constant over the whole system, but it is useful for calculation to find its average over all states

$$\ln\xi(f)=\frac{1}{N}\sum_s f_s\left[\beta(\epsilon_s+E_s)+\ln f_s-\ln(1+\delta f_s)\right].$$

Last expression can be substituted into free energy (4) and thus we get

$$\beta F[f]=\beta\sum_s f_s\epsilon_s+\frac{\beta}{2}\sum_{s,s'}V_{ss'}f_sf_{s'} + \sum_s\left[f_s\ln f_s-(f_s+\delta)\ln(1+\delta f_s)\right]. \tag{6}$$

Equation (6) can be easily interpreted. First two terms are kinetic and potential energy respectively. Third term is a contribution of entropy. This contribution should be equal to zero if $T=0$.

Before finishing this part let us check the compliance of our theory to the classical results. If we consider grand canonical ensemble with fixed chemical potential we can get from (3) generalization of the well-known distribution

$$f_s=\frac{1}{e^{\beta(\epsilon_s-\mu_s)}-\delta}, \tag{7}$$

with a generalized chemical potential

$$\mu_s=\mu-E_s.$$

It is obvious that the saddle point approximation is equivalent to the mean field approximation in this case. If we consider ideal gas ($\mu_s\equiv\mu$) classical statistical distributions can be obtained.

With these equations we can wind up general consideration of interacting particles system and move on to some more specific cases. As main results of this subsection we should point out equation (6) (will be used in next sections) and (7) that can be used to obtain particles distribution in grand canonical ensemble.

2.1 Two-dimensional system of Fermi particles

Formalism developed in Section 2 is rather general and can be applied to various systems with different dimensionality. But since we are interested in certain system some specification of presented equations is needed. In this subsection we aim obtaining free energy of 2-dimensional Fermi particles system basing on results from Section 2.

Let us use equation (6) to obtain the free energy of such system. For Fermi particles we use $\delta = -1$. Dimensionality of the system is $d = 2$. Using continual approach we can write

$$\begin{aligned}\beta F[\mu] = &\int \frac{d^2\boldsymbol{p}\, d^2\boldsymbol{r}}{(2\pi\hbar)^2}\frac{\beta\boldsymbol{p}^2}{2m}\frac{1}{e^{\beta(\boldsymbol{p}^2/(2m)-\mu(\boldsymbol{r}))}+1}\\ &+\frac{\beta}{2}\iint\frac{d^2\boldsymbol{p}\, d^2\boldsymbol{r}}{(2\pi\hbar)^2}\frac{d^2\boldsymbol{p}'\, d^2\boldsymbol{r}'}{(2\pi\hbar)^2}V(|\boldsymbol{r}-\boldsymbol{r}'|)\\ &\times\frac{1}{e^{\beta(\boldsymbol{p}^2/(2m)-\mu(\boldsymbol{r}))}+1}\frac{1}{e^{\beta(\boldsymbol{p}'^2/(2m)-\mu(\boldsymbol{r}'))}+1}\\ &+\int\frac{d^2\boldsymbol{p}\, d^2\boldsymbol{r}}{(2\pi\hbar)^2}\left(\frac{1}{e^{\beta(\boldsymbol{p}^2/(2m)-\mu(\boldsymbol{r}))}+1}\right.\\ &\times\ln\frac{1}{e^{\beta(\boldsymbol{p}^2/(2m)-\mu(\boldsymbol{r}))}+1}\frac{1}{e^{-\beta(\boldsymbol{p}^2/(2m)-\mu(\boldsymbol{r}))}+1}\\ &\left.\times\ln\frac{1}{e^{-\beta(\boldsymbol{p}^2/(2m)-\mu(\boldsymbol{r}))}+1}\right). \qquad (8)\end{aligned}$$

We can integrate this expression over the momentum. It is necessary to notice that performing integration in the case of the Bose statistics we will loose the Bose-condensation effects. In our case we do not have this problem. Using integral form of dilogarithm definition (Li_2, polylogarithm of the second order) we get (Appendix A)

$$\begin{aligned}\beta F[\mu] = &\frac{m^2}{8\pi^2\hbar^4\beta}\iint d^2\boldsymbol{r}\, d^2\boldsymbol{r}'V(|\boldsymbol{r}-\boldsymbol{r}'|)\\ &\times\ln\left(1+e^{\beta\mu(\boldsymbol{r})}\right)\ln\left(1+e^{\beta\mu(\boldsymbol{r}')}\right)\\ &+\frac{m}{2\pi\hbar^2\beta}\int d^2r\left[\mathrm{Li}_2\left(\frac{1}{1+e^{\beta\mu(\boldsymbol{r})}}\right)-\frac{\pi^2}{6}\right.\\ &\left.+\frac{1}{2}\ln^2\left(1+e^{\beta\mu(\boldsymbol{r})}\right)\right]. \qquad (9)\end{aligned}$$

Now let us step aside from expression for free energy and look for connection between chemical potential $\mu(\boldsymbol{r})$ and particles density $\rho(\boldsymbol{r})$. Expression we will obtain can be used later to simplify (9).

As a first approximation we can suppose that dispersion relation has quadratic form

$$\varepsilon_s = \varepsilon(p) = p^2/(2m). \qquad (10)$$

Of course it will be slightly different in the presence of an external field, but looks quite reasonable, since the gas of Fermi particles under consideration is highly degenerated [3]. Introducing dimensional constant – thermal length

$$\lambda_T = \sqrt{2\pi^2\hbar^2\beta/m}, \qquad (11)$$

and using (7) we get what follows

$$\begin{aligned}\rho(\boldsymbol{r}) &= \int\frac{d^2\boldsymbol{p}}{(2\pi\hbar)^2}\frac{1}{e^{\beta(\boldsymbol{p}^2/(2m)-\mu(\boldsymbol{r}))}+1}\\ &= \frac{\pi}{\lambda_T^2}\ln\left(1+e^{\beta\mu(\boldsymbol{r})}\right).\end{aligned}$$

Last equation can be applied to (9)

$$\begin{aligned}F[\rho] = &\frac{1}{2}\iint d^2\boldsymbol{r}\, d^2\boldsymbol{r}'V(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r})\rho(\boldsymbol{r}')\\ &+\frac{\pi}{\beta\lambda_T^2}\int d^2\boldsymbol{r}\left[\mathrm{Li}_2\left(e^{-\pi^{-1}\lambda_T^2\rho(\boldsymbol{r})}\right)-\frac{\pi^2}{6}\right]\\ &+\frac{\lambda_T^2}{2\pi\beta}\int d^2\boldsymbol{r}\rho^2(\boldsymbol{r}). \qquad (12)\end{aligned}$$

This functional can be easily used to determine thermodynamically stable states, that have to minimize it. We obtain such states directly in the form of the particle density and no additional mathematical transformations are needed.

With equation (12) we have achieved the goal of this subsection. It should be mentioned that presented method allows to find analogous expressions for different number of dimensions. Free energy for Bose-statistic can be found as well, but with precautions mentioned above (we are loosing Bose-condensation).

3 Fermi particles arranged in a lattice

In Section 2, we have discussed general features of the system of interacting particles. Adhering "from general to concrete" principle we will narrow a set of possible systems to the ones with periodical particles arrangement. This section will provide some general discussion of such systems.

For the sake of simplicity it is highly desirable to treat system as infinite. This will simplify all following equations. On the other hand to do so we need to switch from calculation of all system's energy to calculation of single particle energy in a lattice. It does not change anything fundamentally in analysis, but avoids us from obtaining infinite values for system's energy (obviously infinite system will have infinite energy). From the mathematical point of view this change will be reflected in integration limits (12). To avoid possible misconceptions we will write these limits explicitly in following equations.

For more convenience we will split general expression for free energy (12) into few parts and treat them separately

$$F_{int} = \iint\limits_{S}\ \iint\limits_{-\infty}^{+\infty} V(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r})\rho(\boldsymbol{r}')d^2\boldsymbol{r}'\, d^2\boldsymbol{r}, \qquad (13a)$$

$$F_s = \iint\limits_{S}\ \iint\limits_{S} V(|\boldsymbol{r}-\boldsymbol{r}'|)\rho(\boldsymbol{r})\rho(\boldsymbol{r}')d^2\boldsymbol{r}'\, d^2\boldsymbol{r}, \qquad (13b)$$

$$F_{ent}^{(1)} = \frac{\pi}{\beta\lambda_T^2}\iint\limits_S \left[\mathrm{Li}_2\left(e^{-\pi^{-1}\lambda_T^2\rho(\boldsymbol{r})}\right) - \frac{\pi^2}{6}\right] d^2\boldsymbol{r}, \tag{13c}$$

$$F_{ent}^{(2)} = \frac{\lambda_T^2}{2\pi\beta}\iint\limits_S \rho^2(\boldsymbol{r})d^2\boldsymbol{r}, \tag{13d}$$

$$F[\rho] = \frac{1}{2}F_{int} - \frac{1}{2}F_s + F_{ent}^{(1)} + F_{ent}^{(2)}. \tag{13e}$$

Here $F_{ent}^{(1)}$ and $F_{ent}^{(1)}$ are contributions caused by entropy. S is a symbol for designating integration over one subset of exact cover of $\mathbb{R}^2$ with similar polygons (relative position of the particle supposed to be the same in every polygon). For example Wigner-Seitz cells provide this kind of covering. But since Wigner-Seitz cell may have complicated shape we should find some workaround to make integration limits in (13) as simple as possible.

The other feature of (13) is separation of interaction energy into two parts (13a) and (13b). This was done because we should omit somehow particles self-interaction for certain types of potential. Reader may think about analogy with discrete case. For example, while calculating Coulomb energy we write $\sum_{i\neq j} e_i e_j / r_{ij}$. So we need to integrate over $\mathbb{R}^2 \setminus S$ that is analogous to $i \neq j$ for discrete case.

One way is directly integrate over $\mathbb{R}^2 \setminus S$. But it seems more convenient to use equation $\iint_{\mathbb{R}^2\setminus S} = \iint_{\mathbb{R}^2} - \iint_{\mathbf{S}}$. So we can integrate over $\mathbb{R}^2$ (13a) and compensate self-interation with additional summand (13b).

By certain types of potential we mean that, for example, Coulomb potential should be compensated this way, because there is no additional term in free energy if there were only one particle in a system. On the other hand, effective potentials caused by particles interaction through medium should be uncompensated, because even in absence of other particles medium will influence this one. Surface distortion is a good example. If particle has distorted a surface it has already done some contribution to system's free energy independently on other particles. Of course it interacts with distortions created by other particles. But at the same time it interacts with "its own" surface distortion that may be treated as "self-interaction". So question about which part of effective potential should be compensated by (13b) must be postponed till we know physical background of considered potential.

Now let us consider (13a) more carefully. All 2D Bravais lattices (i.e. oblique, rectangular, centered rectangular (rhombic), hexagonal, and square) can be treated as constructed from parallelograms. We will use this property and simplify integration region S to parallelogram. For example let us illustrate this for hexagonal lattice (Fig. 1).

Supposing, that some probability distribution function for a single particle ($\rho_{SP}(\boldsymbol{r})$) is already known, we can easily compute F_s, $F_{ent}^{(1)}$ and $F_{ent}^{(2)}$ from equations (13). This is possible because integration in this expressions is performed over S, so we can just substitute ρ_{SP} instead of ρ. But computation of F_{int} needs knowing of probability distribution function for the whole system. We will try to find some relation between F_{int} and $\rho_{SP}(\boldsymbol{r})$ that may simplify our equations.

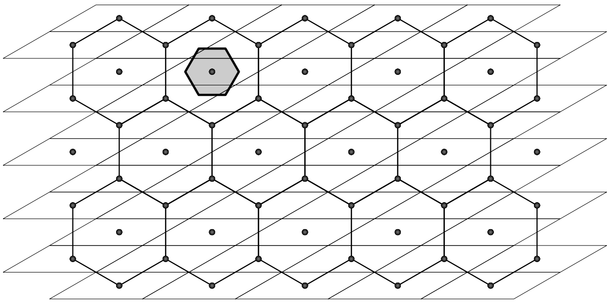

Fig. 1. Hexagonal lattice. Parallelograms show sufficient exact cover. With gray color one Wigner-Seitz cell is shown.

Basing on the inverse lattice vectors (B.1) we will provide following decomposition for probability distribution function $\rho(\boldsymbol{r})$

$$\rho(x;y) = \sum_{n,m=-\infty}^{\infty} \rho_{n,m} f_{n,m}(x;y) \tag{14a}$$

$$\rho_{n;m} = \frac{1}{ab\sin(\alpha)}\iint\limits_S f_{n,m}(x;y)\rho_{SP}(x;y)d^2\boldsymbol{r} \tag{14b}$$

$$f_{n,m}(x;y) = e^{2\pi\mathbf{i}\varphi_{n,m}(x;y)} \tag{14c}$$

$$\varphi_{n,m}(x;y) = \frac{n}{a}\left(x - y\cot(\alpha)\right) + \frac{m}{b}y\csc(\alpha). \tag{14d}$$

It may be shown that presented series are actually Fourier series for ρ_{SP} and ρ is periodic along vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (Appendix B). So we may treat $\rho(x;y)$ as function on $\mathbb{R}^2$ "composed" from single particle distributions "arranged in a lattice". This principle is demonstrated by Figure 3. Since ρ has the same symmetry as the lattice and locally is a good approximation for ρ_{SP} it will be treated as probability distribution function for the whole lattice.

If we use equations (14) we can express (13a) in terms of ρ_{SP} (Appendix C)

$$F_{int} = F_{int}^{(0)} + F'_{int}, \tag{15a}$$

$$F_{int}^{(0)} = 2\pi\bar{\rho}\int\limits_0^\infty V(r)rdr, \tag{15b}$$

$$F'_{int} = \frac{1}{\bar{\rho}^2}\sum_{\substack{n,m=-\infty\\ n^2+m^2\neq 0}}^{\infty} \left|\rho_{n,m}\right|^2 V_{n,m}, \tag{15c}$$

$$V_{n,m} = 2\pi\bar{\rho}\int\limits_0^\infty \mathrm{J}_0(r\lambda_{n,m})V(r)rdr, \tag{15d}$$

$$\lambda_{n,m} = \frac{2\pi}{\sin(\alpha)}\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} - 2\frac{mn}{ab}\cos(\alpha)}, \tag{15e}$$

where $\bar{\rho}$ is mean particle density and J_0 is Bessel function of the first kind.

We should mention that $F_{int}^{(0)}$ is dependent on mean particle density, but not on their distribution. So it will not influence any minimization process if mean charge density does not change (this assumption was introduced in Sect. 2).

Moreover, comparing two systems with equal mean charge we can ignore $F_{int}^{(0)}$. This is a very useful observation since we may work with "catastrophic" potentials like Coulomb where $F_{int}^{(0)}$ is infinite. In this cases we will neglect it due to mentioned reasons.

With these equations we have achieved the goal of the present section. Before we move further let us glance over obtained results again. We started with equation (12) and separated summands (13). Then we assumed we already have an expression for probability distribution function for one particle (it will be presented in following sections). This means that $F_{ent}^{(1)}$ and $F_{ent}^{(2)}$ can be calculated directly from (13c) and (13d) if we substitute ρ_{SP} instead of ρ. Much sophisticated is calculation of F_{int}. But it can be done using (15) and (14) if we substitute known parameters and calculate them one by one.

Last thing to mention in this section are possible generalizations and extensions. If we add some δx and δy to the equation for $\varphi_{n,m}$ (14d) it is possible to calculate interaction between current particle and a different sublattice. So it is possible to treat some quasi-periodical cases as well.

4 Electrons on the liquid helium surface

4.1 Potential

The main goal of this subsection is to introduce the explicit form of potential in order to use it further. We will find $V_{n,m}$ for introduced potential as well.

According to [3,17,35] the interelectron potential in the presence of liquid helium film on the substrate is:

$$\begin{gathered} V^{(ee)}(r) = \frac{e^2}{r} - \frac{\varepsilon \bar{e}^2}{\sqrt{r^2 + (2d)^2}}, \\ \varepsilon = \frac{\varepsilon_s - \varepsilon_{\mathrm{He}}}{\varepsilon_s + \varepsilon_{\mathrm{He}}}. \end{gathered} \tag{16}$$

Here ε_s and $\varepsilon_{\mathrm{He}}$ are substrate and the liquid helium dielectric constants respectively, r is the distance between electrons, d is the helium film thickness. The first term is related to the ordinary Coulomb interaction, the second one is the result of the liquid helium and substrate polarization. These two terms describe direct interaction between electrons. Notice, that the second term is different from that one used in reference [15] and takes into account polarization of substrate and provides additional attraction.

An effect of surface-buckling instability can be interpreted using the idea of competition between the attractive and the repulsive interactions. But polarization attraction is rather weak comparing to direct Coulomb interaction. To enhance it we should involve another attractive force between electrons. In our model it will be effective attraction produced by helium surface deformation.

In the presence of an external field electrons can be pressed against the helium surface with the force that exceeds gravitational one by many orders. On the other hand, they cannot just go through this surface as being pushed out due to quantum effects [3,17,35,36]. So it should be considered with imminence that electrons can act on the liquid helium surface with significant force.

In the introduction it was assumed that the system under consideration is two-dimensional. But taking helium surface deformations into account "electron layer" deflection should be considered as well, which means that third dimension is also active and should in some way be included in calculation.

This problem is solved by adding to (16) effective capillary interaction. The lateral capillary interaction between two electrons on the helium surface was calculated in reference [15]. It can be presented in the form:

$$V^{(\mathrm{cap})}(r) = -\frac{\bar{e}^2 E^2}{2\pi\sigma} \mathrm{K}_0(r/l_0). \tag{17}$$

Here eE is the actual force that acts on each electron by the external field (in our case it is electrical clamping field), σ is the surface tension of liquid helium, r is the distance between particles, K_0 is the modified Bessel function, and $l_0 = \sqrt{\sigma/g\rho_{\mathrm{He}}}$ [15] is the capillary length that depends on the fluid properties only.

As result of all previous calculations we get potential of electron-electron interaction

$$V(r) = \frac{\bar{e}^2}{r} - \frac{\varepsilon \bar{e}^2}{\sqrt{r^2 + (2d)^2}} - \frac{\bar{e}^2 E^2}{2\pi\sigma} \mathrm{K}_0(r/l_0) \tag{18}$$

and from Appendix D

$$V_{n,m} = \frac{2\pi \bar{e}^2 \bar{\rho} \left(1 - \varepsilon e^{-2d\lambda_{n,m}}\right)}{\lambda_{n,m}} - \frac{\bar{e}^2 E^2}{\sigma} \frac{l_0^2 \bar{\rho}}{1 + l_0^2 \lambda_{n,m}^2}. \tag{19}$$

Equations (18) and (19) are aimed result of this section. But we should mention that some extensions for other physical systems are possible. It is convenient to work with potentials that can be integrated with J_0.

4.2 Probability distribution function for single particle

Nevertheless we have done a lot of calculations, explicit expression for free energy is still missing. The reason is absence of equations for $\rho(\boldsymbol{r})$ and $\rho_{n,m}$, respectively. In this subsection we will introduce these quantities and use them to obtain entropy part of the free energy. Obtaining of interaction energy will be postponed until Section 5.

First of all we suppose that all particles are arranged in a lattice. This means that particle position is quite determined. But since temperature differs from zero we can expect that particle position fluctuates near its lattice site.

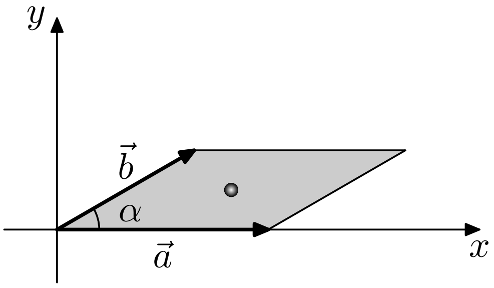

Fig. 2. One cell from exact cover. Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are base vectors for Bravais lattice. Angle between them is supposed to be α. Region of integration S is shown with gray. One particle is shown in the center.

Probability distribution function of finding particle near lattice site we will assume as follows:

$$\rho_{\mathrm{SP}}(\boldsymbol{r}) = \frac{1}{2\pi s^2} e^{-r^2/(2s^2)}. \tag{20}$$

This assumption seems quite reasonable since this is normal distribution.

Introduced constant s means dispersion of presented distribution. From physical point of view it can be treated as localization distance. Later we will find its value and compare it with localization distance obtained in different works.

If we substitute (20) into expression for $F_{ent}^{(1)}$ and $F_{ent}^{(2)}$ from (13) analytic expression for entropy part of the free energy can be found. But as reader may see this expressions will be quite complicated. So we will find some approximate equation.

First of all we suppose that temperature is low and thus particle position fluctuations are small. This means that s is smaller then any characteristic distance in the parallelogram cell, i.e. $s \ll a$, $s \ll b$ (see Fig. 2). So peak presented in Figure 3 will be quite thin. This means that we can approximate $F_{ent}^{(1)}$ and $F_{ent}^{(2)}$ by integrating over $\mathbb{R}^2$ instead of cell S. So we immediately find

$$F_{ent}^{(2)} \approx \frac{\lambda_T^2}{2\pi\beta} \iint\limits_{-\infty}^{\infty} \rho_{SP}^2(\boldsymbol{r}) d^2\boldsymbol{r} = \frac{\lambda_T^2}{8\pi^2 s^2 \beta}. \tag{21}$$

Finding of $F_{ent}^{(1)}$ is a bit more sophisticated. We return once more to the assumption that (20) represents a very "sharp" Gaussian, i.e localization radius is very small. This means that we can think about ρ_{SP} as small on distances more than localization radius $\rho_{SP}(r > s) \approx 0$ and $\rho_{SP}(r < s) \approx 1/(4\pi s^2)$. As result we get following approximation:

$$\mathrm{Li}_2\left(e^{-\pi^{-1}\lambda_T^2 \rho(\boldsymbol{r})}\right) \approx \begin{cases} 0, r \leq s; \\ \pi^2/6, r > s. \end{cases} \tag{22}$$

Last assumption can be substituted into (13) and we get final result

$$F_{ent}^{(1)} \approx -\frac{\pi^4 s^2}{6\beta\lambda_T^2}. \tag{23}$$

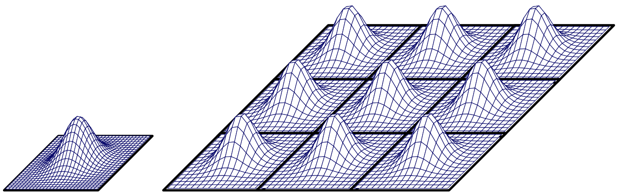

Fig. 3. Function $\rho_{SP}(\boldsymbol{r})$ defined for S only (on the left figure) and obtained from (14) $\rho(\boldsymbol{r})$ (on the right) which is defined for all $\mathbb{R}^2$.

Now approximation for the whole entropy part can be explicitly written:

$$F_{ent} = F_{ent}^{(1)} + F_{ent}^{(2)} \approx \frac{\lambda_T^2}{8\pi^2 s^2 \beta} - \frac{\pi^4 s^2}{6\beta\lambda_T^2}. \tag{24}$$

At this point we have finished with free energy part that is caused by entropy of the system. Now let us return to the interaction energy. It seems we are still missing $\rho_{n,m}$. Using again that $\rho_{SP}(r)$ is a very "sharp" Gaussian we expand integration limits in expression for $\rho_{n,m}$ from (14) and write

$$\rho_{n;m} \approx \bar{\rho} \iint\limits_{-\infty}^{\infty} f_{n,m}(x;y) \rho_{SP}(x;y) d\boldsymbol{r}. \tag{25}$$

Last equation can be easily integrated. As result we get

$$\rho_{n;m} \approx \bar{\rho} e^{-s^2 \lambda_{n,m}^2/2}. \tag{26}$$

Here $\lambda_{n,m}$ is same as in (15e) so we will not write it here explicitly again.

At this point we already know expression for interaction potential (18). But as we have mentioned in Section 3 there is one compensatory term we need to calculate. This term will exclude Coulomb self-interaction as was already discussed. If we omit technical details (see Appendix E) explicit form of F_s can be obtained

$$F_s = \frac{1}{2s^2} \int\limits_0^{\infty} e^{-r'^2/(4s^2)} V(r') r' dr', \tag{27}$$

or substituting Coulomb potential as V

$$F_s = \frac{\sqrt{\pi}\bar{e}^2}{2s} \left(1 - \varepsilon e^{d^2/s^2} \operatorname{erfc}\left(\frac{d}{s}\right)\right). \tag{28}$$

Here erfc is the complementary error function.

Last equation finishes current section. Once more, we have introduced probability distribution function for single particle near lattice site. Basing on this distribution (20) and assumption of "sharpness" of this distribution we have calculated entropy part of free energy (24). Besides, we have calculated compensatory part of the potential (28) and we have enough data to find interaction part if we know lattice parameters. To do this we should use (19), (26) and (15).

5 Triangular vs. square lattice comparison

At this point we already have expression for free energy that can be applied to any Bravais lattice. But most interesting for us are square and triangular (hexagonal) lattices. In this section we will obtain equations describing their free energy.

We will start from square lattice, since it is more easy for analysis. Obviously $a = b = L$, where L is interelectron distance (see Fig. 2). Between L and mean particles density $\bar{\rho}$ exists simple connection $\bar{\rho} = 1/L^2$. Angle between vectors $\alpha = \pi/2$. Thus from (15e) we get

$$\lambda^{(s)}_{n,m} = \frac{2\pi}{L}\sqrt{n^2+m^2}. \tag{29}$$

Here (s) denotes square lattice.

Similar calculations can be provided for triangular lattice. From Figure 1 and geometry we can get $\alpha = \pi/6$ (see Fig. 2 as well). In this case we will have $a = \sqrt{3}\,b$ (see again Figs. 1 and 2 for geometrical reasoning). Since we want both lattices to have equal particles density we should suppose $L^2 = ab\sin(\alpha)$. Combining all together and performing some simplification (see Appendix F) we get

$$\lambda^{(t)}_{n,m} = \frac{2\pi}{\sqrt[4]{12}L}\sqrt{3\left[m-n\right]^2+\left[m+n\right]^2}. \tag{30}$$

For more convenience we will provide here equation combined from (15), (19), (26) and (28) as was mentioned in Section 4.2

$$\begin{aligned} F[\rho] = {} & \frac{\lambda_T^2}{8\pi^2 s^2\beta} - \frac{\pi^4 s^2}{6\beta\lambda_T^2} + \frac{1}{2}F^{(0)}_{int} \\ & + \bar{\rho}\sum_{\substack{n,m=-\infty \\ n^2+m^2\neq 0}}^{\infty} e^{-s^2\lambda^2_{n,m}}\left(\frac{\pi\bar{e}^2\left(1-\varepsilon e^{-2d\lambda_{n,m}}\right)}{\lambda_{n,m}}\right. \\ & \left. - \frac{\bar{e}^2E^2}{2\sigma}\frac{l_0^2}{1+l_0^2\lambda^2_{n,m}}\right) \\ & - \frac{\sqrt{\pi}\bar{e}^2}{4s}\left(1-\varepsilon e^{d^2/s^2}\operatorname{erfc}\left(\frac{d}{s}\right)\right). \end{aligned} \tag{31}$$

Here $\lambda_{n,m}$ is substituted from (29) or (30) depending on lattice type we want to examine.

Last equation can be easily used for lattices comparison and finding phase transition between them. Unknown variable s, which plays role of localization distance should be found for every lattice by minimization of $F[\rho]$, i.e. solving

$$\frac{\partial F}{\partial s} = 0. \tag{32}$$

Nevertheless we are interested in two concrete types of the lattice results can be extended to any type in an obvious way.

5.1 Analytical approach

We have already obtained equation for free energy for different lattice types. It is sufficient for computer calculations and some estimations. But it may seem a good idea to make some approximations and obtain expression that is less precise, but does not contain summation. Besides we aim finding conditions when $F^{(s)} < F^{(t)}$ and $F^{(t)} < F^{(s)}$. This regions and their borderline may be treated as some approximation of phase diagram, so we will use this and related terms refering to obtained plots.

We expect particle density to be $\bar{\rho} = 10^9$ cm$^{-2}\ldots$ 10^3 cm^{-2} [3], so L is at most $\sim 10^{-5}$ cm. On the other hand we can expect $d \sim 0.1$ cm [17] and $l_0 \sim 1.1$ cm [15]. Basing on this data we can provide some approximations of (31) (Appendix G)

$$\begin{aligned} F^{(s)}[\rho] = {} & \frac{\lambda_T^2}{8\pi^2 s^2\beta} - \frac{\pi^4 s^2}{6\beta\lambda_T^2} - \pi\bar{e}^2\sqrt{\bar{\rho}} \\ & + \frac{\varepsilon\bar{e}^2}{4d} + \frac{\bar{e}^2E^2}{8\pi\sigma}\left(\gamma + 2\ln\left(2\pi s\sqrt{\bar{\rho}}\right)\right), \end{aligned} \tag{33a}$$

$$\begin{aligned} F^{(t)}[\rho] = {} & \frac{\lambda_T^2}{8\pi^2 s^2\beta} - \frac{\pi^4 s^2}{6\beta\lambda_T^2} - \sqrt[4]{\frac{3}{4}}\pi\bar{e}^2\sqrt{\bar{\rho}} \\ & + \frac{\varepsilon\bar{e}^2}{4d} + \frac{\bar{e}^2E^2}{8\pi\sigma}\left(\gamma + 2\ln\left(\sqrt[4]{12}\pi s\sqrt{\bar{\rho}}\right)\right). \end{aligned} \tag{33b}$$

As a test for obtained equation lets find localization distance s. So approximately we get

$$\frac{\partial F}{\partial s} = -\frac{\lambda_T^2}{4\pi^2\beta s^3} - \frac{\pi^4 s}{3\beta\lambda_T^2} + \frac{\bar{e}^2E^2_{\perp}}{4\sigma\pi s} \tag{34}$$

independently on lattice we are considering (e.g. we can use either $F^{(s)}$ or $F^{(t)}$). Solving equation $\partial F/\partial s = 0$ we get

$$s^2 = \frac{3\hbar^2\beta^2\bar{e}^2E^2}{4\pi^3\sigma m}\left(1-\sqrt{1-\frac{16\sigma^2\pi^4}{3\bar{e}^4E^4\beta^2}}\right). \tag{35}$$

If temperature tends to be small $T \to 0$ (e.g. $\beta \to \infty$)

$$s = \sqrt{\frac{2\pi\sigma\hbar^2}{m\bar{e}^2E^2}}. \tag{36}$$

This result does not differ from the classical one presented in reference [3].

If we consider (35) more carefully some interesting properties may be obtained. For the sake of simplicity we will consider low densities so that any value of s may be treated as much lower than interelectron distance L. Then we may notice that solution for s exists if

$$\frac{E^2}{T} \geq \frac{4\sigma\pi^2 k}{\sqrt{3}\bar{e}^2} \approx 2100\,\frac{\text{dyn}}{\text{cm K}}. \tag{37}$$

Last equation means that there is no lattice if temperature T is too high, or if field E is too weak. This statement is in agreement with our physical intuition.

As a practical application of obtained equations let us compare two lattices and explore if there exist phase transition between them. To do this we find difference between their free energies

$$F^{(t)} - F^{(s)} = \left(1-\sqrt[4]{\frac{3}{4}}\right)\pi\bar{e}^2\sqrt{\bar{\rho}} - \frac{\bar{e}^2E^2}{4\pi\sigma}\ln\left(\frac{2}{\sqrt[4]{12}}\right) \tag{38}$$

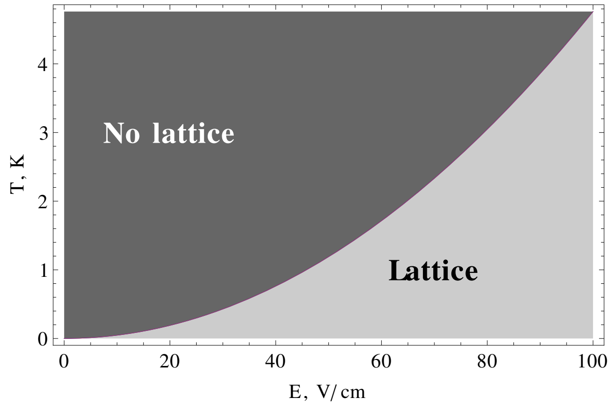

Fig. 4. Melting of Wigner crystal.

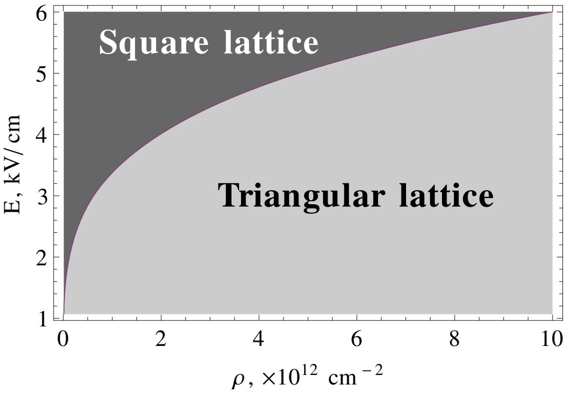

Fig. 5. Phase transition between two types of lattice.

and compare it with zero. Negative sign means that triangular lattice is more beneficial from the energetic point of view and positive – that square.

Line of phase transition is then described by equation

$$\frac{E^2}{\sqrt{\rho}} = \frac{4\left(1-\sqrt{3/4}\right)\pi^2\sigma}{\ln\left(2/\sqrt[4]{12}\right)} \approx 11.4 \text{ dyn}, \tag{39}$$

which can be plotted as shows Figure 5.

Obtained plots finish our consideration of square and triangular lattices started few sections ago. We will make a short overview of obtained results in the next section. Now let us discuss some possible generalizations and extensions.

In this subsection, we have done many assumptions. Actually, more precise expressions may be obtained for free energy, because all functions we use are integrable in terms of special functions. But even without making (G.1) more precise, we still can use more precise expressions for free energy of the square and triangular lattices (e.g. do not use Taylor series). This means that proposed method does not stop on "first-order effects" but may be considered further.

6 Conclusions

Here we present an overview of provided calculations and list results we have obtained.

We started with a brief calculation for inhomogeneous system of interacting particles. Without any specification of the system we have obtained expressions for free energy (6). This results may be applied to the wide variety of physical systems.

Aiming to explore system of the electrons on the liquid helium surface we have applied obtained results to two-dimensional Fermi-particles system. As result we have got expression for free energy (12), but without using explicit form of potential V. It should be mentioned that presented method allows to find analogous expressions for different number of dimensions. Free energy for Bose-statistic can be found as well, but with precautions mentioned above (we are loosing Bose-condensation).

Then we mentioned that our goal is to deal with a very specific system. So we took into account that electrons are arranged in a lattice. Starting with equation (12), separating summands (13) and assuming that we somehow know probability distribution function for single particle ρ_{SP} we found a way of obtaining free energy of the system. Result is that $F^{(1)}_{ent}$ and $F^{(2)}_{ent}$ can be calculated directly from (13) if we substitute ρ_{SP} instead of ρ. Much sophisticated is calculation of F_{int}. But it can be done using (14) and (15) if we substitute known parameters and calculate them one by one.

Last steps we took to specify our system are obtaining potential (18) and calculating $V_{n,m}$ (19) to be used in equation for free energy and distribution (20). Basing on this distribution and assumption of "sharpness" of the distribution we have calculated entropy part of free energy (24). Besides, presented results allow to find interaction part if we know lattice parameters. To do this we should use (19), (26) and (15).

Last section winds up analysis by presenting expressions for triangular and square lattices free energy. This equations are then approximated and treated analytically. As result we get expression for localization distance (35) and may see that for $T \to 0$ it is the same as classic result (36). Moreover, we have obtained conditions of existence one of this two lattices. It is perfectly consistent with our intuition – lattice exists if field is stronger and temperature is lower then some values (37), Figure 4. And we have compared energy of square and triangular lattices (39), Figure 5.

We suppose that with modern experimental technique verification of presented results can be provided.

Appendices

In following subsections we aim covering some technical details. They are necessary since we claim rigorous exposition of presented ideas, but may be omitted if reader is interested in physical sole rather than "mathematical tricks" behind the scene.

Appendix A: Integration over momentum in $\beta F[\mu]$

We start with (8). First two summands can be easily integrated if we mention

$$\frac{1}{e^x+1} = \frac{e^{-x}}{1+e^{-x}},$$

but last summand should be integrated in terms of special functions, namely dilogarithm [37]

$$\mathrm{Li}(z) = -\int_0^z \frac{\ln(1-z)}{z} dz,$$

if we note that

$$\frac{1}{1+e^x} = 1 - \frac{1}{1+e^{-x}}.$$

As result we get

$$\begin{aligned}\beta F[\mu] = \frac{m^2}{8\pi^2\hbar^4\beta} \int\int d^2\boldsymbol{r}\, d^2\boldsymbol{r}' V(|\boldsymbol{r}-\boldsymbol{r}'|) \\ \times \ln\left(1+e^{\beta\mu(\boldsymbol{r})}\right) \ln\left(1+e^{\beta\mu(\boldsymbol{r}')}\right) \\ + \frac{m}{2\pi\hbar^2\beta} \int d^2r \Big[\beta\mu(\boldsymbol{r}) \ln\left(1+e^{\beta\mu(\boldsymbol{r})}\right) \\ - \frac{\beta^2\mu^2(\boldsymbol{r})}{2} - \mathrm{Li}_2\left(-e^{-\beta\mu(\boldsymbol{r})}\right) - \frac{\pi^2}{6}\Big], \end{aligned} \tag{A.1}$$

but this equation can be simplified. First of all we will use Landen's identity

$$\begin{aligned}&\mathrm{Li}_2\left(1-z\right) + \mathrm{Li}_2\left(1-\frac{1}{z}\right) = -\frac{1}{2}\ln^2(z), \\ &z \in \mathcal{C}\backslash]-\infty;0]\end{aligned} \tag{A.2}$$

with $z = 1+e^{-\beta\mu(\boldsymbol{r})}$ and expand $\ln^2\left(1+e^{-\beta\mu(\boldsymbol{r})}\right)$. This leads to (9).

Appendix B: On the properties of $\mathbf{f_{n,m}(x;y)}$

In (14) we present how to calculate probability distribution function for the whole system basing on single-particle probability distribution function. It is based on the decomposition of ρ_{SP} using $f_{n,m}(x;y)$ (14c) as basis functions in inverse lattice space (basis vectors $\boldsymbol{k}_x$ and $\boldsymbol{k}_y$)

$$\begin{aligned}\boldsymbol{k}_x &= \frac{1}{a}\boldsymbol{e}_x - \frac{1}{a\tan(\alpha)}\boldsymbol{e}_y, \\ \boldsymbol{k}_y &= \frac{1}{b\sin(\alpha)}\boldsymbol{e}_y,\end{aligned} \tag{B.1}$$

where $a = |\boldsymbol{a}|$, $b = |\boldsymbol{b}|$ and $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ are unitary vectors along x and y coordinates. Here we add some consideration about connection between Fourier series and presented one.

Suppose we have a function $g(x;y)$ defined on unit square $S_1 \equiv [0;1]\times[0;1]$. This function can be expressed in terms of Fourier series

$$\begin{aligned}g(\boldsymbol{r}) &= \sum_{\boldsymbol{k}} g_{\boldsymbol{k}} e^{2\pi\mathbf{i}(\boldsymbol{k}\cdot\boldsymbol{r})}, \\ g_{\boldsymbol{k}} &= \iint_{S_1} g(\boldsymbol{r}) e^{2\pi\mathbf{i}(\boldsymbol{k}\cdot\boldsymbol{r})} d^2\boldsymbol{r}.\end{aligned} \tag{B.2}$$

Let us consider a bijection $\boldsymbol{s}$ from S_1 to S and its inverse $\bar{\boldsymbol{s}}$ (see Fig. 2 for geometrical reasoning and Eqs. (B.1) for connection with inverse lattice)

$$\begin{aligned}\boldsymbol{s}(\boldsymbol{r}) &= \{ax + by\cos(\alpha); by\sin(\alpha)\}, \\ \bar{\boldsymbol{s}}(\boldsymbol{r}) &= \{[x - y\cot(\alpha)]/a; y\csc(\alpha)/b\},\end{aligned} \tag{B.3}$$

and consider the case $g(\boldsymbol{r}) = \rho_{SP}(\boldsymbol{s}(\boldsymbol{r}))$. Since we know that $\rho_{SP}(\boldsymbol{s}(\bar{\boldsymbol{s}}(\boldsymbol{r}))) = \rho_{SP}(\boldsymbol{r})$ one can immediately write

$$\rho_{SP}(\boldsymbol{r}) = \sum_{\boldsymbol{k}} g_{\boldsymbol{k}} e^{2\pi\mathbf{i}(\boldsymbol{k}\cdot\bar{\boldsymbol{s}}(\boldsymbol{r}))}. \tag{B.4}$$

And same way we consider second equation from (B.2)

$$g_{\boldsymbol{k}} = \iint_S \rho_{SP}(\boldsymbol{s}(\boldsymbol{r})) e^{2\pi\mathbf{i}(\boldsymbol{k}\cdot\bar{\boldsymbol{s}}(\boldsymbol{s}(\boldsymbol{r})))} \frac{d^2\boldsymbol{s}(\boldsymbol{r})}{\nabla\boldsymbol{s}(\boldsymbol{r})}. \tag{B.5}$$

Now if we mention that $\nabla\boldsymbol{s}(\boldsymbol{r}) = ab\sin(\alpha)$ we immediately get from (B.4) and (B.5) equations (14). Moreover, obtained result means that all properties of Fourier series can be applied to decomposition (14).

Last thing to mention are periodical properties of presented series. One may note that we are expanding ρ_{SP} defined on S and then use this series for the whole $\mathbb{R}^2$. One may see from (14d)

$$\varphi_{n,m}(\boldsymbol{r} + k\boldsymbol{a} + l\boldsymbol{b}) = \varphi_{n,m}(\boldsymbol{r}) + kn + lm \tag{B.6}$$

which means with regard to (14c)

$$f_{n,m}(\boldsymbol{r} + k\boldsymbol{a} + l\boldsymbol{b}) = f_{n,m}(\boldsymbol{r}) \tag{B.7}$$

and from (14a)

$$\rho(\boldsymbol{r} + k\boldsymbol{a} + l\boldsymbol{b}) = \rho(\boldsymbol{r}). \tag{B.8}$$

Last equation justifies our view of connection between ρ and ρ_{SP} as it is presented in Figure 3.

Appendix C: On the expression for $\mathbf{F_{int}}$

Let us start with (13a). Inner integral has infinite borders so we may rewrite this expression as follows:

$$F_{int} = \iint_S \iint_{-\infty}^{+\infty} V(|\boldsymbol{r}'|)\rho(\boldsymbol{r})\rho(\boldsymbol{r}+\boldsymbol{r}') d^2\boldsymbol{r}'\, d^2\boldsymbol{r}. \tag{C.1}$$

Now we can substitute ρ from (14a) and mention that $f_{n,m}(\boldsymbol{r}+\boldsymbol{r}') = f_{n,m}(\boldsymbol{r})f_{n,m}(\boldsymbol{r}')$ (14c). So if we designate

$$V_{n,m} = \bar{\rho}\iint\limits_{-\infty}^{+\infty} f_{n,m}(\boldsymbol{r}')V(|\boldsymbol{r}'|)d^2\boldsymbol{r}' \tag{C.2}$$

it can be written

$$F_{int} = \frac{1}{\bar{\rho}}\sum_{n,m}\rho_{n,m}V_{n,m}\sum_{k,l}\rho_{k,l}\iint\limits_{S} f_{n,m}(\boldsymbol{r})f_{k,l}(\boldsymbol{r})d^2\boldsymbol{r}.$$

Last equation can be simplified if we perform integration. From Appendix B we expect orthogonality of $f_{n,m}$ functions (this result may be obtained by integration $f_{n,m}$ from (14c))

$$\iint\limits_{S} f_{n,m}(\boldsymbol{r})f^*_{k,l}(\boldsymbol{r})d^2\boldsymbol{r} = ab\sin(\alpha)\delta_{n,k}\delta_{m,l}. \tag{C.3}$$

Mentioning that $ab\sin(\alpha) = 1/\bar{\rho}$, where $\bar{\rho}$ is mean particle density (one particle per S, see Figs. 1 and 2 for geometrical reasoning) we get

$$F_{int} = \frac{1}{\bar{\rho}^2}\sum_{n,m=-\infty}^{\infty}|\rho_{n,m}|^2V_{n,m}. \tag{C.4}$$

We will split last expression as in (15a) into two parts where one is with $n = 0, m = 0$ and other includes the rest of the sum. So from (C.4) we immediately get (15c).

Now we see from (14d) and (14c) that $f_{0,0}(\boldsymbol{r}) \equiv 1$. If we use (C.2)

$$V_{0,0} = \bar{\rho}\iint\limits_{-\infty}^{+\infty} V(|\boldsymbol{r}|)d^2\boldsymbol{r}, \tag{C.5}$$

and from (14b) we get $\rho_{0,0} = \bar{\rho}$. Combining this all together we get (15b).

Now we can provide some simplification for $V_{n,m}$ from (C.2). We will change Cartesian coordinates to polar in (14c) and (14d) and then use trigonometric identities to get the following

$$f_{n,m}(\boldsymbol{r}) = e^{\mathbf{i}\lambda_{n,m}r\cos(\theta-\delta\theta)},$$

where $\lambda_{n,m}$ is as in (15e) and $\delta\theta = \cos^{-1}(a/(n\lambda_{n,m}))$.

Now we use that V depends on $|\boldsymbol{r}|$ and integrate (C.2) over angles in terms of modified Bessel functions of the first kind. We will use following relation [38]

$$\mathrm{I}_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos(\theta)}d\theta, \tag{C.6}$$

and get

$$V_{n,m} = 2\pi\bar{\rho}\int_0^{\infty}\mathrm{I}_0(\mathbf{i}r\lambda_{n,m})V(r)rdr.$$

Mentioning that $\mathrm{J}_0(\mathbf{i}x) = \mathrm{I}_0(x)$ [38] we immediately get (15d).

Appendix D: Calculation of $\mathbf{V_{n,m}}$ for electrons on the liquid helium surface

To find $V_{n,m}$ we only need to substitute (18) into (15d) and provide integration. This is quite technical part of work, so we will provide only keysteps.

First of all if $a > 0$ and $y > 0$ [39]

$$\int_0^{\infty}\frac{x\mathrm{J}_0(yx)dx}{\sqrt{x^2+a^2}} = \frac{e^{-ay}}{y}, \tag{D.1}$$

and we can immediately write $V_{n,m}$ for direct interelectron interaction

$$V^{(ee)}_{n,m} = \frac{2\pi\bar{e}^2\bar{\rho}}{\lambda_{n,m}} - \frac{2\pi\varepsilon\bar{e}^2e^{-2d\lambda_{n,m}}\bar{\rho}}{\lambda_{n,m}}. \tag{D.2}$$

In this equation first term corresponds to Coulomb interaction and second to substrate polarization.

On the other hand if $a > 0$ and $b > 0$ it is known [39]

$$\int_0^{\infty}x\mathrm{J}_0(ax)\mathrm{K}_0(bx)dx = \frac{1}{a^2+b^2}. \tag{D.3}$$

With this equation we easily obtain $V_{n,m}$ for capillary interaction part

$$V^{(\mathrm{cap})}_{n,m} = -\frac{\bar{e}^2E^2}{\sigma}\frac{\bar{\rho}}{\lambda^2_{n,m}+1/l_0^2}. \tag{D.4}$$

Combining all together we get

$$V_{n,m} = V^{(ee)}_{n,m} + V^{(\mathrm{cap})}_{n,m}, \tag{D.5}$$

which is same as (19).

Appendix E: Calculation of $\mathbf{F_s}$

We will start with equation (13b) and assume that ρ_{SP} is a very "sharp" function. With this assumption we can change integration limits with infinite and this in turn will allow us perform variable exchange as we did in (C.1)

$$F_s = \iint\limits_{-\infty}^{+\infty}\iint\limits_{-\infty}^{+\infty} V(|\boldsymbol{r}'|)\rho_{SP}(\boldsymbol{r})\rho_{SP}(\boldsymbol{r}+\boldsymbol{r}')d^2\boldsymbol{r}'\,d^2\boldsymbol{r}. \tag{E.1}$$

First of all we perform some mathematical transformations with (20) (in Cartesian coordinate system) and write the following

$$\rho_{SP}(\boldsymbol{r})\rho_{SP}(\boldsymbol{r}+\boldsymbol{r}') = \frac{1}{4\pi^2s^4}e^{-[\boldsymbol{r}+\boldsymbol{r}'/2]^2/s^2-\boldsymbol{r}'^2/(4s^2)}.$$

Now we can substitute this expression into (E.1) and perform integration over $\boldsymbol{r}$. Writing result in polar coordinate system and integrating over angle we immediately get (27).

Since we know explicit expression for V (16) we can substitute it into (27) and perform integration using definition of the complementary error function [38]

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \tag{E.2}$$

As result we will get (28).

Appendix F: Alternative representation for $\lambda^{(t)}_{n,m}$

Substitution of $\alpha = \pi/6$, $a = \sqrt{3}\, b$ and $L^2 = ab \sin(\alpha)$ into (15e) leads to the following

$$\lambda^{(t)}_{n,m} = \frac{4\pi}{\sqrt[4]{12} L} \sqrt{n^2 + 3m^2 - 3mn}, \tag{F.1}$$

where (t) denotes triangular lattice.

But current equation is quite inconvenient for further manipulations especially for approximation of infinite sum by Euler-Maclaurin formula (used in Appendix G). Thus it is highly desirable to find some alternative expression for $\lambda^{(t)}_{n,m}$.

We know $\lambda^{(t)}_{n,m}$ is needed for calculation (15c) only. General view of (15c) is as follows

$$F'^{(t)}_{int} = \sum_{\substack{n,m=-\infty \\ n^2+m^2 \neq 0}}^{\infty} f\left(n^2 + 3m^2 - 3mn\right), \tag{F.2}$$

at this point we are not interested in expression for f. If we suppose $n = k + m$ it can be written

$$F'^{(t)}_{int} = \sum_{\substack{k,m=-\infty \\ k^2+m^2 \neq 0}}^{\infty} f\left(\frac{3}{4}[m-k]^2 + \frac{1}{4}[m+k]^2\right). \tag{F.3}$$

Reader should note that $k^2 + m^2 \neq 0 \Leftrightarrow n^2 + m^2 \neq 0$. Which means we can rewrite $\lambda^{(t)}_{n,m}$ as (30)

Appendix G: Approximation of F'_{int}

First of all we should mention from assumptions in Section 5.1 that $l_0 \gg L$ and we can suppose that $l_0 \lambda_{n,m} \gg 1$. This means we can neglect 1 in denominator of (31). Analogous we can think about d and $\lambda_{n,m}$. Since $d \gg L$, $d\lambda_{n,m} \gg 1$ and we can suppose exponent from (31) to be zero. This leads to the following

$$F'_{int} = \bar{\rho} \sum_{\substack{n,m=-\infty \\ n^2+m^2>0}}^{\infty} e^{-s^2 \lambda^2_{n,m}} \left(\frac{2\pi\bar{e}^2}{\lambda_{n,m}} - \frac{\bar{e}^2 E^2}{\sigma \lambda^2_{n,m}}\right). \tag{G.1}$$

If we mention [40] that $\operatorname{erfc}(x \to \infty) \sim e^{-x^2}/(x\sqrt{\pi})$ we can as easily estimate F_s

$$F_s = \frac{\sqrt{\pi}\bar{e}^2}{2s}\left(1 - \frac{\varepsilon s}{d\sqrt{\pi}}\right). \tag{G.2}$$

Equation (G.1) is still too complicated since it contains sum. To get some simplification we will substitute $\lambda_{n,m}$ from (29) and (30) into (G.1) and use Euler-Maclaurin equation to achieve approximation of sum with an integral. Besides, we note that $\bar{\rho} = 1/L^2$ for both lattices. We will use substitution $m = \xi \cos(\varphi)$, $n = \xi \sin(\varphi)$ for square lattice and $m = \xi \cos(\varphi - \pi/3)$, $n = \xi \sin(\varphi - \pi/6)$ for triangular to change coordinate system to polar. After integration over angle we get

$$\begin{aligned} F'^{(s)}_{int} &= \frac{2\pi\bar{e}^2}{L} \int_1^\infty e^{-s^2\xi^2 4\pi^2/L^2} d\xi \\ &\quad - \frac{\bar{e}^2 E^2}{2\pi\sigma} \int_1^\infty \frac{e^{-s^2\xi^2 4\pi^2/L^2}}{\xi} d\xi, \end{aligned} \tag{G.3a}$$

$$\begin{aligned} F'^{(t)}_{int} &= \frac{\pi\bar{e}^2 \sqrt[4]{12}}{L} \int_1^\infty e^{-s^2\xi^2\pi^2\sqrt{12}/L^2} d\xi \\ &\quad - \frac{\bar{e}^2 E^2}{2\pi\sigma} \int_1^\infty \frac{e^{-s^2\xi^2\pi^2\sqrt{12}/L^2}}{\xi} d\xi. \end{aligned} \tag{G.3b}$$

Lower limit of presented integrals is caused by the fact we omit point $m = 0, n = 0$ (G.1).

If we use definition of the incomplete gamma function [40]

$$\Gamma(a; x) = \int_x^\infty e^{-t} t^{a-1} dt, \tag{G.4}$$

and complementary error function (E.2), we can easily integrate (G.3) in terms of special functions

$$F'^{(s)}_{int} = \frac{\sqrt{\pi}\bar{e}^2}{2s} \operatorname{erfc}\left(\frac{2\pi s}{L}\right) - \frac{\bar{e}^2 E^2}{4\pi\sigma} \Gamma\left(0; \frac{4\pi^2 s^2}{L^2}\right),$$

$$F'^{(t)}_{int} = \frac{\sqrt{\pi}\bar{e}^2}{2s} \operatorname{erfc}\left(\frac{2\sqrt[4]{3}\pi s}{\sqrt{2}L}\right) - \frac{\bar{e}^2 E^2}{4\pi\sigma} \Gamma\left(0; \frac{\sqrt{12}\pi^2 s^2}{L^2}\right).$$

Reader should note that Jacobian for case $F'^{(t)}_{int}$ is $\xi/2$, not ξ as for $F'^{(s)}_{int}$.

If we note that $s \ll L$ we can approximate F_{int} using equations $\Gamma(0; x \to 0) \sim -\gamma - \ln(x)$ [38] and $\operatorname{erfc}(x \to 0) \sim 1 - 2x/\sqrt{\pi}$ [40] we can easily simplify previous equations. After simplification we will use that $L = 1/\sqrt{\bar{\rho}}$ for both lattices and substitute obtained equations as well as (G.2) into (31, 13e). As result we immediately get (33). Reader may notice that we neglected $F^{(0)}_{int}$ due to reasons mentioned in Section 3.

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