Eur. Phys. J. A (2017) **53**: 29
DOI 10.1140/epja/i2017-12218-5

THE EUROPEAN PHYSICAL JOURNAL A

Regular Article – Theoretical Physics

Scattering state solutions of the Duffin-Kemmer-Petiau equation with the Varshni potential model

O.J. Oluwadare[1,a] and K.J. Oyewumi[2,b]

[1] Department of Physics, Federal University Oye-Ekiti, P. M. B. 373, Oye-Ekiti, Ekiti State, Nigeria
[2] Department of Physics, Federal University of Technology, Minna, Niger State, Nigeria

Received: 21 January 2017
Published online: 17 February 2017 – © Società Italiana di Fisica / Springer-Verlag 2017
Communicated by M. Anselmino

Abstract. The scattering state of the Duffin-Kemmer-Petiau equation with the Varshni potential was studied. The asymptotic wave function, the scattering phase shift and normalization constant were obtained for any J states by dealing with the centrifugal term using a suitable approximation. The analytical properties of the scattering amplitude and the bound state energy were obtained and discussed. Our numerical and graphical results indicate that the scattering phase shift depends largely on total angular momentum J, screening parameter β and potential strengths a and b.

1 Introduction

There has been an increasing interest in finding the analytical solutions of a linear relativistic wave equation (Duffin-Kemmer-Petiau equation known as DKP equation) for some physical potential models in quantum mechanics. This is due to the fact that its analytical solutions can be used to describe the behaviour of spin-one and spin-zero particles [1–4]. The equivalence of this equation with the Klein-Gordon equation has raised a lot of arguments by various researchers in times past: see the detail reviews in [5–8] and the references therein.

The DKP equation has a wide range of application in cosmology and theoretical nuclear physics and its formalism has been used to study the deuteron-nucleus scattering [9] and quark confinement problems of quantum chromodynamics (QCD) theory in particle physics [10]. The search for scattering state solutions within different type of potential models has been a subject of interest to researchers in relativistic and non-relativistic quantum mechanics [11–19].

Furthermore, the scattering and bound states of the DKP equation have been solved using various methods within some interesting potential models including Deng-Fan Interaction [5], Hulthén potential [6,20], Hellman potential [7,21], coupled Hulthén-Woods-Saxon potentials [8], non-minimal vector double-step potential [22], non-minimal vector smooth step potential [23], Sextic oscillator [24], Yukawa potential [25–27], Coulomb interaction [28], Deformed Woods-Saxon potential [29], Harmonic oscillator and Coulomb potential [30], pseudoharmonic potential [31], Smorodinsky-Winternitz potential [32], hyperbolical potential [33], Kratzer potential [34], Manning-Rosen potential plus a ring-shaped–like potential [35] among others.

The purpose of this work is to investigate the scattering state solutions of the DKP equation with the Varshni potential model, obtain the phase shift, normalization constant, bound state energy formula and also discuss the possible limiting case.

The Varshni potential is given as [36–38]

$$V(r) = a\left[1 - \frac{b}{r}e^{-\beta r}\right], \tag{1}$$

where r is the internuclear distance, a and b are the strengths of the potential and β is the screening parameter which controls the shape of the potential energy curve.

This potential is a short range repulsive potential energy function which has been investigated within the formalism of the Schrödinger equation and it also plays a fundamental role in chemical and molecular physics [36, 37]. The Varshni potential was also studied by Lim using the 2-body Kaxiras-Pandey parameters [37]. In his work, he reported that Kaxiras and Pandey used this potential to describe the 2-body energy portion of multi-body condensed matter. In 2014, Arda and Sever investigated the pseudospin and spin symmetric solutions of the Dirac equation with the Hellmann potential, the Wei-Hua potential and the Varshni potential [38]. The relativistic

[a] e-mail: oluwatimilehin.oluwadare@fuoye.edu.ng (corresponding author)
[b] *On Sabbatical Leave from*: Theoretical Physics Section, Department of Physics, University of Ilorin, Ilorin, Nigeria; e-mail: kjoyewumi66@unilorin.edu.ng

bound state energies and spinor wave function were also reported.

The organization of this work is as follows: section 2 contains the scattering states of the DKP equation with the Varshni potential. Section 3 contains the discussion on the numerical and graphical results while the conclusion is given in sect. 4.

2 Scattering states of the Duffin-Kemmer-Petiau equation with the Varshni potential

The DKP equation with energy $E_{n,J}$, total angular momentum centrifugal term and the mass m of the particle is given as [6,28,29]:

$$F''_{n,J}(r)-J\,(J+1)\,r^{-2}+\left[\left(E_{n,J}+U_v^0\right)^2-m^2\right]F_{n,J}(r)=0. \quad (2)$$

Due to the total angular momentum centrifugal term, eq. (2) cannot be solved analytically for $J \neq 0$ states. Therefore, we employ the following suitable approximation scheme [5,14,21,38–40]:

$$\frac{1}{r^2}\approx\frac{\beta^2}{(1-e^{-\beta r})^2}\,, \quad (3)$$

to overcome the effect of this centrifugal. This approximation has been reported to be valid for $\beta r \ll 1$ [40]. Inserting eqs. (1) and (3) into eq. (2) and transforming by using the variable $y = 1 - e^{-\beta r}$, yields

$$F''_{n,J}(y)-\frac{1}{(1-y)}F'_{n,J}(r)+\frac{1}{y^2(1-y)^2}\left[-h_1y^2+h_2y-h_3\right]F_{n,J}(y)=0, \quad (4)$$

where

$$-h_1=\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-a^2b^2-J(J+1)-\frac{k^2}{\beta^2}\,, \quad (5)$$

$$h_2=\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-2a^2b^2, \quad (6)$$

$$-h_3=J(J+1)-a^2b^2, \quad (7)$$

and $k = \sqrt{(E_{n,J}^2-m^2)+a^2-2aE_{n,J}-J(J+1)\beta^2}$, which is the asymptotic wave number. Assuming the wave function of the form

$$F_{n,J}(y)=y^{\lambda}(1-y)^{-i(k/\beta)}f_{n,J}(y), \quad (8)$$

and inserting it into eq. (4), yields the following hypergeometric equation [41]:

$$y(1-y)f''_{n,J}(y)+\left[2\lambda-\left(2\lambda-2i\frac{k}{\beta}+1\right)y\right]f'_{n,J}(y)+\left[\left(\lambda-i\frac{k}{\beta}\right)^2+h_1\right]f_{n,J}(y)=0, \quad (9)$$

where we have used the following phase shift parameters:

$$\lambda=\frac{1}{2}+\sqrt{\frac{1}{4}+J(J+1)-a^2b^2}\,, \quad (10)$$

$$\eta_1=\lambda-i\frac{k}{\beta}-\sqrt{\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-a^2b^2-J(J+1)-\frac{k^2}{\beta^2}}\,, \quad (11)$$

$$\eta_2=\lambda-i\frac{k}{\beta}+\sqrt{\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-a^2b^2-J(J+1)-\frac{k^2}{\beta^2}}\,, \quad (12)$$

$$\eta_3=2\lambda. \quad (13)$$

The radial wave functions for any arbitrary J-wave scattering states for the Varshni potential are obtained as

$$F_{n,J}(r)=N_{n,J}\left(1-e^{-\beta r}\right)^{\lambda}e^{ikr}{}_2F_1\left(\eta_1,\eta_2,\eta_3;1-e^{-\beta r}\right), \quad (14)$$

where $N_{n,J}$ is the normalization constant to be determined.

2.1 The scattering phase shifts and normalization constant

We can obtain the phase shifts δ_J and normalization constant $N_{n,J}$ by employing the recurrence relation of the hypergeometric function or analytic-continuation formula [41]:

$$\begin{aligned}{}_2F_1\left(\eta_1,\eta_2,\eta_3;y\right)&=\frac{\Gamma(\eta_3)\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}{}_2F_1\\&\times(\eta_1;\eta_2;1+\eta_1+\eta_2-\eta_3;1-y)\\&+(1-y)^{\eta_3-\eta_1-\eta_2}\frac{\Gamma(\eta_3)\Gamma(\eta_1+\eta_2-\eta_3)}{\Gamma(\eta_1)\Gamma(\eta_2)}\\&\times{}_2F_1(\eta_3-\eta_1;\eta_3-\eta_2;\eta_3-\eta_1-\eta_2+1;1-y). \quad (15)\end{aligned}$$

Equation (15) with the condition that ${}_2F_1(\eta_1,\eta_2,\eta_3;0)=1$, when $r\to\infty$, leads to

$$\begin{aligned}{}_2F_1\left(\eta_1,\eta_2,\eta_3;1-e^{-\beta r}\right)&\xrightarrow[r\to\infty]{}\Gamma\left(\eta_3\right)\\&\times\left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}+e^{-2ikr}\left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right|^*\right|, \quad (16)\end{aligned}$$

where we have used the following phase shift relations:

$$\eta_3-\eta_1-\eta_2=\left(\eta_1+\eta_2-\eta_3\right)^*=2i(k/\beta), \quad (17)$$

$$\eta_3-\eta_2=\lambda+i\frac{k}{\beta}-\sqrt{\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-a^2b^2-J(J+1)-\frac{k^2}{\beta^2}}=\eta_1^*, \quad (18)$$

$$\eta_3-\eta_1=\lambda+i\frac{k}{\beta}+\sqrt{\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-a^2b^2-J(J+1)-\frac{k^2}{\beta^2}}=\eta_2^*. \quad (19)$$

Table 1. Scattering phase shifts for the DKP equation under Varshni potential as a function of the screening parameter β with $E_{n,J} = m = 1$.

J	β	δ_J for $a = b = 0.15$	δ_J for $a = b = 1$
0	0.2	−1.72757	−12.23212
	0.4	2.49849	−2.24510
	0.6	2.92309	0.13057
	0.8	2.83386	1.02819
	1.0	2.66856	2.79961
1	0.2	−5.10924	−11.57255
	0.4	−0.74219	−1.92768
	0.6	0.08681	0.29818
	0.8	0.38141	1.13872
	1.0	0.51906	1.54044
2	0.2	−10.00448	−11.76819
	0.4	−5.66079	−2.68247
	0.6	−4.78870	−0.67640
	0.8	−4.47597	0.06604
	1.0	−4.43997	1.51863
3	0.2	−16.16141	−12.72669
	0.4	−12.03657	−4.27730
	0.6	−11.22290	−2.48504
	0.8	−10.93385	−1.83279
	1.0	−10.79946	−1.52596

Table 2. Scattering phase shifts for the DKP equation under Varshni potential as a function of the potential strength b with $\beta = 0.2$ and $E_{n,J} = m = 1$.

J	b	δ_J for $a = 0.15$	δ_J for $a = 0$
0	−2	−1.70082	1.57080
	−1	−1.79581	1.57080
	0	−1.74554	1.57080
	1	−1.57351	1.57080
	2	−1.25398	1.57080
1	−2	−5.25693	0.76042
	−1	−5.20360	0.76042
	0	−5.12355	0.76042
	1	−5.01663	0.76042
	2	−4.88149	0.76042
2	−2	−10.13493	−4.07243
	−1	−10.08067	−4.07243
	0	−10.01526	−4.07243
	1	−9.93844	−4.07243
	2	−9.84973	−4.07243
3	−2	−16.26480	−10.56258
	−1	−16.22031	−10.56258
	0	−16.16957	−10.56258
	1	−16.11245	−10.56258
	2	−16.04873	−10.56258

Now, by taking

$$\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)} = \left|\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right| e^{i\delta}, \tag{20}$$

and substituting this into eq. (16), we have

$$ {}_2F_1\left(\eta_1,\eta_2,\eta_3;1-e^{-\beta r}\right) \xrightarrow[r\to\infty]{} \Gamma(\eta_3)\left[\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right] \times e^{-ikr}\left[e^{i(kr-\delta)}+e^{-i(kr-\delta)}\right]. \tag{21}$$

Consequently, we obtain the asymptotic form of eq. (14) for $r\to\infty$ as

$$F_{n,J}(r) \xrightarrow[r\to\infty]{} 2N_{n,J}\Gamma(\eta_3)\left[\frac{\Gamma(\eta_3-\eta_1-\eta_2)}{\Gamma(\eta_3-\eta_1)\Gamma(\eta_3-\eta_2)}\right] \times \sin\left(kr+\delta+\frac{\pi}{2}\right). \tag{22}$$

Finally, with the appropriate boundary condition, eq. (22) yields [42]

$$F_{n,J}(\infty) \to 2\sin\left(kr+\delta_J-\frac{l\pi}{2}\right). \tag{23}$$

The phase shifts expression and the normalization constant are obtained, respectively, as follows:

$$\delta_J = \frac{\pi}{2}+\frac{J\pi}{2}+\delta = \frac{\pi}{2}(J+1)+\arg\Gamma\left(2i(k/\beta)\right) - \arg\Gamma\left(\eta_2^*\right) - \arg\Gamma\left(\eta_1^*\right) \tag{24}$$

and

$$N_{n,J} = \frac{1}{\sqrt{\eta_3}}\left|\frac{\Gamma(\eta_1^*)\Gamma(\eta_2^*)}{\Gamma(2i(k/\beta))}\right|. \tag{25}$$

2.2 Bound state energy at the pole of scattering amplitude

Here, we consider the analytical properties of the partial-wave s-matrix to obtain the bound state energy at the poles of the s-matrix in the complex energy plane. And therefore, we need to discuss the following property $\Gamma(\eta_3-\eta_1)$ [42] as

$$\eta_3-\eta_1 = \lambda+i\frac{k}{\beta} + \sqrt{\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-a^2b^2-J(J+1)-\frac{k^2}{\beta^2}}. \tag{26}$$

The first-order poles of $\Gamma(\lambda+i\frac{k}{\beta}+\sqrt{\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-a^2b^2-J(J+1)-\frac{k^2}{\beta^2}})$ are situated at

$$\Gamma\left(\lambda+i\frac{k}{\beta}+\sqrt{\frac{2abE_{n,J}}{\beta}-\frac{2a^2b}{\beta^2}-a^2b^2-J(J+1)-\frac{k^2}{\beta^2}}\right) +n = 0 \quad (n=0,1,2,\ldots). \tag{27}$$

Table 3. Scattering phase shifts for the DKP equation under Varshni potential as a function of the total angular momentum J with $a = b = 0.15$ and $E_{n,J} = m = 1$.

J	δ_J for $\beta = 0.1$	δ_J for $\beta = 0.2$	δ_J for $\beta = 0.3$	δ_J for $\beta = 0.4$	δ_J for $\beta = 0.5$	δ_J for $\beta = 0.6$
0	−17.41433	−1.72757	1.49590	2.49849	2.83698	2.92309
1	−20.43171	−5.10924	−1.89091	−0.74219	−0.20652	0.08681
2	−24.78812	−10.00448	−6.83683	−5.66079	−5.09919	−4.78870
3	−30.32460	−16.16141	−13.14598	−12.03657	−11.51143	−11.22290
4	−36.88289	−23.39202	−20.57391	−19.55010	−19.06889	−18.80557
5	−44.38034	−31.53057	−28.91089	−27.96819	−27.52710	−27.28630
6	−52.55930	−40.43820	−38.00118	−37.13003	−36.72360	−36.50205
7	−61.48027	−50.00450	−47.73031	−46.92116	−46.54440	−46.33922
8	−71.01726	−60.14278	−58.01231	−57.25685	−56.90556	−56.71438
9	−81.10528	−70.78461	−68.78088	−68.07212	−67.74287	−67.56377
10	−91.68882	−81.87517	−79.98354	−79.31569	−79.00567	−78.83709
11	−102.72046	−93.36988	−91.57789	−90.94613	−90.65303	−90.49369
12	−114.15964	−105.23192	−103.52896	−102.92926	−102.65115	−102.50000
13	−125.97162	−117.43047	−115.80746	−115.23644	−114.97172	−114.82787
14	−138.12644	−129.93937	−128.38855	−127.84333	−127.59064	−127.45334
15	−150.59817	−142.73617	−141.25085	−140.72897	−140.48716	−140.35578

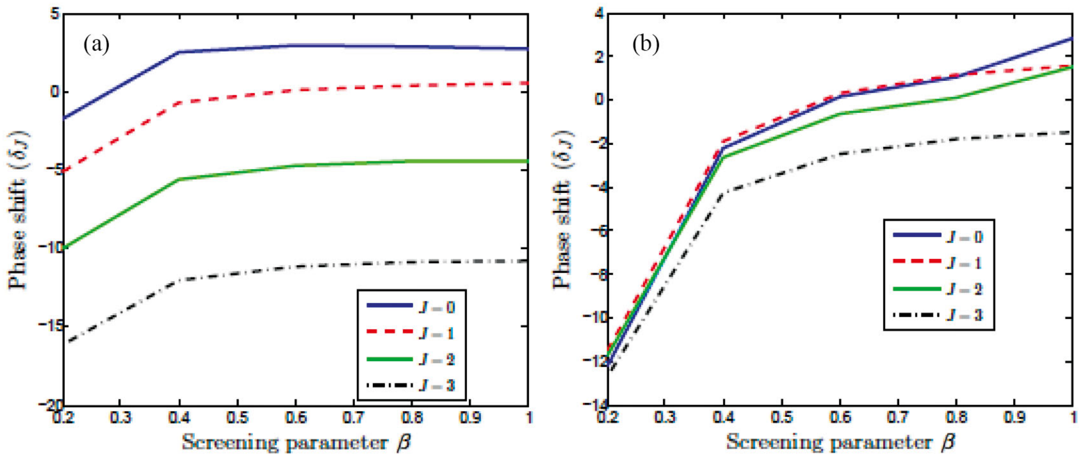

Fig. 1. (a) Scattering phase shifts for the DKP equation with the Varshni potential as a function of the screening parameter β with $a = b = 0.15$ and $E_{n,J} = m = 1$. (b) The same as (a) with $a = b = 1$ and $E_{n,J} = m = 1$.

Consequently, the bound state energy equation for the Varshni potential under the DKP equation is obtained as

$$k^2 = -\beta^2 \left[\frac{(n+\lambda)^2 - \frac{2abE_{n,J}}{\beta} + \frac{2a^2 b}{\beta^2} + a^2 b^2 - J(J+1)}{2(n+\lambda)} \right]^2 . \tag{28}$$

3 Numerical results and discussions

Table 1 shows that the scattering phase shift increases with increasing screening parameter β for all total angular momentum J. Column 3 (δ_J for $a = 0.15$) of table 2 clearly indicates that the scattering phase shift increases linearly with increasing potential strength b for all total angular momentum J. Column 4 (δ_J for $a = 0$) of table 2 shows that the scattering phase shift does not really depend on the potential strength b but is significantly dependent on the total angular momentum J.

Obviously, table 3 shows that the scattering phase shift depends largely on the total angular momentum J as it decreases exponentially with the increase in the total angular momentum J for the selected potential parameters. A linear increase in the values of the phase shift is ob-

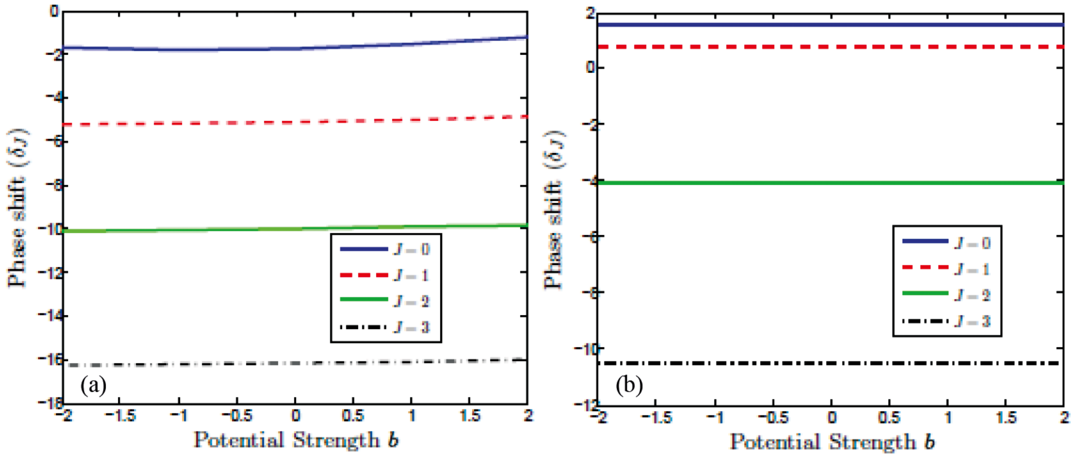

Fig. 2. (a) Scattering phase shifts for the DKP equation with the Varshni potential as a function of the potential strength b for $a = 0.15$, $\beta = 0.2$ and $E_{n,J} = m = 1$. (b) The same as (a) for $a = 0$, $\beta = 0.2$ and $E_{n,J} = m = 1$.

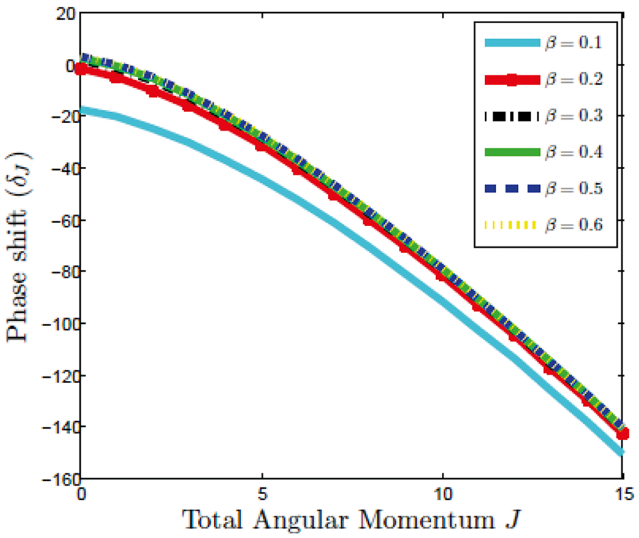

Fig. 3. Scattering phase shifts for the DKP equation under Varshni potential as a function of the total angular momentum J with $a = b = 0.15$ and $E_{n,J} = m = 1$.

served as the screening parameter increases from 0.1 to 0.3. But a minor linear increase is noticeable as the screening parameter increases from 0.4 to 0.6. This shows that the scattering phase shift depends mainly on the total angular momentum J than any other potential parameters. To see the trends and the beauty of our results, we displayed the graphical solutions in figs. 1–3. All the figures confirmed that the scattering phase shift largely depends on the total angular momentum J, screening parameter β and potential strengths a and b.

4 Conclusion

We have studied the scattering state solutions of the DKP equation with the Varshni potential by applying a suitable approximation scheme within the formalism of the functional analytical method. The approximate scattering phase shift, normalization constant and the corresponding asymptotic wave function have been obtained. The approximate bound state energy at the poles of the scattering amplitude has been reported.

The numerical values of the scattering phase shift using some selected values of potential parameters and other related quantities have been presented in tables 1–3. The graphical results have been presented to see the trend, clarity and dependence of the phase shift on the aforementioned parameters. It is evident and observed from both the numerical and graphical results that the scattering phase shift is dependent on the total angular momentum J, screening parameter β and potential strengths a and b. Our results find applications in chemical and nuclear physics where the scattering of particles is of importance.

OJO declared that this paper represents an output of his PhD Research Thesis supported by the Tertiary Education Trust Funds (TETFunds) through the Federal University Oye Ekiti, Ekiti State, Nigeria. The authors also declared that there is no conflict of interest regarding this paper.

References

1. G. Petiau, Acad. R. Belg. Cl. Sci. Mem. Collect. **8**, 16 (1936) PhD Thesis, University of Paris.
2. N. Kemmer, Proc. R. Soc. A **166**, 127 (1938).
3. R.J. Duffin, Phys. Rev. **54**, 1114 (1938).
4. N. Kemmer, Proc. R. Soc. A **173**, 91 (1938).
5. S. Hassanabadi *et al.*, Adv. High Energy Phys. **2012**, 804652 (2012).
6. S. Zarrinkamar *et al.*, Eur. Phys. J. Plus **128**, 109 (2013).
7. A.N. Ikot *et al.*, Few Body Syst. **55**, 211 (2014).
8. A.N. Ikot *et al.*, Z. Naturforsch. A **70**, 185 (2015).
9. R.E. Kozack *et al.*, Phys. Rev. C **40**, 2181 (1989).
10. V. Gribov, Eur. Phys. J. C **10**, 71 (1999).
11. N.F. Mott, H.S.W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, Cambridge, 1965).

12. O.J. Oluwadare, K.E. Thylwe, K.J. Oyewumi, Commun. Theor. Phys. **65**, 434 (2016).
13. K.E. Thylwe, O.J. Oluwadare, K.J. Oyewumi, Commun. Theor. Phys. **66**, 389 (2016).
14. K.J. Oyewumi, O.J. Oluwadare, Eur. Phys. J. Plus **131**, 295 (2016).
15. K.J. Oyewumi, O.J. Oluwadare, Adv. High Energy Phys. **2017**, 1634717 (2017).
16. S.H. Dong, M. Lozada-Gassou, Phys. Lett. A **330**, 168 (2004).
17. A.D. Alhaidari, J. Phys. A: Math. Gen. **38**, 3409 (2005).
18. G.F. Wei, Z.Z. Zhen, S.H. Dong, Cent. Eur. J. Phys. **7**, 175 (2009).
19. G.F. Wei, W.C Qiang, W.L. Chen, Cent. Eur. J. Phys. **8**, 574 (2010).
20. Z. Molaee *et al.*, Chin. Phys. B **22**, 060306 (2013).
21. C.A. Onate *et al.*, Afr. Rev. Phys. **9**, 0062 (2014).
22. L.P. de Oliveira, A.S. de Castro, Can. J. Phys. **90**, 481 (2012).
23. L.B. Castro, T.R. Cardoso, A.S. de Castro, Nucl. Phys. B Proc. Suppl. **199**, 207 (2010).
24. F. Yasuk, M. Karakoc, T. Boztosun, Phys. Scr. **78**, 0031, 045010 (2008).
25. M. Hamzavi, S.M. Ikhdair, Few Body Syst. **54**, 1753 (2013).
26. N. Salehi, H. Hassanabadi, Eur. Phys. J. A **51**, 100 (2015).
27. C.A. Onate, Afr. Rev. Phys. **9**, 0033 (2014).
28. H. Hassanabadi *et al.*, Phys. Rev. C. **84**, 064003 (2011).
29. M. Hamzavi, S.M. Ikhdair, Few. Body Syst. **53**, 461 (2012).
30. I. Boztosun *et al.*, J. Math. Phys. **47**, 062301 (2006).
31. H. Hassanabadi, Z. Molaee, A. Boumali, Found. Phys. **43**, 225 (2013).
32. M.K. Bahar, F. Yasuk, Ann. Phys. **344**, 105 (2014).
33. H. Hassanabadi, Z. Molaee, A. Boumali, Chin. Phys. C. **37**, 073104 (2013).
34. H. Hassanabadi *et al.*, Phys. Part. Nucl. Lett. **10**, 699 (2013).
35. H. Hassanabadi, M. Kamali, B.H. Yazarloo, Can. J. Phys. **92**, 1 (2014).
36. Y.P. Varshni, Rev. Mod. Phys. **31**, 839 (1959).
37. T.C. Lim, J. Serb. Chem. Soc. **74**, 1423 (2009).
38. A. Arda, R. Sever, Z. Naturforsch. A **69**, 163 (2014).
39. H. Hassanabadi, S. Zarrinkamar, H. Rahimov, Commun. Theor. Phys. **56**, 423 (2011).
40. W.A. Yahya *et al.*, Int. J. Mod. Phys. E **22**, 1350062 (2013).
41. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (U.S. Department of Commerce, National Bureau of Standards, New York, 1965).
42. L.D Landau, E.M. Lifshitz, *Quantum Mechanics, Non-Relativistic Theory*, 3rd edition (Pergamon, New York, 1977).