Eur. Phys. J. A (2014) **50**: 169
DOI 10.1140/epja/i2014-14169-7

THE EUROPEAN PHYSICAL JOURNAL A

Regular Article – Theoretical Physics

A study of a relativistic quark-diquark model for the nucleon

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Received: 1 July 2014 / Revised: 17 October 2014
Published online: 11 November 2014 –
Communicated by S. Hands

Abstract. We develop a specific quark-diquark constituent model for the study of the nucleon and its excited states. A relativistic kinetic energy operator is used for both the quark and the diquark. Interaction terms depending on the spin and angular momentum operators are introduced to describe in detail the mass spectrum. Different values of the parameters in the interaction operators are taken for the scalar and axial-vector diquark states. The tensor interaction is also considered in the present model. The baryon spectrum is calculated up to resonance masses of 2 GeV.

1 Introduction

The study of hadronic systems by means of phenomenological models represents a topic of high interest in particle physics due to the difficulties that are found when deriving from *Quantum Chromo-Dynamics* (QCD) a truly complete and detailed theoretical explanation of hadron properties.

We recall that QCD, which is assumed to be the correct underlying, fundamental, field theory, allows for a quantum and relativistic study of the strong interactions. Furthermore, the gauge character of this theory insures the renormalizability for the physical observables. The study of hadronic matter in deep inelastic scattering experiments corroborated the QCD description of the hadrons in terms of quarks and gluons and confirmed that QCD can be really considered as the correct theory of the hadronic interactions.

The recent numerical studies performed by means of lattice QCD have reached interesting results in order to explain the hadron dynamics; see, for example, refs. [1–3]. Many nonperturbative field-theoretical approaches have been developed to understand the quark dynamics in the low-energy domain where confinement represents the most relevant physical effect.

However, a detailed interpretation of the precise experimental data of the hadronic spectra (that belong to this low-energy domain) still represents a theoretical challenge that leaves open the possibility of using and developing QCD inspired constituent models.

More precisely, different versions of *Constituent-Quark Models* (CQM) have been developed in order to give a phenomenological description of hadronic bound states. In this regard we cite here the historical work of ref. [4] and some relevant review papers [5–10]. The CQM are constructed assuming the same global symmetries of QCD. Although they do not represent a fundamental theory of hadronic systems, the CQM can offer a phenomenological description of these systems in terms of quark degrees of freedom, allowing to correlate many experimental data of hadron spectroscopy.

In general, CQM are based on a Hamiltonian operator for the hadronic system in which a specific effective potential (or quasi-potential, in relativistic treatments) is chosen to represent the quark interaction. In this way, solving the eigenvalue equation for the Hamiltonian, it is possible to calculate the main properties of hadrons, in particular, their mass spectrum, their wave functions and, in consequence, their electroweak response functions. In more detail, many specific forms of CQM have been proposed in order to represent quark-antiquark ($q\bar{q}$) and quark-quark (qq) interactions [11,12]. These models, using a reasonable number of free parameters, reproduce with satisfactory accuracy the hadronic mass spectra.

Focusing our attention on the baryon spectroscopy (that will be studied in this work by means of our quark-diquark model) we first mention the Hypercentral Constituent Quark Model (HCQM). This model includes the three-body quark interactions in the theoretical framework of the CQM by introducing the spatial hyperspherical quark variables. The HCQM has also allowed to calculate with high accuracy the electromagnetic elastic form factors and the helicity transition amplitudes of the nucleon [6,13–24].

Models based on the symmetry properties of the baryonic states have been developed [25].

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Furthermore, the possibility of introducing in the CQM the *continuum*, or pair-creation effects has been explored in the so-called *unquenched* quark model [26–30], also including strange ($s\bar{s}$) pairs [31–33]. This model can explain in a natural way the decays and the resonance widths, also trying to solve the *missing resonance* problem (to be discussed in the following) of the baryon spectrum.

Finally, we point out that the *diquark* hypothesis (on which the present work is based) has also been introduced for the study of the baryon spectroscopy. The concept of diquark was first proposed by Gell-Mann and soon afterward constituent quark-diquark models for baryons were developed by Ida and Kobayashi [34] and Lichtenberg *et al.* [35–37] also studying, within the model, the electromagnetic properties of the baryons. This area of investigation grew rapidly and, at the moment, is extremely active. We cite here only a small, and not exhaustive, sample of the published articles [38–47]. We note that in the more recent works this model has been studied in a relativistic theoretical framework.

In this respect, we previously proposed a nonrelativistic version of model with relativistic corrections [48]. That model has been deeply revised and generalized in the present relativistic work.

Very recently, a parallel but independent investigation has been developed about a relativistic quark-diquark model with spin-isospin transition amplitudes [49].

For completeness we also mention that many other different developments and applications of the quark-diquark model and of the diquark hypothesis have been studied.

A simple hybrid three-quark and quark-diquark model [50] has been proposed with the aim of analyzing in detail the *missing resonance* problem of the baryon spectrum. A model in which the quark-diquark configurations are immersed in a pion cloud has been also studied [51].

The quark-diquark model has been used to study the nucleon elastic scattering [52,53].

On a more fundamental level, diquark condensates have been studied by means of a Nambu-Jona-Lasinio–type model [54,55].

Furthermore, for hadron spectroscopy, recent studies have shown that the diquark can be very helpful for the description of the exotic mesons known as tetraquarks that can be represented as bound-states of a diquark and an antidiquark [56–61].

Finally, a nonconventional description of the meson spectra in terms of quarks and flavor-antisymmetric diquarks has been also proposed [62,63].

We now direct our attention on the main goal of the present work that is the construction of a quark-diquark model for the nucleon and its resonances. To this aim, we previously recall that the diquark is considered as a strongly correlated quark pair, with *frozen* internal spatial excitations. More exactly, these excitations are assumed to occur at higher energies than the upper limit of the resonance mass scale under examination. The two quarks (of the diquark) are identical fermions that must satisfy the exclusion Pauli principle. Since the intrinsic spatial wave fuctions of the diquarks are considered to be *symmetric* under quark interchange (recall that the diquarks are states with no spatial excitations) their color-spin-flavor wave functions must be *antisymmetric*. In consequence, the permitted color-spin-flavor representations must be antisymmetric in color and symmetric in spin-flavor (or, hypothetically, symmetric in color and antisymmetric in spin-flavor). In order to obtain a colorless baryon, only the antisymmetric color representation is allowed. Therefore we have *scalar* ($s_1 \equiv t_1 = 0$) and *axial-vector* ($s_1 \equiv t_1 = 1$) diquarks, where s_1 and t_1 represent the spin and isospin of the diquark, respectively. Here and in the following, the index 1 is used for the diquark, while the index 2 will identify the (other) quark of the baryon.

As it will discussed in the paper, a peculiar aspect of the present model is that the two diquark states, namely the scalar diquark and the axial-vector one, are assumed to have *different* dynamical properties and, in consequence, different interactions with the quark.

The use of the diquark model for baryons presents several advantages. In particular, one has to face a two-body problem that, in the Center of Mass (CM) of the baryon, requires *one* spatial relative coordinate $\boldsymbol{r}$, instead of the *two* spatial relative coordinates (usually $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$) of the standard three-quark CQM. On a practical level, the solution of a two-body problem is mathematically much simpler than the original three-body one. Furthermore, a relevant improvement of the baryonic spectroscopy is also obtained because the use of *only one* relative coordinate allows to reduce the number of spatial excitations permitted by the model and, in consequence, of the number of the predicted baryon resonances. In this way, as anticipated above, a possible solution to the so-called *missing resonance* problem can be found. Considering only the nonstrange sector, we recall that, up to an excitation energy of 2.41 GeV, about 45 nucleon states are predicted, in general, by the standard three-quark CQM. On the other hand, experimentally, only 12 *established* and 7 *tentative* resonances are found. This discrepancy represents a major failure of all the three-quark CQM, while it is not found when quark-diquark models are used [49].

We note that, although the diquark hypothesis is very interesting for the study of hadron spectroscopy, a completely satisfactory microscopic explanation of the diquark is not yet available and its relationship with QCD remains under examination. In this work, we only study the phenomenological aspects of the quark-diquark model without attempting to analyze its properties on a fundamental level.

In the present paper, as a first step, we construct a Hamiltonian for our model of the nucleon, considered as a bound state of quark-diquark. As shown in the next sect. 2, the form of the model Hamiltonian is chosen in order to reproduce the main quantum mechanical symmetries of the problem. For each term of this operator a possible phenomenological interpretation will be suggested. Considering that for low constituent-mass values the relativistic effects may be important, the kinetic operators for both quark and diquark are taken in relativistic form, as shown in subsect. 2.1. This choice, as discussed in the

following, allows for constructing a covariant model. At a phenomenological level, it takes into account the relativistic *kinetic* effects due to the motion of the quark and of the diquark, improving the consistency of the model with respect to the previous semirelativistic version [48].

The confining part of the interaction is represented by a modified Cornell potential. In this concern, a preliminar analysis has shown that a standard Cornell potential does not allow to reproduce accurately high excitation resonances. A better reproduction of the spectra may be obtained with the modified Cornell potential that will be introduced in subsect. 2.2. The modification to the Cornell potential that is performed in the present work is purely phenomenological. However, a more detailed investigation on a (possibly) first-principle inspired quark interaction should be performed by considering a *simpler* system, like Charmonium (for which, a pure Cornell potential is also insufficient).

Spin- and isospin-dependent interaction terms are added in a conventional way in order to reproduce the detailed structure of the spectrum. The form of the corresponding operators will be given in subsect. 2.3.

In subsect. 2.4, an angular-momentum–dependent interaction will be introduced in order to enhance the energy difference of states with the same orbital angular momentum. In the same subsect. 2.4, a tensor interaction term will be also introduced. In this respect, we recall that an interaction of this kind has given good results in the study of meson spectra and it is interesting to analyze its effect in the present nucleon model.

In order to determine the eigenvalues and the eigenfunctions of the Hamiltonian, a numerical method of diagonalization and minimization is implemented. The wave functions of the variational basis will be introduced in sect. 3 and the method for the numerical solution will be briefly described in sect. 4. The mass spectrum is then obtained by choosing, with a fit procedure, the appropriate values for the free parameters of the Hamiltonian. This point will be discussed in sect. 5 for the baryonic resonances given by the scalar and by the axial-vector diquark states.

Finally, in sect. 6, a discussion is performed comparing the results of the present work with those given by other approaches.

Before discussing the details of our Hamiltonian, we make a brief comparison of the present model with that of the work [49]. We notice that, at a fundamental level, in that investigation, a spin-isospin transition interaction has been introduced in order to mix scalar and axial-vector diquark states. That transition interaction is not considered here. Furthermore, a different parametrization of the quark-diquark interaction has been used. In ref. [49] a regularized Coulomb potential plus a linear confining term is taken, while here the *modified Cornell potential* is used. As for the spin-isospin–dependent interaction, ref. [49] considers an *exchange* operator, while here only one term of a spin-isospin dependent operator is used. Furthermore, we have here an angular-momentum–dependent interaction and a tensor interaction operator, inspired by the reduction of a vector field theory. Finally, as stated before, here we take different interaction parameters for the two diquark states.

As a very relevant remark, we refer the interested reader to ref. [49] for a detailed discussion of the relativistic transformation properties of the Hamiltonian eigenstates. In both models, the CM Hamiltonian, or better, *mass operator*, is obtained by adding to the free Hamiltonian the interaction operator specifically obtained by means of the Bakamjian-Thomas construction [64]. In this way, *both models* are invariant under the *point form* transformations of the Poincaré group [43,65–68].

2 Hamiltonian of the model

The CM Hamiltonian (or mass operator) chosen for our quark-diquark model can be written in the following form

$$H = E_0 + T + V_c + V_{st} + V_{lq}\,. \tag{1}$$

In the previous equation, E_0 represents a *zero point* constant energy that is customarily considered in quark-diquark models (see, for example, [39]). The other terms of the Hamiltonian will be discussed in the following subsections. The specific form of these terms gives rise to a total operator H that is invariant under spatial translations, spatial rotations and "rotations" in the isospin space. The use of relativistic kinetic energy operators and the specific form of the interaction allow for Poincaré invariance, as discussed above. Formally, our Hamiltonian does not depend on the color Gell-Mann matrices $\hat{\boldsymbol{\lambda}}_i$ of the quarks. In more detail, for any quark pair i, j one has, in the colorless, antisymmetric, state, $\langle \hat{\boldsymbol{\lambda}}_i \cdot \hat{\boldsymbol{\lambda}}_j \rangle = -2/3$ so that this *constant* mean value can be implicitly taken into account by means of the effective coupling constants of the phenomenological interaction operators that will be introduced in the following subsections.

2.1 The kinetic energy

The first term, T, represents the kinetic energy operator of the quark and diquark. Due to the presence of light quarks (u, d), we take for this operator a relativistic expression. Our aim is to improve the corresponding nonrelativistic version of the model [48] in which the relativistic corrections of the kinetic operators were treated perturbatively. There, we showed that the kinetic relativistic effects are numerically large, so that an exact calculation is preferred.

In the present work, we use the following standard expression:

$$T = \sqrt{m_1^2 + \boldsymbol{p}^2} + \sqrt{m_2^2 + \boldsymbol{p}^2}\,, \tag{2}$$

where, in the CM, we have taken $\boldsymbol{p}_1 = -\boldsymbol{p}_2 = \boldsymbol{p}$. This relative momentum operator has canonical commutation rules with the relative coordinate $\boldsymbol{r}$. Furthermore, m_1 and m_2 respectively represent the mass of the diquark and the mass of the quark.

2.2 The modified Cornell potential

A very well-known expression of the quark interaction, widely used in the CQM, is given by the Cornell potential [69,70], in the following form:

$$V_{Corn.}(r) = -\frac{\tau}{r} + \beta r, \tag{3}$$

where the first term represents a short-range Coulombic interaction, inspired by the Fermi-Breit reduction for a vector interaction [5,71]. We recall that this reduction also gives the corresponding spin-spin and the tensor interaction terms.

The second, linear, term gives rise to the quark confinement. This term, originally introduced in a phenomenological way, has been also supported by lattice calculations. However, our preliminar numerical analysis and other studies in the CQM framework have suggested that the confining interaction term should not necessarily have a linear behavior. For this reason, to obtain a better reproduction of the spectrum, we introduce a *modified Cornell potential* in the following form:

$$V_c(r) = -\frac{\tau}{r} + \beta_1 \left(\frac{r}{d}\right)^{k_1} + \beta_2 \left(\frac{r}{d}\right)^{k_2}, \tag{4}$$

where the *dummy* constant $d \equiv 1\,\mathrm{fm}$ has been introduced. In this way, it is possible to define the adimensional variable $x = \frac{r}{d}$ and the exponents k_1 and k_2 can take (with no difficulty) any real, positive, value. Furthermore, the coupling parameters β_1 and β_2 have the units of energy. For clarity, we also recall that in the first term of the previous equation a factor $\hbar c$ is understood, so that, as customary, the parameter τ represents an *adimensional* coupling constant, that, as discussed at the beginning of the present section, also takes into account the mean value of the Gell-Mann color matrices.

The modified Cornell potential of the previous equation is used in the eq. (1) for the present work.

2.3 Spin- and isospin- dependent interaction

Within the three-quark CQM, in order to reproduce the baryonic spectrum and, in particular, the energy difference between the nucleon $N(939)$ and Delta $\Delta(1232)$ states, it is necessary to introduce the spin-spin interaction that, from a fundamental point of view, appears in the Fermi-Breit reduction of a vector field theory. An interaction of this kind is generalized to the case of the quark-diquark models with the introduction of spin-spin– and also isospin-isospin–dependent operators [39]. In the present work, we take an operator of the following form:

$$V_{st} = U_{st}(r) \cdot (\boldsymbol{s_1} \cdot \boldsymbol{s_2})(\boldsymbol{t_1} \cdot \boldsymbol{t_2}), \tag{5}$$

with

$$U_{st}(r) = \eta e^{-\rho^2 r^2}. \tag{6}$$

In eq. (5) $\boldsymbol{s_1}$, $\boldsymbol{s_2}$ are the spin operators for the diquark and for the quark, respectively. Analogously, $\boldsymbol{t_1}$, $\boldsymbol{t_2}$ represent the corresponding isospin operators.

The coupling constant η and the spatial spread parameter ρ represent the free parameters of this interaction.

As it will be shown in the following, this isospin-dependent interaction is important to fit accurately the baryon mass spectrum.

2.4 Angular-momentum–dependent and tensor interactions

The last term in the Hamiltonian of eq. (1), denoted as V_{lq}, contains the interactions that explicitly depend on the angular momentum. For clarity, we split this term into two parts:

$$V_{lq} = V_l + V_q. \tag{7}$$

The first operator is

$$V_l = (\boldsymbol{l}^2)^{\frac{1}{4}} \beta_3 \cdot \left(\frac{r}{d}\right)^{k_3}, \tag{8}$$

where $\boldsymbol{l}$ represents the orbital angular momentum of the system. The second operator is

$$V_q = \frac{\sigma}{r^2} S_{12}, \tag{9}$$

where S_{12} represents the standard tensor operator that, written by means of spherical operators, has the form

$$S_{12} = \sqrt{\frac{8\pi}{15}} \sum_{m=-2}^{2} Y_{2,m}(\hat{r})[s_1 \otimes s_2]_{2,-m}(-1)^m; \tag{10}$$

finally, the free parameters of these interaction terms are the coupling constants β_3 and σ and the exponent k_3.

From a phenomenological point of view, the angular-momentum–dependent term V_l is important in order to enhance the mass difference between states with different angular momentum [72]. Theoretically, terms of this kind can be found in quantum field theory reductions [71]; in this case, momentum-dependent operators give rise to a dependence on $\boldsymbol{l}^2$. In the present work, to improve the fit of the spectrum, we use the specific form shown in eq. (8). On the other hand, the tensor term of eq. (9) has the standard form given by the Fermi-Breit reduction for a vector field interaction.

Finally, we note that for a scalar diquark state, *i.e.* with $s_1 = 0$, the spin-spin interaction of eq. (5) and the tensor interaction of eq. (9), are vanishing.

3 The trial wave functions of the model

In general, the total wave functions for our model (that is defined by the Hamiltonian of eq. (1)) are constructed by means of the tensor products of spatial, spin, isospin and color wave functions. Schematically one has

$$\Psi_{\mathrm{tot}} = \psi_{\mathrm{spatial}} \otimes \chi_{\mathrm{spin}} \otimes \Phi_{\mathrm{isospin}} \otimes \Psi_{\mathrm{color}}. \tag{11}$$

We assume for Ψ_{tot} the same structure that is taken for the case of the standard three-quark CQM. In particular, the

total antisymmetry (with respect to quark interchange) is given by the color factor $\Psi_{\rm color}$. However, this color factor gives no contribution to the matrix elements of the observable quantities that will be calculated in the following and, for this reason (as costumary in CQM) it will be always omitted. Furthermore, considering the diquark as pair of strongly correlated quarks, as discussed in the introduction, its spin and isospin *intrinsic* wave functions must have the same symmetry. We recall that, in consequence, we have $s_1 \equiv t_1 = 0, 1$.

For the spatial wave functions we factorize, as usual, the radial wave functions and the spherical harmonics:

$$\psi_{n,l,m}(\boldsymbol{r}) = R_{n,l}(r) Y_{l,m}(\hat{r}). \tag{12}$$

As radial *trial* wave functions, for the variational calculation, we choose the well-known Harmonic Oscillator (HO) wave functions [73]:

$$R_{n,l}(r) = \left(\frac{1}{\bar{r}}\right)^{\frac{3}{2}} C_{n,l} \cdot \exp\left[-\frac{1}{2}\left(\frac{r}{\bar{r}}\right)^2\right] L^{l+\frac{1}{2}}_{\frac{(n-l)}{2}}\left[\left(\frac{r}{\bar{r}}\right)^2\right], \tag{13}$$

where $\bar{r}$ represents the (variational) dimensional parameter. The possible values for n and l, in the previous equation, are:

$$n = 0, 1, 2, 3, \ldots \qquad l = n, n-2, n-4, \ldots (l \geq 0). \tag{14}$$

The normalization constant is

$$C_{n,l} = \left[\frac{2\left[\frac{(n-l)}{2}\right]!}{\Gamma(\frac{n+l+3}{2})}\right]^{\frac{1}{2}} \tag{15}$$

and the Laguerre polynomials have the standard form,

$$L^{l+\frac{1}{2}}_{\frac{(n-l)}{2}}\left[\left(\frac{r}{\bar{r}}\right)^2\right] = \sum_{m=0}^{\frac{(n-l)}{2}} \frac{(-1)^m}{m!}\binom{\frac{n+l}{2}+\frac{1}{2}}{\frac{n-l}{2}-m}\left(\frac{r}{\bar{r}}\right)^{2m}. \tag{16}$$

We point out that the HO wave functions, whose properties are studied in a comprehensive textbook [73], have been selected to perform the present calculations because the corresponding radial wave functions in the momentum space, that are denoted as $\hat{R}_{n,l}(p)$, have the same analytic form, apart from a phase factor, as the $R_{n,l}(r)$. In the same textbook [73] it is shown that those wave functions can be successfully used to study many different bound-state problems. As a side remark, we also note that the Coulombic wave functions would not be suitable for the present study. In fact, the asymptotic behavior of these wave functions (at least in a nonrelativistic framework) is not compatible with a confining potential.

We anticipate that, as will be shown in the next sect. 4, the matrix elements of the relativistic kinetic energy operators (that depend on p^2) can be easily calculated in the momentum space, while the matrix elements of the interaction operators (that depend on r) are standardly calculated in the coordinate space.

The spin (isospin) wave functions are obtained by coupling the spinors (isospinors) of the diquark and of the quark:

$$\chi^{s_1,s_2}_{S,S_z} = [\chi_{s_1} \otimes \chi_{s_2}]_{S,S_z} \qquad \phi^{t_1,t_2}_{T,T_z} = [\phi_{t_1} \otimes \phi_{t_2}]_{T,T_z}, \tag{17}$$

with $s_1 \equiv t_1$=0,1 and $s_2 \equiv t_2 \equiv \frac{1}{2}$. Finally, the total wave functions are written as

$$\Psi_{n,\{\nu\}} = R_{n,l}(r) \cdot [Y_l(\hat{r}) \otimes \chi^{s_1,s_2}_S]_{J,M} \cdot \phi^{t_1,t_2}_{T,T_z}, \tag{18}$$

where, in the r.h.s., J, M represent the total angular momentum quantum numbers of the of the system; furthermore, in the l.h.s., the shorthand notation $\{\nu\}$ has been introduced to denote all the quantum numbers of the wave function, with the exception of n. Note that the quantum numbers $\{\nu\}$ represent *good* quantum numbers for the Hamiltonian of our model, introduced in eq. (1).

4 Solution method

The eigenvalue equation for the Hamiltonian of eq. (1) does not admit analytical solutions. Therefore, as customary in hadronic bound-state problems, a numerical method must be employed in order to find the approximate energy eigenvalues and eigenfunctions. In this work, a procedure of *diagonalization and minimization* will be implemented. This method consists in diagonalizing the Hamiltonian matrix for each set of states identified by the same *good* quantum numbers $\{\nu\}$, that explicitly appear in the r.h.s. of eq. (18). Furthermore, the energy of the lower-lying state is minimized with respect to the *dimensional* parameter $\bar{r}$, on which the radial wave functions depend, as shown in eq. (13). In this way, the approximate eigenvalues and eigenstates of the Hamiltonian matrix are obtained.

The method has been tested by reproducing, in the nonrelativistic limit, the spectrum of a Coulombic potential, for which the analytic form of the eigenvalues is known. Also in this case (of a nonconfining potential) very high numerical accuracy has been obtained.

We now give some more details about the calculation of the Hamiltonian matrix elements. As stated above, for each set of physical states, identified by their *good* quantum numbers $\{\nu\}$, we have to diagonalize the following Hamiltonian matrix:

$$\langle \Psi_{n',\{\nu\}} | H | \Psi_{n,\{\nu\}} \rangle = H_{n',n}(\{\nu\}). \tag{19}$$

These matrix elements obviously depend also on the *good* quantum numbers $\{\nu\}$.

We now discuss the explicit calculation of the different terms of H in eq. (19). The terms of the relativistic kinetic energy (see eq. (2)) are calculated numerically with the HO radial wave functions in the momentum space. We have:

$$\langle \Psi_{n',\{\nu\}} | \sqrt{\boldsymbol{p}^2 + m_i^2} | \Psi_{n,\{\nu\}} \rangle = \int_0^\infty \mathrm{d}p\; p^2 \hat{R}_{n',l}(p) \sqrt{p^2 + m_i^2}\; \hat{R}_{n,l}(p). \tag{20}$$

These matrix elements depend on n, n' and l.

The Cornell modified potential matrix elements (see eq. (4)) are calculated in the coordinate space, as follows:

$$\langle \Psi_{n',\{\nu\}} | V_c(r) | \Psi_{n,\{\nu\}} \rangle = \int_0^\infty \mathrm{d}r\ r^2 R_{n',l}(r) V_c(r) R_{n,l}(r). \tag{21}$$

Also in this case the matrix elements depend on n, n' and l.

As for the matrix elements of the spin- and isospin-dependent operator V_{st} of sect. 2.3, we previously calculate (with the coupled wave functions of sect. 3) the mean values $\langle \boldsymbol{s_1} \cdot \boldsymbol{s_2} \rangle$ and $\langle \boldsymbol{t_1} \cdot \boldsymbol{t_2} \rangle$ by means of the standard *Landè formula*:

$$\langle \boldsymbol{s_1} \cdot \boldsymbol{s_2} \rangle = \frac{1}{2}[S(S+1) - s_1(s_1+1) - s_2(s_2+1)]. \tag{22}$$

An analogous expression holds for the isospin term. Then, the radial matrix elements of $U_{st}(r)$ are calculated in the same way as in eq. (21).

As for the interaction V_{lq} of subsect. 7, the matrix elements of V_l are easily calculated replacing $\langle \boldsymbol{l}^2 \rangle = l(l+1)$ and then performing the usual radial integration. The matrix elements of the tensor interaction V_q are calculated by means of the standard angular momentum algebra [74]. In the present calculation, only the *diagonal* matrix elements of the tensor interaction are considered, by taking, for the Hamiltonian matrix elements, both *bra* and *ket* states with $l' = l$ and $S' = S$. The mixing terms are numerically negligible. We finally recall that, for this interaction, the matrix elements for the state with $S = \frac{1}{2}$ are vanishing [74], so that we only have the contribution of state $S = \frac{3}{2}$.

5 The mass spectrum

In order to obtain the specific mass spectrum of our quark-diquark model, defined by the Hamiltonian of eq. (1), we have to determine the values of the free parameters, namely E_0, m_1, m_2, τ, β_1, β_2, β_3, k_1, k_2, η, ρ and σ. This objective is reached by performing the best fit of the experimental data of the mass spectrum, by varying the free parameters of the model.

As anticipated in the Introduction, the two states of the diquark (scalar and axial-vector) are assumed to have different dynamical properties. In consequence, in the fitting procedure, the corresponding interaction parameters are allowed to take different numerical values. Obviously, for the quark mass m_2 the same value is taken in the two cases. We also recall that in the present model no mixing between the two states of diquark is considered.

5.1 States of the scalar diquark ($s_1 \equiv t_1 = 0$)

We point out that, for these states, the V_{st} interaction of eq. (5) and the tensor interaction of eq. (9) are vanishing.

Table 1. Comparison between the experimental values [75] of the masses of the nonstrange baryon resonances with $s_1 = 0$ (up to 2 GeV) and the theoretical ones. In the second column the status of each resonance is reported according to the classification given by PDG [75].

Baryon	Status	M^{exp} (MeV)	J^P	l^P	S	M^{th} (MeV)
$N(939)$	****	938	$\frac{1}{2}^+$	0^+	$\frac{1}{2}$	938
$N(1440)$	****	1430–1470	$\frac{1}{2}^+$	0^+	$\frac{1}{2}$	1463
$N(1520)$	****	1515–1530	$\frac{3}{2}^-$	1^-	$\frac{1}{2}$	1503
$N(1535)$	****	1520–1555	$\frac{1}{2}^-$	1^-	$\frac{1}{2}$	1503
$N(1680)$	****	1675–1690	$\frac{5}{2}^+$	2^+	$\frac{1}{2}$	1690
$N(1720)$	****	1650–1750	$\frac{3}{2}^+$	2^+	$\frac{1}{2}$	1690

By using the following values for the parameters: $E_0 = 558.7\,\mathrm{MeV}$, $m_1 = 400\,\mathrm{MeV}$, $m_2 = 200\,\mathrm{MeV}$, $\tau = 1.42$, $\beta_1 = 1750\,\mathrm{MeV}$, $\beta_2 = 500\,\mathrm{MeV}$, $\beta_3 = 480\,\mathrm{MeV}$, $k_1 = 0.7$, $k_2 = 0.9$ and $k_3 = 0.16$, we obtain the result presented in table 1.

As we can see, the results can be considered successful for these states: the predicted masses belong to the experimental uncertainty intervals.

5.2 States of the axial-vector diquark ($s_1 \equiv t_1 = 1$)

In the same way we use the complete Hamiltonian to fit the states with $s_1 = 1$. With the following values of the free parameters: $E_0 = 718.6\,\mathrm{MeV}$, $m_1 = 450\,\mathrm{MeV}$, $m_2 = 200\,\mathrm{MeV}$, $\tau = 1.33$, $\beta_1 = 1300\,\mathrm{MeV}$, $\beta_2 = 200\,\mathrm{MeV}$, $\beta_3 = 530\,\mathrm{MeV}$, $k_1 = 0.9$, $k_2 = 1.3$, $k_3 = 0.11$, $\eta = 270\,\mathrm{MeV}$, $\rho = 0.304\,\mathrm{fm}^{-1}$, $\sigma = 19.04\,\mathrm{MeV} \cdot \mathrm{fm}^2$, we obtain the results shown in table 2.

For Δ ($T = 3/2$) states some discrepancies are found, suggesting that further investigations are needed for the quark-diquark model of the baryonic spectrum. In any case we highlight that the spin- and isospin-dependent interaction plays a crucial role to reproduce the masses of these states.

The final results are also shown graphically in figs. 1 and 2.

For completeness, we also discuss here the technical point related to our definition of the quantity χ^2 that is minimized in the fit procedure. One has

$$\chi^2 = \sum_i (M_i^{\mathrm{exp}} - M_i^{\mathrm{th}})^2 \frac{1}{\Delta_i}\,. \tag{23}$$

The sum is performed over all the states which are being reproduced by the quark-diquark model. The M_i^{exp} and M_i^{th} represent the experimental and the calculated theoretical values of the i-th resonance mass, respectively.

Table 2. Comparison between the experimental values [75] of the masses of the nonstrange baryon resonances with $s_1 = 1$ (up to 2 GeV) and the theorerical ones. In the second column the status of each resonance is reported according to the classification given by PDG [75].

Baryon	Status	$M^{\rm exp}$ (MeV)	J^P	l^P	S	$M^{\rm th}$ (MeV)
$N(1650)$	****	1640–1680	$\frac{1}{2}^-$	1^-	$\frac{3}{2}$	1661
$N(1675)$	****	1670–1685	$\frac{5}{2}^-$	1^-	$\frac{3}{2}$	1674
$N(1700)$	***	1650–1750	$\frac{3}{2}^-$	1^-	$\frac{3}{2}$	1687
$N(1710)$	***	1680–1740	$\frac{1}{2}^+$	0^+	$\frac{1}{2}$	1688
$\Delta(1232)$	****	1230–1234	$\frac{3}{2}^+$	0^+	$\frac{3}{2}$	1232
$\Delta(1600)$	***	1550–1700	$\frac{3}{2}^+$	0^+	$\frac{3}{2}$	1577
$\Delta(1620)$	****	1615–1675	$\frac{1}{2}^-$	1^-	$\frac{1}{2}$	1677
$\Delta(1700)$	****	1670–1770	$\frac{3}{2}^-$	1^-	$\frac{1}{2}$	1677
$\Delta(1900)$	***	1850–1950	$\frac{1}{2}^-$	1^-	$\frac{3}{2}$	1792
$\Delta(1905)$	****	1870–1920	$\frac{5}{2}^+$	2^+	$\frac{3}{2}$	1979
$\Delta(1910)$	****	1870–1920	$\frac{1}{2}^+$	2^+	$\frac{3}{2}$	1977
$\Delta(1920)$	***	1900–1970	$\frac{3}{2}^+$	2^+	$\frac{3}{2}$	1978
$\Delta(1930)$	***	1920–1970	$\frac{5}{2}^-$	1^-	$\frac{3}{2}$	1797
$\Delta(1950)$	****	1940–1960	$\frac{7}{2}^+$	2^+	$\frac{3}{2}$	1978

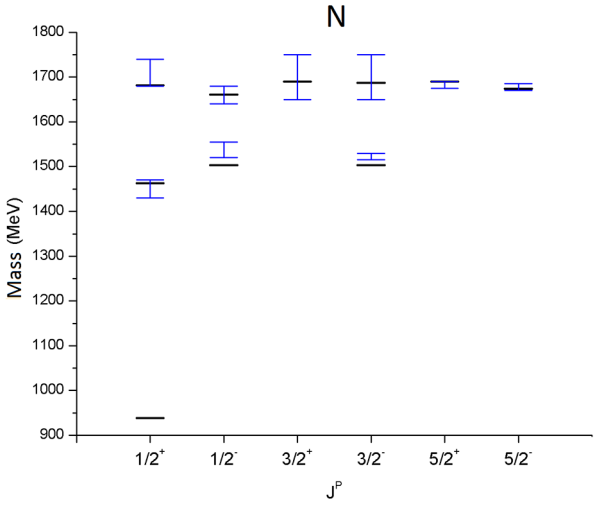

Fig. 1. Comparison between the masses calculated (black lines) and the experimental masses (blue intervals), for N ($T = 1/2$) resonances.

The quantity Δ_i is associated with the weight of each resonance. In the present work, it has been defined as

$$\Delta_i = \sqrt{(\Delta_i^{\rm nat})^2 + (\Delta_i^{\rm exp})^2}, \qquad (24)$$

where $\Delta_i^{\rm nat}$ is the Breit-Wigner *natural* width of the resonance and $\Delta_i^{\rm exp}$ is given by the experimental error on the resonance mass value.

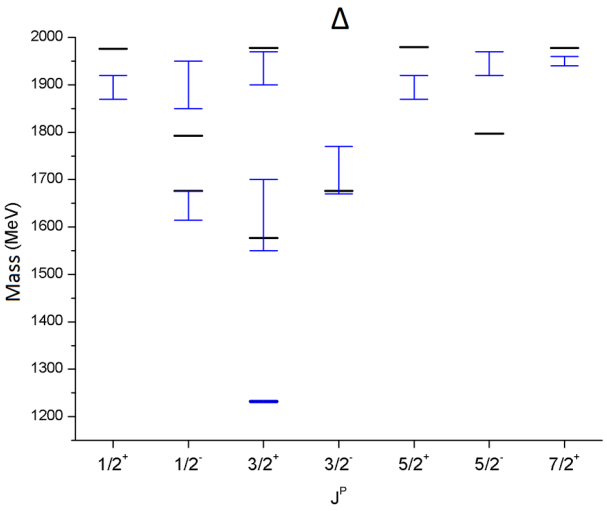

Fig. 2. Comparison between the masses calculated (black lines) and the experimental masses (blue intervals), for Δ ($T = 3/2$) resonances.

6 Results and discussion

We have developed a relativistic quark-diquark model for the baryon spectrum using the Hamiltonian of eq. (1).

As for the numerical technique of calculation, we have adopted a minimization-diagonalization procedure. The use of HO basis as trial functions has been advantageous to calculate the relativistic kinetic energy operator matrix elements. We tested the convergence of the numerical procedure obtaining a satisfactory result.

From a theoretical point of view, we recall that the Hamiltonian of eq. (1) is invariant under the main physical symmetries of the system. In particular, to this aim, the relativistic kinetic energy operators for both the quark and the diquark have been introduced.

The model reproduces the baryon spectrum with an accuracy that is better than our previous semirelativistic model [48] and of the same order of other nonrelativistic models [39].

We have verified that, as in other quark-diquark models, the spin and isospin dependent interaction is strictly necessary to fit the spectrum. We also highlight that the introduction of the angular-momentum–dependent interaction is very important for the description of the mass spectrum for N as well as for Δ resonances. In particular this interaction is very beneficial to obtain the correct mass difference between the states $N(939)$ and $N(1440)$. On the other hand, the tensor interaction gives only small contributions.

The specific (original) hypothesis of this work consists in considering, for the two diquark states (the scalar state and the axial-vector one), different numerical values of the parameters that appear in the interaction operators.

The model can be globally considered in agreement with other similar approaches, even though a comparison of the parameters of the various models cannot be easily done in a sensible way, due to the different parametrizations that are adopted for the Hamiltonian operators.

Table 3. Comparison between some parameters of our work with those of other similar models. The indices S and AV denote the values for the scalar and axial-vector states, respectively. Our parameter η is compared to A_{SI} of the other models. With the exception of τ, all the values are in MeV.

Parameter	This model	Ref. [43]	Ref. [49]
E_{0S}	558.7	154	826
E_{0AV}	718.6	154	826
m_2	200	200	140
m_{1S}	400	600	150
m_{1AV}	450	950	360
τ_S	1.42	1.25	1.23
τ_{AV}	1.33	1.25	1.23
η	270	375	350

However, to study this point in some more detail, we show in table 3 the values of some parameters, compared with the corresponding ones of other two similar models, namely, refs. [43] and [49].

Particular interest is attributed to the diquark mass, in the scalar and axial vector states, that is m_{1S} and m_{1AV}, respectively. A comparison of the values of this quantity, obtained in other different models, is given in ref. [49]. We note that in our model the axial-vector diquark mass is greater than the scalar diquark mass, according to fundamental physical reasons. However we find a *smaller* difference (between the two masses) with respect to the majority of the other models. This behavior can be explained taking into account that in our model we have *two* values, E_{0S} and E_{0AV}, for the zero point energy constant, while the other models use *only one* value for the two cases.

The comparison of τ is only tentative, due to the difference of the spatial parametrization of the Coulomb-like terms in the different models.

Purely tentative is the comparison of our spin-isospin parameter η with the A_{SI} of the exchange interaction in refs. [43] and [49].

Finally, as a possible development of the present study, we recall that the wave functions obtained in this work can be also used to study the electromagnetic response functions of the nucleon, *i.e.*, the elastic form factors and the helicity amplitudes. In this way it would be possible to test in more detail the physical accuracy of the model, and, if necessary, to introduce new degrees of freedom, beyond the quark and the diquark states [33].

In order to improve the Hamiltonian of the model, one could use an interaction directly given by a relativistic field model, requiring a numerical calculation in the momentum space to solve the corresponding integral equation [76].

One of us (CG) would like to thank the CAPES Brazilian agency for partial support.

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