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The econ in econophysics

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Abstract. Modern authors have identified a variety of striking economic patterns, most importantly those involving the distribution of incomes and profit rates. In recent times, the econophysics literature has demonstrated that bottom incomes follow an exponential distribution, top incomes follow a Pareto, and profit rates display a tent-shaped distribution. This paper is concerned with the theory underlying various explanations of these phenomena Traditional econophysics relies on energy-conserving "particle collision" models in which simulation is often used to derive a stationary distribution Those in the Jaynesian tradition rely on entropy maximization, subject to certain constraints, to infer the final distribution. This paper argues that economic phenomena should be derived as results of explicit economic processes For instance, the entry and exit process motivated by supply decisions of firms underlies the drift-diffusion form of wage, interest and profit rates arbitrage. These processes give rise to stationary distributions that turn out to be also entropy maximizing. In the arbitrage approach, entropy maximization is a result. In the Jaynesian approach, entropy maximization is the means.

1 Introduction

In what follows, I will concentrate on various different explanations of three key observations in econophysics: The near-exponential distribution of the bottom 97 percent of incomes representing labor incomes: the Pareto distribution of the top 3 percent representing property incomes [21,24], and the tent-shaped distribution of the profit rates of firms [1,16,17].

With mean-normalized incomes (r) on the horizontal axis and the cumulative probability distribution from above C(r) on the vertical axis, an exponential distribution will be a straight line on a log-linear scale, while that of a Pareto Distribution will be a straight line on a log-log scale[1]. Figure 1 displays IRS adjusted gross income data of individuals in 2011, with the bottom 97 percent in the left panel and the top 3 percent in the right panel. These patterns have been shown to hold in every year

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[1] For an exponential distribution with mean-normalized income (r), the cumulative probability distribution from above is $C(r) = e^{-r}$ so that $\ln C(r) = -r$ is a straight line in r. For a power law, the cumulative distribution from above is $C(r) \propto r^{-\alpha}$ so $\ln C(r) = -\alpha \ln r$ is a straight line in $\ln(r)$.

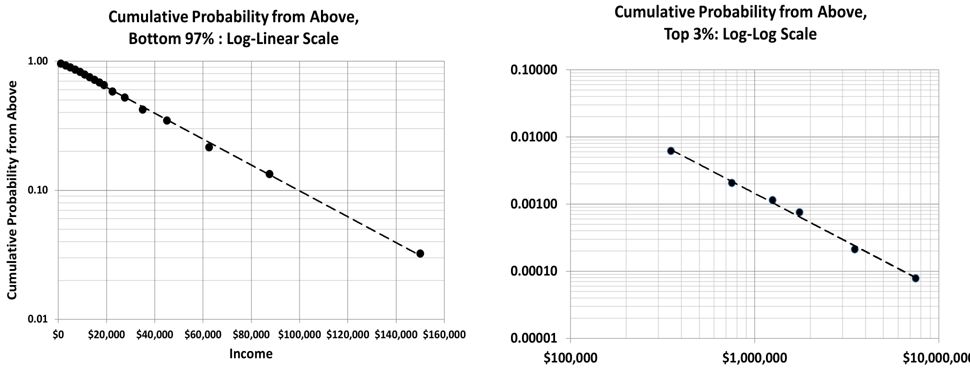

Fig. 1. Cumulative distribution from above, bottom 97 % (log-linear) and top 3% (log-log).

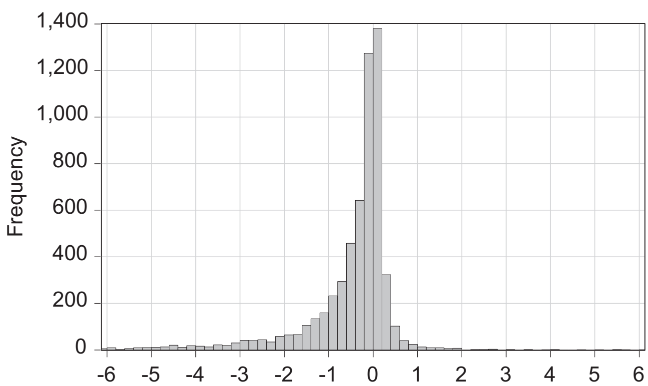

Fig. 2. Rate of return on assets.

in the US, as well as in other advanced countries [4,21,25]. Figure 2 depicts the third striking pattern – the tentshaped distribution of profit rates[2].

2 Economic arbitrage and the distribution of wages, property incomes and profit rates

Arbitrage is a fundamental principle of economic analysis. In a growing economy, the supply and demand for labor and capital are also growing. Then, if there is a difference in real wage rates in two regions, the supply of labor will accelerate relative to demand in the high wage region and bring the wage down. In the low wage region, the supply will decelerate relative to demand and bring the wage up. Financial capital flows do the same the work in the case of interest rates, and real capital do it in the case of profit rates. In all instances, participants will experience unexpected positive and negative shocks, and their intentions may or may not be realized. A long-standing representation of turbulent arbitrage, as in CIR and related models [2] is in terms of entry and exit movements induced by differences in variables (drift) and the effects of ongoing shocks (diffusion). Drift alone leads to the equalization of the variables to their means, but the additional presence of diffusion leads to a persistent distribution. Because the means are changing over time, we normalize

[2] See Figure 7.14 and Appendix_7.2_data_tables.xlsx, tab = iropdataUSind from [18], available on at http://realecon.org/data/.

the variables by the appropriate mean, so that mean-normalized variables gravitate around 1. This general approach can directly explain the observed patterns of wages, property incomes and profit rates, as developed in Shaikh and Jacobo [19].

2.1 Labor income

There are many possible drift-diffusion models. Since labor incomes (alternately the excess of labor incomes over some minimum wage) must be positive, all wage models must reflect this constraint. The Cox-Ingersoll-Ross (CIR) model is the iconic one for interest rate analysis, widely used in empirical work and the fount of many theoretical extensions [5,11]. It posits a linear mean reverting process for the drift term and a diffusion term that depends on the square root of the interest rate so that volatility goes to zero as the interest rate goes to zero. The log-linear and log-log specifications of the mean reverting process are nonlinear ones that keep the ith wage w_i and rate of return on assets ρ_i positive without any restrictions on the volatility because ln(x) only exists for $x > 0$. Let $\theta =$ the strength of drift, $\sigma =$ the standard deviation of diffusion and $W_t =$ a Wiener process. Then at time t,

$$\frac{d\left(\ln w_{i_t}\right)}{dt} = -\theta_i\left(w_{i_t} - 1\right) + \sigma_i \frac{dW_{i_t}}{dt} \quad \text{[log-linear model]} \tag{1}$$

$$\frac{d\left(\ln w_{i_t}\right)}{dt} = -\theta_i\left(\ln w_{i_t} - \ln 1\right) + \sigma_i \frac{dW_{i_t}}{dt} = -\theta_i \ln w_{i_r} + \sigma_i \frac{dW_{i_t}}{dt}, \quad \text{since } \ln 1 = 0 \text{ [log-log model]}. \tag{2}$$

The arithmetic mean is the appropriate one in the log-linear model, while the geometric mean (the mean of the logs) is appropriate in the log-log model. Arbitrage is meant to apply to the wages of given types of labor, so that partitions of the overall distribution by race and gender, as well as consolidations by occupation, will likely exhibit the same patterns [20]. The log-linear model of equation gives rise to a stationary gamma distribution, while the log-log model of equation yields a stationary lognormal distribution[3] (see [19] for details). Figure 3 displays the fits of both potential models to the labor income data in Figure 1. The log-linear model provides a good fit, but the log-log model clearly does not.

2.2 Property income

Property income $\pi \equiv \rho a$ is the product of the stock of financial assets (a) and its rate of return (ρ), both measured relative to their respective means. The rate of return represents the average holding rate of return on portfolios of interest-bearing and dividend-paying assets. It is therefore modeled as a positive return that like labor income, is subject to arbitrage, so we can make use of the form of wage equations and as models for rates of return. For financial assets per person, it makes sense to posit that when the rate of return of the ith portfolio is above the mean, the portfolio will increase because its value will have gone up and/or because more savings will flow into it. This gives us two possible expressions for the reaction of property income to the rate of return: $d\left(\ln a_{i_t}\right)/dt = -\gamma_i\left(\rho_{i_t} - 1\right) + \sigma_i \frac{dW_{i_t}}{dt}$, or

[3] Even though the variance of the lognormal distribution increases linearly in time, we include it because it "is still widely used for fitting income distribution" and provides a good fit to our data [24].

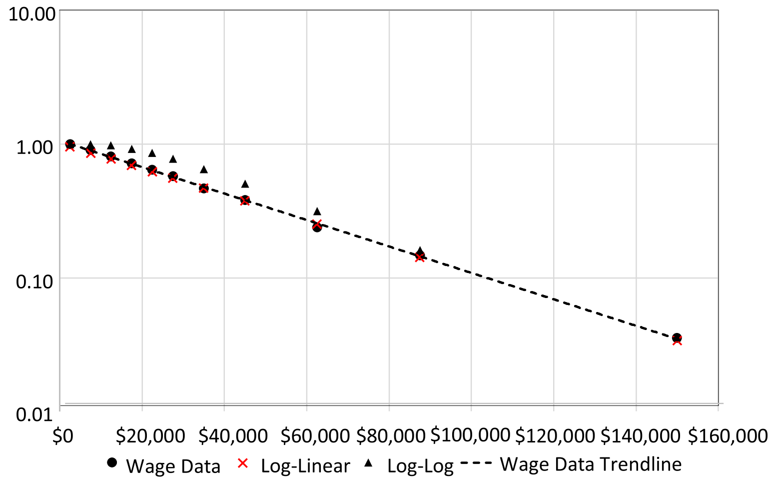

Fig. 3. Cumulative probability from above, labor income and fitted log-linear and log-log models.

$d\left(\ln a_{i_t}\right)/dt = -\gamma_i \ln \rho_{i_t} + \sigma_i \frac{dW_{i_t}}{dt}$. Because $\pi \equiv \rho a$ is an algebraic identity, we can write $d\left(\ln \pi_{i_t}\right)/dt = d\left(\ln \rho_{i_t}\right)/dt + d\left(\ln a_{i_t}\right)/dt$. Combining these with the two adjustment forms for the rate of return gives us two property income models (see [19] for details).

Log-linear Property Income Model:

$$d\left(\ln \rho_{i_t}\right)/dt = -\theta_i\left(\rho_{i_t} - 1\right) + \sigma_i \frac{dW_{i_t}}{dt} \tag{3}$$

$$d\left(\ln \pi_{i_t}\right)/dt = d\left(\ln \rho_{i_t}\right)/dt + d\left(\ln a_{i_t}\right)/dt = \left(-\theta_i + \gamma_i\right)\left(\rho_{i_t} - 1\right) + \sigma_i \frac{dW_{i_t}}{dt} \tag{4}$$

Log-log Property Income Model:

$$\frac{d\left(\ln \rho_{i_t}\right)}{dt} = -\theta_i \ln \rho_{i_t} + \sigma_i \frac{dW_{i_t}}{dt} \tag{5}$$

$$d\left(\ln \pi_{i_t}\right)/dt = d\left(\ln \rho_{i_t}\right)/dt + d\left(\ln a_{i_t}\right)/dt = -\left(\theta_i + \gamma_i\right)\ln \rho_{i_t} + \sigma_i \frac{dW_{i_t}}{dt} \tag{6}$$

As previously noted, the rate of return (ρ) represents an average rate of interest on income-paying assets, and hence is positive. Since only the top 1–3 percent of the overall income distribution is dominated by property income with a power-law distribution (see the second panel of Fig. 1). In references [6,7] Gabaix argues that the top portion of a lognormal distribution is essentially a power-law, and suggests using a lower reflecting barrier to restrict the domain in such cases. We therefore set the reflecting barrier at 95 percent in order to restrict the property income distribution in equations (5)–(6) to the top five percent.

In the log-linear model, equation has the same form as the wage equation, so the rate of return (ρ) follows a gamma distribution. In equation, at $\rho_{i_t} \approx 1$ the growth rate of financial assets becomes a random growth process in which a lower reflecting barrier yields an approximate Power Law in Figure 4. In the log-log model the dynamics of asset returns in equation (6) are similar to those of wages in (3), so that the stationary distribution of returns is log-normal (see [19], Appendix). As

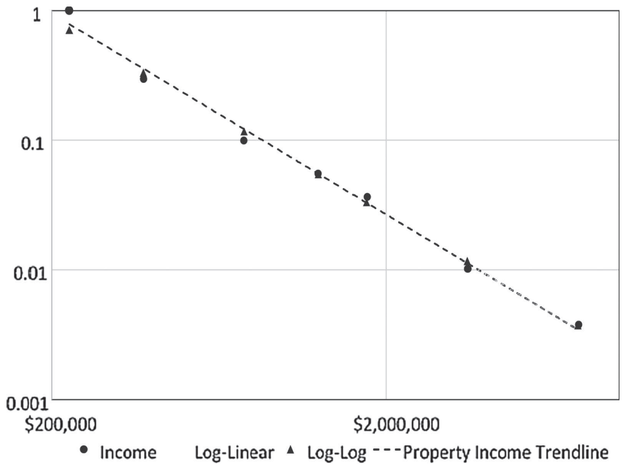

Fig. 4. Cumulative probability from above, property income and fitted log-linear and log-log models, 2011.

noted, the signature of a power law is a straight-line plot of the log of the cumulative distribution from above versus the log of income. Figure 4 shows that in both models the top 95% of both theoretical distributions approximate power laws that provide good fits to actual top income data, with the log-log being the better of the two since its upper reaches are close to a power law [12].

2.3 Profit rates

Financial assets of a given type such as stocks and bonds have no vintages. A share purchased 10 years ago will have the same current value as a share purchased yesterday. This is not true of machines, buildings and other fixed capital: an older plant embodies an older technology, and hence has a different capital value from a new plant. More importantly, the older plants generally have higher costs per unit out – precisely why new investment is embodied in newer plants. It follows that in the case of real capital, it is the rate of return on *new* investment that motivates inter-industrial capital flows. Let $P =$ total profit, $K =$ total capital stock, and $I \equiv \Delta K =$ investment, all in real terms. Then the average rate of profit $r = \frac{P}{K}$ while the rate of return on new investment (the incremental rate of profit) is $r_I = \frac{\Delta P}{\Delta K} \approx \frac{\Delta P}{I}$, and it is *this* rate that is equalized by competition between industries. Figure 5 shows these rates for US manufacturing industries from 1988–2005 are indeed equalized in a turbulent manner (see [18], Fig. 7.17), and Figure 6 depicts their empirical distribution along with a fitted normal curve.

The goal here is to show that the observed tent-shaped distribution of profit rates [10,16] can be derived from three distinct processes. Inter-industry mobility of capital that turbulently equalizes the profit rates of firms having best-practice (regulating) conditions of production (Fig. 5) yielding a roughly Gaussian distribution (Fig. 6); intra-industry equalization of selling prices that leads to unequal profit rates across firms within any given industry; and ongoing technical change that creates an persistent spectrum of intra-industry conditions of production (Fig. 7). The object is not to find the "best fit" to each of the observed distributions. Rather, it is to show that the simplest analytical representation of each process is sufficient to derive a tent-shaped distribution of regulating and non-regulating profit rates. This sort of

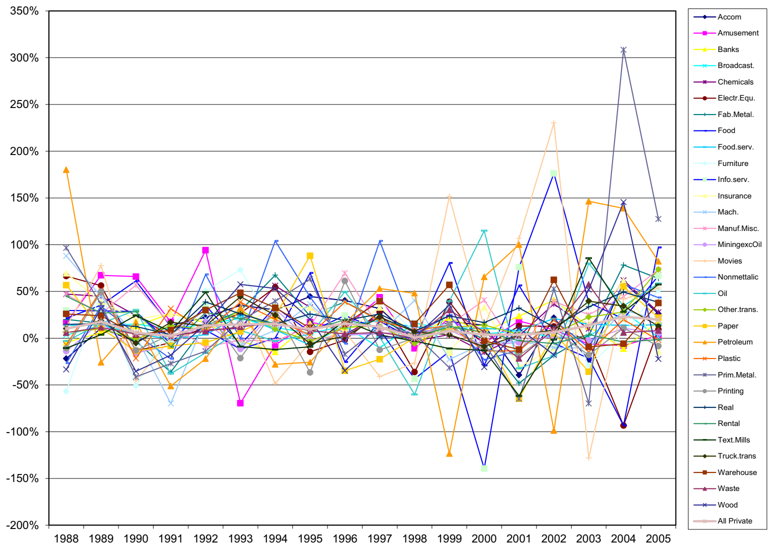

Fig. 5. Incremental rates of profit in US industries, 1987–2005.

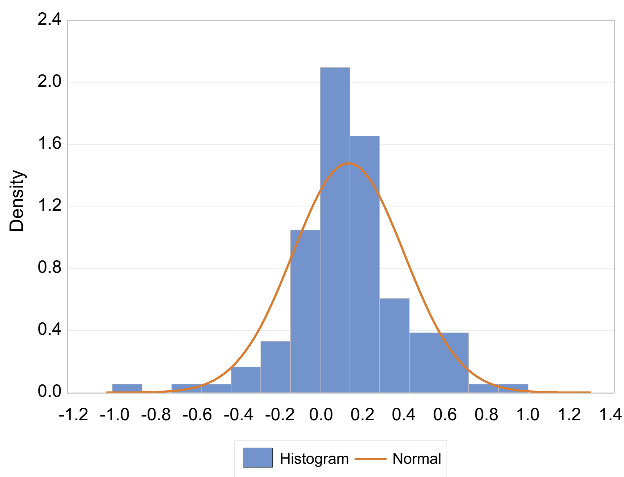

Fig. 6. Distribution of regulating profit rates and fitted normal curve.

approach is standard in the econophysics literature (see [3], Figs. 4–5), and I would argue that it is also the proper theoretical procedure in this case.

In each industry, the rate of return on new investment is the profit rate of the capitals representing the best reproducible conditions of production for entering capitals. Competition then equalizes the profit rates of these *regulating* capitals (r^*) across k industries [18]. Since these profit rates can range from negative to positive, the linear Ornstein–Uhlenbeck (OU) process in equation applied to a cohort of N industry regulating capitals, is a good representation of the inter-industry competitive process. As is well known, the resulting stationary distribution is Gaussian [14] – as is roughly the empirical case in Fig. 6. Since the distribution is stationary, it is also the entropy maximizing one – derived here as the *result* of a turbulent dynamic process.

$$\frac{dr_k^*}{dt} = -\theta_k \left(r_k^* - 1\right) + \sigma_k \frac{dW_{i_t}}{dt} \tag{7}$$

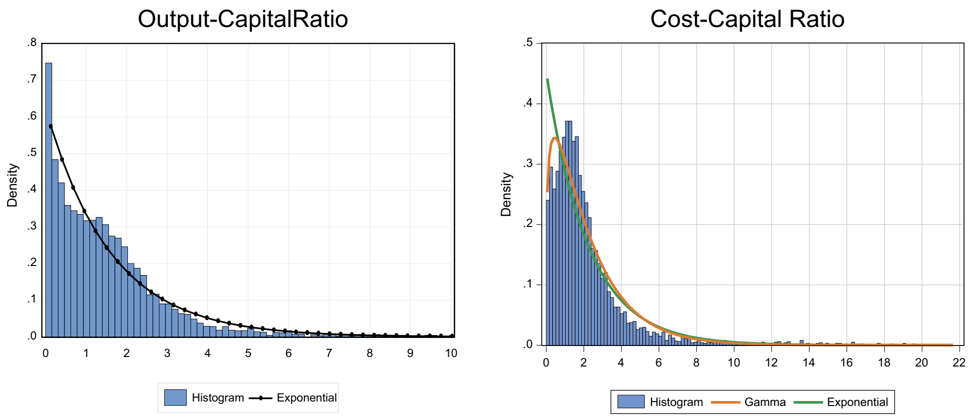

Fig. 7. Cost-capital ratio and output-capital ratio.

In the theory of real competition, one must account not just for capitals with the best reproducible conditions of production (regulating capitals), but also for various vintages of non-regulating capitals. Ongoing cost-reducing technical change constantly introduces new conditions and constantly removes ones that are longer viable. This gives rise to a *spectrum* of production conditions existing any given time. "Gross investment is the vehicle of new techniques" (65) and "the plants in existence at any one time, are, in effect, a fossilized history of technology over the period spanned by their construction dates – the capital stock represents a petrified chronicle of the recent past" [15]. While it is legitimate to focus on regulating capitals for the analysis of inter-industrial profit rate equalization, it is not legitimate to leave out the others in the consideration of empirical evidence. For the kth capital (including regulating ones) in an industry, let r_k = the profit rate, P_k = the total profit, X_k = total sales, C_k = total costs (materials, depreciation and wages), K_k = total assets, c_k = the ratio of firm's costs to its assets, and x_k = the ratio of firm's sales to its assets. Then

$$r_k \equiv \frac{P_k}{K_k} = \frac{X_k - C_k}{K_k} = x_k - c_k. \tag{8}$$

Figure 7 shows that exponential pdfs can be used as first approximations to the empirical distributions of the sales/capital and costs/capital ratios [18]. It follows from equation that the distribution of the kth profit rate is the difference of two roughly exponential pdfs, i.e. roughly a Laplacian tent-shaped distribution [10,16]. From this point of view, the persistent distribution of vintages is the key factor in the empirical dispersion of profit rates shown in Figure 2[4].

3 Simulation approaches to the distribution of incomes

3.1 Economics

In reference [13] Ragab adopts an agent-based approach to labor incomes. The mean wage is assumed to be determined by macroeconomic forces, and individual wages are normalized relative to this mean, so that the sum of normalized wages is conserved. Lower wageworkers challenge higher wage ones, subject to random factors. The former

[4] The derivation of the distribution of vintages will be the subject of a separate paper.

may succeed and move up, or stay where they are; the latter may hold on or move down, perhaps all the way down to the zero wage (unemployment). The simulation result of these interactions indicates a stationary exponential distribution. As in the drift-diffusion approach, the stationary (entropy maximizing) distribution is the result of an explicit economic process.

3.2 Physics

Just as arbitrage is a fundamental principle in economics, energy conservation is a fundamental principle of physics. With a given amount of total energy and a given number of particles interacting in a manner that conserves this total, the stationary equilibrium distribution of energy corresponding to the maximum-entropy state is Boltzmann–Gibbs. Yakovenko et al. adapt this particle-collision story to monetary flows: they assume a closed economy with a fixed amount of money and a fixed number of agents whose individual monetary transactions conserve total money, and who always spend less than the possession so that money holdings are always positive. In this base case, they derive the resulting stationary exponential distribution in one of three ways: by simulation, by probability arguments from physics, and by means of entropy maximization In subsequent extensions, they model production, sales and debt, property incomes and wealth stocks, using simulations to identify the stationary distributions [3,24]. Finally, diffusion models that only approximately conserve total money or even have declining total money, as well as those in which negative shocks are partitioned out among agents in a manner that keeps individual money stocks positive, are shown through simulation to yield stationary exponential distributions [8]. An important caveat is that these physics models treat money exchanges among agents as income flows, which is not generally true. They also assume that the total money stock, and hence total income, is essentially constant even though "there is no fundamental reason why the sum of incomes ... must be conserved" [4,23].

4 Entropy maximization approaches

In his book, Venkatasubramanian [22] derives an "ideal" distribution of income in perfect capitalism. He operates within standard neoclassical economic theory, assuming utility maximizing consumers, perfect markets and fully equalized wages and rates of return on capital assets He defines the effective utility from work, which he calls happiness, as utility from pay *minus* disutility from effort *plus* utility arising from a sense of a fair opportunity (49–51). Entropy is re-interpreted as a measure of fairness, and the entropy maximization is used to derive the ideal income distribution in "utopia". Under his particular specification of effective utility, the ideal income is lognormal (49–51, 103–115). There is no process as in drift-diffusion or particle collision models, only an entropy-maximizing specification of the ideal state.

In reference [16] Scharfenaker and Foley consider profit rate equalization within the classical Smithian theory of competition. They note that while there is an observed tent-shaped distribution of profit rates (see Fig. 2), the entry and exit decisions of firms are not observed (4) They argue that "when observed variables depend non-trivially on unobserved variables the joint distribution of profit rates and entry and exit decisions is underdetermined due to incomplete information" (Abstract). In reference [9] Jaynes argues that "when we make inferences based on incomplete information, we should draw them from that probability distribution that has the maximum entropy permitted by the distribution we have". Hence, Scharfenaker and Foley turn to MaxEnt as the appropriate means to infer the underdetermined joint distribution, using constraints motivated by theoretical considerations. They argue that the unobserved entry and exit of firms in response to profit rate differences can be represented as "quantal entry and exit decisions" (8). Entropy maximization subject to quantal constraints then yields a tent-shaped (Laplacian) distribution (16).

Two comments are in order here. First, unobserved decisions do not require recourse to entropy maximization. They note that while we have data on the distribution of wage incomes, we generally do not on the movements of labor from low wage regions to high wage ones. Yet the latter type arbitrage is an integral part of economic theory and can be naturally represented as a dynamic adjustment process – as in the drift-diffusion models of Section 2. Secondly, Scharfenaker and Foley make no distinction between regulating and non-regulating capitals, so the well observed cost variations among firms play no role in their analysis I have argued that real competition only equalizes the profit rates of *regulating* capitals, and that it is *precisely* the cost variations that give rise to the observed tent-shaped distribution of profit rates (Sect. 2.3).

5 Summary and conclusions

Modern econophysicists have established that bottom individual incomes follow an exponential distribution, top incomes follow a Pareto, and profit rates display a tent-shaped distribution. This paper surveys three approaches to these patterns. First, physics energy-conserving "particle collision" models applied to economics, in which simulation is often used to derive a stationary distribution. Second, Jaynesian models that invoke unobserved processes to justify the use of entropy maximization subject to particular constraints in order to derive a stationary distribution. However, there is a long-standing third tradition, as in the analysis of interest rates via the CIR and other models, which derives the final distribution from an explicit treatment of the arbitrage process. For instance, the unobserved entry and exit decisions of firms motivated by arbitrage considerations can be sensibly modeled as drift-diffusion processes whose mathematically derived stationary distributions match the observed ones of wage, interest and profit rates. These derived distributions are also entropy maximizing. The physics and arbitrage approaches formulate explicit models of unobserved processes, while the Jaynesian approach uses the existence of unobserved processed to justify entropy maximization. In the first two cases, the entropy maximizing distribution is the result. In the Jaynesian approach, it is an assumption. The distinction between these approaches is a methodological and philosophical one.

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