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Statistical equilibria in economic systems: Socio-combinatorial or individualist-reductionist characterizations?

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Abstract. This paper motivates the distinctive analytical usefulness of socio-combinatorial characterizations of statistical equilibria in economic systems. It does so by drawing on Gibbs' approach to thermodynamic ensembles and on Jaynes' epistemic characterization of probabilities, entropy, and associated concepts. The resulting approach is contrasted with two classes of individualist-reductionist characterizations of statistical equilibria that are influential among economists and physicists: Micro-econometric cross-sectional and drift-diffusion models. Two illustrative applications contrast the insights each approach offers into the economic and social content of observed statistical equilibria, involving distributions of individual incomes and rates of return for individual enterprises.

1 Introduction

> "At the stage in development of a theory where we already have a formalism successful in one domain, and we are trying to extend it to a wider one, some kind of philosophy about what the formalism 'means' is absolutely essential to provide us with a sense of direction."
>
> – E.T. Jaynes, "The Delaware Lecture," 1967

Contributions from physicists and political economists over the past three decades have established that the frequency distributions for many economic variables are consistently well approximated by known functional forms [1–3]. This includes a number of quantities that are central to the functioning of financial markets and broader capitalist economies: changes in financial asset prices, their correlations over different time horizons, and financial-market trading volumes [4–9]; individual income and wealth [10–14]; corporate rates of growth and profitability [15–20]; the measure of corporate security prices given by Tobin's q [21,22]; and daily changes in foreign exchange rates [23].

These findings are a promising development for political economy and economics. Those fields confront complex social systems shaped by evolving interactions between large numbers of non-linearly coupled individual members whose own characteristics

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and behavior are conditioned by their socio-economic relations [24]. Functional stability in frequency distributions for several important economic variables suggests that despite the complex detail of individual economic behavior and interactions, competition in decentralized, capitalist economies yields outcomes that may be usefully understood as *statistical equilibria*. The characteristics of such equilibria may provide important clues about the economic and social content of market outcomes in those economies.

An important difficulty arises in the pursuit of this potentially fruitful line of inquiry. There are significant ambiguities and unresolved disagreements among physicists concerning the phenomenological content of statistical equilibria and the evolution of physical systems toward them [25]. Making sense of the significance and nature of statistical equilibria in economic and social systems requires a clear resolution of those ambiguities and disagreements.

Some physicists favor *kinetic* characterizations of statistical equilibria, emphasizing detailed descriptions of the microscopic dynamics and evolutions toward stationary states. Kinetic interpretations appeared very early in the development of statistical mechanics: Boltzmann's "H-Theorem" effectively characterized the Second Law of Thermodynamics (and thus the evolution of certain thermodynamic systems toward and at statistical equilibria) as a consequence of a particular specification of patterns of collisions between particles. They are also common in more recent work, most often in uses of the Fokker–Planck equation to characterize statistical equilibria and the disequilibrium paths toward them in terms of drift-diffusion micro-kinetic models of the dynamic evolution of individual non- or weakly-interacting particles in a system.

Others have more forcefully and deliberately emphasized the *combinatorial* content of statistical equilibria. Taking their cue from the work of Gibbs, they characterize statistical equilibria simply as the combinatorially dominant states across the phase spaces defined by the physical laws governing the system in question [26,27]. Under this view, statistical equilibria embody something significantly more general than the particular kinetic paths or collisions experienced by particles in a system. They are the systemic expression of the relevant laws or regularities governing the functioning of the system.

It should be obvious that whenever our knowledge of the laws or regularities bearing upon a system can yield reasonably successful descriptions of its microscopic evolution, the conceptual differences between these two approaches will not always have practical implications. But in systems where we lack such knowledge, those conceptual differences will generally be a matter of great practical importance as well.

This is particularly true in analysis of economic or social systems. We do not possess any significant behavioral postulates with the empirical success and generality of the principles of conservation of energy and momenta in analysis of physical systems. The nearest thing we have is the aggregate accounting identity between individual expenditures and revenues over any given time period. While economic outcomes can be shaped by that identity in interesting and paradoxical ways [28,29], they are also shaped by a much broader range of other behavioral, institutional, and social factors that remain poorly understood.

More significantly, the conceptual differences between the kinetic and combinatorial emphases of Boltzmann and Gibbs have important parallels with a fundamental schism at the heart of economic analysis for more than a century.

Much of contemporary economic thought has settled on a very strong form of individualist reductionism. It seeks to characterize the functioning of economic systems on the basis of detailed descriptions of the behavior of "representative" individuals, or of similarly detailed game-theoretic representations of interactions between small numbers of agent types. This methodological approach has its roots in the "marginalist revolution" of the 1870s [30–32], and took its contemporary form under the influence of the "micro-foundations" revolution of the 1970s [33]. Like kinetic characterizations of statistical equilibria, this approach conceives of observable economic outcomes in terms of detailed descriptions of individual behavior and dynamic evolutions.

A diverse range of contributions have offered a radically different conceptual approach to economic analysis [34–39]. Despite very important theoretical and normative differences among them, these contributions have all effectively recognized that discernible regularities in the functioning of competitive, decentralized economies are not generally reducible to detailed descriptions of individual actions and evolutions. Some of those contributions identify regularities in the functioning of economic systems with forms of spontaneous self-organization, ensuring that competition and price systems yield outcomes that can be understood as socially desirable, independently of the objectives and knowledge animating individual actions. Others have sought to show how the institutional framework and systems of property rights of capitalist political economies ensure that interactions in markets for labor-power, capital, and goods reproduce uneven, exploitative relationships between aggregate, social groups. In all cases, these contributions suggest observable economic outcomes are reduced-form, emergent results of the interplay between individual agencies, interactions, and systemic realities and interdependences [40].

Drawing on these latter traditions of economic analysis, and on the combinatorial emphasis of Gibbs, a number of recent contributions have shown how observed statistical equilibria in socio-economic systems can be usefully understood as emergent, systemic regularities [24,41–43]. Those regularities can inform work grappling with the economic processes generating the observed data and with their social content.

This brief paper outlines and motivates the approach underlying those contributions. It also contrasts the conceptual and practical merits of that approach in analysis of economic systems to those of two important micro-individualist frameworks used widely in economics and physics: Micro-econometric models used in empirical work based on cross-sectional economic data, and micro-kinetic, drift-diffusion characterizations of statistical equilibria used in recent work in economics. The paper does this in four substantive parts. Section 2 offers a positive exposition of the socio-combinatorial approach taken by these contributions; Section 3 takes on the difficulties posed by conventional micro-econometric approaches to cross-sectional data; Section 4 offers a critical discussion of drift-diffusion characterizations of statistical equilibria in socio-economic systems; Section 5 offers two illustrative comparisons of what may or may not be learned from observed statistical equilibria by taking each approach; and a brief Section 6 concludes.

2 A systemic, Gibbs–Jaynes approach to statistical equilibria

The approach to statistical ensembles and phase spaces developed by Gibbs can be adapted to assist quantitative, observational inquiry into economic and social systems. Invariances or *regularities* in the functioning of those systems rule out some conceivable micro-level configurations of individual states. Inquiry starts by drawing on prior knowledge and observation to identify those regularities, which appear mathematically as constraints on the system's phase space. Robustly and formally identified regularities then set the observational burden that must be met by any successful theory of the economic nature and social significance of the processes generating what we observe.

2.1 Micro-level phase spaces, macroscopic states, and entropy measures

Consider an economic or social system as composed of a large number N of members, which may be actual individuals or functional units. At any given point in time, each of those members has an individual state defined over a set of $v = 1, 2, \ldots, d$ degrees of freedom, $X^v = \{X_0, X_1, \ldots, X_d\}$ that exhaustively describes their economic situation. Individual degrees of freedom may describe quantifiable individual characteristics – contemporaneous or past – as well as macroscopic quantities that take the same value across a large number of individuals in the system. They may also describe qualitative or categorical individual characteristics, including descriptions of an individual's institutional or relational situations. Coding schemes mapping the latter characteristics onto distinct real numbers allow individual states to be represented by vectors $\mathbf{x_v} = \{x_0, x_1, \ldots, x_d\}$, with the set of all such individual states denoted by $T \subseteq \mathbb{R}^d$.

In all practical social inquiry involving measurement of individual states, the space T is effectively "coarse grained." Consider a coarse graining into $i = 1, 2, \ldots, s_v$ bins or effective individual states, defining a total of s_v^N mathematically conceivable micro-level configurations for the system. The system's *phase space* Γ_v contains all of the $N \times d$ matrices γ describing micro-level configurations that are phenomenologically possible – i.e., those generated by the functioning of the system. A deterministic and fully reductionist understanding of the functioning of the system would be given mathematically by a full characterization of $\gamma(t)$. Seeking such an understanding of socio-economic systems is impractical. It is also unnecessary and besides the point in *social* inquiry.

The task of micro-level quantitative social inquiry can be understood as the identification of the relevant phase space Γ_v defined by the economic or social system in question. The existence of invariances, laws, or regularities in the system allows description of its phase state in term of statements involving functions of the system's micro-configurations. Formally, each of these may be indexed by j and expressed as,

$$\mathcal{H}_j(\gamma) \leq 0. \tag{1}$$

The set of m regularities in the functioning of the system yield a formal representation of its phase space,

$$\Gamma_v = \{\gamma \mid \mathcal{H}_j(\gamma) \leq 0, j = 1, 2, \ldots, m\}. \tag{2}$$

Micro-level quantitative social inquiry seeks to identify and find mathematical expressions for those regularities so as to inform theorization about their economic or social content – that is, to inform phenomenological characterizations of the economic or social laws and regularities in the system.

Regularities can involve any $\mathcal{H}_j(\gamma)$ establishing functional relationships between any set of elements in the matrix γ and equalities or inequalities in relation to κ_j. In a complex economic system with large numbers of interactions between dynamically evolving members, regularities are generally systemic, involving relationships between the states of large numbers of individuals. Important examples involve technologically necessary input-output relationships between enterprises or industries [44], or the presence of other networks of interaction between individual agents in the economy.

A particular, common type of systemic regularity has traditionally been associated with statistical equilibria – those defined by sums of functions $g_j : T \to \mathbb{R}$ of individual states and scalars κ_j,

$$\mathcal{H}_j(\gamma) = \sum_{n=1}^{N} g_j(\mathbf{x_n}) = \kappa_j. \tag{3}$$

These include simple regularities like $\sum_n x_c = \kappa$, which may reflect a basic scarcity in the total quantity of a good X_c, a basic accounting identity if X_c denotes changes in individual net monetary positions, or, as discussed below, the fact that competition ensures X_c is in fact a "socially scaled" measure of another individual degree of freedom [24]. As will be clear below, regularities like (3) also include the presence of given values for variances and covariances for the system's individual degrees of freedom. These regularities embody irreducibly systemic interdependences between individual states.

It is often more convenient and effective to take a macro-level approach to social inquiry. Since X^v is by definition an exhaustive description of all individual characteristics relevant to economic and social interactions in question, it is possible to consider that the functioning of the system is entirely indifferent between individuals with the same $\mathbf{x}_i$. They are indistinguishable. The only thing that matters to systemic functioning is the total number of members or occupancy n_i in each of the s_v bins in T. It is thus possible to represent the macroscopic state of the system as a frequency function $\mathbf{f} = f(\mathbf{x}_i)$ describing the normalized occupancy of each bin.

The functioning of a system defines a space Φ_v containing all macroscopic states $f(\mathbf{x}_i)$ the system may occupy. The macroscopic laws and regularities that define a system can be understood to be expressed in the shape of Φ_v.

While not all regularities can be simply represented both at the micro and macroscopic level, micro-level regularities of the form (3) have very simple macroscopic expressions,

$$\sum_{n=1}^{N} g_j(\mathbf{x_n}) = N \sum_{i=1}^{s_v} f(\mathbf{x_i})\, g_j(\mathbf{x_i}) \Rightarrow \langle g_j(\mathbf{x}) \rangle_{\mathbf{f}} = \kappa_j. \tag{4}$$

They are moment constraints on the system's macroscopic state $\mathbf{f}$.

Entropy measures can be most usefully understood in socio-economic inquiry in terms of the relationship between a system's micro-level phase-space Γ_v and its set Φ_v of possible macroscopic states. Clearly every $\gamma \in \Gamma_v$ supports a unique macroscopic state $\mathbf{f} \in \Phi_v$. But in general, each macroscopic state $\mathbf{f}$ is supported by a multiplicity of micro-level configurations γ. Entropy functionals offer informational measures of those multiplicities or phase-space volumes supporting specific macroscopic states.

Entropy is useful in analysis of systems with large $N \gg s_v$. In those systems the combinatorial dominance of the distribution $f^*(\mathbf{x_i})$ achieving maximum entropy over all other macroscopic states in Φ_v is overwhelming. This conclusion can guide the iterative process of observational inquiry into the functioning of such systems. If we have a set of knowledge, beliefs, or hypotheses K suggesting that the functioning of the system keeps it within a phase-space Γ^K and a corresponding set Φ^K of macroscopic states, we should expect to observe macroscopic behavior in line with the state $f^*(\mathbf{x}|K)$ that maximizes entropy over that set. Why? Because that distribution is the most common macroscopic state across all possible micro-level configurations in ΓK. This is the *Principle of Maximum Entropy* (PME).

It is important to note that the PME is not a behavioral hypothesis and is entirely independent of the elements in set K. In fact, if we observe macroscopic behavior at variance with $f^*(\mathbf{x}|K)$, the PME tells us that K is either incomplete or wrong, informing subsequent inquiry [45]. What the Principle offers is a distinctive and logically robust way to link knowledge we may have about the micro-level functioning of a system and what basic combinatorial considerations lead us to conclude about its observable macroscopic states. This is a very different conceptualization of the relationship between micro- and macro-level functioning than that which grounds most contemporary economic thinking.

2.2 Observational inquiry and statistical equilibria

The preceding considerations allow a formal statement of the kind of inverse, ill-posed problem typically faced by observational social inquiry. It is possible to observe individual values taken by $o \ll d$ individual degrees of freedom across a large number $N_o \leq N$ of members of the system. Coarse graining the observable individual state space $T_w \subset T$ into $k = 1, 2, \ldots, s_w$ bins allows the construction of frequency histograms $f(\mathbf{x}_k)$ over the values taken by the vector $\mathbf{x}_k$ of observed individual states. Social scientists have limited knowledge about the micro-level interactions driving the functioning of the system. Even the full set of relevant degrees of freedom X^v is not known.

Observational social inquiry seeks to draw on observed cross sections $f(\mathbf{x}_k)$ to infer as much as possible about regularities in the functioning of the social or economic system at hand. Those distributions reflect the accumulated results of repeated interactions between evolving economic agents over the period of time defining the quantities observed. They do not reflect the full detail of those interactions and individual evolutions, much of which is lost between the annual, quarterly, or at best monthly observations available to economists. The regularities they may embody are reduced-form, systemic results of competitive interactions. Information theory offers robust concepts and tools allowing a general approach to emergent systemic associations between individual degrees of freedom. Measures of mutual, joint, and incremental or conditional information between those degrees of freedom provide valuable, non-parametric tools in economic analysis [40].

Sometimes an observed cross-sectional frequency $f(\mathbf{x_k})$ is persistently and ubiquitously well described by known, closed-form functional forms. Such observations suggest those frequencies are the most common macroscopic state across all possible micro-level configurations of individuals across all coarse-grained values of $\mathbf{x_k}$. That is, they suggest the observed cross sections are entropy maxima. This opens interesting possibilities for inquiry into the functioning of economic systems.

In some instances it is possible to identify the moment constraints $\langle g_j(\mathbf{x}) \rangle_{\mathbf{f}} = \kappa_j$ defining the sets Φ_w over which a persistently observed distributional form maximizes entropy. This allows a converse application of the PME [17,18,21,41,46,47]. It is possible to infer that those moment constraints offer good systemic descriptions of laws or regularities present in the processes conditioning values of $\mathbf{x}_k$. They give mathematical expression to the outcomes of interactions involving all observed and non-observed degrees of freedom in X^v that shape Φ_w and the corresponding subset $\Gamma_w \in \Gamma_v$ of the system's overall phase space.

Those constraints can provide important formal clues about the macroscopic or *social* content of the micro-processes at hand. They define the observational burden on any successful economic or social theories of the nature and content of the processes shaping measures of $\mathbf{x_k}$ [24,41,42]. The kinds of observationally grounded theorizations and distinctive systemic insights this approach enables are illustrated in Section 5.

3 Parametric micro-econometric models

The approach outlined above offers a distinctive, critical perspective on the relative usefulness of parametric micro-econometric models to guide inference from aggregate or systemic patterns of statistical organization present in cross sections of economic data, including statistical equilibria observed over certain domains of individual degrees of freedom.

While not widely understood as such, micro-econometric models are statistical-equilibrium models, founded on the supposition that regularities in economic systems are *individual*. Individual regularities involve stable relationships between individual degrees of freedom holding homogeneously across all individuals in a system (up to a set of possible parametric variations across small numbers of sub-groups of individual members of the system in question). They are generally more restrictive than the broader range of possible systemic regularities depicted in (1). Each of them imposes N constraints on the phase space of the economic system in question, thus embodying a far stronger restriction on the system's phase space than a single systemic constraint.

Despite their practical usefulness across a variety of settings, micro-econometric models founded on individual regularities generally offer a poor conceptual foundation for grappling with the economic or social significance of statistical regularities involving all moments of observed distributions – like statistical equilibria over marginal distributions of certain individual degrees of freedom. In the estimation exercises defined by such models, observed statistical regularities are only relevant inasmuch as they condition the estimation of values taken by the parameters of statistical equilibria defined by individual regularities supposed by researchers. Inasmuch as regularities in economic data are generally systemic or social, as was argued in the previous section, this approach makes a category error. It also considers what is known (the data) only in terms of what is conjectured (the individualist model) – a striking and persistent instance of what E.T. Jaynes termed the "mind projection fallacy" common in scientific inquiry [26]. This section discusses these difficulties in turn.

3.1 Individual regularities

Formally, parametric micro-econometric models assert that for each and every individual in a system, a specific relationship exists between a "dependent" degree of freedom x_0; a vector $\mathbf{x_m}$ of "independent" degrees of freedom $m \in L$, a set of modelled degrees of freedom; and a sum of unspecified functions u involving the large number of $i \notin L$ degrees of freedom not explicitly considered in the model,

$$g\left(\mathbf{x_n}\right) = x_0 - h\left(\mathbf{x_m}, \theta\right) + \sum_{i \notin L} u_i\left(x_i\right) = 0, \ \forall n. \tag{5}$$

The function $h\left(\mathbf{x_m}, \theta\right)$ may be linear or non-linear on the degrees of freedom $\mathbf{x_m}$. The sum in (5) is understood as a compound, "disturbance" individual degree of freedom ϵ, whose distribution across individuals is taken as well defined, with $\langle \epsilon \rangle = 0$ across all individuals.

By focusing exclusively on a possible relationship between each individual's own degrees of freedom, models of this type offer individualist-reductionist accounts of the processes generating observed data. They allow no scope for *explicit* consideration of interactions between individuals that may define irreducibly systemic or macro-level regularities. This is sometimes motivated with the claim that regularities like (5) are reduced-form, average relationships. Other times a far stronger set of claims is made: That individual regularities are defined by the intentions and actions of "rational," optimizing individuals; that it is possible to successfully characterize those intentions and actions; and that equilibrium prices across all markets ensure that observable individual outcomes reflect the successful pursuit of those intentions by all individuals, as characterized in economists' models.

In all cases, models predicate the value of an individual's "dependent" degree of freedom on functions of their other degrees of freedom exclusively. This tends to present individual outcomes not as results of complex patterns of social interaction, but as reflections of an individual's own characteristics. Existing differential socioeconomic outcomes are presented as simple expressions of differentials in given individual characteristics, and not as results of social interdependences conditioning differences in "independent" individual degrees of freedom and in their effects on the "dependent" degree of freedom in question.

To illustrate the difficulties arising as a result, consider the relationship between the profitability of individual enterprises and their investment behavior. Many contributions have developed and estimated models where investment is a positive function of profitability – either because higher measures of profitability make investment more attractive, or because more profitable firms have more internal funds available to sustain investment [48–50]. But in any setting where the cost of capital to enterprises reflects the expected rates of returns investors can access across all enterprises, realized levels of investment will tend to respond not to absolute individual measures of profitability but comparative, social ones [41]. It is also well known that the total measure of investment in an economy conditions total value added, which includes total profits. The resulting systemic interdependences between all individual measures of investment and profitability may give rise to irreducibly *systemic* regularities that can shape the full distributions of the quantities involved. Approaching analysis by assuming individual regularities will clearly yield limiting, and at times entirely misleading results.

More generally, abstraction from explicit consideration of social interdependences between individuals gives rise to models exhibiting the defining characteristic of individualist-reductionist approaches in science: According to them, macroscopic regularities are nothing but scaled up versions of the specified regularities in individual states. Put differently, they suppose that systemic averages obey the same relationship specified for each individual. In this case,

$$\langle x_0 \rangle = \langle h(\mathbf{x_m}, \theta) \rangle . \tag{6}$$

These are very strong and generally inappropriate suppositions for *social* systems. This is especially so given the well-understood and formidable epistemological and practical difficulties in grappling scientifically with details of ever-evolving individual intentions, actions, and agencies [36,40], and how limited economists' knowledge about them really is as a result.

3.2 Strongly specified statistical equilibria

Once the existence of homogeneous, individual regularities is accepted, it is straightforward to see how micro-econometric models along the lines of (5) are in fact statistical-equilibrium accounts of the conditional distribution $f(x_0|\mathbf{x_m};\theta)$. They can be understood as statements that two things are known or conjectured: That for every given $\mathbf{x_m}$, x_0 has an average value of $h(\mathbf{x_m};\theta)$ across all individuals, and that the influence of large numbers of unspecified individual degrees of freedom resolves itself into a statistically regular distribution for ϵ, belonging to a family $M(\mu,\Sigma)$ of distributions centered on μ and exhibiting a spread parametrized by a vector Σ.

Formally, this results in models of the conditional distribution of x_0 of the form,

$$f^*(x_0|\mathbf{x_m};\theta) = M(h(\mathbf{x_m};\theta), \Sigma) . \tag{7}$$

This is a statement of statistical equilibrium: No matter what happens in the interactions shaping the degrees of freedom $i \notin L$, researchers expect a distribution $M(h(\mathbf{x_m};\theta), \Sigma)$ to be observed. This supposes that distribution is the most common macroscopic state across all possible micro-level configurations of individual across values of all x_i defining ϵ – that is, it maximizes entropy over the phase space defined by the processes shaping those degrees of freedom. All distributional forms used in econometric analysis are special cases of the Lambert-W function, all of which can be understood as entropy maxima representing statistical equilibria over well defined phase spaces [51].

This framing of cross-sectional econometric models is not widely understood. Yet it makes clear how economists have for a long time been effectively pointing to the existence of very strongly (and implausibly) specified statistical-equilibria in economic systems. It also helps identify a further set of difficulties in the accepted approach taken to estimation of these models and its conventional use to sustain inferences from cross-sectional economic data.

At the most fundamental level, the problem with that approach is that it does not start with the identification of regularities in the frequencies of observed degrees of freedom $f(\mathbf{x_k}) = f(x_0, \mathbf{x_m})$. That is the correct, observational approach for any non-experimental field of inquiry. Instead, work starts by assuming that all suppositions conditioning maximum entropy models like (7) are true. This creates a series of difficulties, which can be seen formally by considering the relationship between the observed joint distribution $f(x_0, \mathbf{x_m})$ and the observed marginal distribution $f(\mathbf{x_m})$ implied by the assumed model,

$$f(x_0, \mathbf{x_m}) = M\left(h(\mathbf{x_m}; \theta), \Sigma\right) f(\mathbf{x_m}). \tag{8}$$

In estimation of the model in (7), the statistical properties of the observed distribution $f(x_0, \mathbf{x_m})$, which embody the statistical relationships between $f(x_0), f(\mathbf{x_m})$, and $f(x_0, \mathbf{x_m})$ established by the functioning of the system in question, are only considered inasmuch as they help shape parameter estimates $\hat{\theta}$ and their distributions. But that is contingent on the model and on the estimation procedure, amounting to a fundamental inversion of the correct logical ordering of observation and model in observational inquiry. No attempts are generally made to establish whether it is reasonable to suppose that individual regularities are in fact generated by the functioning of the system; whether there are other, systemic regularities bearing on the conditional distributions being estimates; or whether there are further regularities involving all moments in the full observed frequencies of individual degrees of freedom.

Practical work nevertheless proceeds to draw on cross-sectional data to obtain estimates $\hat{\theta}$ of model parameters, and to use the cross-sample statistical properties of particular estimates to consider pairs of mutually exclusive hypotheses about parameter values. The possibility that certain elements defining the statistical-equilibrium being estimated may not be warranted is only considered *ex post*, implicitly, and ad hoc: Not in relation to the strong restrictions on the system's phase space resulting in models like (5), but only inasmuch as estimation of the model yields evidence suggesting problems with the specification of $h(\mathbf{x_m}; \theta)$, "non-spherical" distributions for "error terms," [52], or associations between the "independent" variables.

Practitioners are trained in ways to adjust their models in light of this evidence, and in the "problems" it creates for the statistical properties of some estimators. The irony in some of this, of course, is that evidence of heteroscedasticity, omitted variables, endogeneity, multicollinearity, etc. actually gives researchers information about the processes generating the data we observe [53]. By considering observed data only in terms defined by rather strong suppositions, parametric micro-econometric inference often ignores this kind of information, going as far as considering it a "problem" in estimation of postulated models.

Finally, it is important to note how some approaches have sought to relax some of the strong constraints on phase spaces imposed by the supposition of simple individual regularities. Approaches based on the Bayesian method of moments use non-parametric, information-theoretic tools to develop data-centred characterizations of the moments of the distribution M [54,55]. While more general than the approach outlined above, those efforts are still defined by an assumed functional form for $h(\mathbf{x_m};\theta)$, which fundamentally shapes all aspects of the ensuing exercises.

Multilevel models consider cases where the function $h(\mathbf{x_m},\theta)$ contains parametric variations θ_j across sub-groups of individuals in the system, where the elements of θ_j are themselves understood as a dependent variable in higher-order regularities of the kind described in (5) [56]. Models of that type are more general, making some allowance for social or institutional categories that may influence economic outcomes, and for possible interdependences between individuals in different groups in the form of covariances between disturbance terms in specifications of elements of θ_j and θ_i for different sub-groups.

But while richer and more general, those models are still founded on the supposition of individualist regularities, in this instance with group-level versions of (5) (which can easily be represented as a single function $h(\mathbf{x_m};\theta)$) and (6). Their estimation also involves consideration of the data and any regularities that may be present in it within terms defined by the strong specifications defining the statistical-equilibrium model, ensuring they will in general also provide poor bases to grapple with observed statistical regularities and equilibria in distributions of economic data.

4 Micro-kinetic, drift-diffusion approaches

Kinetic characterizations of statistical equilibria in physical systems offer a different kind of individualist approach in economic analysis. This involves dynamic drift-diffusion models that characterize observed statistical regularities in distributions for certain economic quantities as stationary states toward which economic systems converge over time. While the analytical emphases are different from those of the micro-econometric exercises just discussed, drift-diffusion characterizations are borne of the same individualist analytical appetite to reduce observed macroscopic regularities to homogeneous regularities holding at the individual level.

Micro-kinetic, drift-diffusion characterizations of statistical equilibria follow a well-trod path. Consider the dynamic evolution of a single individual degree of freedom x for a member of a system. Suppose that evolution is Markovian: The future evolution of x depends only on present values of all relevant covariates. This ensures the process is "memoryless," in the sense that past values of all relevant covariates are irrelevant to the future evolution of x.

A particular Markovian process is often used in these models, Itô processes, according to which, the evolution of x obeys,

$$dx = a(x,t)\,dt + b(x,t)\,dz \tag{9}$$

where z is a Weiner process continuously generating Gaussian diffusion increments with mean zero and a given variance.

In these cases the Chapman–Kolmogorov equation is satisfied and ensures that the frequency distribution $f(x,t)$ describing the values of x taken by a large number of individual members of a system evolving according to (9) evolves dynamically according to,

$$\frac{\partial}{\partial t} f(x,t) = -\frac{\partial}{\partial x} a(x,t)\, f(x,t) + \frac{\partial^2}{\partial x^2} D(x,t)\, f(x,t) \tag{10}$$

where $D(x,t) = \frac{1}{2} b(x,t)^2$.

For time-homogeneous drift and diffusion, a stationary state f^* must obviously solve,

$$\frac{\partial}{\partial x} a(x) f^*(x) = \frac{\partial^2}{\partial x^2} D(x) f^*(x) . \tag{11}$$

Drift-diffusion accounts of an observed, persistent distributional form $f^*(x)$ as a stationary state can be constructed by identifying $a(x)$ and $D(x)$ satisfying this condition.

This line of reasoning poses at least three deep problems in analysis of economic systems. Those problems make it very difficult to attach economic or social significance to particular pairs of drift and diffusion functions capable of generating persistently observed distributional forms.

First, the supposition that the evolution of economic variables is Markovian is excessively restrictive and generally wrong. The dynamic accumulation of *stocks* of productive assets, financial assets, work-in-progress, inventories, and liabilities is inherent to economic functioning in a capitalist economy. This creates important path dependences. Even variables like market prices of financial assets whose evolution is over many time horizons usually Markovian [37], often follow non-Markovian paths as a result of stock (or balance-sheet) effects [57].

Second, drift-diffusion representations of given statistical equilibria are not generally unique. This is obvious from condition (11), which imposes only one equation on the search for two functions $a(x)$ and $D(x)$ for a given $f^*(x)$. This search is an underdetermined problem, introducing an important measure of arbitrariness in any pair of drift and diffusion functions capable of generating observed distributions.

Third and most importantly, drift-diffusion characterizations consider that all members of a system obey the same dynamic rule, as given in (9). This is a strong and deeply unsatisfying individualist supposition. The “memoryless” nature of Markov processes ensures that at the steady-state distribution, all individuals are indistinguishable. Information about differences in initial conditions will have been lost by the time the system comes to statistical equilibrium.

This symmetry ensures that at the steady-state distribution, the functional form taken by the cross-sectional, macroscopic distribution of x is the same as the distribution of values of x taken by each individual over sufficiently long periods of time. In fact, under this kind of account of statistical equilibria, all individuals occupy all allowed values of x at the same frequency over time as that with which those values are instantaneously occupied by the entire population. There is no social significance to different x outcomes for different individuals in this abstraction. Everybody eventually comes to occupy every state with the same frequencies as everybody else. As in individualist micro-econometric models, macroscopic patterns of organization are taken as nothing but scaled up versions of micro-level, individual regularities.

While formally capable of accounting for some observed distributional forms, this approach cannot generally sustain social or economic theory, which concerns itself precisely with the reasons and significance for differences in outcomes.

5 Two contrasting examples

This section considers two specific examples illustrating the importance of the differences between the three approaches to statistical equilibrium outlined above. The examples involve the persistent observation of Boltzmann–Gibbs exponential cross-sectional frequencies for individual income over long ranges of income distributions in a number of different national economies, and the persistent observation of double power-exponential functions in cross-sectional frequencies for different measures of rates of return. These are discussed in turn.

5.1 Boltzmann–Gibbs exponentials and wage income

Observed cross-sectional frequencies of individual income have suggested income distributions follow Boltzmann–Gibbs exponential functions very closely for wide ranges of income across a number of economies [24,58,59]. Formally, this may be conveniently and approximately put as a statement that over a domain $[0, \infty)$, individual annual income x follows the distribution,

$$f(x) = c\, e^{-cx}, \quad c > 0. \tag{12}$$

This observation elicits very different conclusions from each of the three approaches.

5.1.1 Drift-diffusion models

Drift-diffusion characterizations would consider these observed distributions as the steady state for a large number of individual evolutions of the type described in (9), with the slight modification of a "reflective boundary" at $x = 0$ [60].

Considering for simplicity only processes with constant drift and diffusion coefficients $a(x, t) = a$ and $D(x, t) = D$, a continuum of pairs (a, D), with $D, -a \geq 0$ can generate a statistical equilibrium at (12) with $c = c(a, D)$. Even when considering only a subset of possible drift-diffusion processes, there is no unique drift-diffusion representation capable of generating the observed distribution. The multiplicity of possible models makes it difficult to take inquiry further on these bases.

More generally, under statistical equilibria for individual incomes generated by a drift-diffusion processes like (9), each individual is dynamically occupying each and every income level over time at frequencies described by the cross sectional distribution. This leaves analysis with no conceptualization of the determinants of differences in individual income levels, and consequently without any economic account for inequalities of income or of their social significance or content.

5.1.2 Mincer equations

The observation of statistical equilibria in marginal distributions of income $f(x)$ highlights important limitations with individualist, micro-econometric approaches to observational work. As already noted, those models are based on the supposition of statistical equilibria elsewhere in the joint distribution of all observable individual degrees of freedom. That strong supposition frames and constrains all empirical work.

Micro-econometric models of income determination were heavily influenced by the early contributions of Jacob Mincer [61]. "Mincer equations" posit the existence of a homogeneous relationship between an individual's wage income x and a vector ϕ of individual characteristics like years of education and potential years of work experience. In its most influential form, Mincer's approach considers a linear relationship or regularity between logarithmic measures of individual labor income and a vector of logarithmic measures of individual characteristics. Formally, for $x \geq 0$,

$$x|\phi = \prod_i \phi_i^{\alpha_i} + \eta \tag{13}$$

where η typically taken as a Gaussian with mean zero and a standard deviation σ, by the Central Limit Theorem.

As noted above, individualist regularities of this kind cast an individual's level of income not as a result of complex patterns of social interaction, but as reducible to homogeneous functions of their own characteristics. They also result in very strongly specified individualist, statistical-equilibrium models of labor income determination,

$$f\left(x|\phi\right)=G\left(\prod_{i}\phi_{i}^{\alpha_{i}},\sigma\right). \tag{14}$$

This *conjectured* statistical equilibrium defines the manner in which *observed* statistical regularities in the marginal distribution of income are considered in most practical micro-econometric work along these lines. The macroscopic regularities in $f\left(x\right)$ are considered not in terms of the clues about the reduced-form outcomes of processes conditioning wage incomes they may provide, but only inasmuch as together with the broader distribution of all observables $f\left(x,\phi\right)$, they help define the best-fitting version of the postulated statistical-equilibrium model in (13).

To see this formally, note that under the assumed model, the observation that (12) provides a good description of the observed distributions of wage income, it follows that,

$$c\,e^{-cx}=f\left(x\right)=\sum_{\phi}f\left(x|\phi\right)f\left(\phi\right)=\sum_{\phi}G\left(\prod_{i}\phi_{i}^{\alpha_{i}},\sigma\right)f\left(\phi\right). \tag{15}$$

The relationship between the observed $f\left(x\right)$ and $f\left(\phi\right)$, and thus $f\left(x,\phi\right)$ is only considered in relation to the supposed model, and to the extent that they help define the set of α_{i}^{*} the offer the best fit (by some specified criterion), under the assumed functional specification.

While imposing a conjectured statistical equilibrium on those conditional distributions, these approaches do not do anything with observed statistical equilibria in marginal distributions of income. Yet those observed equilibria offer a rather unusual opportunity to develop formal characterizations of the systemic, reduced-form results of the complex processes determining individual incomes, despite the prohibitive practical and conceptual difficulties faced in observing and adequately theorizing the detailed actions and interactions involved. Those systemic characterizations can cast new light on the social significance of the processes conditioning individual measures of income.

5.1.3 First-moment constraints and social scaling

The socio-systemic approach developed in Section 2 offers a different route to the economic and social significance of Boltzmann–Gibbs exponential statistical equilibria (12) for cross sections of individual income.

As is well known, those distributions maximize entropy across all distributions with support on $\mathbb{R}^{+}$, subject to a first-moment constraint. Physicists who first verified the presence of these distributional form mooted the possibility of a "conservation of money," analogous to the way the conservation of energy results in a Boltzmann–Gibbs exponential distribution for the energy levels of individual particles or systems in the microcanonical and canonical ensembles. While recognizing the significance of the empirical discovery made by those contributions, most economists engaging with this literature did not find this argument convincing. After all, much economic analysis concerns itself precisely with variations in individual and aggregate fluctuations in quantities like income.

A more recent contribution pointed to a different possibility – that the first-moment constraints or zero-sum interactions implicit in the persistent observation of distributions like (12) may reflect the presence of competitive processes of *social* *scaling* [24]. The basic concept of social scaling is simple. It follows from the hypothesis that individual outcomes of many competitive, socio-economic processes depend on the measure of certain individual characteristics relative to social or aggregate measures of themselves. For example, suppose an individual's valuation of their well being w_i is defined on average by a measure of their consumption level l_i and the average measure of consumption in their community,

$$w_i = \frac{l_i}{\langle l \rangle} + \eta_i \tag{16}$$

where η_i has a population mean of zero.

It is trivial to see that in such a setting, the measure of well being is subject to a first-moment constraint, $\langle w \rangle = 1$. This does not reflect the presence of any conservation principle. It simply reflects the socio-referential content of the measure of well being in this example.

Under this account the observation of distributions like (12) for individual income would suggest that individual income is the product of processes of competitive scaling of this kind: Complex and unobservable competitive processes in labor and capital markets generate individual wage outcomes reflecting the normalization of individual characteristics relative to social measures of themselves. One possibility is that competition in labor and capital markets ensures individual wages reflect a social scaling of the effective bargaining power workers across different labor-market segments defined by skill, experience, region, social identity, etc, have in securing shares of trend measures of money value added by the enterprises employing them. Bargaining power can be understood as a compound, effective measure of the ability of a group of wage earners to move wages in their favor, conditioned by their economic and broader socio-political characteristics.

An improvement in bargaining power by one group of wage earners leaving them with a greater share of the value added by the enterprises employing them puts pressure on earnings for those enterprises. Capital-market competition tends to spread the resulting losses across all capitals. The generalized losses may induce all enterprises to intensify bargaining pressure on all wage earners, resulting in dynamic income losses for those not enjoying the same kind of improvement in bargaining power as the first group of workers. Individual wage incomes under these condition can be understood to be shaped by the social scaling of measures of bargaining power by wage earners in each labor-market segment.

Under this account the observed statistical equilibria for individual incomes betray how competition in capital markets not only shapes the distribution of income across capitalist enterprises, but also creates competitive interdependences that have been neglected by much political economy. Between wage earners, whose incomes under this account depend on a socially scaled measure of their bargaining power. The socio-combinatorial approach here leads inquiry into a more complete picture of the inter- and intra-class conflicts over the market distribution of the social product.

5.2 Power-exponentials and rates of return

A number of recent contributions have identified persistent double power-exponential functional forms in cross sections for realized and implicit expected rates of return on assets. This includes Subbotin distributions for measures for returns on assets for U.S. corporations [18,19]; double stretched-exponential distributions for returns on assets for very large samples of corporate and private enterprises across twenty European economies [20]; and Asymmetric Laplace distributions for cross sections of the logarithm of "Tobin's q" for U.S. non-financial corporations [41]. In all cases, cross sections are strongly and persistently organized around a modal value $\bar{x}$ that may be readily understood as a measure of the cost of capital.

These are truly remarkable findings given the complexity of the competitive interdependences involved in the determination of realized and expected rates of return on assets. Each of the approaches to statistical equilibria discussed above sustains radically different conclusions from these observations.

5.2.1 The law of one price

While there is no standing micro-econometric literature trying to grapple with the statistical distribution of profitability, an extensive literature has considered the distribution of the measure valuation of corporate securities given by Tobin's q [62,63]. In a nutshell, Tobin's q measures the rate of return investors expect a corporation to generate on its assets, divided by the expected rate of return they demand to receive on their holdings of that corporation's liabilities. Any value different from one implies an opportunity for managers to generate profits for incumbent shareholders, either by investing or divesting in line with market valuations.

Salient contributions to economic theory have proposed this creates tendencies keeping all individual measures of Tobin's q close to one [64]. Work seeking to test this instance of the "Law of One Price" has broadly taken two versions of the individualist, micro-econometric approach outlined above. Some influential contributions have sought to test the hypothesis that security valuations have an independent influence on investment on the basis of posited individualist regularities involving valuations, "fundamental" measures of profitability and liquidity, and investment [62]. Others have taken an aggregate approach, estimating aggregate or average versions of those regularities [63]. In both cases no attention is paid to the full distribution of Tobin's q. Given the formal stability and decidedly non-Gaussian form of those distributions, this effectively ignores important information about the competitive regulation of values of Tobin's q.

5.2.2 A drift-diffusion approach

A recent contribution set out to consider the significance of Subbotin statistical equilibria, proposing a drift-diffusion model of a "representative" evolution of the profitability x of each enterprise [65],

$$dx = -\frac{D}{2\sigma}\operatorname{sgn}(x-\bar{x})\left|\frac{x-\bar{x}}{\sigma}\right|^{\alpha-1} dt + \sqrt{D}dz \tag{17}$$

which for $D, \alpha, \sigma > 0, m \in \mathbb{R}$ yields a Subbotin stationary distribution with exponent α.

As discussed above, accounts of this type consider that all individual enterprises follow the same dynamic evolution. They each occupy all possible levels of profitability (or of logarithms of Tobin's q) over time with the same frequency at which those levels are contemporaneously occupied by different enterprises. This ensures the model offers no room for consideration of competitive *interactions*, since the distribution appears simply as a scaled up version of the micro-level evolution of independent enterprises. This is a very poor conceptual basis for a theory of competitive processes, which inherently involve complex interactions between large numbers of heterogeneous enterprises. Even as a reduced-form relationship, it is very difficult to associate any economic reasoning or significance to processes like (17).

The authors also offered a model of the observed statistical equilibria based on a systemic description of the phase space over which the observed distributions maximize entropy. That approach points to a much more fruitful line of inquiry.

5.2.3 Arbitrage and the spontaneous, competitive pricing of information

A more recent set of contributions has applied the socio-combinatorial approach of Section 2 to the development of economic characterizations of the observed distributional forms for measures of profitability and the logarithm of Tobin's q [41,42]. They account for observed power-exponential statistical equilibria as an emergent result of the actions of arbitrageurs acting in competitive capital-market systems.

The account is based on an *informational accounting* of the effects of complex competitive processes on distributions of realized or expected returns [66,67]. While neither observers nor economic agents know what the effects of generic competitive actions by heterogeneous enterprises, investors, and other agents in a decentralized, market economy will be, it can be reasonably expected that those actions will generally disorganize or increase the entropy of those distributions. This follows from simple combinatorial considerations in systems with large numbers of agents.

The only economic actions consistently capable of contributing to the organization of those distributions involve the pursuit of profits latent in any heterogeneity in rates of return. Those are the actions of arbitrageurs seeking to generate profits in capital markets by moving capital value and competitive efforts away from low-yielding and toward high-yielding allocations. By so doing, the actions of arbitrageurs tend to reduce heterogeneity in rates of return. As a result, it is possible to understand the shape of distributions of profitability or of Tobin's q as reflecting the economic imperatives governing the outcome of the pursuit of arbitrage profits amid the broader competitive process.

Along lines proposed by Austrian political economists [36,68,69], it is possible to consider that quite apart from the actions, knowledge, and intentions of any given individual, the price system in competitive capital markets tends to ensure that resources are allocated to would-be arbitrageurs in ways that maximize the aggregate net return their operations realize over any given period.

Aggregate gross arbitrage returns realized over a short period are falling on end-of-period measure of foregone arbitrage returns, which is proportional to $\langle|x - \bar{x}|\rangle$. Aggregate costs incurred by arbitrageurs during the period can be understood as rising on the gross entropy reductions their actions effect. Those reductions are, in turn, falling on the end-of-period entropy of the relevant distribution of returns. The competitive maximization of aggregate arbitrage returns creates a tradeoff between these two observable features of that distribution. The observed distributions can be understood to express this competitive tradeoff and to represent the emergent pecuniary pricing of organization effected by competitive arbitrage in capital markets.

The model of aggregate arbitrage profit maximization has a dual entropy-maximization problem, highlighting the fact that the observed distributions can be understood as statistical equilibria over phase spaces defined by emergent, systemic regularities created by the competitive pricing of information in capital markets. This is the first explicitly *economic* account of competitive processes capable of generating power-exponential distributions for realized or expected rates of return.

6 Some conclusions

Drawing on observed statistical equilibria in cross sections for important economic quantities to cast light onto the functioning of socio-economic systems requires addressing longstanding ambiguities in how those equilibria are characterized.

Micro-kinetic or individualist-reductionist characterizations based on strong specifications of homogeneous individual regularities or dynamic evolutions of non- or weakly-interacting individuals are deeply unsatisfactory. Whether in micro-econometric or drift-diffusion guise, those approaches embrace the same error. The contention that the macroscopic functioning of a *social system* is nothing but a scaled up version of the functioning of more or less equivalent individual members that effectively do not interact. This results in the rather patent category error of attributing to individuals and their characteristics observed regularities that are generally the result of the interplay between individual agencies and macroscopic or social interactions and interdependences.

The inherent difficulties posed in trying to acquire detailed knowledge about all aspects of individual economic behavior pose an additional, practical problem for all individualist-reductionist approaches. Those difficulties will continue to ensure economic and social analysis has many gaps in its ability to develop empirically successful detailed characterizations of the behavior and specific economic evolution of individuals. Without that kind of knowledge, detailed specifications of individual behavior, actions, and patterns of interactions between individuals, are generally arbitrary. Efforts to interpret observed statistical equilibria in relation to arbitrary individualist models lacking in observational foundation commit a form of the "mind-projection fallacy." This includes the use of game-theoretic or agent-based models, which offer potentially useful *thought exercises* about the consequences of certain specified patterns of interaction between small numbers of individuals or individual types, to develop interpretations of regularities detected in observed data.

The socio-combinatorial characterization of observed statistical equilibria and the broader approach to observational inquiry with which it is associated offer a superior approach, pointing inquiry to significantly more interesting, systemic directions.

Analysis starts from recognition of the significance of statistical equilibria over parts of the observed joint frequency of individual states, not from the supposition of statistical equilibria elsewhere in those frequencies. This allows inquiry to center not on estimation of parameters in very strongly specified individualist models, but on extracting as much information as possible from what is observed. In characterizing those equilibria, the approach emphasizes the identification of formal statements of emergent social regularities suggested by observation, not detailed kinetic characterizations of "representative" evolutions or homogeneous individual regularities.

When applied to distributions of income, this approach suggests processes conditioning measures of individual wage incomes are irreducibly *social*. They reflect not individual characteristics in themselves, but their measures relative to social averages. When applied to distributions of returns on assets, the approach conceives of the outcomes of capital-market competition not in terms of individual evolutions or characteristics, but as an instance of spontaneous self-organization created by the actions of arbitrageurs and by the operation of the capital-market price system. Capital-market prices create competitive aggregate tradeoffs between costs and returns to arbitrage, resulting in the pecuniary pricing of organization in distributions of profitability, which is what we observe in those frequencies.

In both cases, the socio-combinatorial approach is founded on the recognition that complex competitive interdependences can give rise to emergent systemic regularities in economic systems. Those regularities are irreducible to the detailed actions, knowledge, or characteristics of individuals in the system. In some instances those regularities define observable statistical equilibria, giving researchers the opportunity to characterize them formally. This in turn informs further efforts to develop observationally founded, systemic theorizations of the economic processes and social significance behind the regularities in question.

It is hoped that the discussion above and these two examples illustrate the greater generality and usefulness to economic and social inquiry of socio-combinatorial characterizations of observed statistical equilibria in economic systems. It is also hoped this paper encourages new, creative uses of the approach to pressing questions in economic and social inquiry.

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