Eur. Phys. J. Special Topics **229**, 1715–1733 (2020)

https://doi.org/10.1140/epjst/e2020-900124-y

THE EUROPEAN PHYSICAL JOURNAL
SPECIAL TOPICS

Regular Article

Maximum entropy approach to market fluctuations as a promising alternative

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Received 29 June 2019 / Received in final form 21 September 2019
Published online 7 July 2020

Abstract. Conventional models studying market fluctuations often suffer from over simplifying assumptions of highly complex economic systems. A brief but critical review is given of "conventional" economic theory and modeling approaches. A detailed discussion contrasts various approaches to modeling market fluctuations which introduce more realistic frameworks than conventional models; namely, Log Periodic Power Law Models (LPPL), Heterogenous Agent Models (i.e. Simple Heterogenous Models and Agent Based Models), and Quantal Response Statistical Equilibrium Models (QRSE). From this review, a clear picture is formed for the capacity for each of these approaches to overcome existing problems in economic theory and modeling of large complex systems. QRSE framework as an "Information Theoretic Maximum Entropy Approach" is presented here and discussed with special emphasis on the explanatory power and physical meaning of the model parameters regarding their economic interpretation and significance, and its simple modeling framework.

1 Introduction

One of the main objectives in Economics and Finance is to explain the forces behind market behavior. Yet, the existence of a large number of boundedly rational market players, their interaction with each other and uncertainty in their decision making process due to imperfect information create a complex nature, such that modeling and studying the causes behind repetitive market fluctuations become a hard task. By ignoring this complex nature of markets for several decades, "The Rational Expectations-Efficient Market Hypothesis" initiated by Lucas [1], which is a neoclassical transformation of "The Efficient Market Hypothesis", first introduced by Samuelson and Fama [2,3][1], has been extensively used in theoretical and empirical

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[1] According to [2,3], price changes are random and unpredictable as long as they incorporate all available information and expectations of all market participants. The random character of the price movements is explained by Fama [2,4] as the result of competition between a large number of rational profit maximizing individuals who compete with each other, trying to predict future market values of individual assets where important current information is almost "freely" available to everyone. Thereby, at any time, competition leads to a situation where actual prices of individual assets already reflect the effects of information based both on an event that has already occurred and on events which the market expects to takes place in the future.

studies to explain the driving forces behind asset price movements. This conventional approach to market efficiency ties risk-averse investors' marginal expected utilities to price changes. According to Lucas [1], all investors have "rational expectations", and utility weighted prices (reflecting all available information) become unpredictable and follow martingales[2] [7]. Therefore, the current price of an asset is the market's best forecast of the future price. In this set up, the aggregated optimization problem of rational individual buyers and sellers with perfect foresight subject to certain constraints allows one to build up demand and supply schedules, whose intersection determines the general equilibrium prices in the market. Hence, with the "rational expectations revolution", the main role of the interaction between the competition and arrival of new information on market efficiency was reduced to the role of perfectly rational representative agent's expectations in a general equilibrium set up[3]. Although some simplification might be necessary for modeling, over-simplification contains fundamental problems as it is the case for the conventional models. Introducing the rational representative agent with rational expectations[4] into their analysis for the sake of providing a micro-foundational explanation for the individuals' behavior, and simplifying the aggregation problem, ignores the very existence of uncertainty, interactions among agents, and the market psychology during the transaction process which are some of the most important drivers of out of equilibrium behaviors observed in reality (i.e. boom-bust cycles and crashes).

Empirical observations from the data, such as excess volatility of prices, fat tails of distributions of price changes, bubbles and crashes have provided enough reasons for researchers to reject the ideas regarding perfect market efficiency and complete randomness of prices. During the early 1980s, the "Rational Expectations Effcient Market Hypothesis" started to be questioned. Shiller [9] argued that stock prices exhibit excess volatility – that is, movements in prices are much larger than movements in underlying fundamentals. For example, the October 1987 Crash could not be explained by the representative agent rational expectations theory [10]. Simon [11] supported the idea of new modeling techniques developing the theory of bounded rationality based on the impossibility of individuals' ability to reach perfect information. With the help of new developments in mathematics and physics, new modeling techniques started to be developed considering the dynamical, complex nature of social systems.

This paper discusses the strengths and weaknesses of three different recently developed modeling techniques: Log Periodic Power Law Models (LPPL), Heterogeneous Agent Models (HAMs) and Quantal Response Statistical Equilibrium Models (QRSE), with a particular focus on housing markets. These models were developed

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[2] The martingale property of prices assumes that price differences are uncorrelated, such that the best predictor of the future prices is the current prices. On the other hand, observed data shows that higher powers of "absolute value" of price changes indicate higher and longer lasting auto-correlation which can be interpreted as high degree of predictability of volatility [5,6].

[3] Fama [8] accepts that testing market efficiency is problematic because if the efficiency is rejected, this could be either because the markets are inefficient or an incorrect equilibrium model is assumed. On the other hand, to overcome this problem, the traditional approach specifies additional structures about individuals' preferences and expectations etc. such that market efficiency is given.

[4] The assumption of rational expectations of individuals with a set of complete, transitive preferences, allows neoclassical theory to overshade the problem of individual's uncertainty in their decision making process, because a rational agent given her fully defined set of preferences would have perfect foresight about her future asset value which would meet the actual price created in the real market in equilibrium.

in order to provide a more realistic approach to modeling market fluctuations by overcoming the shortcomings of the conventional models which rely on the "Rational Expectations Efficient Market Hypothesis" assumptions. An argument is presented in support of QRSE models – which employ an information theoretic maximum entropy approach – as a promising alternative to modeling real dynamics of complex systems. In particular, QRSE models provide a logical and efficient approach to characterizing macroeconomic systems since they are compact (few model variables) but able to capture the behavior of complex systems in order to analyze real phenomena. Parameters recovered from QRSE models provide "economically" meaningful explanations for the market behavior. The maximum entropy principle employed in QRSE models is "the least biased" method of inference, because it avoids imposing unnecessary assumptions. Most importantly, QRSE models reject the assumption of rational representative agent, replacing it with boundedly rational typical agent whose preferences do not have to be well-defined, complete, and transitive. Lastly, QRSE models also introduce a platform to model social interactions among heterogeneous individuals, uncertainties in individuals' decision making processes, and endogeneity of the system dynamics. Sections 2–4 will present the basic framework and discuss the strengths and weaknesses of the three modeling approaches: LPPL, HAMs, QRSE. Section 5 will focus on the advantages of QRSE models.

2 Log periodic power law models of market crashes

The Log Periodic Power Law (LPPL) model, which is also known as Johansen-Ledoit-Sornette (JLS) model, carries the idea of critical point and phase transition from statistical physics[5]. The main purpose of this type of models is to predict the crises in different markets based on the "weaker efficient market hypothesis" [14][6].

In this model framework, a crash is not a critical (regime shift) point itself but the proximity of the system to the critical point, in which the hallmark of a crisis is believed to follow a power law acceleration of prices [12,14–18]. Furthermore, LPPL argues that additional properties and ingredients related to crash should be expected beyond the power law acceleration of prices[7].

The behavior of the system is modeled based on the positive feedback mechanism, which results from cooperation between and within imitative noise traders and rational traders by creating a network effect. According to Sornette [6], market crashes are caused by the slow build up of long range correlations leading to a global cooperative behavior of the market and eventually ending in a collapse in a short,

[5] "Critical points" are described in mathematical parlance as singularities associated with bifurcation and catastrophe theory. At critical points, scale invariance holds and its signature is the power law behavior of observables [12,13].

[6] They assume the process of the "emergence of intelligent behavior at a macroscopic scale" that individuals at the "microscopic scale" do not have.

[7] Price has two components governed by different rules. The fundamental component, follows a random walk based on martingales, and a log-periodic power law structure is added into it which is used to detect the bubbles. Dynamics of the bubble component, on the other hand, satisfies a simple stochastic equation: $\frac{dp/p}{\mu_{(t)}dt+\sigma dW-\kappa dj}$, where $\mu_{(t)}$ is the drift, dW is the increment of the Wiener process, dj is the jump process such that $j = 0$ before the crash and 1 after, and κ is the loss amplitude related to crash. Dynamics of the jump is governed by a crash hazard rate, $h_{(t)}$ such that expectations of jumps are given by $E_t[dj] = h_{(t)}dt$ such that aggregate effect of noise traders' imitation can be followed by the dynamics of the hazard crash rate, $h_{(t)} = B'|t_c - t|^{m-1} + C'|t_c - t|^{m-1}\cos(\omega\ln(|t_c - t|) - \phi$. The cosine part captures the existence of possible "hierarchical cascades" of accelerating panic leading to bubble [19].

critical time interval. His key assumption, hence, is that a crash may be caused by a "local self-reinforcing imitation process". This process leads to the bubble if the tendency for traders to imitate their neighbors increases up to a certain point called the "critical point", where many traders may place the same order (i.e. sell) at the same time, thus causing a crash. A crash is not a certain outcome of the bubble but can be realized by its "Hazard rate", which is the probability per unit time that the crash will occur in the next instant provided it has not occurred yet [6,19–21]. Since the crash is not a certain deterministic outcome of a bubble – which could be price-driven or risk-driven, it is rational for investors to remain in the market given that they are compensated by a higher rate of growth of bubble for taking the risk of crash, because there is a finite probability of "landing smoothly"[8].

In LPPL models, two forces affect agents' opinions. First, they have idiosyncratic signals that they themselves receive, and second, agents tend to imitate the opinions of their neighbors. The first force creates disorder while the second creates order in the market. The crash occurs when order takes over (when everyone has the same opinion for which action to take). In normal times, disorder becomes dominant (when buyers and sellers disagree with each other and roughly balance each other out). Macro level coordination can arise from micro level imitation [20]. To describe the imitation process they employ mean field theory through the hazard rate which is the collective result of the interaction between the agents. The higher the price the higher the hazard rate, and the probability of crash. This process is assumed to reflect the self fulfilling nature of the crises. The idea behind this is that when systems get closer to critical points, local changes propagate over longer distance and the average state of the system becomes more vulnerable to small perturbations because different parts of the system becomes highly correlated (just like the spread of infectious diseases). At the vicinity of its critical point, the system becomes scale invariant, which means different scales (from smallest scale to the largest) become alike. This is why when the imitation strength of the overall market becomes really strong, local imitation cascades through the scales towards global coordination.

The model tries to capture these characteristics of the complex systems (mostly financial markets) by a power law with an oscillatory expansion, which is called LPPL for the logarithm of price. The simplest version of the LPPL function is

$$\ln[p_{(t)}] = A + B|t_c - t|^m + C|t_c - t|^m \cos[\omega * \log(t_c - t) - \phi] \tag{1}$$

where t_c is an estimate of the end of a bubble so that $t < t_c$, A, B, C, m are coefficients, ϕ is a phase constant and ω is the angular log-frequency. If the exponent m is negative, $\ln[p_{(t)}]$ is singular when $t \to t^-$ and $B > 0$ ensuring that $\ln[p_{(t)}]$ increases. If $0 < m < 1$, $\ln[p_{(t)}]$ is finite but its first derivative $d \ln[p_{(t)}]/dt$ is singular at t_c and $B < 0$ ensuring that $\ln[p_{(t)}]$ increases [21].

This equation shows that potential crash at the end of the bubble is proportional to price[9]. The key part of the model is that oscillations become visible in the price dynamics right before the critical date.

Using this framework, they state that bubbles are characterized not by an exponential increase in prices but rather by a faster than exponential growth of price. This phenomenon is basically generated by behaviors of investors and traders that create positive feedback in the valuation of assets leading to unbounded growth ending with

[8] Main building block of LPPL is "Rational Bubbles" model introduced by Blanchard and Watson [22].

[9] Sornette and Johansen [14] propose a more general and complex formula with additional degrees of freedom to better capture behavior away from the critical point based on a finer analysis of the "renormalisation group theory" but it is not discussed here. For further information please see [6,14,20].

a critical shift at a finite time t_c. This positive feedback mechanism could be created first, via technical strategies of investors such as option hedging and insurance portfolio strategies etc., or second, they could be caused by imitative behaviors of individuals which leads to a network effect and herding in the system. Critical time t_c is interpreted as the end of the bubble, which is often (but not necessarily) the time of the crash. During the phase of the bubble build-up, the competition between noise traders and rational investors creates deviations around the "hyperbolic power law growth" in the form of oscillations that are periodic in the logarithm of the time of t_c. Therefore, prices during the bubble process, are described as log periodic (hyperbolic) power law (LPPL). The main objective of the model is to predict the most probable time of the crash.

LPPL has been employed to detect different types of market crashes in different regions[10] For example, Zhou and Sornette [21] shows that 22 US states (mostly the North East and the West) exhibit faster-than-exponential growth signaling a fast growing real estate bubble between 1992 and 2006. Using LPPL model parameter estimations they argue that mid-2006 signals the critical point of the system. Similarly they found two different turning points for the UK real estate market which are end of 2003 and mid-2004 [24].

These models fit the time series data to an LPPL function in order to calibrate the model parameters for the detection of the critical point, t_c. In order to check the robustness of the calibration process, they first (generally) employ a GARCH (1, 1) model estimated from the time series of log prices. From this GARCH (1, 1) model, a large number of "independent data sets" (varies from 2000 to 10 000) are generated. After, they employ a Tabu-Search[11] to find the initial values for non-linear parameters[12]. Using the obtained initial values, they find an ensemble of local minima of parameters[13] by using "the Levenberg–Marquardt algorithm". During the fitting process, they truncate the time interval used in the fitting by removing points and re-starting the fitting process. The procedure is repeated until the full time period is recovered. They found that a year from the crash is not enough to give any reliable results. After one year from the crash, fit starts to signal the best fitted date for the crash, t_c.

Although the theoretical argument behind LPPL framewok, which analyzes complex dynamical systems as evolutionary, self-correcting mechanisms by taking into account the interaction between heterogeneous rational and noise traders, is promising, the estimation techniques of LPPL could be criticized on several grounds. First, parameter values heavily depend on the time scale and starting date, and they do not provide a convincing argument about their method of choosing the start date for the fitting process. Second, finding the local minima of the parameters to approximate the global minima in a high dimensional parameter space of LPPL is highly problematic given the high complexity of the systems under consideration. As a result, randomness in the calibration and fitting process, causes inconsistency in the model applications[14]. Moreover, even though LPPL models seem to be successful at

[10] For example, 2006–2008 oil bubble [23], Las Vegas real estate bubble [13], UK stock market bubble [24].
[11] See [25].
[12] Parameters; A, B and C are expressed as functions of non linear parameters to reduce the dimensionality of search space.
[13] Sornette [19] claims that this method is more robust than searching for a single minimum.
[14] To illustrate, since there is not any special rule while choosing the starting date and/or time interval in the fitting process, the results may differ drastically even between the applications of the same data.

approximating critical times for some crises, they failed to approximate the timing for others[15].

3 Heterogeneous agent models (HAMs)

Heterogenous Agent Models consist of two branches, one of which is called "Simple or Stylized Heterogenous Agent Models (SHAMs), while the other is called "Agent Based Models (ABMs)". The main objective of both models is to provide an alternative way of representing the real market dynamics, such as highly volatile prices, boom-bust cycles, and crashes, which are believed to be caused by the interaction among boundedly rational heterogeneous agents. Agent-Based Models (ABMs) are computationally more demanding HAMs, which are complex, dynamical systems comprising autonomously interacting heterogeneous individuals. While SHAMs are analytically tractable to some extend, ABMs are impossible to follow analytically; therefore, they are constructed as simulations of real world dynamics like markets in small scale.

Simple Heterogeneous Agent Models (SHAMs) are a new branch of simple stochastic models of interacting traders which are inspired from the models of "multiparticle" interaction in physics [26]. The idea is to explain the observed regularities in complex systems such as financial markets through the microscopic interactions of the individuals. Since the emergent macroscopic behavior of complex systems are different from each microscopic component, SHAMs support the least detailed determination of individuals' behaviors by focusing only on studying a few plausible rules to define microscopic behavior of a large body of interacting individuals, and the emergence of macroscopic behavior of the overall market with a large number of traders. Their main purpose is to explain the "stylized facts" of financial markets such as fat tailed distributions of prices and their time-variations as a response to incapabilities of existing models which take efficient market hypothesis as their starting point.

SHAMs do not have a foundation for utility maximization or any other mechanism for decision making process of individuals. Instead, they employ certain behavioral rules in their models where individuals are assumed to be boundedly rational. The starting point is to set a finite number of rules to distinguish the types of agents and to bring the heterogeneity in to the model. The authors generally choose two types of rules to define the forecasting behavior of the agents, such as fundamentalists vs. chartists[16]. Fundamentalists set their future expectations for assets and trading strategies based on market fundamentals such as dividends, interest rate, GDP growth, etc. They tend to invest in assets that are undervalued, whose values are below a benchmark fundamental value. Chartists[17] on the other hand, do not take market fundamentals into account, instead, they base their expectations about future asset prices and their trading strategies upon historical patterns in past prices such as trends, and exploit these patterns. Once the set of rules whether fundamentalist, chartist, or some combination of the two is established, how agents decide amongst

[15] See [6] for more detailed explanation for the results of LPPL application to different crises periods.

[16] Some other SHAM models prefer rational arbitrageurs vs. noise traders [27,28], informed arbitrageur vs. positive feedback traders or rational optimizer vs. simple imitator [29–32].

[17] Chartists in SHAMs are represented as noise traders in LPPL models of previous section but imitative behavior of trend following noise traders in LPPL is epidemiological where agents infect each other and cause network clusters. On the other hand, in SHAMs, they are introduced as an alternative behavioral rules for the switching mechanism based on relative performances of different strategies.

the rules is specified. Their demands and the set of market prices or equilibrium prices are determined, depending on what type of equilibrium concept they are using. The choice of forecasting rule is based on the success of such rules in the past, but that success is determined by how many people follow the rules. The interaction between these different types of individuals and switching between different rules, therefore, could set mechanisms such as herding behaviors, and positive feedback loops, which can cause endogenous fluctuations in prices, returns and market crashes [10].

Estimating an heterogeneous agent model with endogenous switching of behaviors introduced by Hommes and Boswijk [30,33] explains the dot.com bubble as being triggered by economic fundamentals (i.e. optimistic news about the markets) amplified by a switch to trend following behaviors of the market participants. Similarly, Hommes [34] applies this framework to housing markets[18] by introducing heterogeneous expectations in to a standard asset pricing model[19] linking housing rentals to buying prices. They show that data for eight different countries support the heterogeneity in expectations with temporary endogenous switching between mean-reverting behaviors of fundamentalists and trend following chartists based on their relative past performances.

SHAMs are often highly non-linear. For example, weights of the fractions of the different traders are time dependent so that they can generate complicated fluctuations for a range of parameters [10]. Chaotic SHAMs with chaotic asset price fluctuations around a fundamental, may explain excess volatility in the market. A disequilibrium model with speculators by Beja and Goldman [37] allows for limit cycles. Zeeman [32] is able to demonstrate the possibility of a sudden market crash in a model similar to [37] with a nonlinear reaction function for chartists. Again, based on [37], Chiarella [38] shows the possibility of periodic oscillations around a fundamental price if chartists excess demand gets sufficiently far from equilibrium case.

Compared to conventional models which take "rational expectations efficient market hypothesis" as given, SHAMs provide a more realistic explanation for price fluctuations in the market by endogenizing demand and supply dynamics through a switching mechanism between different behaviors of two types of heterogeneous agents. This way, they are also able to bring the interaction among individuals into the picture. However, introducing a restricted amount of stationary ad-hoc behavioral rules – generally not more than two, to impose only two types of agents to represent a more complex interactions in the real markets, still fails to provide an adequate framework to capture and understand the dynamics of social systems.

3.1 Agent Based Models (ABMs)

Agent Based Models[20] are known as more complex computational heterogeneous agent models, which are not analytically tractable compared to SHAMs. Since ABMs are hard to follow analytically, they are constructed as simulations of real world dynamics like markets in small scale [45]. The main purpose of the ABMs is to explain the macrodynamics of a system via modeling detailed interactions between a large number of individuals, and capture how the observed patterns are formed when the economy is out-of-equilibrium [46][21]. Imitating the real world, ABM of a market

[18] For more housing market applications, see [35,36].

[19] In the model the annual cost of home ownership ("imputed rent") is equal to the housing rent, and future excess return on investing in housing is the the "sum of capital gains minus mortgage/maintenance costs and saving on rent" [34].

[20] See [39–44].

[21] Out-of Equilibrium markets may converge or shows patterns of equilibrium but this could be a special case [46].

includes a large number of heterogeneous buyers and seller who act independently, making decisions of buying and selling based on their information and behavioral traits. For example, if an ABM has 20 000 agents, there will be 20 000 of actual separate entities engaging in trade with one another. This means that equilibrium in ABMs is not imposed or assumed, but arises by itself. Agents in these models are equipped with a set of properties and algorithmic behavioral rules. The properties reflect the state they are in, and rules determine their actions. For example, an economic agent may have income and consumption as properties, while her search strategy on the market is a behavioral rule. Agents do not necessarily know others' behavioral rules or properties. Based on the setting, behavioral rules let individuals to be goal-seeking and adaptive to changes in the environment of the model.

ABM researchers criticize SHAMs for their representation of noise traders vs. rational traders [40,41,46]. They argue that the stationary decision making process of noise traders in which they do not learn from their previous mistakes, is not a realistic representation of the real world. They also believe that the assumption of rational agents' perfect insight about behaviors of noise traders and other rational agents, is far from reality. Instead, ABMs aim to bring an evolutionary framework to explain how heterogeneous individuals' set their expectations by adapting the changes in the market formed mutually by others' heterogeneous expectations. Hence, dynamic heterogeneity which is represented by "a distribution of agents across a fixed or changing set of strategies" plays a critical part in ABMs [45]. Individuals try to optimize their actions by taking into account others' decisions but the state space of the decisions are too complicated to find an optimal solution most of the time, which in turn, brings the existence of bounded rationality of individuals and the modelers into ABM framework. These heterogeneity of individuals' expectations may give rise to either homogeneous rational expectations and efficient markets or a collective behavior, such as herding can occur causing boom-bust cycles and following crashes.

Designing an ABM necessitates certain steps [45]. Modelers first choose what types of preferences will be available to agents (e.g. simple mean, variance preferences, myopic vs. inter-temporal etc.) with certain behavioral components (e.g. loss aversion). Second, they decide the mechanism for price determination[22]. Then they choose a learning process for individual decision making. The most commonly used tool is called "Genetic Algorithm (GA)" with evolutionary strategies which is an optimization technique[23]. After setting up a learning mechanism for agents, the most delicate issue is how to present a large data set into a trading plan for individuals. They generally prefer to define different variables to convey the information as trading plans in the models. Besides, how to signal informational changes to which agents and how frequent it should be cause other complexities in the model[24]. Another model question is how to introduce social learning to the model. In some cases, agents only interact with each-other via prices but some other cases some form of communication rules are introduced. After these steps, model parameters are calibrated based on real data so that the model would match the observed historical patterns. Finally, in order to control and validate the evolution of the model, modelers set up a baseline model [51], which converges to a well-defined path (i.e. rational expectations, market clearing equilibrium), to compare more realistic scenarios for the market dynamics.

[22] i.e. Slow price adjustment which market is never in equilibrium [47]; market clearing price mechanism [40]; a mechanism in which individuals declare their offers for transaction [48]; mechanism with a randomly existing and trading agents of trade is beneficial [49].

[23] See [50] for more detailed explanation.

[24] To overcome these difficulties some uses artificial intelligence technologies such as "classifiers" [40].

3.2 A basic framework for ABMs: artificial market models

Palmer et al. [41] and Arthur et al. [40] employ an evolutionary approach[25], which agents generalize patterns from their past to be guided for their future decisions by taking into account others' expectations and adopt to new conditions in their environment using their inductive reasoning[26]. Here, expectations endogenously co-evolve and are co-created based on individuals adaptations to the changes in market environment. As a result, the market may either converge to the rational expectations efficient market outcome or more complex self-organizing patterns such as bubbles.

In this example, following a neoclassical asset pricing theory, market price is first declared by a market maker to all participants such that participants can find a matching rule for their decision to sell or buy. It is then set to adjust market state reflecting the excess demand-supply conditions based on

$$p_{t+1} = p_t + \lambda(B_t - S_t) \tag{2}$$

where λ measures the sensitivity of agents to market depth (i.e. how easy to find a match), B_t and S_t represent buyers and sellers.

There are N heterogeneous agents who decide on their portfolio independently based of a utility function. There are only two assets, risk free bond with a constant interest rate r_f and infinite supply and risky stock paying a stochastic dividend following an AR(1) process as:

$$d_t = \bar{d} + \rho(d_{t-1} - \bar{d}) + \mu_t + \epsilon_t \tag{3}$$

where $\mu_t = N(0, \sigma_\mu^2)$ such that price of a stock p_t is endogenously determined. Individuals demand for a stock based on

$$x_t^i = \frac{\hat{E}_t^i(p_{t+1} + d_{t+1}) - p_t(1 + r_f)}{\gamma \hat{\sigma}_{p+d,i}^2} \tag{4}$$

where γ represents the risk aversion and agents forecast[27] their future returns based on a linear rule,

$$\hat{E}_t^i(p_{t+1} + d_{t+1}) = a_j(p_t + dt) + b_j \tag{5}$$

where j is the rule chosen by individual i. They get a temporary price equilibrium by setting the total number of shares to a fixed value; therefore, after the price is set agents are allowed to change their portfolio and trading volumes, which are recorded at each time.

After the rules for preferences and forecasting are set, they introduce a learning mechanism for agents using "Genetic Algorithm (GA)", which is implemented with a certain probability by agents. This parameter tells us about the learning speed of the agents who update their rules by estimating the strength of each rules. This

[25] Evolutionary approach cannot be solved analytically, therefore, only way to reach a solution is computational with possible algorithms for learning and adaptation.

[26] In contrast to deductive reasoning of normative orthodox models which they dictate how the markets should behave, ABMs take inductive reasoning approach (bottom-up) to show how markets behave in reality.

[27] Agents use classifier systems to predict mean and variance of the stock returns. A classifier rule is represented as bit-string and a parameter vector such as $(\#, 0, 1\#; a_j, b_j, \sigma_j^2)$ where the first part represent the current condition in the market, 1 would match the true condition, 0 represents the false, # means no match (neither false nor true) [41,51]. Given the rule, model searches to find an actual match to the system. Through time the matched rules are updated based on the forecast accuracy.

produces an evolutionary dynamic which result in mutation or extinction of the rules. Simulations completed by Arthur et al. [40] runs for 250 000 periods with $N = 25$ agents and $M = 100$ predictors based on $J = 12$ market descriptors. Their results show that inductively rational agents can exist both in two different regimes. In the first regime, the market settles into rational expectations equilibrium if the rate of exploration of different forecast is low. In the second regime, with a more realistic rate of exploration market dynamics become more complex such that psychological components play a significant role, and bubbles and crashes are observed indicating more realistic features of price movements.

Agent based models mostly focus on simulating stock market behavior, but there are some other applications whose main focus is to explain housing market behaviors. Axtel et al. [52] develop an ABM of the housing market for Washington DC metropolitan area by simulating a one-to-one map of the actual housing market to understand the housing market boom and crash of 1997–2009 by encompassing the empirical heterogeneity in households' expenditures on housing, borrowing conditions, and demographic variables. They initialize their model by making an initial guess at matching households to houses and running the model for a number of periods using 1997 data until it settles down into a more or less steady state. Then they simulate the period of 1997–2009. Their results show that the main factor of 1997–2009 boom and bust was the leverage in the market. Erlingsson et al. [42] develop a macroeconomic ABM model including a housing market which is susceptible to endogenous crashes due to over borrowing by households. A paper by Ge [43] proposes an agent-based model of housing markets, which centers on the behavior of speculators and leniency of the mortgage system.

ABMs have been criticized in several aspects. First, they are too ambitious in a sense that they cannot be analyzed analytically. For example, they create strong non-linearity and stochasticity in the individuals' behaviors which are made of multiple components connected through complex interactive networks, such that it is hard to capture causality between the behavior and the outcome [53]. Moreover, due to a large number of parameters used to define the rules for individuals and the states of the model, calibration process suffers from high dimensionality [54], which in turn, causes amplification of simple data errors in calibrated parameters [51]. Additionally, since the results of these models are generated by the rules and structures, which are highly depended on the modeler's personal taste, different ABMs generate different results for the same problem. This, as a result, causes biases and lack of robustness in the model.

4 Quantal response statistical equilibrium models (QRSE)

Quantal Response Statistical Equilibrium models (QRSE), recently introduced by Scharfenaker and Foley [55], provide a method to explain the observed regularities in highly complex macroeconomic systems by taking into account social interactions between a large number of heterogeneous market participants and their reactions to changes in macroeconomic market variable(s). Their main objective is to analyze the market behavior in a simple but realistic way without relying on ad-hoc normative assumptions. They aim at capturing and representing the reality based on available information gained from the data and related economic theory. In order to achieve this goal, they employ the "Information Theoretic Maximum Entropy Program (MEP)" introduced by Jaynes [56–58], in which the concept of statistical equilibrium from the field of Statistical Mechanics [59–61] along with Information Theory [62] are combined in order to explain the observed behaviors of complex social systems.

QRSE models are built upon three main concepts: Statistical Equilibrium of targeted economic variable(s) representing the macroeconomic state of the system, the assumption of Quantal response behavior of the market participants, and the maximum entropy principle as a method of inference.

Deviating from the traditional concept of general equilibrium, information theoretic *statistical equilibrium*[28] takes equilibrium as a probability distribution of a macroeconomic variable (i.e. prices, profit rates) over all combinations of the possible states of the system. This probabilistic approach is thereby, able to capture the "endogenous fluctuations" around a central tendency simultaneously, as the result of the equalization process. The statistical equilibrium distribution of the macroeconomic variables therefore, takes the form of a single peaked (laplace-like) distribution. The fluctuations around the peak (i.e. mode of the distribution) are argued to be resulted from the transactions taken place at the out of equilibrium prices. This means, total gains and losses from market transactions do not cancel each other out[29]. The main goal of information entropic statistical equilibrium approach is to offer a general framework to model macroscopic (macroeconomic) behavior of a system without relying on strong assumptions about the detailed microscopic behavior of the individual components (i.e. market participants)[30].

The second important concept in QRSE is the maximum entropy principle (MEP)[31] [56–58], which serves as a method of inference, allows modeler to estimate the statistical equilibrium distribution of the system. In implementation, MEP can be expressed as a constrained maximization problem. For the discrete case, where the observed discrete variables are x_k, $x = 1, 2, \ldots, X$, the maximum entropy program can be set by maximizing the entropy of the distribution of the variables subject to m different constraints on the expectations of the functions $f_m(x_k)$ as:

$$\begin{aligned} \underset{p_k}{\text{Maximize}} \quad & -\sum p_{(k)} \log(p_{(k)}) \\ \text{subject to} \quad & \sum_{k}^{K} p_{(k)} = 1 \\ & \sum_{k=1}^{K} p_{(k)} f_m(x_k) = F_m, m = 1, 2, \ldots, M; M < K \end{aligned} \tag{6}$$

[28] It is called information theoretic statistical equilibrium because, as it is first realized by Jaynes [56–58], information entropy, provides an alternative way to interpret entropy concept from the statistical mechanics. Entropy in statistical mechanics represents a measure of the number of micro-states of the components of a system (e.g. gas molecules) consistent with a macro-state described by the distribution of a variable, while it is interpreted as a measure of uncertainty in information theory. Shannon (Information) Entropy of a probability measure, $p_i \geq 0$, on a finite set of variable $X = x_1, x_2, \ldots, x_n$ is defined as the "expected value of information content", $I[p] = \log[1/p]$: $H(p) = -\sum_{i=1}^{n} p_{(x_i)} \log[p_{(x_i)}]$, where $\sum p_i = 1$. As a result, the higher the entropy is, the higher the uncertainty in the system. For more detailed information see [63].

[29] Walrasian equilibrium process with an auctioneer, on the other hand, is the case where gains and losses from the market transactions cancel each other out such that market clears. Therefore, fluctuations can only be caused by random shocks (i.e. policy changes, or arrival of news) to the system, but they would not be lasting long since the market instantaneously corrects and clears itself.

[30] For example, models based on traditional economic theory assume complete and transitive set of preferences for "homogeneous" individuals so that they can reduce the problem of providing a micro-foundation for the macroeconomic system to a simple aggregation of all "representative agents" decisions.

[31] MEP program is argued to be the least biased method among all available techniques, because it only allows the use of available information about the system from the theory and observed data in the form of constraints to the maximization problem. See [64] for further details, and applications of maximum entropy principle to different economic problems.

where K (such that $k = 1, 2, \ldots, K$) represents all possible states of the system, M (such that $m = 1, 2, \ldots, M$) represents the the constraints of the maximization problem, $p_{(k)}$ is all probability distributions on the finite variables x_k, and F_m is the expected values of $f_m(X)$.

The solution to this entropy maximization gives the inferred statistical equilibrium frequency distribution with M parameters is

$$p_{(k)} = \frac{e^{\lambda_1 f_1(x_1)+\ldots+\lambda_m f_m(x_k)}}{\sum_{k=1}^{K} e^{\lambda_1 f_1(x_1)+\ldots+\lambda_m f_m(x_k)}} \tag{7}$$

where λ_m represents the Lagrangian multipliers. The most crucial part of setting up an entropy maximization program is to come up with a set of constraints that are relevant to the system under consideration. The constraints could be obtained from the available data series representing the system (e.g. introducing mean or variance of the data as a constraint), or the theory that helps to explain the dynamics observed in real world (e.g. theory of competition can be introduced as a constraint). This, in turn, helps researchers to deal with the problems related to data (i.e. lack of high quality data)[32].

The maximum entropy statistical equilibrium approach has been used to model different markets. A maximum entropy-statistical equilibrium model of commodity markets as an alternative to Walrasian equilibrium models is introduced by Foley [65], in which the assumption of market auctioneer is removed from the picture. By doing so, he is able to show how complex interactions of economic agents in a "decentralized" commodity market, where transactions can take place at different price ratios than market clearing prices, give rise to observed regularities in transactions[33]. Stutzer [70] develops a statistical equilibrium model of asset pricing to find the most likely "Gibbs Canonical distribution" of risk neutral asset prices. Assuming no arbitrage opportunities, he first estimates empirical distribution of risk neutral asset prices based on the data to satisfy the martingale measure of the prices. Then, by minimizing the KL divergence[34] subject to the risk-neutral (martingale) measure of the asset price, he estimates the statistical equilibrium distribution of the asset prices, which allows him to replicate the results from Black-Scholes tests in a simpler way.

Finally, the last concept used in QRSE models is the assumption of quantal response behavior of individuals represented as a logit function[35], obtained from maximizing individual's expected payoff function given a mixed strategy over the choices

[32] For example, the main difficulty of modeling the effect of individuals' actions in the market, is to get a detailed data representing individual's actions, e.g. as buying and selling assets. Moreover, economic phenomena are unique in their nature most of the time, such that it is not possible to have repetitive events to have a large enough data sets to get robust results.

[33] See [66] for the labor market application of his general theory of statistical market equilibrium, Farjoun and Machover [67] and Scharfenaker and Semieniuk [68] for an analysis of firm level profit rate, and [69] for the application on the Tobin's q.

[34] It minimizes the "entropy deficiency" between the empirical distribution and the estimated probabilities [64].

[35] If the agent chooses a mixed strategy, $f[a|x] : AxX \to (0, 1)$ over binary actions $a \in A$ (i.e. buying and selling) to maximize her expected payoff, $\sum_a f[a|x]u[a, x]$, subject to a constraint on the entropy of the frequency distribution describing the mixed strategy which represents the uncertainty in their decisions, the result gives the logit (Gibbs) function; $f[a|x] \propto e^{\left[\frac{u[a,x]}{T}\right]}$ where $u[.]$ is the payoff function of the agent, $f[a, x]$ is the joint frequency distribution, $f[a|x]$ is the conditional action frequency distribution and T is the responsiveness parameter of the individual which tells us how responsive the agent is to reacting the any changes in x, outcome variable (i.e. price changes).

(i.e. buying or selling) subject to a a lower bound on the informational entropy of the frequency distribution describing the mixed strategy by using the maximum entropy program introduced above[36]. By doing so, QRSE introduces bounded-rationality into the model as a characteristic of economic agents without over-imposing it.

As a result of combining these three concepts, QRSE models introduce a new way of modeling the endogenous dynamics of complex social systems which result from the interactions among a large number of heterogeneous, goal oriented individuals' behaviors without using any ad-hoc assumptions. Earlier applications of maximum entropy statistical equilibrium approach mentioned above, have been criticized due to their lack of providing economically meaningful reasoning for the choice of the constraints. The main motive behind this criticism is the necessity of emphasizing the difference between modeling the behavior of physical components of the physical world (i.e. gas molecules) and the behavior of the goal oriented purposeful socially interacting agents of the social systems. QRSE aims to overcome this deficiency by introducing economically/socially meaningful constraints in the form of conditional action and outcome frequencies representing the real world dynamics of observed systems based on the theory and data which in turn, provides meaningful interpretation of model parameters.

4.1 Basic framework for QRSE models

Different applications of QRSE model have been studied for different economic problems [71,72] where it seems to produce promising results to explain complex dynamics of different social phenomena. Applying the same QRSE model framework to the US housing market between 2000 and 2015 based on the relationship between buying/selling actions of agents and regional house price changes, Ömer [72] was able to capture different characteristics of the most recent US housing market boom-bust cycles and following market crash.

Ömer [72] set up the model based on the relationship between selling/buying decisions of the individuals in each metropolitan areas in the USA, $A \rightarrow \{a = sell, \bar{a} = buy\}$ as the binary action variable, and the monthly house price growth rate, x as the outcome variable. The maximum entropy-statistical equilibrium was, thus, inferred as the joint distribution of the x and A, $\hat{f}[A, x]$ such that the marginal distribution of the monthly house price changes (x), corresponding conditional outcome and conditional action distributions; $f[A|x]$ and $f[x|A]$ were also derived by maximizing the entropy of the joint distribution of the system, $f[A, x]$.

To build up the maximum entropy program for the housing market case, Ömer [72] first, assumes – as the first constraint in MEP, that during their decision making process, individuals have some uncertainty about assessing the house price changes in a particular region, and they express this as a constraint on their action, which is represented by a quantal response logit function as; $f[a|x] = \frac{1}{1+e^{-\frac{x-\mu}{T}}}$. Parameter μ represents the expected fundamental rate of price increase for the house which the agent is willing to to sell/buy. In this case, μ determines the indifference point of the house price growth rate in which the probability of selling is 50%. It provides useful information about the willingness of market participant's to sell at a certain house price growth rate. Employing this constraint, it was argued that when the regional price increase is well below sellers' estimate of fundamental rate of price increase of the house they would almost never sell. But, when the regional price increase gets closer

[36] According to Foley, this maximization problem can be explained as a process of "satisfising" [11] (rather than optimizing) the expected payoff in which the individual gains some level of satisfaction from her action.

to their expected rate of fundamental increase of the house, they sell more frequently depending on the regional price increase; even though their expectations about the price increase of the house do not perfectly coincide with the actual price increase in the region[37]. T, on the other hand, signals how responsive are the individuals to changes in house prices to sell-buy in a particular region.

In the model, another constraint was introduced based on the theory of competition of Adam Smith [74]. It was argued that individuals' selling/buying decisions create a negative feedback mechanism in the region such that if individuals sell their houses in a region to maximize their expected returns from the transaction, it causes a decline in the house price growth in that region. The opposite is true when the individuals buy. This constraint is represented as the difference between the expected house price growth rates conditional on selling and buying actions weighted by the marginal probability of selling and buying actions by setting up a bound on the degree of negative feedback (effectiveness of competition), δ

$$E[x|a]f[a] - E[x|\bar{a}]f[\bar{a}] = \delta. \tag{8}$$

Combining these two constraints and including an additional mean constraint to the maximum entropy program, the following statistical equilibrium (marginal) distribution of the house price changes was inferred:

$$\hat{f[x]} = \frac{e^{H_{\mu,T}[x]} e^{-\beta \operatorname{Tan} h \left[\frac{x-\mu}{2T}\right] x} e^{\gamma}}{\sum_x e^{H_{x,\mu,T}} e^{-\beta \operatorname{Tan} h \left[\frac{x-\mu}{2T}\right] x} e^{\gamma}} \tag{9}$$

where $H_{\mu,T}[x] = -\sum_{A=\{a,\bar{a}\}} f[A|x] \log[f[A|x]]$ is the binary entropy function, and $\operatorname{Tan} h\,[.]$ is the hyperbolic tangent function. The resulting recovered statistical equilibrium distribution (Eq. (9)) fits the observed data pretty well[38]. The estimation results were argued to prove that the observed patterns in the monthly house price growth rate data can be explained as the result of the relationship between mechanisms produced by the interaction of a large number of competitive individuals' quantal actions and the resulting feedback effect of those actions on the price changes in the regions.

The estimation results show the existence of a statistical equilibrium for before (2000–2006) and after the crash (2010–2015) periods, while a drastic shift in the statistical equilibrium is captured for the crisis period (2007–2009). Model fits and the estimated parameters signal the existence of the housing bubble before the crash of 2007 as a result of the interrelation between increasing price appreciation due to wrong expectations of individuals' about the expected price increase in their houses and worsening negative feedback mechanism from their selling/buying actions on

[37] According to traditional theory [1,73], when utility maximizing rational individuals face with a decision to buy/sell a house in a particular market based on their payoff functions (i.e. the difference between the rate of increase in prices in the region and their estimate of the fundamental rate of price increase of the house), they would almost never sell (buy) the house when prices in the region rise slower (faster) than their expected fundamental rate of price increase of the house, and would wait until their expectations about house price increase coincide with actual price increase in the market. Introducing quantal response behavior of individuals as a constraint, QRSE model manages to disregard this assumption of the traditional theory.

[38] The "information distinguishability (ID) statistics introduced by Soofi and Retzer [75] was used to calculate how well the fitted frequency distribution, $\hat{f}[x]$ captures the observed frequency distribution, $\bar{f}[x]$. The fitted results were shown to be able to capture approximately 97% of the information content of the observed distribution.

house price increase. Based on the estimated parameters, which explain the catastrophic shifts in individuals' behaviors and market reaction to them, the crisis period is shown to be an example against the "Rational Expectations Efficient Market Hypothesis" and " Perfect Foresight of Representative Agents" argument by explaining the causes of existing negative fat-tails in the data [72]. As a result, different from previous housing market models, QRSE models provide economically meaningful behavioral parameters (μ, β, and T) which help the modeler to explain different characteristics of housing market behaviors before, during and after the crash.

Besides all positive features that have been mentioned above, there is a point which is open to criticism of QRSE models which is that QRSE models require discretization of the data in the form of coarse-grained bins in order to recover statistical equilibrium joint distribution of action A, and outcome x. This issue raises a criticism since the estimation of the results are sensitive to bin size chosen by the modeler.

5 Discussion

Modeling highly complex social systems such as housing markets is difficult because their dynamics evolve through interactions among a large number of heterogeneous individuals under the influence of each others' decisions and possible external impacts (e.g. policy changes, natural events etc.). This is mostly because market interactions and individuals actions are often not observable to the researcher, and the parameter space necessary to model them has high dimensionality. Moreover, a lack of high quality data to represent unique non-repetitive social events makes modeling more difficult. Conventional theory and techniques used to model market behavior seem to ignore reality, and rely heavily on idealized assumptions. Therefore, unrealistic ad-hoc assumptions, like perfect rationality of individuals or perfect efficiency of market clearance, over simplify real market behavior and thus, obscure more complex behaviors which frequently occur.

Emphasizing the inability of the conventional economic theory and models to explain extremely destructive market fluctuations (i.e. crises), Log Periodic Power Law (LPPL) models attempt to predict the timing of market collapse as a critical/regime shift point. Although perfect prediction of the timing of highly complex market crashes are likely impossible, LPPL models provide some plausible explanations of the main reasons behind their model choices (i.e. imitation between noise and rational traders), and the resulting network effect, in which prices follow a scale-free power law distribution. However, the LPPL modeling technique is not well established, that is, the choice of parameterization and the estimations of the parameters are quite random, thus, weaken the explanatory power of the model, and lead to strong inconsistencies between model applications.

Simple Heterogeneous Agent Models (SHAMs) carry the analysis one step beyond the conventional models – which take rational representative agent as their starting point – by introducing two or three different types of individuals and their interactions with each other into the model. They are able to drop the assumption of utility maximizing, perfectly rational representative agents. As a result, SHAMs are able to reproduce endogenous fluctuations caused by the interactions between heterogeneous individuals. However, they use some other unrealistic simplifications, such as using few stationary rules to represent behaviors of individuals, their interactions, the movements of prices. The use of just "a few" rules to describe a large and highly complicated (real) system raises questions about how realistic the model results can be in explaining the real causes behind the market behavior. Computational Agent Based Models (ABMs) solve the problems of SHAMs by providing an evolutionary process for the determination of behaviors of heterogeneous agents; therefore, they are able to model a larger amount of more complex interactions between the agents as the main cause of endogenous out-of-equilibrium dynamics of the markets. ABMs are strong tools particularly when it comes to studying out-of-equilibrium behaviors of the markets; however, their highly complex and ambitious agenda complicates the estimation process, which in turn, weakens the robustness of the estimation results. Additionally, since the ABMs are structurally complex, they blur the causal relationships behind the market behaviors.

QRSE models may be a more promising approach than the others to understand the market fluctuations. Combining the most innovative approaches from information theory and statistical mechanics, QRSE models provide a simple and powerful alternative to deal with complex systems. In statistical mechanics, equilibrium takes the form of a distribution representing all possible states of the system. Using this, QRSE models are able to calculate the central tendencies of the system and the fluctuations around them. Studying the central tendency of a distribution (e.g. mean or mode of the distribution) and the fluctuations around it (e.g. how disperse or asymmetric the distribution is etc.) release important information about market behavior. As a result, QRSE models introduce a "general framework" for equilibrium where the conventional market clearing general equilibrium exists just as a special and highly unlikely state of the economic systems.

Furthermore, the maximum entropy principle as a method of inference, is the "least biased" method among all because it maximizes the uncertainty of the system and gets the least informative state with no additional assumptions other than the existing knowledge of the researcher and observed data[39]. By utilizing the economic theories in the form of model constraints, QRSE models manage to overcome the problems of "lack of high quality data" and "incomplete information" which all the models studied in this paper suffer from. Another important feature of the QRSE models is that the estimations by using maximum entropy (MAXENT) program are highly reliable. To explain, if the model constraints did not include enough useful information about the system, the MAXENT program would not recover the observed distribution based on those constraints. This means, the recovered distribution would not be a good fit to the observed data frequencies. However, this does not mean the model is bad and should be discarded. Instead, the modeler gains some new information from this failure which tells her/him that there is some incomplete information which should be included in the model as a constraint[40]. Additionally, the number of constraints or the existence of misinformative constraints do not affect the performance of the model because maximum entropy program only takes into account the constraints with useful information about the system.

Producing economically meaningful parameters is one of the most powerful feature of the QRSE models which makes them attractive. For example, in a housing market model of QRSE [72], the parameter μ captures how willing the individuals are to sell/buy. It also shows whether their expectations from their actions are completely fulfilled. β provides information about the strength of the feedback mechanism from individuals' actions on price changes, and T releases information about the responsiveness of the individuals to react to price changes. As a result, changes in values of estimated parameters for different time periods carry valuable information to explain what pushes markets towards the crash and what happens during and after the crash.

Finally, QRSE models deal with modeling heterogeneity of interacting agents by introducing an assumption on the behavior of "purposeful" individuals represented

[39] The only assumption used in the QRSE models is the assumption of quantal response behavior of individuals, which has also been tested and supported by experiments in the literature (see [76]).

[40] See [57] for an answer to "Wolf's Dice Example".

as quantal response logit functions in the form of an additional constraint. Therefore, individuals maximize their expected payoffs subject to a lower bound on the entropy of their mixed strategies such that their information processing capacities are subject to some degree of uncertainty. By doing so, QRSE rejects the assumption of rational representative agent, and replaces it with boundedly rational typical agent whose preferences do not have to be well-defined, complete, and transitive.

Overall, QRSE models can capture large fluctuations, such as boom-bust cycles and out-of-equilibrium tendencies of the real economies endogenously, and produce economically meaningful parameter estimation results in a simple but comprehensive way. As a result, they introduce a platform to model social interactions among heterogeneous individuals, uncertainties in individuals' decision making processes, and endogeneity of the system dynamics without imposing any ad-hoc assumptions.

6 Conclusion

This paper first introduces the existing problems in conventional economic theory and models of market. A discussion on LPPL models, HAMs and QRSE models is presented to demonstrate how some more recent approaches handle the complexity of real economic systems – with an emphasis on housing markets. This paper details a clear picture for each model's strengths and weaknesses. It is argued that comparing to conventional models, each model discussed in the paper provides a more realistic framework to represent market behaviors while explaining the causal relationships which give rise to large market fluctuations such as boom and bust cycles mostly followed by crises. They question the unrealistic rational expectations hypothesis and the applicability of efficient market hypothesis to the real markets. In order to do so, they introduce some forms of heterogeneity of individuals and interactions among economic agents to their framework to varying degrees of success.

After introducing a basic framework for each model, QRSE models are discussed in further detail to demonstrate their unique value to modeling complex systems in a clear and concise format. In terms of providing economically meaningful parameters, QRSE models overcome the challenge of "incomplete information" in a simple and less biased framework, and offer a clear explanation for causality since model parameters have clear physical interpretations within the system. Therefore, QRSE models are argued to have a promising potential in future economic modeling.

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