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Review

Open quantum walks

A mini review of the field and recent developments

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Abstract. Open quantum walks (OQWs) are a class of quantum walks, which are purely driven by the interaction with the dissipative environment. In this paper, we review theoretical advances on the foundations of discrete time OQWs, continuous time OQWs and a scaling limit of OQWs called open quantum Brownian motion. The main focus of the review is on the results and developments of discrete time OQW, covering general formalism, quantum trajectories for OQWs, central limit theorems, the microscopic derivation as well as possible generalisations and applications of OQWs.

1 Introduction

Classical random walks (CRWs) [1] are an important tool for understanding various physical phenomena, with wide applications in computer science, biology and economics. The trajectory and probability of finding a CRW walker is fully determined by the transition matrix of the underlying graph. Unitary quantum walks (UQWs), a quantum counterpart of CRWs were introduced almost three decades ago [2,3] and they found numerous applications in quantum information processing and communication science [4]. In this case, the trajectory of the UQW walker depends on both the transition matrix of the underlying graph and the state of the internal degree of freedom of the walker, i.e. spin or polarisation. The probability of finding the UQW walker is the result of quantum interference between different trajectories. This leads to a very different asymptotic distribution of UQWs as compared to the CRW case.

Discrete time open quantum walks (OQWs) were introduced as a quantum Markov chain on an underlying graph [5,6]. In OQWs, the transition between the nodes is driven purely by the dissipative interaction with an environment. As for the case of UQWs, the probability of finding the walker on a particular node is determined by the state of the internal degree of freedom and the transition matrix of the underlying graph. The crucial difference between UQWs and OQWs is that OQWs do not

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rely on the quantum interference between the nodes and admit central limit theorems. Essentially, OQWs start as quantum walks and in a long time limit become CRWs. Utilisation and generalisation of methods of classical Markov chains to OQWs led to recent results in the field, which include various central limit theorems, site recurrence criteria, definition of exit and passage times. Discrete time OQWs have been generalised to continuous time OQWs and the scaling limit of OQWs has been formulated.

The review has the following structure. In Section 2, we review the discrete time OQWs. We start by reviewing the formalism of OQWs and introducing the unravelling of OQWs. We continue by reviewing the results of the application of methods from the theory of classical Markov chains to the quantum trajectories of OQWs. This includes central limit theorems, ergodic properties, passage time, exit time and site recurrence criteria. We review possible generalisations and applications of OQWs. We conclude Section 2 by outlining the microscopic derivation of OQWs. In Section 3, we briefly introduce continuous time OQWs and review recent results. In Section 4, we briefly describe the scaling limit of OQWs – open quantum Brownian motion (OQBM). In Section 5 we conclude.

2 Discrete-time open quantum walks

2.1 Discrete-time open quantum walks: general formalism

Discrete-time OQWs were formulated by Attal et al. as a quantum Markov chain on a graph [5,6]. Physically, OQWs are quantum walks where the transitions between the nodes are driven by the dissipative interaction with an environment. Mathematically, OQWs are defined on the finite or countable set of vertices or nodes $\mathcal{V}$ with oriented edges $\{(i,j) : i,j \in \mathcal{V}\}$. The space of states corresponding to the dynamics on a set of nodes $\mathcal{V}$ will be denoted by $\mathcal{K} = \mathbb{C}^{\mathcal{V}}$. If the number of nodes in $\mathcal{V}$ is countably infinite then $\mathcal{K}$ is any separable Hilbert space with an orthonormal basis $\{|i\rangle\}_{i\in\mathcal{V}}$ indexed by $\mathcal{V}$. The internal degree of freedom of the quantum walker, e.g. the spin, OAM, polarisation or n-energy levels, is described by a vector in a separable Hilbert space $\mathcal{H}$ attached to each node of the graph, such that any state of the walker at any time is described by a density matrix ρ on the direct product of the Hilbert spaces $\mathcal{H} \otimes \mathcal{K}$.

To describe the dynamics of the walker, for each edge (i,j) we introduce a bounded operator $B_i^j \in \mathcal{B}(\mathcal{H})$. This operator describes the change in the internal degree of freedom of the walker due to a "quantum jump" from vertex i to vertex j (see Fig. 1). By restricting that for each node j,

$$\sum_i B_j^{i\dagger} B_j^i = I_j, \tag{1}$$

we make sure that for each node $j \in \mathcal{V}$ there is a corresponding completely positive trace-preserving map on the operators of $\mathcal{B}(\mathcal{H})$:

$$\mathcal{M}_j(\tau) = \sum_i B_j^i \tau B_j^{i\dagger}. \tag{2}$$

The transition operators B_j^i act only on the internal state Hilbert space $\mathcal{H}$ and do not perform transitions of the quantum walker from the node j to node i, therefore they can be straightforwardly dilated to operators $M_j^i \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ acting on the total Hilbert space as $M_j^i = B_j^i \otimes |i\rangle\langle j|$. It is obvious that, if the transition operators B_j^i

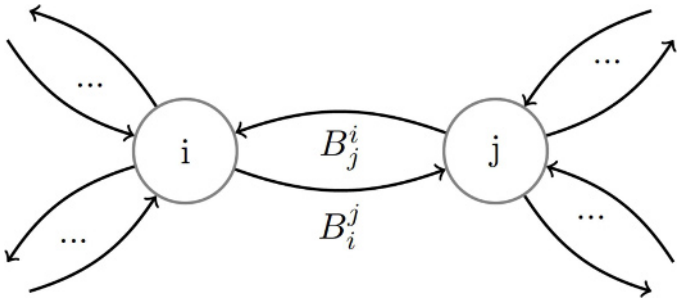

Fig. 1. A schematic representation of a discrete time OQW on a graph with nodes i and j. The operators B^j_i and B^i_j represent the transition operators of the walk.

satisfy equation (1), then $\sum_{i,j} M^{i\dagger}_j M^i_j = I$. This normalisation condition defines a completely positive trace preserving map for density matrices on $\mathcal{H} \otimes \mathcal{K}$, i.e.,

$$\mathcal{M}(\rho) = \sum_{i,j} M^i_j \rho M^{i\dagger}_j. \tag{3}$$

This map defines the discrete time OQW [5–7]. It is easy to see that, for an arbitrary initial state the density matrix $\sum_{i,j} \rho_{i,j} \otimes |i\rangle\langle j|$ will take a diagonal form after one step of OQW, namely,

$$\begin{aligned}
\mathcal{M}\left(\sum_{p,q} \rho_{p,q} \otimes |p\rangle\langle q|\right) &= \sum_{i,j,p,q} B^i_j \otimes |i\rangle\langle j| \left(\rho_{p,q} \otimes |p\rangle\langle q|\right) B^{i\dagger}_j \otimes |j\rangle\langle i| \\
&= \sum_{i,j,p,q} B^i_j \rho_{p,q} B^{i\dagger}_j \otimes |i\rangle\langle i| \delta_{j,p}\delta_{j,q} \\
&= \sum_i \left(\sum_j B^i_j \rho_{j,j} B^{i\dagger}_j\right) \otimes |i\rangle\langle i|.
\end{aligned} \tag{4}$$

Therefore, we will assume that the initial state of the system is given by the density matrix diagonal in the node space, i.e., $\rho = \sum_i \rho_i \otimes |i\rangle\langle i|$. One can see that the iteration formula for OQWs from the step $[n]$ to the step $[n+1]$ is as follows: $\rho^{[n+1]} = \sum_i \rho_i^{[n+1]} \otimes |i\rangle\langle i|$, where $\rho_i^{[n+1]} = \sum_j B^i_j \rho_j^{[n]} B^{i\dagger}_j$. This iteration formula provides a clear physical meaning of the map $\mathcal{M}$: the state of the walker at the node i is determined by the conditional "quantum jumps" from all connected nodes j and the state of the internal degree of freedom of the walker on that node described by the operator ρ_j.

2.2 Example: open quantum walk on $\mathbb{Z}$

As an illustration of the OQW, we consider OQWs on the line where the walker is allowed to jump only between adjacent sites (Fig. 2a). In this case, the generic OQW iteration formula reduces to $\rho^{[n+1]} = \sum_i \rho_i^{[n+1]} \otimes |i\rangle\langle i|$, where $\rho_i^{[n+1]} = B^i_{i+1}\rho^{[n]}_{i+1}B^{i\dagger}_{i+1} + B^i_{i-1}\rho^{[n]}_{i-1}B^{i\dagger}_{i-1}$. For simplicity we consider a homogenous OQW (which means that $\forall i$, $B^{i+1}_i \equiv B$ and $B^{i-1}_i \equiv C$). For a walker initially localized

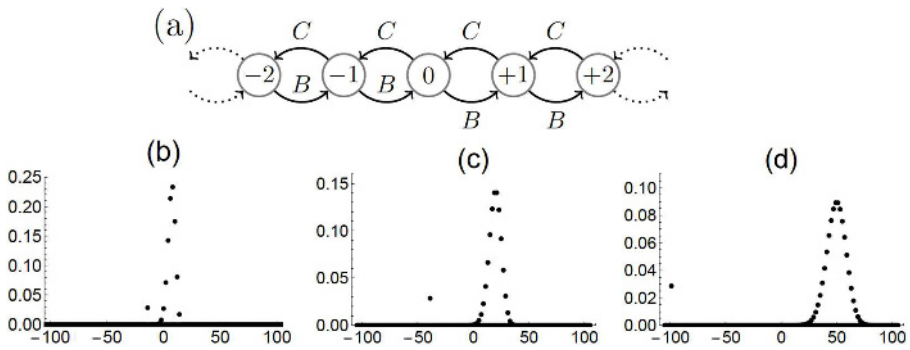

Fig. 2. OQWs on $\mathbb{Z}$. (a) A schematic illustration of the homogeneous OQWs on $\mathbb{Z}$: all transitions to the right are induced by the operator B, while all transitions to the left are induced by the operator C; (b–d) the probability distribution to find the walker on a particular site for initially localised walker $\rho^{[0]} = |-\rangle\langle -| \otimes |0\rangle\langle 0|$ and transition operators given by equation (5) after 15, 40, 100 steps, respectively.

at the vertex 0: $\rho^{[0]} = \rho_0^{[0]} \otimes |0\rangle\langle 0|$, the total density matrix after one and two steps is given by,

$$\rho^{[1]} = B\rho_0^{[0]}B^\dagger \otimes |1\rangle\langle 1| + C\rho_0^{[0]}C^\dagger \otimes |-1\rangle\langle -1|,$$

and

$$\rho^{[2]} = B\rho_1^{[1]}B^\dagger \otimes |2\rangle\langle 2| + \left(C\rho_1^{[1]}C^\dagger + B\rho_{-1}^{[1]}B^\dagger\right) \otimes |0\rangle\langle 0| + C\rho_{-1}^{[1]}C^\dagger \otimes |-2\rangle\langle -2|.$$

To illustrate the dynamics of the probability distribution of finding the walker on a particular node, we chose B and C as

$$B = \frac{1}{6}\begin{pmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 5 \end{pmatrix}, \quad C = \frac{1}{2\sqrt{3}}\begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}, \tag{5}$$

so that the normalisation condition equation (1) is satisfied, i.e. $B^\dagger B + C^\dagger C = I$. If we pick the initial state of the walker to be localised at the site 0 with the internal degree of freedom in the "minus" state ($|-\rangle = (|1\rangle - |0\rangle)/\sqrt{2}$), i.e. $\rho^{[0]} = |-\rangle\langle -| \otimes |0\rangle\langle 0|$, then by iteration we can find the state of the walker after an arbitrary number of steps. In Figures 2b–2d, we show the probability of finding a walker on a particular node for a various number of steps. After 40 steps (Fig. 2c), one can clearly see the formation of a "soliton"-like distribution and a Gaussian moving in different directions. For the greater number of steps (100 on Fig. 2d), this behaviour is even more obvious.

2.3 Unravelling OQWs

An interesting property of OQWs is that they can be simulated by means of quantum trajectories [5,7]. To demonstrate the formalism we choose the initial state of the walker to be localised on the node i with an arbitrary internal state, namely $\rho^{[0]} = \rho \otimes |i\rangle\langle i|$. After one step of the OQW, the state of the walker is $\rho^{[1]} = \sum_j \left(B_i^j \rho B_i^{j\dagger}\right) \otimes |j\rangle\langle j|$. The probability of finding the walker at the node j is as follows $p_j = \mathrm{Tr}\left(B_i^j \rho B_i^{j\dagger}\right)$. If we measure the position of the walker at node j

the reduced state walker is given by $\frac{1}{p_j}\left(B_i^j \rho B_i^{j\dagger}\right) \otimes |j\rangle\langle j|$. Repetition of this procedure gives rise to a classical Markov chain, valued in the set of states of the form $\rho \otimes |i\rangle\langle i|$.

Let us calculate the average over the trajectories generated by the procedure described above and show that it will simulate an OQW map $\mathcal{M}$ defined in equation (3). We assume that in the step $[n]$ the walker is localised at the node i and the state of the walker is given by $\rho^{[n]} = \rho_n \otimes |i\rangle\langle i|$. If we monitor the position of the walker, after one step the walker will jump randomly to the node j with probability $p_j = \mathrm{Tr}\left(B_i^j \rho_n B_i^{j\dagger}\right)$ and the state of the walker will be given by $\rho^{[n+1]} = \rho_{n+1}(j) \otimes |j\rangle\langle j| = \frac{1}{p_j}\left(B_i^j \rho_n B_i^{j\dagger}\right) \otimes |j\rangle\langle j|$. The ensemble average over all possible trajectories will simulate an OQW map $\mathcal{M}$:

$$\begin{aligned}\mathcal{E}[\rho^{[n+1]}] &= \sum_j p_j \rho_{n+1}(j) \otimes |j\rangle\langle j| \\ &= \sum_j p_j \frac{1}{p_j}\left(B_i^j \rho_n B_i^{j\dagger}\right) \otimes |j\rangle\langle j| = \sum_j \left(B_i^j \rho_n B_i^{j\dagger}\right) \otimes |j\rangle\langle j| = \mathcal{M}[\rho^{[n]}].\end{aligned}$$

A quantum trajectory of the OQW after n steps can also be denoted as $(\rho_n, X_n)_{n\geq 0}$, where ρ_n is the density matrix of the internal degree of freedom and X_n is a random variable tracing the position of the walker.

The initial pure state $\rho^{[0]} = |\phi\rangle\langle\phi| \otimes |i\rangle\langle i|$ will remain in the pure state for the whole realisation of the OQW. It is clear that an arbitrary initial pure state $|\phi\rangle \otimes |i\rangle$ will randomly jump to a state $\frac{1}{\sqrt{p_i^j}} B_i^j |\phi\rangle \otimes |j\rangle$ with probability $p_i^j = ||B_i^j|\phi\rangle||^2$. This procedure leads to a classical Markov chain valued in the space of wavefunctions of the form $|\phi\rangle \otimes |i\rangle$. On average, this random walk simulates an OQW master equation driven by $\mathcal{M}$. Examples of unravelling of OQWs can be found in [5,7].

2.4 Connection to classical random walks and unitary quantum walks

It is interesting to mention that the OQWs contain, as a special case, CRWs [5,7]. To illustrate this, let us consider the case $\mathcal{H} = \mathcal{K} = \mathbb{C}^{\mathcal{V}}$ and define a stochastic matrix $P = \{P_{i,j}\}$ of classical transition probabilities on the graph $\mathcal{V}$ with standard normalisation condition $\sum_i P_{j,i} = 1$ and add an arbitrary set of unitary operators $U_i^j \in \mathcal{B}\left(\mathbb{C}^{\mathcal{V}}\right)$. To recover CRWs, we need to consider the transition operators B_i^j to be $B_i^j = \sqrt{P_{i,j}} U_i^j$. For an arbitrary initial state $\rho^{[0]} = \sum_k \rho_k \otimes |k\rangle\langle k|$, the probability of finding a walker after one step on the node i is as follows,

$$p^{[1]}(i) = \mathrm{Tr}\left(\mathcal{M}(\rho^{[0]})|i\rangle\langle i|\right) = \sum_k P_{k,i} \mathrm{Tr}\left(\rho_k\right), \tag{6}$$

after two steps, the probability of finding a walker at the node i reads,

$$p^{[2]}(i) = \mathrm{Tr}\left(\mathcal{M}(\mathcal{M}(\rho^{[0]}))|i\rangle\langle i|\right) = \sum_{k,m} P_{m,k} P_{k,i} \mathrm{Tr}\left(\rho_m\right). \tag{7}$$

One can see that the probability of finding a walker on the site i does not depend on the internal degree of freedom and is fully determined by the stochastic matrix P, as expected for a CRW.

UQWs can be recovered for the set of transition operators B_i^j satisfying an extended normalisation condition $\sum_i B_j^{i\dagger} B_{j'}^i = \delta_{j,j'} I_j$ and using the "realisation" procedure. The detailed description of the link between Hadamard quantum walks and OQWs can be found in [5].

2.5 Central limit theorem, reducibility and ergodic properties of OQWs

Several numerical experiments with various QOWs [5–7] seem to indicate that the asymptotic distribution of positions of the OQW walker converges to a Gaussian distribution or sum of Gaussian distributions. For the case of nearest neighbour homogeneous OQWs on $\mathbb{Z}^d$ with a unique invariant state Attal et al. proved the central limit theorem (CLT) [8].

Here, we demonstrate the application of the CLT to an OQW on $\mathbb{Z}$. Consider a homogeneous OQW on a line with a generic setup as described in Section 2.2. The state of the walker for the arbitrary step $n+1$ is given by the iteration formula

$$\rho^{[n+1]} = \sum_i \rho_i^{[n+1]} \otimes |i\rangle\langle i|, \text{where} \quad \rho_i^{[n+1]} = C\rho_{i+1}^{[n]} C^\dagger + B\rho_{i-1}^{[n]} B^\dagger. \tag{8}$$

To satisfy the CLT, OQW should have a unique invariant state which means that the density matrix acting on the Hilbert space of the internal degree of freedom $\rho_\infty \in \mathcal{B}(\mathcal{H})$ admits the unique solution for the equation $\rho_\infty = C\rho_\infty C^\dagger + B\rho_\infty B^\dagger$.

For a quantum trajectory $(\rho_n, X_n)_{n\geq 0}$ corresponding to the OQW equation (8), where ρ_n denotes the state of the internal degree of freedom and X_n the position of the walker after n steps, the CLT guarantees that $(X_n - nm)/\sqrt{n}$ converges to a Gaussian distribution $\mathcal{N}(0, \sigma^2)$, where parameter m is as follows $m = \lim_{n\to\infty} \frac{X_n}{n}$. The CLT also allows the calculation of the parameters of the asymptotic distribution. For an OQW given by equation (8) the parameter m is given by $m = \mathrm{Tr}(B\rho_\infty B^\dagger) - \mathrm{Tr}(C\rho_\infty C^\dagger)$ and the variance σ^2 reads $\sigma^2 = 1 - m^2 + 4\left(\mathrm{Tr}(\rho_\infty B^\dagger L B) - \mathrm{Tr}(B\rho_\infty B^\dagger)\mathrm{Tr}(\rho_\infty L)\right)$, where operator L satisfies the following equation $L - B^\dagger L B - C^\dagger L C = 2B^\dagger B - (1+m)I$.

For example, an OQW with transition operators

$$B = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \tag{9}$$

admits a unique invariant state $\rho_\infty = \frac{1}{2} I$, which leads to a parameter $m = 0$ and variance $\sigma^2 = \frac{8}{9}$ [8].

Using the CLT [8] Konno and Yoo studied limit distributions for various OQWs [9]. They introduced a dual process for the OQWs in the Fourier space, which allows one to find the formal expression for the probability distribution of the OQWs. Konno and Yoo apply the developed formalism to a range of OQWs and explicitly calculate the mean and variance of the corresponding asymptotic distributions [9].

Sadowski and Pawela consider a generalisation of the CLT for the case of non-homogenous OQWs [10]. They consider OQWs on $\mathbb{Z}^d$ with a finite number of classes of nodes. Different classes of nodes are characterised by a different set of $2d$ transition operators. First, they assign the classes in a regular pattern. Second, they assign to each node a random class with transition invariant distribution. In both cases the CLT is proven.

Sinayskiy and Petruccione studied the properties of OQWs on $\mathbb{Z}$ for the case of simultaneously diagonalisable transition operators B and C [11]. They derive a

general form of the probability distribution to find the walker on a particular site. They found that the asymptotic distribution consists of maximally two "soliton"-like distributions and a certain number of Gaussian distributions. They establish a connection between the explicit form of the transition operators B and C and the number of Gaussian distributions as well as their mean and variance.

Using the fact that OQW quantum trajectories can be seen as classical Markov chains, Carbone and Pautrat introduce notions of irreducibility, period, communicating classes for OQWs [12] and apply these generic definitions to a homogeneous OQW on $\mathbb{Z}^d$ [13]. In [12], Carbone and Pautrat apply the notion of the irreducibility and aperiodicity of positive maps to the case of OQWs. This leads to a proof of the ergodic behaviour of irreducible OQWs. They continue the analysis for the case of reducible OQWs and derive a general form of stationary states for OQWs. In the follow-up study, they apply these generic results to the case of homogeneous OQWs on $\mathbb{Z}^d$ [13]. They prove the CLT and formulate the large deviation principle for a quantum trajectories of OQWs. They fully characterise OQWs on $\mathbb{Z}^d$ with a two-dimensional internal degree of freedom.

2.6 Passage times, exit times, site recurrence, hitting times for OQWs

The strong analogy between OQWs and Markov chains is the main driver of the results, which are summarised in this section. Bardet et al. studied the probability of visiting a given node in a finite time, expected number of visits, expected return time, and their relation with the Dirichlet problem, as well as exit probabilities and exit times for any finite subset of the graph [14].

Lardizabal and Souza considered the special case of OQWs, where transition operators B_j^i are given by the Kraus operators of PQ-channels [15]. For this scenario, they studied return probability, recurrence and positive recurrence. In the follow-up study, they utilised the analogy between OQWs and quantum Markov chains [16,17] and prove an ergodicity criterion in terms of singular values. By using quantum trajectories Lardizabal and Souza defined a notion of hitting time for OQWs (first time visit) and calculated it explicitly for certain cases [18]. Furthermore, Lardizabal continued the study of the relations between the mean hitting time formula from classical probability theory and quantum hitting time for OQWs [19]. He derived a relation between mean hitting time for OQWs and the fundamental matrix of an ergodic OQWs. Carvalho et al. investigated similarities and differences in site recovery for UQWs and OQWs [20]. They proved an equivalence between monitored-recurrence and SJK-recurrence for OQWs. By constructing orthogonal matrix polynomials Jacq and Lardizabal studied OQWs on $\mathbb{Z}^+$ and described transition probability for certain classes of OQWs via the matrix Karlin–McGregor formula [21]. They discussed an open quantum version of Foster's theorem for the expected returns.

Dhahri and Mukhamedov [22] studied recurrence properties of OQWs from the perspective of quantum Markov chains [23]. Grünbaum and Velázquez introduced an FR-function as a generalisation of Schur functions [24]. They demonstrated the application of FR-functions to the study of recurrence for UQW and OQW.

2.7 Generalisations and applications of OQWs

Xiong and Yang introduced a generalisation of OQWs called the partially open quantum random walks (POQRW) [25]. POQRWs are parametrised by a parameter p ($0 \leq p \leq 1$), such that for $p = 0$ POQRWs become UQWs, while for $p = 1$ POQWs recover the OQW formalism. With the help of the quantum Fourier transform and

basis of the generalised Gell-Mann matrices they proved a limit theorem for POQRWs on $\mathbb{Z}$.

Pawela et al. studied a generalised OQW on Apollonian Networks [26]. Their formalism assumes for every edge (i, j) of the graph one can introduce a completely positive trace non-increasing map $\mathcal{E}_{i,j}$. Maps $\mathcal{E}_{i,j}$ are defined such that the sum of all maps leaving one node j of the graph is completely positive trace-preserving map. Conventional OQWs are recovered in the case when maps $\mathcal{E}_{i,j}$ are rank one.

Wang et al. studied OQWs from the perspective of quantum Bernoulli noise [27]. They have shown that for an initial localised state the limit probability distribution coincides with the probability distribution for a corresponding CRW. They also established a connection of quantum Bernoulli based open walks with a UQW. In a sequence of papers, Ampadu studied the time-dependent generalisation of OQWs [28–30]. He considered examples of OQWs on $\mathbb{Z}$ with transition operators $B(n)$ and $C(n)$ changing for every step. In the investigated cases, the limit distribution and return probability were found. Liu suggested a scheme for a dilation of OQWs on a lattice and a finite graph into a UQW [31]. Using this approach, he studied the mean probability of finding a walker at a node. In a recent study, Lardizabal used a matrix representation of completely positive maps to study OQWs and associated quantum trajectories of the position of the walker [32]. In the case when the CP-map acting of the internal degree of freedom of the walker is given by the primitive quantum channel he derived expressions for the mean hitting time and expected return time.

It is well known that the dissipative effects can be used to create complex entangled states [34–38] and to perform universal quantum computation [39]. Sinayskiy and Petruccione have demonstrated that OQWs can be used to create complex quantum states [6] and to implement a dissipative quantum computing model [33]. It is remarkable that the proposed OQW implementation of the dissipative quantum computing model outperforms the traditional implementation of this model of quantum computation.

2.8 Microscopic derivation of OQWs

Originally, OQW have been formulated as a particular type of a completely positive trace-preserving map [5,6]. According to Stinespring's theorem every CPTP map can be dilated in an extended space into a unitary operation, which in principle can be implemented experimentally. However, this theorem only guarantees the existence of such a physical system but does not give a recipe how this system can be constructed. Sinayskiy and Petruccione suggested two possible ways to implement OQWs: first by using an effective operator formalism they suggested a quantum optics implementation of OQWs [40], second they followed the traditional theory of open quantum systems approach and derived OQWs based on the microscopic system-environment model [41,42].

In the quantum optical implementation of OQWs, Sinayskiy and Petruccione used an example of the two-level system in the cavity in the dispersive regime [40]. This setup realised an OQW on the line with a two level system as the internal degree of freedom and the Fock states of the cavity mode as a 1D lattice.

In the remaining part of this section, we will outline the basic idea of the microscopic derivation of OQWs [42]. OQWs are quantum walks where the transition between the nodes is fully driven by the interaction with the dissipative environment. This implies that one could apply methods of the theory of open quantum systems to derive OQWs [43]. From the microscopic point of view the Hamiltonian of the total system is given by

$$H = H_S + H_B + H_{SB}, \tag{10}$$

where H_S, H_B and H_{SB} denote the Hamiltonian of the system, bath and system-bath interaction, respectively. OQW is defined as a completely positive trace-preserving map equation (3) on both internal and external degrees of freedom of the quantum walker. This means that the Hamiltonian of the system H_S should describe the local free evolution of the internal degree of freedom of the quantum walker and underlining graph, namely,

$$H_S = \sum_i \Omega_i \otimes |i\rangle\langle i|. \tag{11}$$

Each vertex of the graph corresponds to a possible position of the walker. The basis of the corresponding Hilbert space ($\mathcal{K}$) is given by the set of orthogonal vectors $\{|i\rangle\}$. The state of the inner degree of freedom of the walker is described by the Hamiltonians in the N-dimensional Hilbert space, i.e. $\Omega_i \in \mathcal{B}(\mathcal{H})$.

OQWs are formulated such that the transitions between different nodes are uncorrelated. To facilitate transition between sites i and j one needs to have at least one local environment between these nodes. This leads to the following form of the Hamiltonian of the bath H_B,

$$H_B = \sum_{i \neq j} \sum_n \omega_{i,j,n} a^\dagger_{i,j,n} a_{i,j,n}, \tag{12}$$

where $a^\dagger_{i,j,n}$ and $a_{i,j,n}$ denote bosonic creation and annihilation operators describing the nth mode of the local environment between nodes i and j.

The system-bath Hamiltonian H_{SB} describes bath assisted transitions of the quantum walker between the sites. The simplest form of such a Hamiltonian includes operators acting on the internal and external degrees of freedom of the walker and degrees of freedom of the local environment in the linearly coupled way. Without loss of generality one could assume that the Hamiltonian of the system-bath interaction is given by,

$$H_{SB} = \sum_{i \neq j} \sum_n A_{i,j} \otimes X_{i,j} \otimes B_{i,j}, \tag{13}$$

where $A_{i,j} \in \mathcal{B}(\mathcal{H})$ denotes operators acting on the internal degree of freedom of the walker. The operator $X_{i,j} \in \mathcal{B}(\mathcal{K})$ describes the transition between the nodes i and j. The simplest Hermitian choice of these operators is given by $X_{i,j} = |i\rangle\langle j| + |j\rangle\langle i|$. The coupling of the quantum walker to a local environment is described by the operator $B_{i,j} = \sum_n g_{i,j,n} a_{i,j,n} + g^*_{i,j,n} a^\dagger_{i,j,n}$, where $g_{i,j,n}$ denotes the coupling constants of the system-bath interaction. The set of these coupling constants $g_{i,j,n}$ satisfies the following condition $\sum_n |g_{i,j,n}|^2 < \infty$ and in the continuum limit this sum converges to a local spectral density $J_{i,j}(\omega)$.

Having specified the Hamiltonian of the total system, one can proceed and derive the reduced master equation for the quantum walker and the lattice. It is assumed that the system is weakly coupled to the local environments, such that the Born–Markov approximation is valid [43]. Under these assumptions, the reduced dynamics of the system in the interaction picture is given by the following equation:

$$\frac{d}{dt}\rho_s(t) = -\int_0^\infty d\tau \mathrm{Tr}_B \left[H_{SB}(t), [H_{SB}(t-\tau), \rho_s(t) \otimes \rho_B]\right], \tag{14}$$

where $\rho_s(t)$ is the reduced density matrix of the system (quantum walker and underlying graph) and ρ_B is the state of the bath. To guarantee that the master equation (14) describes a completely positive trace-preserving evolution one needs to perform an additional rotating-wave approximation [43]. This rotating wave approximation can be easily implemented if one decomposes the system-bath Hamiltonian H_{SB} in the basis of eigenoperators of the system Hamiltonian H_S. For each site $|i\rangle$ one can introduce the set of orthonormal Hermitian projectors $\{\Pi_i(\lambda^{(i)})\}$ onto the eigenvalues $\lambda^{(i)}$ of each Hamiltonian Ω_i, such that

$$\Omega_i = \sum_{\lambda^{(i)}} \lambda^{(i)} \Pi_i(\lambda^{(i)}). \tag{15}$$

In this notation, the system-bath Hamiltonian H_{SB} in the interaction picture reads,

$$\begin{aligned} H_{SB}(t) = &\sum_{i,j}\sum_{\omega} e^{it\omega} A^{\dagger}_{i,j}(\omega) \otimes |i\rangle\langle j| \otimes B_{i,j}(t) + \text{h.c.} \\ &+ \sum_{i,j}\sum_{\omega'} e^{-it\omega'} A_{i,j}(\omega') \otimes |i\rangle\langle j| \otimes B_{i,j}(t) + \text{h.c.}, \end{aligned} \tag{16}$$

where the operator $B_{i,j}(t)$ is given by

$$B_{i,j}(t) = \sum_n g_{i,j,n} a_{i,j,n} e^{-it\omega_{i,j,n}} + \text{h.c.} \tag{17}$$

and the operators $A^{\dagger}_{i,j}(\omega)$ and $A_{i,j}(\omega')$ are defined as,

$$\begin{aligned} A^{\dagger}_{i,j}(\omega) &= \sum_{\lambda^{(i)}-\lambda^{(j)}=\omega>0} \Pi_i(\lambda^{(i)}) A_{i,j} \Pi_j(\lambda^{(j)}), \\ A^{\dagger}_{i,j}(\omega) &= A_{i,j}(-\omega). \end{aligned} \tag{18}$$

Using the explicit form of the system-bath Hamiltonian equation (16) one can substitute it into the master equation for the reduced density matrix equation (14) and trace out the environment degrees of freedom. Here, one assumes that the environment is in a thermal equilibrium state at the inverse temperature $\beta = (k_B T)^{-1}$, namely $\rho_B = \exp(-\beta H_B)/\text{Tr}[\exp(-\beta H_B)]$. After a straightforward transformation and the application of the rotating-wave approximation for the transition frequencies ω and ω' [43,44] the master equation (14) can be written as,

$$\begin{aligned} \frac{d}{dt}\rho_s(t) = &\sum_{i,j}\sum_{\omega} \gamma_{i,j}(-\omega) \mathcal{L}\left(A_{i,j}(\omega) \otimes |j\rangle\langle i|\right) \rho_s(t) + \gamma_{i,j}(\omega) \mathcal{L}\left(A^{\dagger}_{i,j}(\omega) \otimes |i\rangle\langle j|\right) \rho_s(t) \\ &+ \sum_{i,j}\sum_{\omega'} \gamma_{i,j}(-\omega') \mathcal{L}\left(A_{i,j}(\omega') \otimes |i\rangle\langle j|\right) \rho_s(t) \\ &+ \gamma_{i,j}(\omega') \mathcal{L}\left(A^{\dagger}_{i,j}(\omega') \otimes |j\rangle\langle i|\right) \rho_s(t), \end{aligned} \tag{19}$$

where $\mathcal{L}(A)\rho$ denotes the semigroup generator in the Gorini–Kossakowski–Sudarshan–Lindblad form (GKSL) [43–46]:

$$\mathcal{L}(A)\rho = A\rho A^{\dagger} - \frac{1}{2}\left\{A^{\dagger}A, \rho\right\} \tag{20}$$

and $\gamma_{i,j}(\omega)$ is the real part of the Fourier transform of the bath correlation functions $\langle B^{\dagger}_{i,j}(\tau)B_{i,j}(0)\rangle$,

$$\gamma_{i,j}(\pm\omega) = \frac{\gamma^{\rm se}_{i,j}}{2}\left(\coth\left(\frac{\beta\omega}{2}\right)\mp 1\right), \tag{21}$$

where $\gamma^{\rm se}_{i,j}$ is the coefficient of the spontaneous emission in the corresponding local environment. In the equation (19), the Lamb-type shift terms are neglected. These terms describe shifts in energy levels of the system due to the dissipative interaction with the thermal bath. Typically, the value of Lamb shifts is smaller than other characteristic parameters of the system Hamiltonian and traditionally they are dropped [43]. If one writes the reduced density matrix $\rho_s(t)$ from equation (19) as $\rho_s(t) = \sum_i \rho_i(t)\otimes|i\rangle\langle i|$, where $|i\rangle\langle i|$ is a projection on the site i, then the quantum master equation (19) reduces to the system of differential equations:

$$\frac{d}{dt}\rho_i(t) = \mathcal{K}_i\left(\{\rho_j\}_j\right), \tag{22}$$

where $\mathcal{K}_i$ are given by

$$\begin{aligned}
\mathcal{K}_i\left(\{\rho_j\}_j\right) = &\sum_{j,\omega}\gamma_{j,i}(-\omega)A_{j,i}\left(\omega\right)\rho_j A^{\dagger}_{j,i}\left(\omega\right) - \frac{\gamma_{i,j}(-\omega)}{2}\{A^{\dagger}_{i,j}\left(\omega\right)A_{i,j}\left(\omega\right),\rho_i\}\\
&+\sum_{j,\omega}\gamma_{i,j}(\omega)A^{\dagger}_{i,j}\left(\omega\right)\rho_j A_{i,j}\left(\omega\right) - \frac{\gamma_{j,i}(\omega)}{2}\{A_{j,i}\left(\omega\right)A^{\dagger}_{j,i}\left(\omega\right),\rho_i\}\\
&+\sum_{j,\omega'}\gamma_{i,j}(-\omega')A_{i,j}\left(\omega'\right)\rho_j A^{\dagger}_{i,j}\left(\omega'\right) - \frac{\gamma_{j,i}(-\omega')}{2}\{A^{\dagger}_{j,i}\left(\omega'\right)A_{j,i}\left(\omega'\right),\rho_i\}\\
&+\sum_{j,\omega'}\gamma_{j,i}(\omega')A^{\dagger}_{j,i}\left(\omega'\right)\rho_j A_{j,i}\left(\omega'\right) - \frac{\gamma_{i,j}(\omega')}{2}\{A_{i,j}\left(\omega'\right)A_{i,j}\left(\omega'\right)^{\dagger},\rho_i\}.
\end{aligned} \tag{23}$$

This system of differential equations (22,23) defines the continuous time OQWs. Continuous time OQWs were introduced by Pellegrini as the continuous time limit of OQWs [47].

To obtain a discrete-time OQW one needs to replace the time derivative by the finite difference with a small time step Δ in equations (22,23) as $d\rho_i/dt \to \left(\rho_i(t+\Delta)-\rho_i(t)\right)/\Delta$. This substitution leads to the following transition operators,

$$\begin{aligned}
B^{i(1)}_j(\omega) &= \sqrt{\Delta\gamma_{j,i}(-\omega)}A_{j,i}(\omega), \quad B^{i(2)}_j(\omega) = \sqrt{\Delta\gamma_{i,j}(\omega)}A^{\dagger}_{i,j}(\omega),\\
B^{i(1)}_j(\omega') &= \sqrt{\Delta\gamma_{i,j}(-\omega')}A_{i,j}(\omega'), \quad B^{i(2)}_j(\omega') = \sqrt{\Delta\gamma_{j,i}(\omega')}A^{\dagger}_{j,i}(\omega'),\\
B^{i}_i &= I_N - \frac{\Delta}{2}\sum_{j,\omega}\left(\gamma_{i,j}(-\omega)A^{\dagger}_{i,j}(\omega)A_{i,j}(\omega) + \gamma_{j,i}(\omega)A_{j,i}(\omega)A^{\dagger}_{j,i}(\omega)\right)\\
&\quad - \frac{\Delta}{2}\sum_{j,\omega'}\left(\gamma_{j,i}(-\omega')A^{\dagger}_{j,i}(\omega')A_{j,i}(\omega') + \gamma_{i,j}(\omega')A_{i,j}(\omega')A_{i,j}(\omega')\right).
\end{aligned} \tag{24}$$

It is clear that this set of transition operators satisfy normalisation conditions equation (1) up to $\mathcal{O}(\Delta^2)$. The iteration formula for the discrete time OQW reads,

$$\rho_i^{[n+1]} = B_i^i \rho_i^{[n]} B_i^{i\dagger} + \sum_{k=1}^{2} \sum_{j,\omega} B_j^{i(k)}(\omega) \rho_j^{[n]} B_j^{i(k)\dagger}(\omega) + \sum_{k=1}^{2} \sum_{j,\omega'} B_j^{i(k)}(\omega') \rho_j^{[n]} B_j^{i(k)\dagger}(\omega'). \tag{25}$$

The transition operators, equation (24), show a connection between the dynamical properties of the OQW and the thermodynamical parameters of the environment.

3 Continuous time open quantum walks

Continuous time OQWs have been introduced by Pellegrini as a continuous time limit of discrete-time OQWs [47]. Pellegrini utilises a correlated projection operator approach [48] and repeated quantum interaction theory [49] to derive continuous-time OQWs as the following system of differential equations:

$$\frac{d}{dt}\rho_j = -i[H_j, \rho_j] + \sum_{\alpha,k} \left(R_k^{\alpha,j} \rho_k \left(R_k^{\alpha,j} \right)^\dagger - \frac{1}{2} \left\{ \left(R_j^{\alpha,k} \right)^\dagger R_j^{\alpha,k}, \rho_j \right\} \right). \tag{26}$$

Here, the positive operators $\rho_j(t) \in \mathcal{B}(\mathcal{H})$ describe the state of the inner degree of freedom of the quantum walker on the site j. These operators satisfy the normalisation condition for the probability to find the walker on the graph $\sum_j \mathrm{Tr}[\rho_j] = 1$. The operators $H_j \in \mathcal{B}(\mathcal{H})$ denote Hamiltonians acting on the internal degree of freedom of the walker on the site j. The set of bounded operators $R_k^{\alpha,j}$ describe transformations of the inner degrees of freedom of the walker during the transition from the node k to the node j, while the index α indicates the number of different ways to perform this transition. Within the microscopic derivation of the discrete-time OQWs Sinayskiy and Petruccione derived continuous-time OQWs and established a direct relation between the transition operators $R_k^{\alpha,j}$ and the thermodynamical parameters of the underlying physical system [42].

Bringuier proved the CLT and formulated the large deviation principle for continuous time OQWs [50]. Similarly, to Attal et al. [8] he considered homogeneous continuous-time OQWs on $\mathbb{Z}^d$ and the case in which continuous-time OQWs equation (26) admit a unique stationary state. Bringuier investigated the dynamics of the quantum trajectories of the position process $(\rho_t, X_t)_{t\geq 0}$. He proved that shifted Markov processes $(X_t - mt)/\sqrt{t}$ will converge to a Gaussian distribution $\mathcal{N}(0, V)$. He also derived an explicit expression for the mean m and variance V of this distribution. Using the Gärtner-Ellis theorem and following [13] Bringuier formulated the large deviation principle for continuous time OQWs.

Liu and Balu studied steady states of continuous time OQWs on finite graphs [51]. They have demonstrated that for all initial states a continuous time OQW always converges to a steady state when the graph is connected. Moreover, if the graph is connected and regular, Liu and Balu demonstrated that the steady state is given by a maximally mixed state. Using explicit examples of continuous time OQWs on a regular (cycle) and irregular (star) graphs they observed very different dynamical behaviour for the quantum walker.

Bardet et al. studied recurrence and transience of continuous time OQWs [52]. They started by defining an irreducibility of the quantum Markov semigroup and relating this semigroup to a corresponding continuous time OQW. The main result which they obtain is the classification of the transience and recurrence of irreducible

continuous time OQWs. As in the case with discrete time OQWs, due to presence of the internal degree of freedom the continuous time OQWs exhibit non-classical behaviour, and the classification of recurrence and transience properties obeys a "trichotomy" rather than a classical dichotomy.

4 Open quantum Brownian motion

Bauer et al. considered a scaling limit of OQWs and obtained a new type of quantum stochastic process called OQBM [53,54]. They started by considering an OQW on $\mathbb{Z}$ and rewrote the iteration rule for a small change in time and space

$$\rho(t+dt,x) = B(x+dx)\rho(t,x)B^{\dagger}(x+dx) + C(x-dx)\rho(t,x)C^{\dagger}(x-dx). \tag{27}$$

They expanded the above expression in dt and dx and chose the scaling relation $\epsilon = dt = dx^2$ as in the classical diffusion equation. This leads to the following differential equation in partial derivatives:

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho(x,t) - \left(N\frac{\partial}{\partial x}\rho(x,t) + \frac{\partial}{\partial x}\rho(x,t)N^{\dagger}\right) - i[H,\rho(x,t)] + N\rho(x,t)N^{\dagger} - \frac{1}{2}\left(N^{\dagger}N\rho(x,t) + \rho(x,t)N^{\dagger}N\right). \tag{28}$$

In equation (28), the positive operator $\rho(x,t)$ describes the state of the internal degree of freedom of the OQBM walker at position x and time t. Naturally, operators $\rho(x,t)$ satisfy the following normalisation condition: $\int dx \mathrm{Tr}[\rho(x,t)] = 1$ which simply means that with probability one the OQBM walker is somewhere on the line. The first term of the OQBM master equation (28) is just a classical diffusion term, the last two terms of equation (28): $-i[H,\rho(x,t)] + N\rho(x,t)N^{\dagger} - \left(N^{\dagger}N\rho(x,t) + \rho(x,t)N^{\dagger}N\right)/2$ describe the unitary and dissipative dynamics of the inner degree of freedom of the walker, where H denotes the Hamiltonian and N denotes a quantum jump operator acting on the internal degree of freedom of the walker. The term $(N\partial_x + \partial_x N^{\dagger})$ is a "decision making" term which describes the influence of the internal state of the walker on its position. The presence of this term makes the quantum Brownian motion "open" and this term plays the role of a "quantum coin", which steers the position of the walker on the line. This term is also responsible for switching between the ballistic and diffusive quantum trajectories observed in [53] as well as the absence of the CLT for generic OQBMs.

Sinayskiy and Petruccione suggested a microscopic derivation of OQBM for a particular case [55]. They considered an OQBM walker with a two-dimensional internal degree of freedom. Both, internal degree of freedom and the position of the walker are coupled to a common decoherent bosonic bath. By utilising a weak system-bath coupling assumption and application of the Born–Markov master equation (14), they obtained the master equation for a OQBM [55]. Sinayskiy and Petruccione studied the steady state properties of OQBM [56]. They calculated analytically mean and variance of the position of the OQBM walker in a long time limit. The combination of the microscopic derivation of the OQBM and the explicit expression for the steady state mean and variance allowed for the relation of the mean and variance of the OQBM walker to the thermodynamical and dynamical parameters of the system. Moreover, they found that the steady state position of the walker can be controlled by the strength of an external classical driving field [56].

5 Conclusion

In this paper, we reviewed the concept of OQWs. We briefly presented the formalism of the discrete time OQWs and demonstrated it on an example of OQWs on $\mathbb{Z}$. We introduced the quantum trajectory approach to OQWs and demonstrated the connection of OQWs to CRWs and UQWs. Afterwards we reviewed the results of the application of methods from the theory of classical Markov chains to the quantum trajectories of OQWs, which include CLTs, ergodic properties, passage time, exit time and site recurrence criteria. We reported on possible generalisations and applications of OQWs. The outline of the microscopic derivation of discrete and continuous time OQWs was presented. We introduced and mentioned some recent results for continuous time OQWs. The scaling limit of OQWs–OQBM was introduced and discussed.

The study of OQWs is a recent research field and it is difficult to predict the domains of application. However, it is clear that the theory of OQWs is finding an application in the generalisations of the theory of classical and quantum Markov chains and quantum probability. Formulation of the microscopic derivation for OQWs allowed a possible application of OQWs to quantum state engineering, dissipative quantum computation and transport in mesoscopic systems. There are a lot of open questions in the field, such as the experimental observation of OQWs, experimental implementation of quantum state engineering and dissipative quantum computing with OQWs, application of the repeated interaction theory to the derivation of OQWs and OQBM, what is a generic form of OQBM, what are the asymptotic properties of OQBM and many others.

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Author contribution statement

IS and FP equally contributed to the manuscript.

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