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Operational dynamical modeling of spin 1/2 relativistic particles

The Dirac equation and its classical limit

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Abstract. The formalism of Operational Dynamical Modeling [Bondar et al., Phys. Rev. Lett. **109**, 190403 (2012)] is employed to analyze dynamics of spin half relativistic particles. We arrive at the Dirac equation from specially constructed relativistic Ehrenfest theorems by assuming that the coordinates and momenta do not commute. Forbidding creation of antiparticles and requiring the commutativity of the coordinates and momenta lead to classical Spohn's equation [Spohn, Ann. Phys. **282**, 420 (2000)]. Moreover, Spohn's equation turns out to be the classical Koopman-von Neumann theory underlying the Dirac equation.

1 Introduction

The Dirac equation is one of the most fundamental building blocks of relativistic quantum theory describing the dynamics of spin 1/2 charged particles. The Dirac equation has found a broad range of applications including solid state physics [42,46,56], optics [1,57], cold atoms [2,14,65], trapped ions [13,35], circuit QED [58], and chemistry of heavy elements [3,52]. In this paper, we revisit the foundations of relativistic quantum and classical mechanics to provide a unified operational derivation of the Dirac equation and its classical counterpart, addressing the role of spinors and antiparticles in the classical limit $\hbar \to 0$.

The procedure of applying the limit $\hbar \to 0$ is fraught with many difficulties. Considering that $\hbar$ is a fundamental constant with the fixed value, this limit is a formal procedure whose physical interpretation needs to be clarified. The classical limit implies two types of analysis: one involving an equation of motion and the other – a quantum state [9]. The limit is mathematically ill-defined requiring auxiliary assumptions that may significantly change the underlying physical picture [10]. A widely used method to remedy mathematical ambiguities is coarse graining, which consists of averaging out features of a quantum state arising from interferences.

This procedure is physically justified by decoherence: erasing quantum coherences by coupling the quantum system to an external bath [45,75]. However, quantum evolution with decoherence recovers irreversible rather than reversible classical dynamics [24]. In addition to the conceptual difficulties, the previous studies of the classical limit $\hbar \to 0$ [15,50,63,64,67] require lengthy algebraic manipulations.

Our approach to the classical limit $\hbar \to 0$ of relativistic dynamics is based on the observation that the commutator between the position and momentum of a quantum particle is proportional to $\hbar$. This encapsulates the Heisenberg uncertainty principle and the experimental fact that the order of measurements affects the measured outcomes [22,60]. However, the position and momentum of a classical particle can be measured simultaneously and observed values do not depend on the measurement order. Mathematically, this implies that the position and momentum of a classical particle commute. Therefore, we define the classical limit as the commutativity of the algebra of observables. Relativity brings an additional constraint that no antiparticles (i.e., negative energy states) should survive the classical limit. This intuition can be formalized by means of Operational Dynamical Modeling (ODM) [18] – a universal and systematic framework for deducing physical models from the evolution of dynamical average values.

To derive equations of motion, ODM needs two inputs: observed data recast in the form of Ehrenfest-like relations and kinematics specifying both the algebra of observables and the definition of averages. As an outcome, ODM guarantees that the resulting equations have the desired physical structure to reproduce the supplied dynamical observations. For example in reference [18], we utilized this method to infer the Schrödinger equation from the Ehrenfest theorems by assuming that the coordinate and momentum operators obey the canonical commutation relation. Otherwise if the coordinate and momentum commute, ODM leads to the Koopman-von Neumann mechanics [28,29,36,48,55,68,69], which is a Hilbert space formulation of non-relativistic classical mechanics where states are represented as complex valued wave functions and observables as commuting self-adjoint operators. ODM has provided a new interpretation of the Wigner function [20,26,33], unveiled conceptual inconstancies in finite-dimensional quantum mechanics [19], formulated dynamical models in topologically nontrivial spaces [74], advanced the study of quantum-classical hybrids [34,59], quantum speed limit [61], yielded new tools for dissipative quantum systems [17,25,70,72,73], and lead to development of efficient numerical techniques [21,23,24,47].

In the non-relativistic case, ODM relied on the fact that quantum and classical states could be represented on an equal mathematical footing – the Hilbert space. For the corresponding relativistic program to be carried out, the state for a spin $1/2$ particle must have a similar representation in the quantum and classical realms. Since spinors represent quantum states, the spinorial formulation of classical mechanics is desired [5,7,16,27,43,64].

It is well known that the Dirac equation incorporates spin, but it is uncommon to associate classical dynamics with spin. The Lorentz group describes a fundamental symmetry of relativistic mechanics. Spinors, also known as "half vectors," are elements of the double cover representation of the Lorentz group [53]. Classical velocities and accelerations can be expressed in the vector basis formed as bilinear constructions of spinors [5,7,43]. Furthermore, there is a specific bilinear combination of these spinors yielding the classical spin, whose physical significance is the subject of an ongoing debate [71]. Note that there is no spinorial formulation of nonrelativistic classical mechanics except for the Kepler problem [44,51].

This paper is organized as follows: Section 2 reviews classical spinorial dynamics. Section 3 provides an ODM derivation of the Dirac equation. Section 4 presents the derivation of the relativistic spinorial Koopman-von Neumann equation. Finally, conclusions are drawn in Section 5.

2 Classical mechanics

The purpose of this section is to review relativistic classical mechanics with a particular emphasis on the spinorial formulation. The time-extended Lagrangian for relativistic classical mechanics with electromagnetic interaction is [6,32,37]

$$\mathcal{L} = \frac{m}{2} u^\mu u_\mu + e A^\mu u_\mu + \frac{1}{2} m c^2, \tag{1}$$

where u^μ is the proper velocity, A^μ is the four-vector potential, m is the mass and c is the speed of light. In this formulation the shell mass $u^\mu u_\mu = c^2$ is not imposed as a constraint but it is instead incorporated as an integral of motion. The Euler-Lagrange equations lead to the relativistic Newton equations

$$\frac{dx^\mu}{d\tau} = u^\mu, \quad m \frac{du_\mu}{d\tau} = e F_{\mu\nu} u^\nu, \tag{2}$$

where τ is the proper time. The canonical momentum, obtained from the Lagrangian is

$$p_\mu = m u_\mu + e A_\mu, \tag{3}$$

where we identify $m u^\mu$ as the kinetic momentum. Note that contravariant indexes are used for physical quantities. The time-extended Lagrangian can be used to obtain the time-extended classical Hamiltonian $\mathcal{H}$ as

$$\mathcal{H} = \frac{1}{2m} \left(p_\mu - e A_\mu \right) \left(p^\mu - e A^\mu \right) - \frac{1}{2} m c^2. \tag{4}$$

Assuming no explicit dependence on the proper time, $\mathcal{H}$ is a conserved integral of motion corresponding to the shell mass condition $\mathcal{H} = 0 \longleftrightarrow u^\mu u_\mu = c^2$. The energy $c p^0$ is extracted from the shell mass as

$$c p^0 = K(p) + c e A_0, \tag{5}$$

with the kinetic energy given by

$$K(p) = \sqrt{(m c^2)^2 + c^2 (p - e A)^k \cdot (p - e A)^k}, \tag{6}$$

where the Latin indices (e.g., k) take values of $1, 2, 3$. The Hamilton equations are derived from equation (4)

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu - e A^\mu}{m}, \quad \frac{dp_\mu}{d\tau} = \frac{e}{m} (\partial_\mu A_\nu) \left(p^\nu - e A^\nu \right), \tag{7}$$

which are equivalent to equation (2).

Classical relativistic mechanics can also be expressed in the spinorial form using two alternative formulations: the Spacetime Algebra by Hestenes [43,44] and the Algebra of Physical Space by Baylis [5,8]. In this paper, we adapt Hestenes' formalism utilizing Feynman's slash notation. The proper velocity is defined as

$$\slashed{u} = u^\mu \gamma_\mu = u_\mu \gamma^\mu \tag{8}$$

where the gamma matrices are 4×4 complex matrices that obey the Clifford algebra in the Minkowski space

$$(\gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\nu) = 2g^{\mu\nu}\mathbf{1} \tag{9}$$

with $g_{\mu\nu} = diag(1, -1, -1, -1)$. In Feynman's notation, the Lorentz inner product is expressed as

$$p^\mu q_\mu = \not{p} \cdot \not{q} = \frac{1}{4}Tr[\not{p}\not{q}], \tag{10}$$

and the shell mass condition reads

$$Tr[(\not{p} - e\not{A} - mc)(\not{p} - e\not{A} + mc)] = 0. \tag{11}$$

A Lorentz transformation of the proper velocity induced by the spinor $L \in \mathbf{Spin}_+(1,3)$, an element of the double representation of the restricted Lorentz group $SO_+(1,3)$ [53], reads as

$$\not{u} \to \not{u}' = L\not{u}L^{-1}. \tag{12}$$

The spinor L redundantly stores the information. In fact, employing the Pauli-Dirac representation of gamma matrices, we have [53]

$$L = \begin{pmatrix} \Psi_1 & -\Psi_2^* & \Psi_3 & \Psi_4^* \\ \Psi_2 & \Psi_1^* & \Psi_4 & -\Psi_3^* \\ \Psi_3 & \Psi_4^* & \Psi_1 & -\Psi_2^* \\ \Psi_4 & -\Psi_3^* & \Psi_2 & \Psi_1^* \end{pmatrix}, \tag{13}$$

where the column spinor Ψ satisfying the Dirac equation is recovered as

$$\Psi = L\Big|_{\text{leftmost column}}. \tag{14}$$

It is shown in Appendix A that

$$\frac{dx^\mu}{d\tau} = u^\mu \implies \frac{dx^\mu}{d\tau} = \Psi^\dagger c\gamma^0\gamma^\mu\Psi. \tag{15}$$

Note that equation (15) is purely classical even though it resembles relativistic Ehrenfest relations. The exclusive role of the gamma matrices is to extract the velocity stored in the spinor

$$u^\mu = \Psi^\dagger c\gamma^0\gamma^\mu\Psi = \frac{1}{4}Tr(cLL^\dagger\gamma^0\gamma^\mu). \tag{16}$$

However, equation (15) does not imply that the particle is moving at the speed of $\pm c$, which are the eigenvalues of $c\gamma^0\gamma^\mu$. The same argument holds in the quantum mechanical case, thus eliminating the controversy attributed to the use of $c\gamma^0\gamma^\mu$ as the velocity operator [31].

In a similar fashion, the relativistic Newton's equations for the Lorentz force in equation (2) can be recast in the two equivalent forms

$$m\frac{du_\mu}{d\tau} = ce\Psi^\dagger\gamma^0\gamma^\nu F_{\mu\nu}\Psi, \quad \frac{dp_\mu}{d\tau} = ce\Psi^\dagger\gamma^0(\partial_\mu\not{A})\Psi. \tag{17}$$

3 The Dirac equation

This section offers a derivation of the Dirac equation employing ODM. According to reference [18], in order to construct a system's dynamical model, ODM requires the following three inputs:

1. The evolution of the average values in the form of Ehrenfest-like relations.

2. The definition of the observables' average.

3. The algebra of the observables.

The classical spinorial equations of motion (17) are parametrized in terms of the proper time τ. Considering that the relation to the time t is

$$\frac{d}{d\tau} = \gamma \frac{d}{dt}. \tag{18}$$

The classical spinorial equations can be written as

$$\frac{dx^\mu}{dt} = \Psi^\dagger c \gamma^0 \gamma^\mu \Psi, \qquad \frac{dp_\mu}{dt} = ce\Psi^\dagger \gamma^0 (\partial_\mu \not{A})\Psi, \tag{19}$$

where the normalization condition $1 = \Psi^\dagger \Psi$ has been imposed, resulting in the absorption of the γ factor in Ψ. Based on these equations, we *postulate* that relativistic dynamics obeys the following Ehrenfest-like relations:

$$\frac{d}{dt}\left\langle \hat{\boldsymbol{x}}^k \right\rangle = \left\langle c\gamma^0\gamma^k \right\rangle, \qquad \frac{d}{dt}\left\langle \hat{\boldsymbol{p}}_k \right\rangle = \left\langle ce\partial_k\hat{A}_\nu\gamma^0\gamma^\nu \right\rangle. \tag{20}$$

Where $\langle \cdots \rangle$ denotes a physical (empirical) average, which needs to be mathematically defined. As per item 2, we represent the expectation values by the Dirac bra-ket "sandwich" in the Hilbert space, $\langle \cdots \rangle = \langle \psi | \cdots | \psi \rangle$. Hence,

$$\frac{d}{dt}\langle\psi|\hat{\boldsymbol{x}}^k|\psi\rangle = \langle\psi|c\gamma^0\gamma^k|\psi\rangle, \qquad \frac{d}{dt}\langle\psi|\hat{\boldsymbol{p}}_k|\psi\rangle = \langle\psi|ce\partial_k\hat{A}_\nu\gamma^0\gamma^\nu|\psi\rangle, \tag{21}$$

where the position x^μ and momentum p_μ variables are replaced by the corresponding operators $\hat{\boldsymbol{x}}^k$ and $\hat{\boldsymbol{p}}_k$ acting on a spinorial Hilbert space of kets $|\psi\rangle$.

According to the Stone's theorem, unitary evolution of $|\psi\rangle$ implies the existence of a self-adjoint operator H such that

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle. \tag{22}$$

Substitution equation (22) into equation (21) leads to

$$\langle\psi|\frac{1}{i\hbar}[\hat{\boldsymbol{x}}^k, H]|\psi\rangle = \langle\psi|c\gamma^0\gamma^k|\psi\rangle, \quad \langle\psi|\frac{1}{i\hbar}[\hat{\boldsymbol{p}}_k, H]|\psi\rangle = \langle\psi|ce\partial_k\hat{A}_\nu\gamma^0\gamma^\nu|\psi\rangle. \tag{23}$$

The expectation values can be dropped assuming that these relations are valid for all initial states

$$\frac{1}{i\hbar}[\hat{\boldsymbol{x}}^k, H] = c\gamma^0\gamma^k, \quad \frac{1}{i\hbar}[\hat{\boldsymbol{p}}_k, H] = ce\partial_k\hat{A}_\nu\gamma^0\gamma^\nu. \tag{24}$$

"Quantumness" is imposed by specifying the commutation relations

$$[\hat{\boldsymbol{x}}^j, \hat{\boldsymbol{p}}_k] = -i\delta^j{}_k \hbar, \tag{25}$$

which specifies item 3 of ODM. Note that the negative sign in the right hand side of equation (25) appears because the nonrelativistic momentum operator is associated with contravariant components $\hat{\boldsymbol{p}}^j$

$$[\hat{\boldsymbol{x}}^k, \hat{\boldsymbol{p}}_j] = -i\hbar\delta^k{}_j \Longleftrightarrow [\hat{\boldsymbol{x}}^k, \hat{\boldsymbol{p}}^j] = i\hbar\delta^{kj}. \tag{26}$$

Assuming that $H = H(\hat{\boldsymbol{x}}^k, \hat{\boldsymbol{p}}_k)$, equation (24) are transformed into the following system of differential equations

$$-\frac{\partial}{\partial \hat{\boldsymbol{p}}_k} H = c\gamma^0 \gamma^k, \quad \frac{\partial}{\partial \hat{\boldsymbol{x}}^k} H = ce\partial_k \hat{A}_\nu \gamma^0 \gamma^\nu. \tag{27}$$

The latter can be readily solved for the unknown generator of motion H

$$H(\hat{\boldsymbol{x}}^k, \hat{\boldsymbol{p}}_k) = -\gamma^0 \gamma^k c \hat{\boldsymbol{p}}_k + \gamma^0 \gamma^\nu ce\hat{A}_\nu + C, \tag{28}$$

where C is a constant matrix. Note that the obtained H has the dimension of energy. Thus, the form of C can be fixed by additionally demanding that the obtained H recovers the classical Hamiltonian when the position and momentum commutes (i.e., the classical limit). As shown in the Appendix of reference [23], this yields

$$C = mc^2 \gamma^0. \tag{29}$$

Finally, note that the equation of motion (22) with (28) and (29) is the sought Dirac equation.

4 Spin 1/2 Koopman-von Neumann theory

Having arrived at the Dirac equation, we now find its classical counterpart.

The classical limit of the nonrelativistic quantum state represented by the Wigner function was identified with the Koopman-von Neumann wavefunction [20]. Consequently, the nonrelativistic classical state belongs to a Hilbert space parametrized by both the position and momentum (i.e., the phase space). The generalization of the Wigner function to the relativistic regime has been accomplished in both relativistic quantum mechanics and quantum field theory [11,12,30,39–41,62]. Now we will construct a formalism where the classical limit of the relativistic Wigner function corresponds to the *spinorial* relativistic Koopman-von Neumann wavefunction.

In this section, the physical averages are represented $\langle \cdots \rangle = Tr[\mathcal{W} \cdots]$ in terms of *the Wigner function* $\mathcal{W}$ for spin 1/2 particles, which is a 4×4 complex matrix [23,26]. In this paper, it is convenient to define the Wigner function of a Dirac spinor $\psi(x)$ as

$$\mathcal{W}(x, p) = \frac{1}{2\pi} \int e^{ip\theta} \psi\left(x - \frac{\hbar\theta}{2}\right) \psi^\dagger\left(x + \frac{\hbar\theta}{2}\right) d\theta. \tag{30}$$
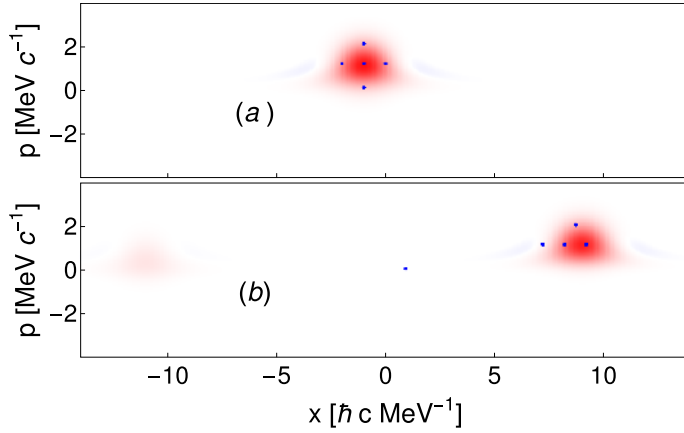
Fig. 1. The Wigner representation of free-particle ($A_\mu = 0$) evolution generated by the (classical) equation of motion (34) and (35) in the phase space. The initial state (a) obtained by the projecting antiparticles out [Eqs. (36) and (37)] from a Gaussian state shown in Figure 2a. The final state at $t = 10\,\hbar/\text{MeV}$ (b) contains antiparticles. The blue dots depict an ensemble of point particles evolving according to the Hamiltonian equation (7). Evolution is restricted to one dimension with $x = x^1$ and $p = p^1$. Red and blue colors represent, respectively, positive and negative values.

The Wigner representation, $\mathcal{W}_{\nu,\nu}$ – the sum of the diagonal elements of the Wigner matrix $\mathcal{W}$, will be used below to visualize dynamics (Figs. 1–3) since the real-valued function $\mathcal{W}_{\nu,\nu}$ is similar to the non-relativistic Wigner function.

Hence, the Ehrenfest relations (20) read

$$\frac{d}{dt}Tr[\mathcal{W}\,\hat{x}^k] = Tr[\mathcal{W}\,c\gamma^0\gamma^k], \quad \frac{d}{dt}Tr[\mathcal{W}\,\hat{p}_k] = Tr[\mathcal{W}\,ce\partial_k A_\nu \gamma^0\gamma^\nu], \tag{31}$$

where the trace is calculated over both the spinorial degrees of freedom and the phase space.

Note that in references [23,26] slightly different definitions are used for the Wigner matrix-valued function and representation; additionally, Tr denotes tracing out the spinorial degrees of freedom only.

"Classicalness" is introduced by the condition

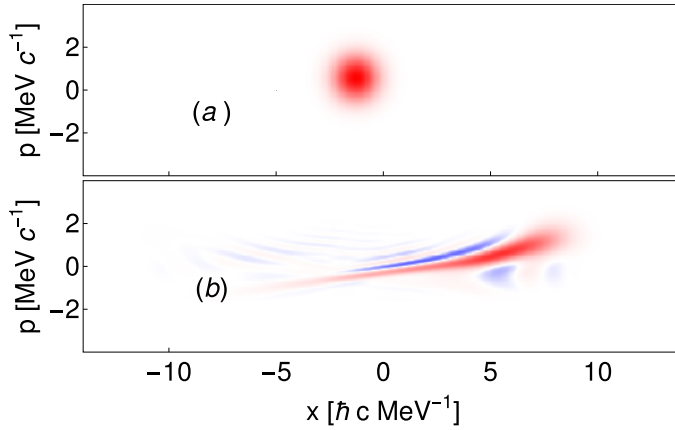$$[\hat{x}^j, \hat{p}_k] = 0. \tag{32}$$

Similar to the nonrelativistic case [18], the classical algebra must be extended to include additional operators $\hat{\theta}^k$ and $\hat{\lambda}_k$ obeying [23,24]

$$[\hat{x}^j, \hat{\lambda}_k] = -i\delta_k^j, \quad [\hat{p}_j, \hat{\theta}^k] = -i\delta_j^k, \tag{33}$$

where all the other commutators vanish.

Assuring unitarity of the dynamics, we propose the following anzats for the equation of motion in the classical case

$$i\frac{\partial}{\partial t}\mathcal{W} = \frac{1}{2}[\gamma^0\gamma^\nu, \hat{K}_\nu\mathcal{W}]_+, \tag{34}$$

Fig. 2. The Wigner representation of quantum free-particle ($A_\mu = 0$) evolution of the initial Gaussian state (a) via the Dirac equations (22) and (28) in the phase space to the final state (b) at time $t = 10\,\hbar/\text{MeV}$. Evolution is restricted to one dimension with $x = x^1$ and $p = p^1$. Red and blue colors represent, respectively, positive and negative values.
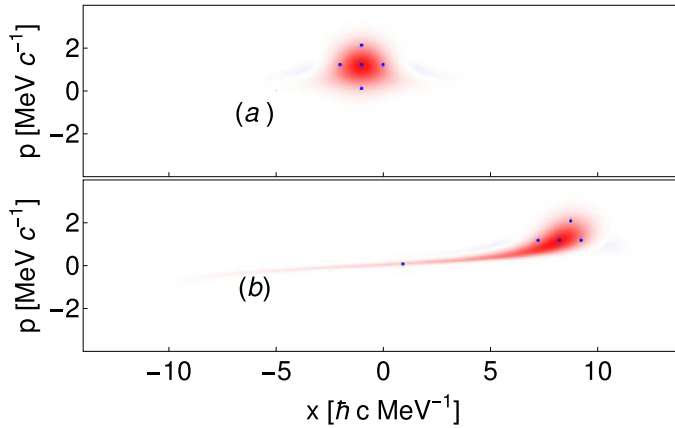
Fig. 3. The Wigner representation of free-particle ($A_\mu = 0$) evolution generated by Spohn's classical equation of motion (39) and (35). The initial state (a) obtained by the projecting antiparticles out [Eqs. (36) and (37)] from a Gaussian state shown in Figure 2a. Unlike the case of Figure 1, the final state at time $t = 10\,\hbar/\text{MeV}$ (b) does not have antiparticles. The blue dots depict an ensemble of point particles evolving according to the Hamiltonian equation (7). Evolution is restricted to one dimension with $x = x^1$ and $p = p^1$. Red and blue colors represent, respectively, positive and negative values.

where $[\cdot, \cdot]_+$ is the anticommutator and $\hat{K}_\nu$ is an unknown self-adjoint generator of motion. Requiring that in the absence of the spinorial degrees of freedom equation (34) should reproduce the non-relativistic Liouvillian equation in terms of the Poisson bracket, we conclude that $\hat{K}_\nu$ must linearly depend on $\hat{\lambda}_k$ and $\hat{\theta}^k$, while remaining an arbitrary function of $\hat{x}^k$ and $\hat{p}_k$. Assuming that $\mathcal{W}$ sufficiently quickly vanishes at infinity, the generator of motion satisfying the Ehrenfest relations (31) is

$$\hat{K}_\nu = -c\hat{\lambda}_\nu - ce(\partial_j A_\nu)\hat{\theta}^j, \tag{35}$$

where $\hat{\lambda}_0 = 0$.

Even though the obtained model fulfills reasonable conditions, a closer inspection reveals that it cannot be a physically valid classical limit. As we will show below, the equation of motion (34) and (34) produces antiparticles.

Antiparticles are convenient to distinguish from particles in the phase space. Since for the latter, the momentum and velocity vectors are parallel. In other words, a particle with a positive (negative) momentum moves into the positive (negative) direction. However, a portion of the phase space distribution of positive (negative) momenta moving into the negative (positive) direction is associated with antiparticles [23]. In other words, antiparticle's momentum and velocity vectors are antiparallel, since according to Feynman's characterization, antiparticles are particles moving backwards in time.

So far, an arbitrary quantum or classical state $\mathcal{W}$ contains both particle and atiparticle components [38] as states with positive and negative energies in the context of relativistic quantum mechanics. However, the application of a projector $\mathcal{P}_+$ can eliminate all the antiparticles [26]

$$\mathcal{W}_0 = \mathcal{P}_+ \mathcal{W} \mathcal{P}_+ \tag{36}$$

where

$$\mathcal{P}_+ = \frac{1}{2}\left(\mathbf{1} + \frac{-\gamma^0\gamma^k c(\hat{p}_k - eA_k) + mc^2\gamma^0}{K(\hat{p})}\right), \tag{37}$$

with $K(p)$ as the classical kinetic energy defined in equation (5). For example, the state depicted in Figure 1a, which is the same as Figure 3a, is obtained by projecting a Gaussian shown in Figure 2a.

Figure 1b shows a result of free-particle evolution (34) of the initial state (Fig. 1a) containing no antiparticles. Even though the equation of motion is valid for an arbitrary vector potential, we depict free-particle evolution as it offers the clearest illustration for antiparticle dynamics since an external electromagnetic field constantly contributes antiparticles. In Figure 1b, one observes two portions of the wave packet containing mostly positive values of momenta but moving into the opposite directions. The left portion consists of antiparticles, whereas particles are on the right. This shows that equation (34) indeed generates antiparticles. As a result, the evolution generated by equation (34) disagrees with the classical Hamiltonian evolution (7) of point particles (see dark blue points in Fig. 1).

Dirac free particle dynamics is shown in Figure 2 for comparison. Since the free Dirac evolution does not create antiparticles, all the antiparticles observed in Figure 2 coming from the non-filtered initial Gaussian state in Figure 2a.

Nevertheless, the problem of antiparticle creation can be fixed by redefining the Ehrenfest relations as

$$\frac{d}{dt}Tr[\mathcal{W}\,\hat{x}^k] = Tr[\mathcal{W}\,c\gamma^0\gamma^k\mathcal{P}_+], \quad \frac{d}{dt}Tr[\mathcal{W}\,\hat{p}_k] = Tr[\mathcal{W}\,ce\partial_k A_\nu\gamma^0\gamma^\nu\mathcal{P}_+]. \tag{38}$$

This leads to the new equation of motion

$$i\frac{\partial}{\partial t}\mathcal{W} = \frac{1}{2}\mathcal{P}_+[\gamma^0\gamma^\nu, \hat{K}_\nu\mathcal{W}]_+\mathcal{P}_+. \tag{39}$$

Rewriting the latter as

$$\mathcal{W}(t+\delta t) = \mathcal{W}(t) - i\frac{\delta t}{2}\mathcal{P}_+[\gamma^0\gamma^\nu, \hat{K}_\nu\mathcal{W}(t)]_+\mathcal{P}_+ + O\left(\delta t^2\right), \tag{40}$$

we see that if the initial state is free of antiparticles [i.e., $\mathcal{W}(t) = \mathcal{P}_+\mathcal{W}(t)\mathcal{P}_+$], so is the final state [i.e., $\mathcal{W}(t+\delta t) = \mathcal{P}_+\mathcal{W}(t+\delta t)\mathcal{P}_+$]. Figure 3 illustrates that the evolution of $\mathcal{W}_{\nu,\nu}$ generated by equation (39) does not create antiparticles. An appearance of the tail in Figure 3b is attributed to the small fraction of the initial wave packet (Fig. 3a) having negative momenta. Furthermore, the wave packet dynamics [Eq. (39)] is in agreement with the classical Hamiltonian evolution (7) of point particles, as expected from a classical statistical theory.

Equation (39) is the classical Koopman-von Neumann theory [18,20] corresponding to the quantum Dirac equation. The numerical methods for the relativistic Wigner function [23] (used for Fig. 2) are directly applicable to propagate equation (39) as well as equation (34).

Equation (39) has been originally derived by Spohn [64] from a different perspective, which established a consistency with the standard classical relativistic mechanics and the Bargmann-Michel-Telegdi (BMT) equation for the classical spin [4].

5 Conclusions

In references [18,20], we have reached the conclusion that the value of the commutator between the position and momentum is the only feature distinguishing non-relativistic quantum from classical mechanics. Here, we have shown that the same conclusion holds in relativistic mechanics. In particular, by starting from the Ehrenfest relations inspired by the spinorial classical mechanics, we deduce the Dirac equation if coordinates and momenta obey the canonical commutation relation. Spohn's equation [64] is arrived at if in addition to the commutativity of coordinates and momentum (i.e., the classical limit) we explicitly forbid generation of anti-particles. From this point of view, Spohn's equation emerges as the classical Koopman-von Neumann theory corresponding to the Dirac equation. The develop methodology can be readily apply to the analysis of other relativistic dynamical systems (e.g., governed by the Klein-Gordon equation [49,66]).

Spin in relativistic classical mechanics plays a critical role in designing experiments uttilzing high energy particle accelerators [54]. In such a setting, the classical spin is conventionally described within the Bargmann-Michel-Telegdi (BMT) equation [4]. Note that even though the BMT equation is only valid for homogeneous electromagnetic fields, it is employed outside this approximation. On the contrary, Spohn's equation [64] is valid for inhomogeneous fields. This opens an opportunity to pursue practical applications in particle acceleration. Moreover, we currently utilize ODM to find spinorial classical equations of motion accounting for a friction forced induced by a dissipative interaction with a bath.

Appendix A: Classical spinor

In classical mechanics, a particle enquires a proper velocity $\not{v}$ by applying a restricted Lorentz transformations $\mathbf{Spin}_+(1,3)$ on the particle at rest $\not{v}_{\text{rest}} = c\gamma^0$

$$\not{v}_{\text{rest}} \rightarrow \not{v} = cLL^\dagger\gamma^0, \tag{A.1}$$

where $\frac{dt}{d\tau} \equiv u^0 > 0$ implies that the direction of time is preserved.

Any element of $\mathbf{Spin}_+(1,3)$ can be decomposed in terms of a Hermitian (B) and a unitary matrix (R)

$$L = BR, \tag{A.2}$$

where B is referred to as a Lorentz boost and R as a rotor. From equation (A.1) we obtain the boost in terms of the proper velocity

$$B = \sqrt{\frac{u\!\!\!/\gamma^0}{c}}. \tag{A.3}$$

The matrix square root can be obtained analytically as

$$B(u\!\!\!/) = \frac{u\!\!\!/\gamma^0 + \mathbf{1}c(\text{sign}u^0)}{\sqrt{2c(\text{sign}u^0)((\text{sign}u^0)c + u^0)}}, \tag{A.4}$$

where $u^0 > 0$ for classical particles.

Multiplying equation (A.1) by γ^ν from the right and taking the trace, we obtain

$$Tr[\frac{d}{d\tau}x^\mu\gamma_\mu\gamma^\nu] = cTr[LL^\dagger\gamma^0\gamma^\nu] = cTr[L^\dagger\gamma^0\gamma^\nu L] = cTr[\gamma^0\gamma^0 L^\dagger\gamma^0\gamma^\nu L]. \tag{A.5}$$

It follows that

$$4\frac{d}{d\tau}x^\nu = cTr[\gamma^0 L^{-1}\gamma^\nu L], \tag{A.6}$$

for spinors belonging to the restricted Lorentz transformations. Adding three traceless terms, we have

$$\begin{aligned}
\frac{4}{c}\frac{d}{d\tau}x^\nu = &Tr[\gamma^0 L^{-1}\gamma^\nu L] + Tr[i\gamma^1\gamma^2 L^{-1}\gamma^\nu L] \\
&+ Tr[L^{-1}\gamma^\nu L] + Tr[i\gamma^0\gamma^1\gamma^2 L^{-1}\gamma^\nu L].
\end{aligned} \tag{A.7}$$

Defining the projector $\mathcal{Q}$ as

$$\mathcal{Q} \equiv \frac{1}{4}(\mathbf{1} + \gamma^0)(\mathbf{1} + i\gamma^1\gamma^2) = diagonal\{1, 0, 0, 0\}, \tag{A.8}$$

obeying $\gamma^0\mathcal{Q} = \mathcal{Q}$ and $i\gamma^1\gamma^2\mathcal{Q} = \mathcal{Q}$, we arrive to

$$\frac{d}{d\tau}x^\nu = cTr[\mathcal{Q}(L^\dagger\gamma^0\gamma^\nu L)\mathcal{Q}], \tag{A.9}$$

which follows from the identity $\mathcal{QQ} = \mathcal{Q}$.

The matrix $L\mathcal{Q}$ contains Ψ in the first column, while the remainder columns are zero. Similarly, $\mathcal{Q}L^\dagger$ contains $\Psi^\dagger$ in the first row, while the remainder rows are zero. Therefore, equation (A.9) leads to

$$\frac{dx^\nu}{d\tau} = \Psi^\dagger c\gamma^0\gamma^\nu\Psi. \tag{A.10}$$

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