Eur. Phys. J. Special Topics **227**, 2171–2182 (2019)

https://doi.org/10.1140/epjst/e2018-800063-2

THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

From von Neumann to Wigner and beyond

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Received 16 April 2018 / Received in final form 2 August 2018
Published online 21 February 2019

Abstract. Historically, correspondence rules and quantum quasi-distributions were motivated by classical mechanics as a guide for obtaining quantum operators and quantum corrections to classical results. In this paper, we start with quantum mechanics and show how to derive the infinite number of quantum quasi-distributions and corresponding c-functions. An interesting aspect of our approach is that it shows how the c-numbers of position and momentum arise from the quantum operator.

1 Introduction

The concept of a quasi-distribution of position and momentum originated with Wigner in 1932, who gave what is now known as the Wigner distribution [22]. Immediately after, in 1933, Kirkwood gave another distribution [14]. Both Wigner and Kirkwood were motivated by the following idea. Since classical quantities such as the second virial coefficient of a gas are expressed in terms of a classical joint distribution of position and momentum, then perhaps we could calculate the quantum corrections to the second virial coefficient if we substituted a joint distribution of position and momentum that somehow included quantum mechanics. Parallel to this development, starting with Born and Jordan [2], and Weyl [21], the question arose as to how to obtain a quantum operator from the classical counterpart. Different procedures were proposed, giving different answers. Somewhat later, in 1948, Moyal saw the connection between the Wigner distribution and the Weyl procedure, and derived the Wigner distribution using it [17]. Subsequently, the quasi-distribution corresponding to the Born-Jordan rule was derived [5]. In these considerations, and in the many subsequent works on quantum quasi-distributions, the framework was how to go from classical machanics to quantum mechanics.

The aim of this paper is to go directly from the density matrix to the derivation of an infinite number of quasi-distributions. We discuss how an infinite number of quantum phase-space distributions could be obtained from first principles. Our treatment does not employ characteristic functions, but is entirely contained within the usual

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formalism of quantum mechanics. Furthermore, we get the distribution and the corresponding c-function in one swoop. This enables us to understand the origin of all terms, the meaning of the phase-space variables, and the origin of the multiplicity of distributions.

Our starting point is quantum mechanics. For an operator $\mathbf{A}$ and density matrix ρ, the expected value of the physical quantity represented by the operator is given by

$$\langle \mathbf{A} \rangle = \mathrm{Tr}\left\{ \rho \mathbf{A} \right\}. \tag{1}$$

We want to express equation (1) so that it can be written as[1]

$$\langle \mathbf{A} \rangle = \iint P(q,p) a(q,p) dq dp, \tag{2}$$

with the very strong conditions that $P(q,p)$ depends *only* on the density matrix ρ, and that the c-function $a(q,p)$ depends *only* on the operator $\mathbf{A}$.

In the next section, we briefly show how we separated equation (1) in [1] for the Wigner distribution, and in Section 3, we show how equation (1) may be separated so that we may write equation (2). There, we use the density matrix approach, and in Appendix A, we give a somewhat different approach by using the wave function directly. In Appendix B, we obtain the inverse general correspondence rule, presenting another approach to getting the c-number $a(q,p)$ from a given operator $\mathbf{A}$.

In quantum optics, the question of how to connect the classical theory of electromagnetism with quantum theory involves phase-space. In looking for analogies between quantum optics and some probabilistic electromagnetic state, we consider the classical probability density $P(E)$ that the electric field has some amplitude between E and $E + dE$. The question is: What is the quantum counterpart of $P(E)$? Since we know that the Glauber coherent states are the most 'classical-like' quantum states of the electric field [19], then quantum-mechanically, we could naïvely say that the analogous quantity to the classical $P(E)$ are the α–β elements of the density matrix, $\langle \alpha | \rho | \beta \rangle$. But those quantities depend on *two* field amplitudes – not one! To correspond to the classical case, the dependence should be on only one field. In quantum optics, this two-versus-one conundrum is easily resolved as follows. In qunatum optics, we may always make use of the completeness of the $|\alpha\rangle$ states, i.e.

$$\frac{1}{\pi} \iint |\alpha\rangle\langle\alpha| \, d^2\alpha = \hat{1}, \tag{3}$$

to write

$$\rho = \frac{1}{\pi^2} \int \langle \alpha | \rho | \beta \rangle \, |\alpha\rangle\langle\beta| \, d^2\alpha d^2\beta. \tag{4}$$

This is the usual result for a quantum system. Alternatively, if we were to write the number state basis analogue to equation (4), then using

$$\sum_n |n\rangle\langle n| = \hat{1}, \tag{5}$$

[1]Operators are indicated by boldface characters. We often represent multiple integrals by a single integration symbol, the differentials indicating the number of integrals. All integrals go from $-\infty$ to ∞.

we write

$$\rho = \sum_{n,m} \langle n|\rho|m\rangle \, |n\rangle\langle m| \, . \tag{6}$$

But in classical optics, we write $\langle f(E)\rangle = \int f(\epsilon)P(\epsilon)d\epsilon$, which involves probability distributions of only one field ϵ – not two, as is the case in $\langle\alpha|\rho|\beta\rangle = \rho(\alpha^*, \beta)$. But it is possible to regain a single-field distribution description as follows: We may write

$$\langle \mathbf{A}\rangle = \mathrm{Tr}\,\{\rho\mathbf{A}\} = \mathrm{Tr}\left\{\rho \sum_{r,s} c_{r,s}(\mathbf{a}^\dagger)^r \mathbf{a}^s\right\}, \tag{7}$$

i.e., we expressed $\mathbf{A}$ in normal-ordered form. Then, we may write

$$\langle \mathbf{A}\rangle = \mathrm{Tr}\left\{\iint \rho\delta(\mathbf{a}^\dagger - \alpha^*)\delta(\mathbf{a} - \alpha) \sum_{r,s} c_{r,s}(\alpha^*)^r \alpha^s d^2\alpha\right\} \tag{8}$$

$$= \iint P(\alpha, \alpha^*) A^{(n)}(\alpha, \alpha^*) d^2\alpha, \tag{9}$$

where the normal-ordered quantity $A^{(n)}$ is $A^{(n)}(\alpha, \alpha^*) = \sum_{r,s} c_{r,s}(\alpha^*)^r \alpha^s$ and $P(\alpha, \alpha^*) = \mathrm{Tr}\left\{\rho\delta(\mathbf{a}^\dagger - \alpha^*)\delta(\mathbf{a} - \alpha)\right\}$ is a phase-space distribution (called the "Galuber–Sudarashan P-distribution") which is the quantum analog of the classical $P(E)$ [10,19,20].

2 Wigner distribution from the density matrix in a few easy steps

In this section, we show how we separated the expectation value of some operator $\mathbf{A}$ in the Wigner distribution case [1].

Noticing that the expectation value of the operator $\mathbf{A}$ is

$$\langle \mathbf{A}\rangle = \frac{1}{2\pi\hbar} \iiiint \langle q_1|\rho|q_2\rangle \langle p_1|\mathbf{A}|p_2\rangle \, e^{i(q_2 p_1 - q_1 p_2)/\hbar} dq_1 dq_2 dp_1 dp_2, \tag{10}$$

we see that the complex exponential is preventing us from separating the integrand into the two quantities, one depending exclusively on ρ, and the other depending exclusively on $\mathbf{A}$. To overcome this, we define the averages and differences of positions and momenta

$$q' = q_1 - q_2 \qquad p' = p_2 - p_1 \tag{11}$$

$$q = \frac{q_1 + q_2}{2} \qquad p = \frac{p_2 + p_1}{2}. \tag{12}$$

We have thus turned the expectation value in equation (10) into

$$\langle \mathbf{A}\rangle = \iint P_{\mathrm{W}}(q,p) A_{\mathrm{W}}(q,p) dq dp, \tag{13}$$

where in equation (13), the distribution P_{W} is

$$P_{\mathrm{W}}(q,p) = \int \left\langle q + \frac{q'}{2} \middle| \frac{\rho}{2\pi\hbar} \middle| q - \frac{q'}{2} \right\rangle e^{-iq'p/\hbar} dq', \tag{14}$$

which does not depend on $\mathbf{A}$, and A_{W} is

$$A_{\mathrm{W}}(q,p) = \int \left\langle p - \frac{p'}{2} \middle| \mathbf{A} \middle| p + \frac{p'}{2} \right\rangle e^{-ip'q/\hbar} dp', \tag{15}$$

which has no dependence on ρ. This means that P_{W} and A_{W} are separated.

3 From quantum mechanics to quasi-distributions

In this section, we show how equation (1) can be separated in general, thus obtaining the general class [5] of phase-space distributions along with their corresponding c-functions. In the position representation, the expectation value of the operator $\mathbf{A}$, equation (1), becomes

$$\langle \mathbf{A} \rangle = \iint \langle q''|\rho|q'\rangle \langle q'|\mathbf{A}|q''\rangle \, dq'dq''. \tag{16}$$

We make a change of variables

$$q'' = u + \hbar\tau/2, \qquad q' = u - \hbar\tau/2, \tag{17}$$

and obtain that

$$\langle \mathbf{A} \rangle = \hbar \iint \left\langle u + \frac{\hbar\tau}{2} \middle| \rho \middle| u - \frac{\hbar\tau}{2} \right\rangle \left\langle u - \frac{\hbar\tau}{2} \middle| \mathbf{A} \middle| u + \frac{\hbar\tau}{2} \right\rangle dud\tau. \tag{18}$$

To separate equation (18) into two terms, one depending only on ρ, and the other depending only on $\mathbf{A}$, we insert the product of two delta functions, $\delta(\tau - \tau')\,\delta(u - u')$,

$$\langle \mathbf{A} \rangle = \hbar \iint \left\langle u + \frac{\hbar\tau}{2} \middle| \rho \middle| u - \frac{\hbar\tau}{2} \right\rangle dud\tau \iint \left\langle u' - \frac{\hbar\tau'}{2} \middle| \mathbf{A} \middle| u' + \frac{\hbar\tau'}{2} \right\rangle \delta(\tau - \tau')\delta(u - u')du'd\tau' \tag{19}$$

where we changed the variables parameterizing the matrix elements of the operator $\mathbf{A}$ from (u, τ) to (u', τ'). We thus have one term depending only on ρ, and another depending on $\mathbf{A}$ only. However, we do not yet have an expression of the form of equation (2) for the expectation value of $\mathbf{A}$. To accomplish that we rewrite the product of Dirac delta functions as

$$\delta(\tau - \tau')\delta(u - u') = \iint \left\{ \frac{1}{(2\pi)^2} \int e^{-i\theta q - i\tau p + i\theta u} \Phi(\theta, \tau) d\theta \right\} \times \left\{ \frac{1}{2\pi} \int e^{i\theta' q + i\tau' p - i\theta' u'} \Phi^{-1}(\theta', \tau') d\theta' \right\} dqdp, \tag{20}$$

where $\Phi(\theta, \tau)$ is *any* two-dimensional function whose significance will be discussed subsequently. Notice that the position and momentum c-variables, q and p, emerge from the separation through the Dirac delta functions.

Substituting equation (20) into equation (19), we have

$$\langle \mathbf{A} \rangle = \iint \left\{ \frac{1}{(2\pi)^2} \iint du d\tau d\theta \left\langle u + \frac{\hbar\tau}{2} \middle| \rho \middle| u - \frac{\hbar\tau}{2} \right\rangle e^{-i\theta q - i\tau p + i\theta u} \Phi(\theta, \tau) \right\} \times \left\{ \frac{\hbar}{2\pi} \iint \left\langle u' - \frac{\hbar\tau'}{2} \middle| \mathbf{A} \middle| u' + \frac{\hbar\tau'}{2} \right\rangle \frac{e^{i\theta' q + i\tau' p - i\theta' u'}}{\Phi(\theta', \tau')} du' d\tau' d\theta' \right\} dq dp. \tag{21}$$

We have thus accomplished our goal: equation (21) has separated the expectation value and we can write

$$\langle \mathbf{A} \rangle = \iint P_\Phi(q,p) a_\Phi(q,p) dq dp, \tag{22}$$

where the quasi-distribution P_Φ is given by

$$P_\Phi(q,p) = \frac{1}{(2\pi)^2} \iiint \left\langle u + \frac{\hbar\tau}{2} \middle| \rho \middle| u - \frac{\hbar\tau}{2} \right\rangle e^{-i\theta q - i\tau p + i\theta u} \Phi(\theta, \tau) du d\tau d\theta \tag{23}$$

which depends only on the density matrix ρ, and the c-function a_Φ is

$$a_\Phi(q,p) = \frac{\hbar}{2\pi} \iiint \left\langle u - \frac{\hbar\tau}{2} \middle| \mathbf{A} \middle| u + \frac{\hbar\tau}{2} \right\rangle e^{i\theta q + i\tau p - i\theta u} \frac{1}{\Phi(\theta, \tau)} du d\tau d\theta \tag{24}$$

which depends only on the operator $\mathbf{A}$. Notice that the function Φ appears in both expressions, and hence the choice of quasi-probability is coupled to the choice of the c-function. The function Φ characterizes the particular quasi-distribution.

Equation (23) was first given by Cohen [5], where $\Phi(\theta, \tau)$ is called the kernel, and parameterizes the totality of quasi-distributions and correspondence rules [15]. Expression (24) gives the c-function $a_\Phi(q,p)$ that corresponds to the operator $\mathbf{A}$. This is in contrast to the usual historical procedure of correspondence rules where one attempts to write the operator for a given classical function. Expression such as equation (24) have been called inverse correspondence rules [13]. In Appendix B, we show the equivalence with the standard formulation.

3.1 Special cases

It is of some interest to consider the derivation of quasi-distributions and their corresponding c-functions for some special cases. As examples, we chose the Wigner and Kirkwood distributions. We note that many variations of the Kirkwood distribution have been studied, both in quantum mechanics and in time-frequency analysis, and among the names that have been used are the Rihaczek, Margenau-Hill, and standard distributions [7].

3.1.1 Wigner case

The Wigner distribution case was derived using these methods in [1]. Here we show that it is a special case of our general approach. The Wigner distribution corresponds to the case where

$$\Phi_W = 1. \tag{25}$$

For equation (20), the delta function product becomes

$$\delta(\tau-\tau')\delta(u-u') = \frac{1}{2\pi}\iint \left[\delta(u-q)e^{-i\tau p}\right]\left[\delta(u'-q)e^{i\tau' p}\right] dqdp \tag{26}$$

where it is important that the integrations over q and p are kept. Therefore, the expectation value of $\mathbf{A}$ as per equation (21) (or (19)) is

$$\langle \mathbf{A}\rangle = \frac{\hbar}{2\pi}\iint \left\{\int e^{-i\tau p}\left\langle q+\frac{\hbar\tau}{2}\middle|\rho\middle|q-\frac{\hbar\tau}{2}\right\rangle d\tau\right\}\left\{\int e^{i\tau' p}\left\langle q+\frac{\hbar\tau}{2}\middle|\mathbf{A}\middle|q-\frac{\hbar\tau}{2}\right\rangle d\tau'\right\} dqdp \tag{27}$$

We can hence write

$$\langle \mathbf{A}\rangle = \iint P_W(q,p)a_W(q,p)dqdp \tag{28}$$

where the quasi-distribution is

$$P_W(q,p) = \frac{1}{2\pi}\int \left\langle q+\frac{\hbar\tau}{2}\middle|\rho\middle|q-\frac{\hbar\tau}{2}\right\rangle e^{-i\tau p}d\tau, \tag{29}$$

which is the Wigner distribution, and the corresponding c-function is

$$a_W(q,p) = \hbar\int \left\langle q+\frac{\hbar\tau}{2}\middle|\mathbf{A}\middle|q-\frac{\hbar\tau}{2}\right\rangle e^{i\tau p}d\tau. \tag{30}$$

In Appendix B we show that equation (30) is the inverse Weyl correspondence for the operator $\mathbf{A}$.

3.1.2 Kirkwood

The Kirkwood distribution corresponds to

$$\Phi_K(\theta,\tau) = e^{i\theta\tau\hbar/2}. \tag{31}$$

Equation (20) for the delta function product becomes

$$\delta(\tau-\tau')\delta(u-u') = \frac{1}{2\pi}\iint dqdp\left(\delta\left[u-\left(q-\frac{\hbar\tau}{2}\right)\right]e^{-i\tau p}\right)\left(\delta\left[u'-\left(q-\frac{\hbar\tau'}{2}\right)\right]e^{i\tau' p}\right), \tag{32}$$

and further, from equation (21) or (19), we have that the expectation value is

$$\langle \mathbf{A}\rangle = \iint \langle q|\rho|p\rangle\langle p|\mathbf{A}|q\rangle dqdp. \tag{33}$$

Inserting an identity into equation (33), we have

$$\langle \mathbf{A}\rangle = \iint \frac{e^{-iqp/\hbar}}{\sqrt{2\pi\hbar}}\langle q|\rho|p\rangle\sqrt{2\pi\hbar}\,e^{iqp/\hbar}\langle p|\mathbf{A}|q\rangle dqdp. \tag{34}$$

We can therefore write the expectation value of $\mathbf{A}$ as

$$\langle \mathbf{A} \rangle = \iint P_K(q,p) a_K(q,p) dq dp, \tag{35}$$

where the quasi-distribution is

$$P_K(q,p) = \frac{e^{-iqp/\hbar}}{\sqrt{2\pi\hbar}} \langle q|\rho|p\rangle, \tag{36}$$

which is the Kirkwood distribution, and the c-function a_Φ is

$$a_K(q,p) = \sqrt{2\pi\hbar} e^{-iqp/\hbar} \langle p|\mathbf{A}|q\rangle. \tag{37}$$

4 Conclusion

An interesting aspect of our derivation is understanding how the separation of the quasi-distribution and c-function is achieved in the expression for the quantum expectation value of an operator. In addition, the separation shows how the position and momentum c-numbers appear.

We emphasize that in the expression for the delta function as given by equations (20) and (A.8), we have assumed that the separation *does not* involve the wave function or the operator, and that is why the quasi-distributions given by equations (23) and (A.11) are called bilinear. One may take the kernel to be a functional of the wave function and/or the operator, in which case, the resulting expressions would not be bilinear. That case has not been studied extensively.

We also note that in our derivation, we have *not* assumed that the operator is Hermitian or that the quasi-distribution has to be real. Indeed, neither has to be the case, as long as the quasi-distribution and c-function are coupled appropriately by way of equations (23)–(24) or equations (A.11)–(A.12). One can impose conditions on the quasi-distribution by imposing conditions on the kernel, and such methods are called kernel design [3,4,6,7,11,12,16,23].

We point out that the same methods may be used to derive the general class of time-frequency distributions for both the deterministic and random cases. Mathematically, the situations become identical if one lets position go into time, treating the wavefunction $\psi(q)$ as the signal $s(t)$ in time. Momentum in the wavefunction case is the frequency ω. For the random case, one replaces the density matrix by the ensemble average of the signal. That is, instead of the density matrix $\rho(q,q')$, one takes the ensemble average $\langle s^*(t)s(t')\rangle$. In this case the Wigner distribution is called the Wigner spectrum. In a future paper, the details of the time-frequency case will be discussed.

We also mention that one can develop the concept of quasi-distributions for variables other than position and momentum. The usual way of studying quasi-distributions for arbitrary variables is by way of the characteristic function [8,9,18]. If α and β are associated with the operators $\mathbf{A}$ and $\mathbf{B}$, then the charactersitic function is

$$M(\theta,\tau) = \langle e^{i\theta\mathbf{A}+i\tau\mathbf{B}}\rangle = \int \psi^*(q) e^{i\theta\mathbf{A}+i\tau\mathbf{B}} \, \psi(q) \, dq, \tag{38}$$

and the quasi probability distribution is

$$P(\alpha,\beta) = \frac{1}{4\pi^2}\iint M(\theta,\tau)\rangle e^{-i\theta\alpha - i\tau\beta} d\alpha d\beta \ . \tag{39}$$

The ambiguity comes in by the fact that instead of $e^{i\theta\mathbf{A}+i\tau\mathbf{B}}$ in equation (38), one can use $e^{i\theta\mathbf{A}}\ e^{i\tau\mathbf{B}}$ or $e^{i\theta\mathbf{A}/2}e^{i\tau\mathbf{B}}e^{i\theta\mathbf{A}/2}$, and each gives a different quasi-distribution, all of which satisfy the marginals for α and β. This method has been applied to a number of variables, including spin. In the case of momentum and position, the derivations presented in this paper avoided the characteristic function method, and it would be of some interest to apply the current approach to the case of arbitrary variables. Preliminary results show that indeed, this can be done, and the results will be presented in a future paper.

JSB and MOS would like to thank the Robert A. Welch Foundation (Grant No. A-1261), the Office of Naval Research (Award No. N00014-16-1-3054), and the Air Force Office of Scientific Research (FA9550-18-1-0141) for their the support. We also thank the referees for their valuable suggestions.

Appendix A: Wave function approach

In this Appendix, we consider the pure case and explicitly use the wave function. In the position representation, the density matrix is

$$\rho(q'', q') = \psi^*(q')\psi(q''). \tag{A.1}$$

The matrix elements, $a_{q'q''}$, of an operator $\mathbf{A}(\mathbf{x}, \mathbf{p}_x)$ in the position representation are

$$a_{q'q''} = \int \delta(q'-x)\mathbf{A}(\mathbf{x}, \mathbf{p}_x)\delta(q''-x)dx. \tag{A.2}$$

The expectation value of $\mathbf{A}$ is then

$$\langle\mathbf{A}\rangle = \mathrm{Tr}(\rho\mathbf{A}) = \iint \rho(q'', q')a_{q'q''}dq'dq'' \tag{A.3}$$

$$= \iiint \psi^*(q')\psi(q'')\delta(q'-x)\mathbf{A}\delta(q''-x)dq'dq''dx. \tag{A.4}$$

In equation (A.4), we change variables according to

$$q'' = u + \hbar\tau/2 \qquad\qquad q' = u - \hbar\tau/2, \tag{A.5}$$

giving that

$$\langle\mathbf{A}\rangle = \hbar\int \psi^*(u-\hbar\tau/2)\psi(u+\hbar\tau/2)\delta(u-\hbar\tau/2-x)\mathbf{A}\delta(u+\hbar\tau/2-x)d\tau du dx. \tag{A.6}$$

To separate the integral into the product of two terms, one depending only on the operator $\mathbf{A}$, and the other only on the quantum state, we insert $\delta(\tau-\tau')\delta(u-u')$, leading to

$$\langle \mathbf{A} \rangle = \hbar \iint \psi^*(u - \hbar\tau/2)\psi(u + \hbar\tau/2) du d\tau \\ \times \iiint \delta(\tau - \tau')\delta(u - u')\delta(u' - \hbar\tau'/2 - x)\mathbf{A}\delta(u' + \hbar\tau'/2 - x) dx d\tau' du'. \tag{A.7}$$

Now, for $\delta(\tau - \tau')\delta(u - u')$ we take

$$\delta(\tau - \tau')\delta(u - u') = \iint \left\{ \frac{1}{(2\pi)^2} \int d\theta e^{i\theta(u-q) - i\tau p} \Phi(\theta, \tau) \right\} \\ \times \left\{ \frac{1}{2\pi} \int d\theta' e^{-i\theta'(u'-q) + i\tau' p} \Phi^{-1}(\theta', \tau') \right\} dq dp \tag{A.8}$$

and insert into equation (A.7), giving

$$\langle \mathbf{A} \rangle = \hbar \iint \left\{ \frac{1}{(2\pi)^2} \iiint \psi^*(u - \hbar\tau/2)\psi(u + \hbar\tau/2) e^{-i\theta q - i\tau p + i\theta u} \Phi(\theta, \tau) d\tau d\theta du \right\} \\ \left\{ \frac{1}{2\pi} \iiiint \delta(u' - \hbar\tau'/2 - x)\mathbf{A}\delta(u' + \hbar\tau'/2 - x) \\ \times e^{i\theta' q + i\tau' p - i\theta' u'} \Phi^{-1}(\theta', \tau') dx d\tau' d\theta' du' \right\} dq dp, \tag{A.9}$$

which achieved the separation. We may therefore write the expectation value of $\mathbf{A}$ as

$$\langle \mathbf{A} \rangle = \iint dq dp \, P_\Phi(q, p) a_\Phi(q, p), \tag{A.10}$$

with the quasi-distribution being

$$P_\Phi(q, p) = \frac{1}{4\pi^2} \iiint \psi^*(u - \tfrac{\hbar}{2}\tau)\psi(u + \tfrac{\hbar}{2}\tau)\Phi(\theta, \tau) \, e^{-i\theta q - i\tau p + i\theta u} \, d\theta \, d\tau \, du, \tag{A.11}$$

depending only on ψ, and the c-function depending only on $\mathbf{A}$ being

$$a_\Phi(q, p) = \frac{\hbar}{2\pi} \int \frac{e^{i\theta q + i\tau p - i\theta u}}{\Phi(\theta, \tau)} \delta(u - \hbar\tau/2 - x)\mathbf{A}\delta(u + \hbar\tau/2 - x) dx d\theta \, d\tau \, du. \tag{A.12}$$

A.1 Wigner distribution

The Wigner case is obtained by taking $\Phi_W(\theta, \tau) = 1$, for which equation (A.8) becomes

$$\delta(\tau - \tau')\delta(u - u') = \iint \left\{ \frac{1}{2\pi} \delta(u - q) e^{-i\tau p} \right\} \left\{ \delta(u' - q) e^{i\tau' p} \right\} dq dp. \tag{A.13}$$

For equations (A.11) and (A.12), we have

$$P_W(q, p) = \frac{1}{2\pi} \int \psi^*(q - \tfrac{\hbar}{2}\tau)\psi(q + \tfrac{\hbar}{2}\tau) \, e^{-i\tau p} du, \tag{A.14}$$

and

$$a_W(q,p) = \hbar \iint e^{i\tau p}\delta(q-\hbar\tau/2-x)\mathbf{A}\delta(q+\hbar\tau/2-x)dx\, d\tau. \tag{A.15}$$

Equation (A.15) corresponds to equation (4.45) of reference [13], and was derived from the Weyl correspondence.

A.2 Kirkwood

Taking the kernel to be

$$\Phi_K(\theta,\tau) = e^{i\theta\tau\hbar/2}, \tag{A.16}$$

and substituting into equation (A.8), we obtain

$$\delta(\tau-\tau')\delta(u-u') = \frac{1}{2\pi}\iint \left(\delta\left[u-\left(q-\frac{\hbar\tau}{2}\right)\right]e^{-i\tau p}\right)\left(\delta\left[u'-\left(q-\frac{\hbar\tau'}{2}\right)\right]e^{i\tau' p}\right)dqdp. \tag{A.17}$$

We insert the expression in equation (A.17) for the delta functions into equation (A.7), and get that

$$P_K(q,p) = \frac{1}{4\pi^2}\iiint \psi^*(u-\tfrac{\hbar}{2}\tau)\psi(u+\tfrac{\hbar}{2}\tau)e^{i\theta\tau\hbar/2}\, e^{-i\theta q-i\tau p+i\theta u}\, d\theta\, d\tau\, du \tag{A.18}$$

$$= \frac{1}{2\pi\hbar}\psi(q)\ e^{-iqp/\hbar}\int \psi^*(\tau)\ e^{i\tau p}\ d\tau, \tag{A.19}$$

which may be written as

$$P_K(q,p) = \frac{1}{\sqrt{2\pi\hbar}}\psi(q)\ e^{-iqp/\hbar}\varphi^*(p), \tag{A.20}$$

where $\varphi(p)$ is the momentum wave function.

For the c-function, equation (A.12) becomes

$$a_K(q,p) = \frac{\hbar}{2\pi}\iiiint e^{-i\theta\tau\hbar/2}\, e^{i\theta q+i\tau p-i\theta u}\delta(u-\hbar\tau/2-x)\mathbf{A}\delta(u+\hbar\tau/2-x)dxd\theta\, d\tau\, du, \tag{A.21}$$

which simplifies to

$$a_K(q,p) = \hbar\int e^{i\tau p}\delta(q-\hbar\tau-x)\mathbf{A}\delta(q-x)dx\, d\tau. \tag{A.22}$$

Appendix B: General correspondence rule and inverse

The general formulation of correspondence rules and quasi-distributions as given by Cohen [5] is that for a quantum operator $\mathbf{A}(\mathbf{q},\mathbf{p})$ and the corresponding c-function

$a(q,p)$ we want the quantum average to be the same as the phase space average,

$$\int \psi^*(q)\,\mathbf{A}\,(\mathbf{q},\mathbf{p})\,\psi(q)\,dq = \iint a(q,p)\,P(q,p)\,\,dq\,dp. \tag{B.1}$$

All bilinear phase space distributions are given by

$$P_\Phi(q,p) = \frac{1}{4\pi^2}\iiint \psi^*(u - \tfrac{\hbar}{2}\tau)\psi(u + \tfrac{\hbar}{2}\tau)\Phi(\theta,\tau)\,e^{-i\theta q - i\tau p + i\theta u}\,d\theta\,d\tau\,du, \tag{B.2}$$

which is equation (A.11), and the generalized correspondence rule for operators is

$$\mathbf{A}\,(\mathbf{q},\mathbf{p}) \;=\; \iint \widehat{a}_\Phi(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta\mathbf{q}+i\tau\mathbf{p}}\,d\theta\,d\tau \tag{B.3}$$

$$= \iint \widehat{a}_\Phi(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta\tau\hbar/2}e^{i\theta\mathbf{q}}\,e^{i\tau\mathbf{p}}d\theta\,d\tau, \tag{B.4}$$

where $\widehat{a}_\Phi$ is the Fourier transform of a_Φ

$$\widehat{a}_\Phi(\theta,\tau) \;=\; \frac{1}{4\pi^2}\iint a_\Phi(q,p)\,e^{-i\theta q - i\tau p}\,dq\,dp. \tag{B.5}$$

In the usual formulation, one starts with the c-function $a_\Phi(q,p)$, while in this paper we started with the operator $\mathbf{A}\,(\mathbf{q},\mathbf{p})$ and we have obtained the c-function, equations (24) and (A.12)

$$a_\Phi(q,p) = \frac{\hbar}{2\pi}\int \frac{e^{i\theta q + i\tau p - i\theta u}}{\Phi(\theta,\tau)}\,\delta(u - \hbar\tau/2 - x)\mathbf{A}\,(x,\mathbf{p}_x)\,\delta(u + \hbar\tau/2 - x)dxd\theta\,d\tau\,du. \tag{B.6}$$

We call the expression in equation (B.6) the inverse general correspondence rule.

We now show that equation (B.6) is indeed the inverse of equation (B.3). We first calculate the Fourier transform of $a(q,p)$. Substituting equation (B.3) into equation (B.5) results in (we drop the subscript Φ to unencumber notation)

$$\widehat{a}(\theta,\tau) \;=\; \frac{\hbar}{2\pi}\int \frac{e^{-i\theta u}}{\Phi(\theta,\tau)}\,\delta(u - \hbar\tau/2 - x)\mathbf{A}\,(x,\mathbf{p}_x)\,\delta(u + \hbar\tau/2 - x)\,dx\,du. \tag{B.7}$$

Substituting $\widehat{a}(\theta,\tau)$ into the left hand side of equation (B.3), we have

$$\iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta\mathbf{q}+i\tau\mathbf{p}}\,d\theta\,d\tau$$
$$= \hbar\frac{1}{2\pi}\iint \frac{1}{\Phi(\theta,\tau)}\,e^{-i\theta u'}\,\delta(u - \hbar\tau/2 - x)\mathbf{A}\,(x,\mathbf{p}_x)\,\delta(u + \hbar\tau/2 - x)\Phi(\theta,\tau)\,e^{i\theta\mathbf{q}+i\tau\mathbf{p}}d\theta d\tau dx du', \tag{B.8}$$

which simplifies to

$$\iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\,e^{i\theta\mathbf{q}+i\tau\mathbf{p}}\,d\theta\,d\tau = \hbar\iint \delta(q - x)\,[\mathbf{A}\,(x,\mathbf{p}_x)\,\delta(q + \hbar\tau - x)]\;e^{i\tau\mathbf{p}}\,dx\;d\tau. \tag{B.9}$$

Changing variables according to

$$y = q + \hbar\tau,$$

we obtain that indeed, equation (B.3) is correct

$$\iint \widehat{a}(\theta,\tau)\Phi(\theta,\tau)\, e^{i\theta\mathbf{q}+i\tau\mathbf{p}}\, d\theta\, d\tau = \iint \delta(q-x)\left[\mathbf{A}\left(x,\mathbf{p}_x\right)\delta(y-x)\right]\, e^{i(y-q)\mathbf{p}/\hbar}\, dx\, dy \tag{B.10}$$

$$= \iint \left[\mathbf{A}\left(q,\mathbf{p}_q\right)\delta(q-x)\right]\, dx \tag{B.11}$$

$$= \mathbf{A}\left(q,\mathbf{p}_q\right). \tag{B.12}$$

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