

# Electrons in Anderson–Mott insulators

Michael Pollak<sup>a</sup>

Department of Physics, University of California, Riverside, CA, USA

Received 13 April 2018 / Received in final form 8 August 2018

Published online 28 January 2019

**Abstract.** The article addresses properties of strongly disordered electronic systems which are Anderson and often also Mott localized. Coulomb interactions are important. In the localized regime Coulomb interactions among electrons are important due to poor screening. Parts of the subject have been controversial, in particular with regard to the importance of dynamic many-body effects and of the role the one-particle density of states plays in transport. The main focus here is an attempt to review the current status of the subject and hopefully to contribute to a reduction in the controversies.

## 1 Introduction

Disordered systems became the accepted term used mainly (but not exclusively) for non-crystalline solids. These solids are of interest because they appear overwhelmingly in nature; every rock is non-crystalline, as are also biological systems. It is fair to say that this entire, now very wide and active field, was initiated by the works of Anderson [1], and Mott and Twose [2]. To do justice to history, let it however be mentioned that years earlier, in 1940s, Schrödinger realized the importance in biology of what we now call disorder [3]. Years before Watson and Crick, he foresaw genetic material to be non-repetitive sequences of molecules which he called “aperiodic solids”. Disorder may be a poor word for genetic material where the specific order is important and a small change in the sequence makes a difference between, e.g. a human and a pig. In that sense “disorder” could be used, slightly paraphrasing Miller [4]: “Disorder is a word we invented for order that is not understood”. Disordered solids are also of practical interest as they now proliferate in technology so it is desirable to understand their physical properties. Electrons in disordered systems are particularly interesting because they are very much quantum particles. Disordered solids are also interesting because, in contrast to traditional solid state physics, they do not possess the symmetries which in crystals determine many of the physical properties. This renders the subject considerably more complicated and mandates new non-traditional approaches, some of which still stir disagreements.

The literature in this field is somewhat disordered, papers often do not connect with each other and some aspects have been controversial for a long time, particularly regarding dynamical properties.

The main purpose of this paper is to review the existing disagreements and where appropriate to argue for one or the other side. Some broader recent reviews of electrons in disordered solids are [5–7].

<sup>a</sup> e-mail: [pollak@ucr.edu](mailto:pollak@ucr.edu)

## 2 Disorder

It is important to distinguish the disorder in a solid from, say, the disordered structure of the liquid. In the liquid, the disorder is such that, averaged over a rather short time, each point in space appears identical, while in the disordered solid the disorder is frozen, often referred to as “quenched disorder”, or changes very slowly in comparison to the electronic system under consideration here. By definition, such materials never reach equilibrium in an observable time and are thus by definition non-ergodic.

For the sake of quantitative treatment, one must be rather specific about the nature of disorder. One important type of disorder is the randomness in the energy of a site; another is the randomness in the distance to other sites. To hope for any quantitative results one needs to know the distributions of the random energies and of the random inter-site spacing. In many cases, on an inter-atomic scale, the material can be nearly ordered while on a longer distance scale it is disordered. Another important consideration is that two macroscopic disordered specimens of the same system should exhibit similar macroscopic physical properties. Thus, statistically speaking, for a large enough system their properties should be identical though microscopically they are different. The scale on which they appear statistically identical is very important. For specimens shorter than this scale one sees so called “mesoscopic effects”, i.e. differences in behavior from specimen to specimen.

An important consequence of disorder is so called frustration. What is meant by the term is that there is no configuration in which every particle can acquire its lowest possible energy. The result is a phase space with many energy minima, a situation referred to as “energy landscapes” in phase space [8] and references therein. It must be noted that frustration is not confined to disordered systems. The classic example of frustration in an ordered system is antiferromagnetically coupled spins occupying a triangular lattice. At least one pair of sites in each triangle must be occupied by parallel spins, violating their preferred mutual orientation. Experimental and theoretical studies of frustration in ordered systems exist in the literature, see e.g. [9,10] but they are apart from the topic of this article.

## 3 Anderson–Mott insulators

In the local function basis set the Hamiltonian can be written

$$H = \sum \varepsilon_k c_k c_k^+ + \sum t_{kl} c_k c_l^+ + \sum V_{klmn} c_k c_l c_m^+ c_n^+ \quad (1)$$

with  $c_k, c_k^+$  the creation and destruction operators on site  $k$ , respectively. The first term in  $H$  expresses the random site energies, the second expresses the quantum tunneling between sites  $k$  and  $l$  and the last term expresses the interaction between two electrons, one spreads (among others) between sites  $m, n$ , the other between sites  $k, l$ . For Coulomb interaction

$$V_{klmn} = a_{klmn} \left[ \langle \chi_k(r_1) \chi_l(r_1) | \frac{1}{|r_1 - r_2|} | \chi_m(r_2) \chi_n(r_2) \rangle - \langle \chi_k(r_1) \chi_m(r_1) | \frac{1}{|r_1 - r_2|} | \chi_l(r_2) \chi_n(r_2) \rangle \right]. \quad (2)$$

The first term in the square bracket is the Coulomb interaction, the second the exchange interaction, and  $|a_{klmn}| < 1$  is a numerical factor to do with normalization. The letter subscripts label sites, the numeral subscripts label electrons. To avoid very

complicated expressions, the preceding notation did not include the electron spin. It also implicitly invoked the tight binding approximation, allowing only a single orbital  $\chi$  on each site, as is commonly used to describe disordered electronic systems. Usually an  $s$  function

$$\chi_k(\mathbf{r}) = \exp[-(|\mathbf{r} - \mathbf{r}_k|/\xi)] \quad (3)$$

is used for simplicity.  $\xi$  is the so-called localization radius. This tight binding approximation is valid when all other local site functions are sufficiently high in energy (e.g. a  $2s$  function well above the  $1s$  function).

It must be pointed out that the set  $\{\chi_i\}$  is not an orthonormal set since  $\langle \chi_k | \chi_l \rangle \neq 0$ . An appropriate orthonormal set is the set of Wannier functions [11,12] formed from  $\{\chi_k\}$ . The Wannier functions tend to be more localized than the local functions because at nearest neighbors of  $k$  the Wannier “correction” to  $\chi_k$  has an opposite sign to that of  $\chi_k$ . Thus, whenever orthogonality of  $\{\chi_k\}$  is important it will be assumed implicitly that  $\{\chi_k\}$  refers to Wannier functions.

An additional approximation customarily made is to reasonably confine an electron to a single site. The tight binding Hamiltonian then becomes

$$H = \sum_i \varepsilon_i n_i + \sum_i u_{i\uparrow i\downarrow} n_{i\uparrow} n_{i\downarrow} + \sum_{j < i; \sigma, \sigma'} u_{i\sigma j\sigma'} n_{i\sigma} n_{j\sigma'} + \sum_{i \neq j} t_{ij} a_i^\dagger a_j. \quad (4)$$

$n_i = a_i^\dagger a_i$  is the occupation number on site  $i$  limited to  $n_i = 0, 1, 2$ ,  $u_{ii}$  is the intra-site interaction energy, and  $u_{ij}$  in the third term is the inter-site interaction energy. The two terms correspond to the last term in (2). Either  $\varepsilon_i$  or  $t_{ij}$  or both can be random variables. A random  $\varepsilon_i$  is often called diagonal disorder, a random  $t_{ij}$  off-diagonal disorder. Notice that the last term in equation (4) is the only term involving quantum mechanics. In many cases  $u_{ii}$  is larger than the other important energies (corresponding to Mott localization) and  $t_{ij}$  is rather small (corresponding to Anderson localization); these conditions referred here as Anderson Mott insulators are the main subject of this paper.

If  $n_k$  is the occupation number of electrons (or holes) then equation (4) violates charge neutrality. To retain charge neutrality it is common to place a positive charge  $eK$  into each site where the compensation  $K = (N - n)/N$  makes the charge on occupied sites  $e(n-K)$  and on unoccupied sites  $-eK$ .  $N$  is the number of sites,  $n < N$  the number of electrons. The last inequality is not compelling but is satisfied in most cases of interest.

## 4 Some specific materials

The classic disordered systems, studied extensively in the fifties and sixties, are doped and compensated crystalline semiconductors of group IV, mainly silicon and germanium at concentrations below the metal–insulator transition. At small  $T$  the carriers are confined to the impurity sites which are disordered both diagonally due to the compensation and off-diagonally due to random impurity location. The most extensive experimental studies were on transport, by Fritzsche’s group e.g. [13], by Davis and collaborators e.g. [14] and at the Yoffe Institute in (then) Leningrad e.g. [15]. The original theoretical hopping model for transport in impurity conduction is due to Mott [16] and Conwell [17]. The most ground breaking theoretical paper is arguably [2] which recognized the importance of Anderson’s now famous work on localization [1] in a disordered system as one of the important papers of the period. Following the incipient understanding of electrons in impurity conduction, studies were broadened

to electrons in many other disordered materials, including group IV and other amorphous semiconductors, granular metals, amorphous chalcogenides including phase change materials, oxide and other glasses, organic semiconductors, conducting polymers, some biological macromolecules, and very recently disordered graphene [18,19]. One of the most extensively studied disordered electronic systems is indium oxide, microcrystalline as well as amorphous, see [20] and references therein, because of the ease to change disorder and carrier concentration. The Hamiltonian of equation (4) is less representative of some of those materials for several reasons, primarily because the localized states are not identical and because they are populated by large numbers of electrons. It must be emphasized that the localization length  $\xi$  in such materials is not the size of the localized states but rather the decay length of the wave function between adjacent localized states [7].

## 5 The one-particle density of states (DOS)

The one-particle DOS in the interacting system is defined as the spectrum of energies into which an extra particle can be inserted while the rest of the system remains in its original ground state. For brevity, the electrons that form the DOS will be called immigrants and those in the ground state will be called residents. The long range Coulomb interaction dramatically modifies the one-particle DOS by depleting states near the Fermi level [21,22]. The basic argument is that inserting a new electron into a system in its ground state is bound to have other electrons move in order to reduce the interaction energy with the immigrant electron (i.e. to form a quasi-particle). Before such relaxation the electron energy was clearly higher, i.e. some finite distance above the Fermi level. Thus, immigrant electrons cannot be accommodated at the Fermi level. After relaxation into a new ground state the Fermi energy cannot have moved a finite distance (in the thermodynamic limit) because the thermodynamic DOS,  $dn/d\mu$  is gapless, as pointed out in [23]. Thus, the immigrants constitute a set of excited electrons.

Efros and Shklovskii [24,25] derived the DOS energy dependence

$$N(E) = \frac{d \cdot |E|^{d-1}}{\pi e^{2d}} \quad (5)$$

from the consideration that the ground state must be stable with respect to one-particle excitations, i.e. that no one particle transition can bring the system to a lower energy. Efros [26] later considered stability to any  $n$ -particle excitation to derive, within some approximations,

$$N(E) \sim \exp \left[ -c \left( \frac{E_g}{E} \right)^{1/2} \right]. \quad (6)$$

The DOS of equation (6) is the preferred equation over equation (5) since it refers, at least approximately, the DOS to the ground state, as required by the definition of the DOS. It describes a much harder Coulomb gap than (5), as a consequence of the lower ground state energy compared to states stable to just one-electron configurations [27].

## 6 Specific heat

In metals the specific heat of electrons is well described by proportionality to  $T$  because the DOS near the Fermi level can be assumed to be constant within the

narrow energy regime of a few  $kT$ . This would be normally also expected to be true in the Anderson localized regime. However, Coulomb interactions greatly affect the low energy states, important for the low temperature specific heat. They tend to be predominantly many-electron excitations [28] with a density quite different than in a non-interacting system, thus strongly affecting the specific heat. Early computer simulations [29] find that the specific heat at low temperatures still increases linearly with temperature. This is confirmed later by more extensive computer simulation studies by Tenelsen and Schreiber [30], by Möbius and coworkers [31,32] and by Yu's group [33]. All these authors find that at low  $T$  the specific heat increases linearly with  $T$  but the magnitude is quite different from what is expected without interaction. Their results show a peak at higher temperature. Past the peak the specific heat decreases, presumably as the system gets into a regime where the temperature randomizes the site occupation. In [33], Overlin et al. consider systems without site energy disorder. They generate disorder by randomly shifting sites from their positions in a perfect lattice. Möbius and coworkers [31,32] on the other hand keep a perfectly ordered lattice and introduce site energy disorder. Interestingly, in both the diagonal and the off-diagonal disorder cases the specific heat at low temperature and large enough disorder appears to be linear with temperature.

As to experiment, there are very few to be found in the recent literature on hopping systems. Two that I found are [34] and [35]. Both found that the specific heat increases linearly with  $T$  and that the resistivity follows Mott's variable range hopping (VRH) law. Reference [34] measured the specific heat in a  $\text{Ti}_x\text{Si}_{100-x}$  ( $x \ll 100$ ) alloy and [35] in doped silicon. The fact that the conductance follows the Mott VRH law indicates a finite DOS at the Fermi level, but a lack of significant electron-electron interaction in this material. Thus, the linear  $T$  dependence would appear to be just the usual result of a constant DOS near the Fermi level and not a support of the computer simulation works on interacting disordered systems. However, in [34] the DOS at the Fermi level obtained from the specific heat is two orders of magnitude larger than that obtained from resistivity measurements. Providing the interpretations of the experiments with the VRH picture and with a specific heat linear in  $T$  apply to the experiment, one must conclude that most excitations do not contribute to dc conductance. One would expect them to contribute to polarization, but unfortunately, no polarization experiments were reported in [34]. In [35], a simple interpretation of the experiments was found as well and was attributed to internal excitations within the localized states. Thus, the few experiments on specific heat in disordered system have at present no theoretical explanation.

## 7 Transport

This subject has been long controversial in relation to systems where Coulomb interactions between electrons are important. The most profound disagreements relate to the importance of many-body effects [7,36] versus [24,25] and to the role the DOS plays in determining transport in the linear response regime [27].

Before proceeding it is useful to briefly summarize transport for cases where electron-electron interactions can be neglected. There is little controversy there, and the methods used can be extended to the interacting systems.

Two theories, percolation theory [37–39] and a scaling theory [40] yield compatible results. Both utilize the exponentially wide distribution of transition rates  $\Gamma_{ij}$  between two localized states,  $i, j$ . Miller and Abrahams (MA) [41] showed that transport in the linear response regime can be mapped onto a system of random resistances  $R$  connected between all pairs of sites. The  $R_{ij}$  are derived from the transition rates  $\Gamma_{ij} = \omega_0 e^{-2r_{ij}/\xi - E_{ij}/kT}$  for  $E > 0$  and  $\omega_0 e^{-2r_{ij}/\xi}$  for  $E < 0$ ,  $\omega_0$  is an attempt frequency, approximately a phonon frequency of order  $10^{-12} \text{ sec}^{-1}$  [41].  $R_{jk}$

is also an exponential function of the random site energies and distances between the sites

$$\begin{aligned} R_{jk} &= (kT/e^2)\omega_0^{-1} \exp(2r_{jk}/\xi + E_{jk}/kT), \\ E_{jk} &= \max(|E_j - E_k|, |E_j - E_F|, |E_k - E_F|). \end{aligned} \quad (7)$$

The exponentially wide distribution suggests that the resistance of any path is dominated by the path's largest resistance and the overall resistivity determined by the smallest of those in the total set of possible paths. Percolation theory determines the value of that resistance. The scaling theory [40] relies on the self similarity of certain distributions and the results are fully compatible with percolation theory. The concept of the scaling theory is easy to understand for completely random spatial and energy distributions. In such a case any realization at one concentration  $\nu$  will have a match at some other  $\nu'$  such that (in the thermodynamic limit) there is a one to one mapping of sites in the two systems. The spacings  $r_{kl}$  between sites will map as  $r_{kl}(\nu) = \alpha r_{kl}(\nu')$  and the site energies as  $E_k(\nu) = \beta E_k(\nu')$ . The energy range  $E_m$  of hopping, and thus  $\nu$ , increase with  $T$  (so one can replace the arguments  $\nu, \nu'$  by  $T, T'$ ). In order that paths at  $T$  and  $T'$  have the same probability the number of active sites should scale as the energy ranges  $E_m$ , which leads to  $[r(T)/r(T')]^{-d} = E(T)/E(T')$  or  $\alpha^{-d} = \beta$  ( $d$  is the dimensionality). Of interest is the path of minimum resistance. This involves some specific  $R_{kl}$ , the largest resistance in the paths. Since  $\ln(x)$  is a monotonically increasing function of  $x$  one can minimize  $\ln(R) = 2\alpha r/\xi + \alpha^{-d} E/kT$  with respect to  $\alpha$ . Substituting the resulting  $\alpha$  into  $R$  leads to the Mott VRH result  $\sigma = \sigma_0 \exp(-T_0/T)^{1/(d+1)}$ .

While the above applies strictly to random distributions, the scaling procedure can be applied to a much wider class of distributions as explained in detail in [40]. In the somewhat more general case of a random spatial distribution and an energy distribution  $\sim E^m$ ,

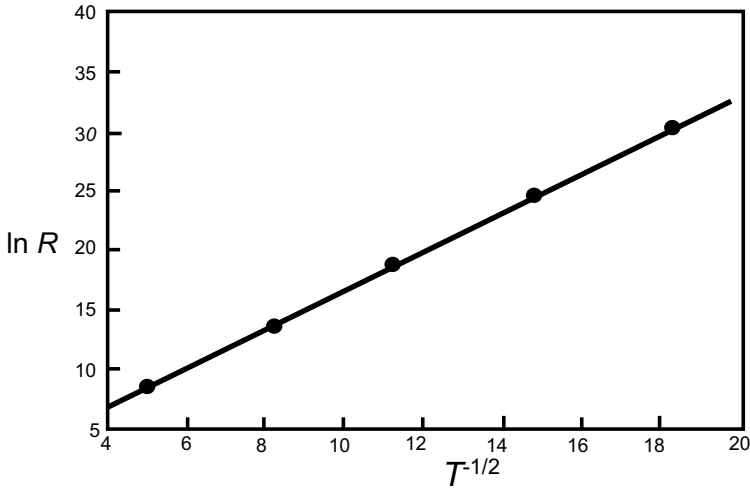
$$\sigma = \sigma_0 \exp(-T_0/T)^{(m+1)/(m+d+1)}. \quad (8)$$

A widely used theory for the resistivity of interacting systems derives dc conduction from the one-particle DOS of equation (5), see [24,25,42] and references therein. Using equation (5), compatible with the pseudo-ground states and percolation theory, one obtains from (8) for the conductivity

$$\sigma = \sigma_0 \exp(-T_0/T)^{1/2} \quad (9)$$

as illustrated by Figure 1. The behavior expressed by (9) has been reported in many experiments (see, e.g., Tab. 1 in [43], and [5]) and is referred to as Efros-Shklovskii (ES) VRH.

Had percolation been applied to the more appropriate equation (6) for the DOS, rather than to (5), the exponent in (9) would be much closer to 1. Furthermore, the one-particle DOS is the energy spectrum of an extra particle inserted into a system in the ground state, i.e. an *excited* particle as explained in Section 5. Such particles relax in energy rather than carry the dc current. (For an experimental verification see Fig. 2 and the discussion around it). Finally, a basic tenet of one-particle transport theories (in this case percolation theory) is that particles move independently of each other but interactions imply correlation in the motions of particles. This is one way interaction reduces the entropy from the more random uncorrelated motion to a less random correlated motion. The correlated motion mandates a description in terms of many-body effects, which is beyond the capability of a one-particle transport theory.



**Fig. 1.** Computer simulation result of percolation for the system resistance  $R$  in configuration space. After Somoza et al. [47].

Two dynamic many-body effects that correlate the motion of the hopping particles are important – many-body transitions in a single quantum mechanical process, for which

$$R_{KL} \propto (kT/e^2) \exp(-E_{KL}/kT) \exp(-\sum r_{kl}/\xi), \tag{10}$$

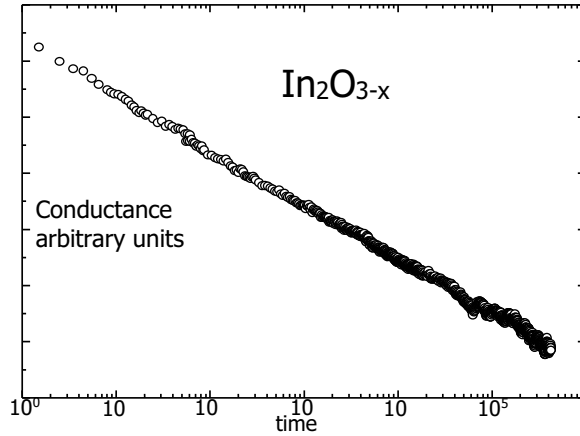
and sequentially correlated transitions [21,44]. In equation (10), the capital subscripts annotate configurations and the lower case subscripts annotate sites in the respective configurations, with  $k \neq l$ , namely sites which differ in  $K$  and in  $L$ . The many-body transitions come about because the interaction part of the Hamiltonian is a two-particle operator, and the sequential correlations occur because a rearrangement of charges, inherent in a transition, changes the interaction energy. Different sequences of transitions traverse different intermediate states, so the characteristic rates for different sequences are different. The general processes are thus sequentially correlated many-body transitions. Neither of these processes can be incorporated in a one-particle theory, e.g., in a real-space percolation theory.

Consider a simple example of a transition of two electrons in four sites. It changes from collective to sequentially correlated with increasing temperature. The resistance for a two electron transition is  $R \sim \exp(E/kT) \exp(r_1/\xi + r_2/\xi)$ . The successive transition resistance is effectively the larger of

$$R_1 \sim \exp(E_1/kT) \exp(r_1/\xi); R_2 \sim \exp(E_2/kT) \exp(r_2/\xi) \tag{11}$$

where  $r_1$  and  $r_2$  are the lengths of the hops of the two electrons and  $E_1, E_2$  are the associated excitation energies. In passing it is noted that the two lengths are not uniquely determined because of different exchange ways to get from the same initial to the same final state. Of all those exchange possibilities the important (multi-electron) jump is the one with the smallest sum of the constituent one-electron hop lengths.

In a two electron transition  $E < E_1, E_2$  when the two electrons can avoid each other and, say,  $R_1 > R_2$ . Then the transition from the collective to the sequentially correlated process occurs at  $(E_1 - E)/kT = r_2/\xi$ . More generally, for a large system, as



**Fig. 2.** Relaxation of conductance after excitation by gate voltage. Courtesy Z. Ovadyahu.

$T$  increases one can expect that a general transition will see an enhanced importance of sequential transitions and a decreasing importance of collective transitions.

There are additional reasons to question the use of the one-particle DOS for calculating transport.

- By definition and derivation, the DOS is the spectrum of energies at which a new particle can be inserted into the bulk of a system in its ground state. Transport measurements are not performed on a system in its ground state.
- Inserting particles through electrodes differs from insertion of electrons into the bulk.
- As already mentioned, the immigrant new particle is in an excited state and thus outside the linear response regime of the conduction measurements.
- After a few transitions the immigrant electron loses energy and its immigrant identity when it becomes absorbed among the resident electrons. The dynamics of the immigrants thus constitutes relaxation rather than dc conduction. This is clearly confirmed by experiment as can be seen in Figure 2 where immigrants are inserted by a change of a gate voltage. The DOS thus cannot be relevant to the dc transport observed in the interacting systems.

A couple of transport theories were proposed that do incorporate many-body effects. One is a generalization of percolation theory to configuration space (i.e. Fock space), the other is a generalization of a scaling theory to configuration space.

It was proposed by Ortuño [45–47] that one can incorporate the many-body effects in a percolation theory in configuration space (i.e. in Fock space) rather than in real space. Clearly such a procedure incorporates collective transitions. A transition between two configurations generally corresponds to a transition of a number of electrons. It also incorporates sequential correlations – changing the sequence of two transitions involves a different intermediate state and thus a different total transition rate. Transition from collective hopping at low  $T$  changes to sequentially correlated hopping at higher  $T$ .

Two necessary conditions in such a percolation process are needed to ensure that in the sequence of transitions which constitutes the percolation path, at least one electron is transferred through the system, and that the final step returns the system into the original configuration. The former guarantees transport through the system,



the latter guarantees that the entire process can be repeated indefinitely, i.e. that the process sets up a dc current.

Such a percolation has been executed by computer simulation [47]. The result is displayed in Figure 1 and fully in agreement with equation (9) and with experiments.

Applying the scaling method to the Fock (or configuration) space in an interacting system [48], the bonds between configurations are given by equation (7). The energies and the distances in that expression need to be scaled. It is convenient to refer the energies to the ground state energy, denoted by the superscript 0  $E_K = \sum_i \varepsilon_i (n_i^K - n_i^0) + \sum_{i<j} (n_i^K n_j^K - n_i^0 n_j^0) / 2r_{ij}$  where  $n_i^K$  is the occupation of site  $i$  in configuration  $K$ .

A difficulty arises with the scaling of the energies  $E_{IJ} = \max(E_I, E_J)$  which depend on random energies *and* on lengths via the Coulomb interactions. The difficulty is alleviated at low  $T$  at which the transport is restricted to sites close to the Fermi level where  $\varepsilon_i$  is small in comparison with interaction energies. At sites further from the Fermi level  $n_i^K - n_i^0 = 0$ , because the occupation at sites far from the Fermi level does not change at low  $T$ . Thus at low  $T$  the term  $\sum_i \varepsilon_i (n_i^K - n_i^0)$  drops out from the last equation. Then  $E_{IJ}$  simplifies to

$$E_{IJ} \approx \sum_{i<j} (n_i^K n_j^K - n_i^0 n_j^0) / 2r_{ij} \tag{12}$$

where  $K = I$  or  $K = J$  in accordance with which is the higher energy. So, at low temperature, both the energy terms  $E_{IJ}$  and the tunneling terms  $r_{IJ}$  of the resistances  $R_{IJ}$  in equation (7) are determined by the random variables  $r_{ij}, i \in I, j \in J, i \neq j$ . The energy term involves distances to sites that change occupation in the transition and the tunneling term,  $2\sum r_{ij} / \xi \equiv 2r_{IJ} / \xi$ , involves the sum of hopping distances, minimized with respect to the different exchange alternatives of electrons transiting between  $I$  and  $J$ . The  $r_{ij}$  in the bonds in the configuration space at  $T'$ ,  $\varsigma(T') = (E_{IJ})_{T'} / kT' + (r_{IJ})_{T'} / \xi$  need to be scaled by some  $c$  to become the bonds at  $T$ :

$$\begin{aligned} \varsigma(T') &= (E_{IJ})_{T'} / kT' + (r_{IJ})_{T'} / \xi \\ \Rightarrow (1/c)(E_{IJ})_{T'} / kT' + c(r_{IJ})_{T'} &= \varsigma(T) = (E_{IJ})_T / kT + (r_{IJ})_T / \xi \end{aligned} \tag{13}$$

at the temperature  $T'$ . The bonds of interest are  $\zeta_m$ , corresponding to the critical resistances. The proper  $c$  is found by minimizing these  $\zeta_m(T)$ .

$$c = \frac{[(E_{IJ})_{T'} / (r_{IJ})_{T'}]^{1/2}}{(\xi kT)^{1/2}} \tag{14}$$

which results in

$$\varsigma_m = 2 \frac{[(E_{IJ})_{T'} / (r_{IJ})_{T'}]^{1/2}}{(\xi kT)^{1/2}} = (E_{IJ})_T / kT + (r_{IJ})_T / \xi. \tag{15}$$

Rewriting the second equality in (13),

$$[(E_{IJ})_{T'} (r_{IJ})_{T'}]^{1/2} = \frac{1}{2} [(E_{IJ})_T / kT + (r_{IJ})_T / \xi] (\xi kT)^{1/2}. \tag{16}$$

The left hand side depends on  $T'$  and the right hand side on  $T$ . So, the equation must be independent of temperature. Thus, the subscripts  $T'$  can be dropped from the product  $E_{IJ} r_{IJ}$  (but not from the individual factors). Furthermore, equation (16) relates  $E_{IJ} r_{IJ}$  to a critical resistance and since all critical resistances are equal,

$E_{IJ}r_{IJ}$  is also independent of  $I, J$  remembering only that  $I, J$  are configurations that bracket a critical resistance. The individual factors can of course differ from one critical resistance to another but their product remains the same. Dropping the subscripts from  $E_{IJ}r_{IJ}$  (but remembering that  $Er$  relates to the critical resistances) (16) finally reduces to

$$\begin{aligned} \varsigma_m(T) &= 2(Er)^{1/2}/(\xi kT)^{1/2} \equiv (T_0/T)^{1/2}, \text{ or} \\ \sigma &= \sigma_o \exp \left[ - (T_0/T)^{1/2} \right], \quad T_0 = 2Er/\xi k \end{aligned} \quad (17)$$

which, except for a different value of  $T_0$ , corresponds exactly to equation (8). For comparison, in the units used here (electron charge = 1, dielectric constant = 1), in the ES theory  $T_0 = \beta/ak$  with  $\beta = 6.2$  for 2d and  $\beta = 2.8$  for 3d.  $Er$  cannot be evaluated by scaling alone and thus an expression for  $T_0$  does not yet exist. All one can say at present is that  $T_0$  here must be smaller than that of Efros and Shklovskii [24,25] because many-electron transitions lower the activation energy.

## 8 High carrier concentration

By high concentration will be understood here carrier concentrations above  $10^{19} \text{ cm}^{-3}$  or so. A large number of such systems [49] were seen to exhibit glassy effects, namely extremely long relaxations from excited states to equilibrium that violate ergodicity, and so-called aging which violates time homogeneity. By time homogeneity is meant that two experiments initiated at different time give similar results, namely to depend only on  $t - t_0$  where  $t_0$  is the initial time of the experiment. It turns out that this expectation is not always fulfilled in disordered systems. The reason for this violation is the violation of a very fundamental hypothesis of statistical mechanics, namely ergodicity. The extremely long relaxation together with aging can be taken to define the modern concept of a glass. In addition to the structural glass, known for thousands of years, there are now a number of disordered systems that show glassy properties, e.g. the widely studied spin glass [50], polymer glass (e.g. [51]) and others. But it is rather unusual for the light electrons to respond so sluggishly as to violate ergodicity. As a group these materials are called electron glasses or sometimes Coulomb glasses. By now there is a long list of such materials [49]. Of these, indium oxide and some granular metals were studied most intensively, the former by Ovadyahu and coworkers, the latter by several groups, including Grenet and Delahaye at Grenoble, Frydman's group at Bar Ilan University and Goldman's at Minnesota. The first experiments that hinted at electron glassiness were by Adkins et al. [52] in Cambridge.

The high density systems are not accurately described by the tight binding Hamiltonian (4) in part because  $n_i$  is not restricted to 0,1,2 but can be much larger, and because  $\xi$  is not necessarily uniform;  $\xi$  here should not be taken to be the size of the localized wave-functions but rather the much smaller inter-site decay length because that determines the value of  $t_{ij}$  (which is still an exponential function of a random variable). In a simplified model, large many-electron localized states can be considered as regions of metallic conductance, that are intermingled with the hopping MA resistances. Since in the metal the resistivity increases with  $T$  while in hopping the opposite is the case, it is possible that above some  $T$  the resistance will look metallic while at lower  $T$  it will look semiconducting. Such effects have been experimentally observed [53] but this experimental phenomenon has not been adequately explored to rule out other possible causes for it.

There is a number of processes that slow down the response of electrons from their very fast motion in crystals. In the first place it is of course the Anderson localization. *Per se*, this is not enough to make them glassy – glassiness has not been observed

in impurity conduction or in amorphous group IV semiconductors. In systems with high electron concentration, sluggishness is strongly enhanced by collective many-body transitions because the second exponent in equation (10) constitutes a product of exponentially small factors. Before a dc current is established all subpercolation clusters must polarize after an electric field is applied. At high electron concentration, the many-electron transitions become important in the polarization of the subpercolation clusters. (It should be mentioned that large subpercolation clusters can be quite slow even in the absence of interaction [54,55]).

An additional, less important effect which can slow down the response (even in non-interacting systems) is the fractal nature of the critical percolation cluster. Thinking of relaxation rates in terms of an  $RC$  product, the conductance is confined to a low fractal dimension, enlarging  $R$ , while the displacement vector is always three-dimensional, enlarging  $C$  in comparison [56]. This enhanced  $RC$  time must be overcome before a dc current can be established.

## 8.1 Experiment

The initial step in experiments on electron glasses is to cool the material to liquid He temperatures, excite electrons far from equilibrium, and measure the conductance – a property measurable with great accuracy. The conductivity in the excited state is higher than the conductivity at equilibrium because excited electrons can decay into more empty states than equilibrium electrons.

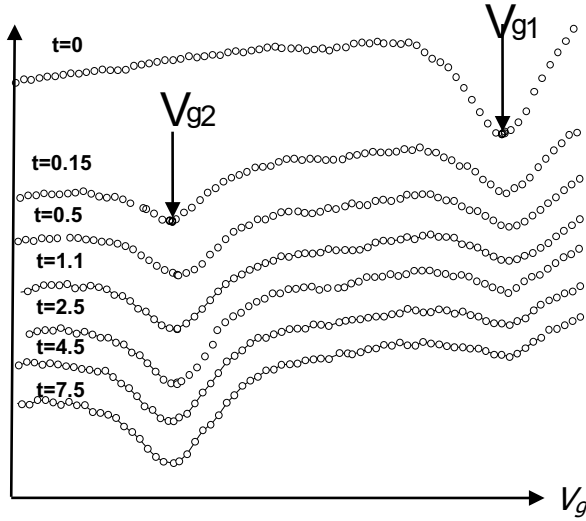
Excitation of the electronic system can be accomplished by several means. The most common is by a sudden change in gate voltage, say by  $\Delta V_g = V_{g2} - V_{g1}$  in MOS structures where the active S layer is the Anderson insulator. The simplest experiment is to measure the time dependent in-plane current following the excitation. Most of these experiments are performed after the sample is for hours or even days at liquid He temperature. The result is a logarithmic decay of the excess conductance towards equilibrium at  $V_{g2}$ . In indium oxide such decay has been observed for  $1 \text{ sec} < t < 10^5 \text{ sec}$  [57]. Even after  $10^5 \text{ sec}$ , of the order of 1 day, there is no hint for  $G(t)$  to level off (i.e. approach equilibrium) – the system is non-ergodic.

A frequently used protocol, called a “two dip experiment” is to hold  $V_g$  at  $V_{g2}$  but scan it rapidly through an interval  $>\Delta V_g$  at times  $t_1, t_2, \dots t_p$ . A typical example of results is shown in Figure 3. Two dips in the conductance  $G$  are observed, a diminishing dip around  $V_{g1}$  and a growing dip at  $V_{g2}$ . The latter is just another way to observe the logarithmic decay towards equilibrium, the former is a gradual “forgetting” that the system was for an extended time at  $V_{g1}$ .

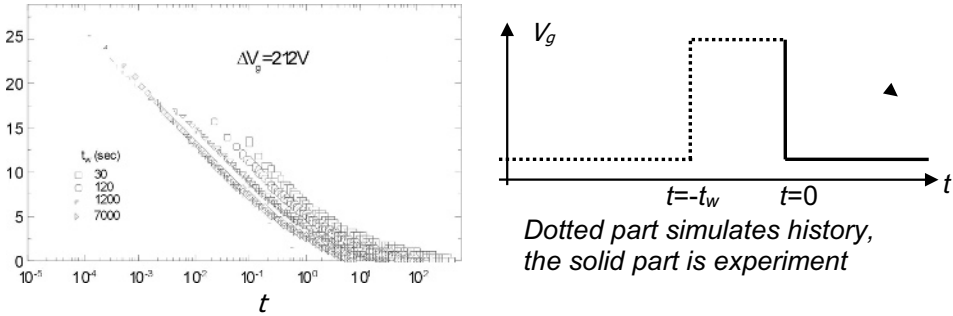
Another common experiment utilizing excitation by  $V_g$ , indicated in Figure 4, is to return the system rapidly from  $V_{g2}$  to  $V_{g1}$  after a time  $t_w$ , following which  $G(t)$  is measured. The interesting result is the dependence of  $G(t)$  on  $t_w$ . The expectation for ergodic systems is  $G(t) = G(t - t_0)$ , i.e. for  $G(t)$  to be independent of the initial time  $t_0$  of the experiment, due to time homogeneity. The violation of time homogeneity is clearly shown in Figure 4. It is referred to as “aging” in the context of glass. Aging is a consequence of the violation of ergodicity – the internal state of the system changes slowly as it spreads through configuration space and the response to applied fields changes with it. Interestingly, it turns out that  $G(t/t_w)$  is independent of  $t_w$ . This behavior is called full aging or simple aging.

More complex experiments have been conducted by a variety of protocols following the excitation [58,59].

Other types of excitations – electromagnetic radiation [60], longitudinally applied high electric field [61], and thermal excitation [61] were also used in experiments. They all show similar glassy relaxation with the exception of excitation by microwaves. A likely cause for the latter is that the energy  $\hbar\omega$  at microwave frequencies is inadequate



**Fig. 3.** The two dip experiment (see text). Courtesy Z. Ovadyahu.



**Fig. 4.** Aging experiment. The lines in the left panel are the time-dependent conductances at  $t \geq 0$  for different  $t_w$ . Left panel courtesy A. Vaknin.

to excite electrons far enough from equilibrium. The enhancement in conductivity by subjecting the system to microwave radiation was attributed to enhancement of the electric field beyond the linear response regime [62]. Recently, a new type of experiment was performed on discontinuous metal films [63] that were prepared at cryogenic temperatures and thus were never brought out of a glassy state. The very interesting new results are that glassy relaxation extends at least up to room temperature and depends not only on the temperature where it is measured but also on the maximum temperature to which the material was warmed.

There is now a large number of disordered materials observed to be electron glasses at low temperature and new ones keep coming up. For a relatively recent list see [49]. To indicate the rate of proliferation, at least three new electron glasses have been added to the list since then, conducting polymers [64], and thin films of non-stoichiometric GeTe [65], GeSbTe [66], and GeBiTe [67].

### 8.2 Theory

Different theoretical approaches have been used by different authors. The results are not always in agreement with each other. The current literature on the subject is

summarized below. Pollak and Ovadyahu [68] developed a theory invoking very slow formation of quasiparticles. This occurs due to relaxation by multi-electron transition equation (10) (see also [7,36]). The crucial property is the exponential dependence of the rates on random variables. First the time dependent energy relaxation  $\Delta E$  from an excited state is evaluated, using the distribution  $N(\Gamma)$  of transition rates  $\Gamma=e^{-\eta}$ .  $\eta$  is a random variable (made up of random hop distances and random energies) with a distribution  $N(\eta)$ . The distribution of  $\Gamma$  then is  $N(\Gamma)=N(\ln\Gamma)/\Gamma$ . The time dependent drop  $\Delta E(t)$  is then

$$\Delta E(t) = \delta E \int_0^{\Gamma_{\max}} \frac{N(\ln\Gamma)}{\Gamma} e^{-\Gamma t} d\Gamma \simeq \text{const.} \int_{\Gamma_{\min}}^{\infty} \frac{e^{-\Gamma t}}{\Gamma} d\Gamma. \quad (18)$$

$\delta E$  is some average drop in energy per decay. The second equality is based on  $N(\ln\Gamma)$  being a very slowly varying function of  $\Gamma$ , except that it must drop off sharply at some  $\Gamma_{\min}$ , and  $\Gamma_{\max}$ . The latter is of the order of a phonon frequency, much larger than measurements allow, so it may be approximated by infinity. The right hand expression will be recognized as the exponent integral. The exact form of the drop-off is immaterial as long as measurements are restricted to  $t \ll 1/\Gamma_{\min}$ . The exponent integral is then very well approximated by  $\beta - \ln(\Gamma_{\min}t)$ . The Euler constant  $\beta$  is time independent, so the energy decay is logarithmic.  $\Gamma$  is a many-electron transition rate; hence  $\Gamma_{\min}$  can have an extremely small value. As an example, for a collective transition of ten electrons, each of them jumping a distance  $2\xi$ ,  $1/\Gamma \sim 10^{-12} \text{ sec} \times e^{40} \approx 2 \times 10^8 \text{ sec}$ , or about 1 year. To appreciate the role of collective transitions: for relaxation by single particle transitions,  $1/\Gamma$  for a hop of length  $2\xi$  is of the order  $10^{-10} \text{ sec}$ .

The measurable quantity is conductance rather than the energy considered so far. A monotonic relation between the two exists – the lower the energy the more collective is transport. Assuming a linear relation, the conductance relaxes logarithmically as well.

Implicit in equation (18) is the assumption that each relaxation in the sequence is to a state from which there is no further relaxation. This is not necessarily accurate – a slow downward transition could be to a state from which a faster downward transition is possible. But that is rather unlikely here; downward transitions become more and more collective, and thus slower, as the energy decreases. In fact, this may be a more appropriate interpretation of the experimental logarithmic relaxation. If the relaxation rates  $\Gamma$  diminish with energy reduction as  $E = \alpha \ln(\Gamma/\Gamma_0)$  (notice that  $E$  is a negative number).

$$\Delta E = \int dE = \alpha^{-1} \int \exp(-\Gamma t) d \left[ \ln \left( \frac{\Gamma}{\Gamma_0} \right) \right] = \frac{\Gamma_0}{\alpha} \int \frac{\exp(-\Gamma t)}{\Gamma} d\Gamma \quad (19)$$

which is basically the same as equation (18). This scenario is also more in keeping with the above theory and is somewhat in the spirit of [69]. This scenario may also account for observed departures from precise logarithmic relaxation in some systems where the relation between  $\Gamma$  and  $E$  is different from the above.

Müller and Pankov [70] (see also [71]) applied Parisi’s theory [72] of the Sherrington and Kirkpatrick mean field spin glass model [73] to disorder-localized electrons. It is a mean field model in the sense that an electron on each site “sees” a similar environment, interacting with all other electrons independently of distance. The work is a tour de force theoretically and includes many-body effects. Interestingly, even though many-body effects are included, it recovers the soft gap of equation (5) rather than the harder gap, equation (6).

The dynamics in [70] do not apply to electron glasses with  $R_{IJ}$  resistances described by (7). As in MA resistances, equation (7) is based on phonon assisted quantum tunneling between states. In [70], the transition rates are assumed to be exponential functions of energies alone rather than of energies and hopping distances. The dynamics in [73] corresponds to processes governed by hops over energy barriers (as in the spin glass model). While such transitions have been proposed for electrons in some disordered materials [74], it is believed that the observed electron glasses obey (7). The quantum aspects of the transition rates contained in  $t_{ij}$  are not included in [70]. In the spin glass model, the distance in the configuration space (in terms of transition times) is the Hamming distance while in the electron glass it is different and, moreover, is temperature dependent. It is also not clear how well one can describe a very disordered real system with the assumption that all sites see a similar environment.

Another mean field theory [75–77] does preserve the local identity of sites. It is mean field in the sense that transitions are governed by the thermodynamic mean site occupation at some local energy minimum rather than by the condition  $n_i + n_j = 1$  for the occupation, mandated for a transition between  $i$  and  $j$ . The theory considers the relaxation of the site occupation from an excited state towards equilibrium occupation and assumes that the occupation is linearly related to the conductance. It does not include the dynamic many-body effects; in this sense it is close to the ES model and yields the soft gap of equation (5). In a later paper [78], the authors recognize that collective transitions must be involved in the very sluggish relaxation. The theory bears some similarity to that of [68] in arriving at equation (18), again due to the exponential dependence of transition rates on a random variable.

Doubts were expressed about the intrinsic glassiness of the very light electrons. The argument made against it is that the conductivity is measurable, so the critical resistance, presumably the largest resistance of importance, does not correspond to very long times. Say  $R_c$  is of order  $10^8$  Ohm. According to equation (10),  $\tau \equiv \omega_0^{-1} \exp(2r_c/\xi) = R_c e^2/kT \approx 10 \mu\text{s}$ . where  $r_c$  is the sum of several hopping distances and  $R_c$  the associated critical resistance. This argument against the very slow relaxation is however not compelling for two reasons. It implicitly assumes that the measured conductance is the equilibrium conductance. But since the system is glassy, the measured conductance is *not* the equilibrium conductance, which cannot be measured and may be orders of magnitude larger. Furthermore, relaxation towards the very specific equilibrium state at low  $T$  may be slower than relaxation towards a less rigidly defined percolating state.

Ovadyahu [57] suggested that the non-ergodic relaxation may be due to quantum friction, discussed extensively by Leggett et al. [79] for a two level systems exposed to a dissipative environment. If this suggestion proves to be correct it will be of considerable interest since the microscopic situations described by the two systems seem totally different, in particular because [79] addresses a symmetrical (or “ordered”) two level system.

### 8.3 Simulations of glassy relaxation

Numerous computer simulations on glassy relaxation of various physical properties are reported in the literature.

The section on specific heat  $c$  implicitly assumed thermodynamic equilibrium, since  $c$  is considered to be an equilibrium property. In reality, however, the very slow transition rates allow only a slow exploration of the phase space. These non-ergodic effects on specific heat were studied in [80]. The computer allowed the system to spread in phase space, while tracking the specific heat. This was done for different temperatures, dimensionalities, disorder models, and degrees of disorder. Under the

conditions simulated by the computer it took orders of magnitude longer than any reasonable experiment to reach the equilibrium value of  $c$ . Interestingly, very slow evolution of the specific heat was observed in an actual system displaying glassy properties [81].

Appreciable computational effort was dedicated to the evolution of the Coulomb gap and generated controversies. For example, Yu [82–84] found that the Coulomb gap evolves very slowly while Tsigankov and Efros [85,86] assert that it forms very rapidly. In good part the controversy has to do with the importance of collective many-body transitions. Arguments against collective transitions arose from a misunderstanding about the range over which collective transitions are effective. They believe that the authors in [87–89] allow for collective hopping over large distances which they deem unphysical. However, equation (10) severely limits the range when the gain in energy due to interaction outweighs the loss due to an increase in total hopping distance. In [90], the authors of [85,86] do attribute the experimental glassy relaxation to collective transitions between pseudoground states.

In a computational effort to assess directly the effect of multi-particle transitions Tsigankov and Efros [85,86], like Möbius et al. [91] and Davies et al. [29] include two-particle transitions in their simulation but do not agree with the others about their effect. The reason for this disagreement is unclear, at least to this author.

The Murcia group did much work on simulation of energy and of conductance relaxation [92–94] and references therein. Collective transitions are clearly involved in those simulations.

## 9 Interaction effect on localization

There has been a wide interest lately in so-called many-body localization (MBL), inspired by the work of Basko et al. [95,102,103]. It relates to the effect of short range and weak interaction on localization. The work is a perturbation expansion to infinite disorder. The basic question was whether such electron–electron interactions can delocalize the system in the absence of electron–phonon interaction or, more generally, in the absence of any thermal bath that the electronic system interacts with. The salient result is that low energy states are collectively localized but above some energy there is a (many-electron) delocalization, i.e. a mobility edge. This review focuses on Coulomb, i.e. *long* range interaction so the MBL problem is outside the scope of this article.

In contrast to MBL, the effect of long range interaction on localization is still an unsolved problem even though a number of efforts were made to tackle the problem by different methods. The basic question is whether Coulomb interaction enhances or impedes delocalization and under what conditions. Some of the research on the subject arose in consequence of an experiment by Kravchenko et al. [96,97] that appeared to indicate a transition from hopping to metallic conduction in a two-dimensional system. For the non-interacting system delocalization in two dimensions is ruled out by the scaling theory of Abrahams et al. [98].

Regarding the effect of the long range Coulomb interactions, early arguments suggested that it must enhance localization because the randomness of  $1/r_{ij}$  between randomly spaced sites brings in an additional disorder energy. This is basically a Hartree argument and leaves out important quantum effects. Inclusion of exchange by solving numerically the Hartree–Fock (HF) equations [99–101] resulted in enhanced localization. HF leaves out the possibility for collective delocalization, namely delocalization by a collective motion of the carriers, as pointed out in [27,102,103] where it was shown that collective transitions can be antilocalizing. In contrast to the HF diagonalization of a single determinant, this requires a many-body wave function consisting of a sum of many Slater determinants – in principle the entire Fock space, or

$N!/n!(N-n)!$  Slater determinants for an exact solution for spinless electrons. This is clearly an impossible task to tackle for any reasonably large  $N$  by computer. The best efforts in that direction so far were by Fleishman and Anderson [104] and by Epperlein et al. [105,106]. These works attempted to choose a basis set close enough to the true eigenfunctions, with the hope that a near exact solution can be achieved with only few Slater determinants. Reference [104] concluded that they likely favor delocalization, while [105,106] concluded that they favor delocalization within a certain range of parameters. Pollak [107] in a different approach, Lowdin's partitioning method [108], found that all states become delocalized, however through a collective motion of about 100 electrons, a highly non-ergodic process excluding any possible measurement to verify the delocalization result [109]. Efros et al. make a similar point on the basis of their computer simulations. Shepelyanski [110] showed that interaction reduces localization for two electrons in a random medium. This spurred a large number of works, confirming his finding by different methods. References [111–115] are examples of such works.

One may ask what exactly constitutes delocalization in the many-body situation: a finite conductivity as  $T \rightarrow 0$ ? A continuum energy spectrum? Coherent collective motion over macroscopic distances? A finite conductivity at  $T=0$ ? These criteria are not necessarily identical in interacting systems.

In considering the problem of whether Coulomb interactions impede or enhance delocalization it seems useful to think in terms of the Rayleigh–Schroedinger perturbation expansion in Fock space

$$\begin{aligned}
 |J\rangle = & |J_0\rangle + \sum_J \frac{\langle J_0|V|I_0\rangle}{E_I - E_J} |I_0\rangle + \sum_{J,K} \frac{\langle J_0|V|I_0\rangle \langle I_0|V|K_0\rangle}{(E_I - E_J)(E_K - E_I)} |K_0\rangle \\
 & + \sum_{J,K,L} \frac{\langle J_0|V|I_0\rangle \langle I_0|V|K_0\rangle \langle K_0|V|L_0\rangle}{(E_I - E_J)(E_K - E_I)(E_L - E_K)} |L_0\rangle + \dots
 \end{aligned} \tag{20}$$

It allows an easy comparison between the interacting and non-interacting system. The size of the successive factors of the form

$$\frac{\langle J_0|V|K_0\rangle}{E_K - E_J}$$

determine whether the series converges (indicating localization) or diverges (indicating delocalization). Interactions enhance both the numerator and the denominator. For the non-interacting system the numerator is finite only when the bra and the ket differ by a single electron. For the interacting system there is no such restriction. The denominator is enhanced by the interaction because it adds to the random site energies the additional effective random energies  $1/r_{ij}$  (the distances being random). The numerators increase when  $t$  is large, the denominators increase when the density is large. The expectation thus is that interactions enhance delocalization when  $\xi$  is large and the density is small, and localization is enhanced in the opposite case.

To study the effect of interaction on the localization problem some authors used quantum field renormalization group theory [116–119]. These studies also led to the conclusion that delocalization may occur in 2d systems. They pointed out as well that the DOS is irrelevant to the dc conductivity and should be replaced by the thermodynamic density  $dn/d\mu$ . These conclusions are similar to those mentioned in the papers discussed above. Unfortunately, no cross references between the two types of approaches exist, so the underlying physical relationship between the two is difficult to assess. Since references [116–119] seem modifications of the free electron transport



( $z = 1$  in their notation) it seems to relate to interaction modified weak localization [122] rather than to the Anderson–Mott localization.

In [99,100], the authors went further than asking merely about delocalization but also investigated the question of the mobility edges and their energy levels. Both works concluded that there exist two mobility edges, the low energy one close to the Fermi level, the upper one at much higher energies. This conclusion is qualitatively intuitive. Since the number of states,  $N!/[n!(N-n)!]$ , increases with volume very much faster than their energy spread, an algebraic function of  $N$ , there must, in the thermodynamic limit of infinite volume exist at least one energy domain where states are delocalized. Quantitatively, as already mentioned, the HF method misses collective phenomena. So, the interesting question is whether collective effects, not accounted for in HF, can push, under certain circumstances, the lower mobility edge below the Fermi level, in which case the Coulomb interaction would render the ground state to be delocalized. Since the Fermi level is an equilibrium concept, collective delocalization by many electrons may be non-ergodic, in which case it would not be observable by definition. A definitive answer to these questions presently still does not exist. Experimentally, the question remains unanswered as well as argued below.

Das Sarma and coworkers expressed reservations regarding the interpretation of the metal–insulator experiments by the renormalization group works. In a very interesting recent paper [120], they explore experimentally the “metallic” regime in Si, for electrons as well as for holes. The authors find that the temperature dependence of the conductivity in the “metallic” regime, similar to those found in the other systems [96,97,123,124], can be quite well accounted for using a theory [121] based on a Boltzmann transport equation. It assumes carrier scattering by screened charged ions, the screening depending on carrier concentration and on temperature. This is basically a one-particle theory. It seemingly constitutes a paradox since [98] does not allow for a metallic conduction in a 2D system in a non-interacting system. The possible resolution is that in all those experiments the sample size is smaller than the localization length. In other words, the samples are in the mesoscopic regime. This seems reasonable because in the regime where  $\beta < 0$  but very small the localization radius is very large. In this case, the interpretation of the [98] theorem should be that most samples will show metallic conduction and only a few will be insulating. Whether such an interpretation is correct could be tested by observing experimentally whether the separatrix between “insulating” and “metallic” depends on the size of the sample. However, such an experiment must be very difficult to realize because of the challenge to produce samples which differ in size alone, while other differences from sample to sample are in the very nature of mesoscopic systems.

If the above is the correct interpretation, then these experiments on high mobility electrons probably relate to a melting of a Wigner solid rather than to the question whether interactions can delocalize carriers in materials where one-particle localization is due to disorder rather than due to the Wigner effect. If so then experiments whether interaction can delocalize Anderson localized electrons do not yet exist.

## 10 Summary

The paper reviews current disagreements on electrons in strongly disordered systems. Interactions in such systems must be important due to lack of screening by localized electrons. A good part of the disagreements centers on the importance of many-body effects in the interacting systems, in particular dynamical correlation effects. Arguments are presented (often in agreement with existing experiments) in support of the need to consider these effects in physical phenomena – specifically in the following.

DOS: by definition the DOS is the spectrum of energies for inserting an extra electron on vacant sites of a system in the ground state. Much of the literature

substitutes for the ground state a state stable to one-particle excitations only, while the ground state is stable to all many-electron excitations. The latter DOS differs substantially from the former as was shown by theoretical studies.

Specific heat: in contrast to non-interacting electrons, where low energy states are dominated by single particle excitation, in the interacting systems the low energy states are dominated by many-electron excitations. The density of low energy excitations thus greatly differs in the non-interacting and in the interacting systems.

Conductance: the prevailing theory is a one-particle theory which ignores stabilization of the ground state by many-particle relaxation, as well as a correlated motion mandated by electron–electron interaction. Both these problems can be resolved only by inclusion of collective transitions.

High carrier concentration: it is shown that only inclusion of collective many-body transitions can account for the non-ergodic relaxation observed in high carrier concentration localized systems.

Many-body delocalization: it has been argued that both Hartree and HF approaches can only enhance localization and that only correlated many-body motion can alleviate localization and account for possible (collective) delocalization in two-dimensional systems.

*Note added in proof.* A very recent paper, [125], reports on relaxation experiments in indium oxide with lower than previously achieved carrier concentrations. In these the relaxation does reach equilibrium during the experimental time.

This paper is written in honor of Sir Michael Pepper's 75th birthday. May we benefit from many more years of his contributions to physics.

## References

1. P.W. Anderson, Phys. Rev. **109**, 1492 (1958)
2. N.F. Mott, W.D. Twose, Adv. Phys. **10**, 107 (1961)
3. E. Schrödinger, *What is Life* (Cambridge University Press, Cambridge, 1944)
4. H. Miller, *Tropic of Capricorn* (Obelisk Press, Paris, 1938)
5. I. Shlimak, *Is Hopping Science* (World Scientific, New Jersey, 2015)
6. M. Palassini, Contrib. Sci. **11**, 163 (2015)
7. M. Pollak, M. Ortuno, A. Frydman, *The Electron Glass* (Cambridge University Press, Cambridge, 2013)
8. D.J. Wales, *Energy Landscapes* (Cambridge University Press, Cambridge, 2003)
9. T. Terao, Eur. J. Phys. B **89**, 209 (2016)
10. L. Rademaker et al., New J. Phys. **20**, 043026 (2018)
11. G.H. Wannier, Phys. Rev. **52**, 191 (1937)
12. G.H. Wannier, Rev. Mod. Phys. **34**, 645 (1962)
13. H. Fritzsche, M. Cuevas, Phys. Rev. **119**, 1238 (1960)
14. E.A. Davis, W.D. Compton, Phys. Rev. A **140**, 2183 (1965)
15. A.G. Zabrodskii, A.G. Andreev, Int. J. Mod. Phys. B **8**, 883 (1994)
16. N.F. Mott, Can. J. Phys. **34**, 1356 (1956)
17. E. Conwell, Phys. Rev. **103**, 1951 (1956)
18. E. Zion et al., Phys. Rev. B **96**, 245143 (2017)
19. F. Gargiulo et al., Phys. Rev. Lett. **113**, 246601 (2014)
20. Z. Ovadyahu, Phys. Rev. B **73**, 214204 (2006)
21. M. Pollak, Discuss. Faraday Soc. **50**, 13 (1970)
22. G. Srinivasan, Phys. Rev. B **4**, 258 (1971)
23. P.A. Lee, Phys. Rev. B **62**, 8030 (1982)
24. A.L. Efros, B.I. Shklovskii, J. Phys. C **8**, L49 (1975)
25. A.L. Efros, B. Skinner, B.I. Shklovskii, Phys. Rev. B **84**, 0642994 (2011)

26. A.L. Efros, J. Phys. C **9**, 2021 (1976)
27. M. Pollak, J. Phys. Solid State **30**, 105602 (2018)
28. M. Pollak, A. Hunt, Philos. Mag. B **52**, 391 (1985)
29. J.H. Davies, P.A. Lee, T.M. Rice, Phys. Rev. B **29**, 4620 (1984)
30. K. Tenelsen, M. Schreiber, Phys. Rev. B **49**, 12662 (1994)
31. A. Mobius et al., Philos. Mag. B **81**, 1101 (2001)
32. A. Mobius, U.K. Rossler, Phys. Rev. B **79**, 174206 (2009)
33. M.H. Overlin, L.A. Wong, C.C. Yu, Phys. Rev. B **70**, 214204 (2004)
34. A.Yu. Rogatchev, U. Mizutani, Phys. Rev. B **61**, 15550 (2000)
35. W. Sasaki, Suppl. Prog. Theor. Phys. **72**, 169 (1982)
36. M.L. Knotek, M. Pollak, Phys. Rev. B **9**, 664 (1972)
37. V. Ambegaokar, B.I. Halperin, J.S. Langer, Phys. Rev. B **4**, 2612 (1971)
38. A.L. Efros, B.I. Shklovskii, Zh. Eksp. Teor. Fiz. [Sov. Phys. JETP] **33**, 468 (1971)
39. M. Pollak, J. Non-Cryst. Solids **11**, 1 (1972)
40. M. Pollak, J. Phys. C **14**, 2977 (1981)
41. A. Miller, E. Abrahams, Phys. Rev. **120**, 745 (1960)
42. B.I. Shklovskii, A.L. Efros, *Electronic Properties of Doped Semiconductors* (Springer Verlag, Berlin, 1984)
43. T.G. Castner, in *Hopping Transport in Solids*, edited by M. Pollak, B.I. Shklovskii (North Holland, Amsterdam, 1991)
44. M.L. Knotek, M. Pollak, Phys. Rev. B **9**, 664 (1972)
45. M. Ortuño et al., in *Proceedings of 22nd International Conference on the Physics of Semiconductors*, edited by D.J. Lockwood (World Scientific, Singapore, 1985)
46. A. Pérez-Garrido et al., Phys. Rev. B **55**, 8630 (1997)
47. A.M. Somoza, M. Ortuño, M. Pollak, Phys. Rev. B **73**, 045123 (2006)
48. M. Pollak, Phys. Status Solidi **5**, 667 (2008)
49. Z. Ovadyahu, C. R. Phys. **14**, 700 (2013)
50. M. Mezard, G. Parisi, M.A. Virasoro (Eds.), *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987)
51. C.B. Roth (Ed.), *Polymer Glasses* (CRC, Boca Raton, 2016)
52. C.J. Adkins, J.D. Benjamin, J.M.D. Thomas, J.W. Gardner, A.J. Geown, J. Phys. C **17**, 4633 (1984)
53. Z. Ovadyahu, personal communication
54. A. Hunt, Philos. Mag. B **81**, 875 (2001)
55. M. Pollak, A. Hunt, in *Hopping Transport in Solids*, edited by M. Pollak, B.I. Shklovskii (North Holland, Amsterdam, 1991)
56. M. Pollak, H. Pohl, J. Chem. Phys. **63**, 2980 (1975)
57. Z. Ovadyahu, Phys. Rev. Lett. **99**, 226603 (2007)
58. T. Grenet, J. Delahay, Eur. Phys. J. B **76**, 229 (2010)
59. T. Grenet, J. Delahay, J. Phys.: Condens. Matter **29**, 455602 (2017)
60. Z. Ovadyahu, Phys. Rev. B **83**, 235126 (2011)
61. Z. Ovadyahu, Phys. Rev. B **73**, 214208 (2006)
62. Z. Ovadyahu, Phys. Rev. B **84**, 165209 (2011)
63. T. Havdala, A. Eisenbach, A. Frydman, Europhys. Lett. **98**, 67006 (2012)
64. M. Ortuño et al., Sci. Rep. **6**, 21647 (2016)
65. Z. Ovadyahu, Phys. Rev. B **94**, 155151 (2016)
66. Z. Ovadyahu, Phys. Rev. Lett. **115**, 046601 (2015)
67. Z. Ovadyahu, Phys. Rev. B **97**, 054202 (2018)
68. M. Pollak, Z. Ovadyahu, Phys. Status Solidi C **3**, 283 (2006)
69. M. Müller, J. Phys. C **18**, 1833 (2006)
70. M. Müller, S. Pankov, Phys. Rev. B **75**, 144201 (2007)
71. M. Müller, L.B. Ioffe, Phys. Rev. Lett. **93**, 256403 (2004)
72. G. Parisi, Physica A **140**, 312 (1986)
73. D. Sherrington, S. Kirkpatrick, Phys. Rev. Lett. **35**, 1792 (1975)
74. S.R. Elliott, Adv. Phys. **36**, 135 (1987)
75. A. Amir, Y. Oreg, Y. Imry, Phys. Rev. B **77**, 165207 (2008)

76. A. Amir, Y. Oreg, Y. Imry, Phys. Rev. Lett. **103**, 126403 (2009)
77. A. Amir, Y. Oreg, Y. Imry, Annu. Rev. Condens. Matter Phys. **2**, 235 (2011)
78. A. Amir, Y. Oreg, Y. Imry, Proc. Natl. Acad. Sci. USA **109**, 1850 (2012)
79. A.J. Leggett et al., Rev. Mod. Phys. **59**, 1 (1987)
80. A. Díaz-Sánchez et al., Phys. Rev. B **59**, 910 (1999)
81. J.C. Lasjaunias, K. Biljakovič, P. Monceau, Phys. Rev. B **53**, 7699 (1996)
82. C.C. Yu, Phys. Rev. Lett. **82**, 4074 (1999)
83. C.C. Yu, Philos. Mag. B **81**, 1209 (2001)
84. C.C. Yu, Phys. Status Solidi (b) **230**, 47 (2002)
85. D.J. Tsigankov, A.L. Efros, Phys. Rev. Lett. **88**, 176602 (2002)
86. D.J. Tsigankov, A.L. Efros, Phys. Status Solidi (b) **230**, 157 (2002)
87. K. Tenelsen, M. Schreiber, Phys. Rev. B **52**, 13287 (1995)
88. A. Pérez-Garrido et al., Phys. Rev. B **55**, 8630 (1997)
89. A. Díaz-Sánchez et al., Phys. Rev. B **59**, 910 (1999)
90. D.J. Tsigankov, E. Pazy, B.D. Laikhtman, A.L. Efros, Phys. Rev. B **68**, 184205 (2003)
91. A. Möbius, M. Richter, B. Drittlar, Phys. Rev. B **45**, 11568 (1992)
92. M. Ortuño, A.M. Somoza, J. Phys. Conf. Ser. **376**, 012007 (2012)
93. A.M. Somoza, M. Ortuno, M. Pollak, Phys. Rev. B **73**, 45123 (2006)
94. J. Bergli, A.M. Somoza, M. Ortuño, Phys. Rev. B **84**, 174201 (2011)
95. D.M. Basko, I.L. Aleiner, B.L. Altshuler, Ann. Phys. (N.Y.) **321**, 1126 (2006)
96. S.V. Kravchenko et al., Phys. Rev. B **51**, 7038 (1995)
97. S.V. Kravchenko, M. Sarachik, Rep. Prog. Phys. **67**, 1 (2004)
98. E. Abrahams et al., Phys. Rev. Lett. **42**, 673 (1973)
99. F. Epperlein, M. Schreiber, T. Vojta, Phys. Rev. B **56**, 5890 (1997)
100. M. Amini, V.E. Kravtsov, M. Müller, New J. Phys. **16**, 015022 (2014)
101. L. Hyun-Yung, K. Ki-Seok, [arXiv:1802.03533v1](https://arxiv.org/abs/1802.03533v1) (2018)
102. M. Pollak, M.L. Knotek, J. Non-Cryst. Solids **32**, 141 (1979)
103. M. Pollak, Philos. Mag. **42**, 781 (1980)
104. L. Fleishman, P.W. Anderson, Phys. Rev. B **21**, 2366 (1980)
105. T. Vojta, F. Epperlein, M. Schreiber, Phys. Rev. Lett. **81**, 4212 (1998)
106. M. Schreiber, T. Vojta, Math. Comput. Simul. **62**, 243 (2003)
107. M. Pollak, J. Phys. C **15**, 1685 (1981)
108. P.O. Löwdin, J. Mol. Spectrosc. **10**, 12 (1963)
109. A.L. Efros et al., Phys. Status Solidi (b) **218**, 17 (2000)
110. D.L. Shepelyanski, Phys. Rev. Lett. **73**, 2607 (1994)
111. Y. Imry, Europhys. Lett. **30**, 405 (1995)
112. J. Talamantes, M. Pollak, L. Elam, Europhys. Lett. **35**, 511 (1996)
113. F. von Oppen, J. Müller, Phys. Rev. Lett. **76**, 491 (1996)
114. Y. Ishai, R. Berkovits, Philos. Mag. B **77**, 1115 (1998)
115. M. Ortuño, E. Cuevas, Europhys. Lett. **46**, 224 (1999)
116. A.M. Finkelstein, J. Exp. Theor. Fyz. **84**, 168 (1983)
117. A.M. Finkelstein, J. Exp. Theor. Fyz. **86**, 367 (1984)
118. C. Castellani et al., Phys. Rev. B **30**, 527 (1984)
119. A. Punnoose, A.M. Finkelstein, Science **310**, 289 (2005)
120. B. Hu et al., Phys. Rev. Lett. **115**, 036801 (2015)
121. S. das Sarma, E.H. Hwang, Phys. Rev. B **69**, 195305 (2004)
122. B.L. Altshuler et al., Phys. Rev. **9**, 225 (1987)
123. Y. Henien et al., Phys. Rev. Lett. **80**, 1288 (1998)
124. A.R. Hamilton et al., Phys. Rev. Lett. **87**, 126802 (2001)
125. Z. Ovadyahu, Phys. Rev. B **97**, 214201 (1918)