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The quantum particle in a box: what we can learn from classical electrodynamics

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Abstract. The problem of a charged particle enclosed in an infinite square potential well is analysed from the point of view of classical theory with the addition of the electromagnetic zero-point radiation field, with the aim to explore the extent to which such an analysis can contribute to enhance our understanding of the quantum behavior. First a proper treatment is made of the freely moving particle subject to the action of the radiation field, involving a frequency cutoff ω_c. The jittering motion and the effective structure of the particle are sustained by the permanent action of the zero-point field. As a result, the particle interacts resonantly with the traveling field modes of frequency ω_c in its proper frame of reference, which superpose to give rise to a modulated wave accompanying the particle. This is identified with the de Broglie wave, validating the choice of Compton's frequency for ω_c. For the stationary states of particles confined in the potential well, the Lorentz force produced by the accompanying field is shown to lead to discrete values for the mean speed and to an uneven probability distribution that echoes the corresponding quantum distribution. The relevance of the results obtained and the limitations of the classical approach used, are discussed in the context of present-day stochastic electrodynamics.

1 Introduction

One of the most elementary problems in any introductory course on quantum mechanics is the particle in a (one-dimensional) box. It serves to illustrate in very simple terms the predictions of the Schrödinger equation with regard to energy quantization and the corresponding spatial distribution of bound particles. The mathematics of the problem is straightforward and leads to a classically unexpected result. Within the quantum framework, normally no attempt is made to provide a physically convincing explanation for the alternating nodes and antinodes of the particle distribution inside the well, nor for the appearance of discrete energy values. The solution readily obtained from the Schrödinger equation, or the appeal to a de Broglie wave of unknown nature, can hardly provide a satisfactory picture of the physics behind the formal solution.

Here, we take the referred example and study it from an unconventional point of view that may help to throw some light on the underlying physics. We consider

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a charged point particle (typically an electron) confined inside the well, subject to the action of a real, Maxwellian fluctuating vacuum or zero-point radiation field (ZPF). Following an entirely classical approach, we analyse the statistical behavior of the particle, for stationary states of the particle + field system. The presence of the vacuum is shown to lead to a particle dynamics that deviates from the standard classical one and produces maxima and minima in the spatial distribution, bearing qualitative resemblance with the quantum distribution.

The point of view followed here originated over fifty years ago with a paper by Marshall [1], which gave birth to stochastic electrodynamics (SED), the theory aiming to explain quantum properties of matter by adding the ZPF to classical physics. Favorable results were obtained for a handful of problems thanks to the work of Marshall, Boyer [2,3] and several others (for reviews and references see [3–5]. Examples of more recent treatments related to the H atom are [6–9]. Other interesting approaches to the quantum problem from a classical random perspective are discussed by 't Hooft [10] and Khrennikov [11,12]).

It is important to distinguish such classical approach to SED (called here CSED for short), which we also follow here, from the general SED theory developed more recently (see Ref. [13] and references therein). In the latter, the particle + plus ZPF system is considered to transit towards a regime in which detailed energy balance holds on the average and the system acquires ergodic properties; under such condition, the description of the dynamics becomes consistent with the quantum rules. Here by contrast, we remain all the time under classical rules and do not consider the transition to the quantum regime. Yet the final results do show some relevant quantum-like features, specifically regarding the probability distribution of particles inside the well and the quantization of the energy in stationary states. Herein lies the interest of the present CSED calculation: while it brings to the fore a physical element that has been shown in other instances to play a key role in explaining quantum phenomena, it reveals some of the limitations of the straightforward classical approach followed.

The paper is organized as follows. In Section 2 we carefully revisit the problem of the particle in interaction with the entire radiation field, including its own. This leads to the identification of de Broglie's clock, as a consequence of the limited response of the particle to high frequencies, which gives rise to a jittering motion sustained by the ZPF. In Section 3, de Broglie's wave is seen to be electromagnetic in nature, as discussed in earlier work (Ref. [13] and related references therein). In Section 4, the analysis of the forces that operate in the stationary states of the particle confined to move within the box under the guidance of de Broglie's wave, is shown to lead to well-defined energy values that coincide with the quantum ones, and to an uneven particle distribution that conforms to the corresponding quantum prediction. The paper ends with a critical discussion in Sections 5 and 6 of the implications of the results obtained.

2 Electrodynamic origin of de Broglie's clock

As anticipated above, the purpose of this section is to establish the electromagnetic nature of de Broglie's clock. For this purpose we revisit the issue of the total electromagnetic force acting on the electron, including the self-force, within a nonrelativistic approach. The topic of the self-force is widely addressed in the literature, using diverse techniques (see e.g. [14,15]). Here we follow a simple and direct procedure, based on physical arguments that allow us to obtain a result free of divergences and acausality. This approach serves to highlight the specific dynamical role played by the cutoff frequency introduced along the derivation.

We start by considering the classical problem of a particle of mass m and charge e, subject to the action of a radiation field with vector potential $\boldsymbol{A}$. Let H_F denote the free-field Hamiltonian; the total Hamiltonian of field and particle is then given by

$$H = H_0 + H_F, \tag{1}$$

$$H_0 = \frac{1}{2m}(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A})^2, \quad H_F = \frac{1}{8\pi}\int d^3x(\boldsymbol{E}^2 + \boldsymbol{B}^2), \tag{2}$$

and the corresponding Hamilton equations are

$$m\dot{\boldsymbol{x}} = \boldsymbol{p} - \frac{e}{c}\boldsymbol{A}, \tag{3}$$

$$\dot{\boldsymbol{p}} = e' \sum_{\boldsymbol{n},\lambda} \left(\dot{\boldsymbol{x}} \cdot \boldsymbol{\epsilon}_{\boldsymbol{n}}^{\lambda}\right) \boldsymbol{k}_{\boldsymbol{n}} \left(q_{\boldsymbol{n}}^{\lambda} \cos \boldsymbol{k}_{\boldsymbol{n}} \cdot \boldsymbol{x} - \frac{p_{\boldsymbol{n}}^{\lambda}}{\omega_{\boldsymbol{n}}} \sin \boldsymbol{k}_{\boldsymbol{n}} \cdot \boldsymbol{x}\right), \tag{4}$$

$$\dot{q}_{\boldsymbol{n}}^{\lambda} = p_{\boldsymbol{n}}^{\lambda} - e' \left(\dot{\boldsymbol{x}} \cdot \boldsymbol{\epsilon}_{\boldsymbol{n}}^{\lambda}\right) \frac{1}{\omega_{\boldsymbol{n}}} \cos \boldsymbol{k}_{\boldsymbol{n}} \cdot \boldsymbol{x}, \tag{5}$$

$$\dot{p}_{\boldsymbol{n}}^{\lambda} = -\omega_{\boldsymbol{n}}^2 q_{\boldsymbol{n}}^{\lambda} + e' \left(\dot{\boldsymbol{x}} \cdot \boldsymbol{\epsilon}_{\boldsymbol{n}}^{\lambda}\right) \sin \boldsymbol{k}_{\boldsymbol{n}} \cdot \boldsymbol{x}\,. \tag{6}$$

Here $e' = e\sqrt{4\pi/V}$, with V the normalization volume; $(\boldsymbol{n}, \lambda)$ describes the modes of the field with polarization vectors $\boldsymbol{\epsilon}_{\boldsymbol{n}}^{\lambda}$, propagation vector $\boldsymbol{k}_{\boldsymbol{n}}$, and frequency $\omega_{\boldsymbol{n}} = ck_{\boldsymbol{n}}$; $q_{\boldsymbol{n}}^{\lambda}$ and $p_{\boldsymbol{n}}^{\lambda}$ represent the quadratures of the field for each mode. In the final calculations it is convenient to perform the transition to the continuous limit, with the conventional rule

$$\frac{1}{\sqrt{V}} \sum_{\boldsymbol{n},\lambda} f_{\boldsymbol{n}}^{\lambda} \to \frac{1}{(2\pi)^{3/2}} \sum_{\lambda} \int d^3k\, f^{\lambda}(\boldsymbol{k}). \tag{7}$$

The equation of motion for the particle becomes

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}_L + \boldsymbol{F}_s, \tag{8}$$

with $\boldsymbol{F}_L$ the (external) Lorentz force and $\boldsymbol{F}_s$ the self-force. The Lorentz force is obtained by integrating equations (5) and (6), and combining with (3) and (4),

$$\begin{aligned}\boldsymbol{F}_L = \frac{e}{\pi\sqrt{2}} \int d^3k \sum_{\lambda} &\left[\boldsymbol{\epsilon}_{\boldsymbol{k}}^{\lambda} + \frac{\dot{\boldsymbol{x}}}{c} \times \left(\hat{\boldsymbol{k}} \times \boldsymbol{\epsilon}_{\boldsymbol{k}}^{\lambda}\right)\right] \\ &\times \left[\omega_k q_k^{\lambda} \cos(\boldsymbol{k} \cdot \boldsymbol{x} - \omega_k t) - p_k^{\lambda} \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega_k t)\right].\end{aligned} \tag{9}$$

As for $\boldsymbol{F}_s$ we get

$$\begin{aligned}\boldsymbol{F}_s = -\frac{e^2}{2\pi^2} \int d^3k \sum_{\lambda} &\left[\boldsymbol{\epsilon}_{\boldsymbol{k}}^{\lambda} + \frac{\dot{\boldsymbol{x}}}{c} \times \left(\hat{\boldsymbol{k}} \times \boldsymbol{\epsilon}_{\boldsymbol{k}}^{\lambda}\right)\right] \\ &\times \int dt' \dot{\boldsymbol{x}}' \cdot \boldsymbol{\epsilon}_{\boldsymbol{k}}^{\lambda} \left[\cos(\omega_k(t' - t) - \boldsymbol{k} \cdot (\boldsymbol{x}' - \boldsymbol{x}))\right],\end{aligned} \tag{10}$$

where $\boldsymbol{x}' = \boldsymbol{x}(t')$.

2.1 The particle self-force

We shall in this section focus on the self-force given by equation (10) and carry out a nonrelativistic calculation, assuming that the particle moves with a velocity $\dot{\boldsymbol{x}}$, with $|\dot{\boldsymbol{x}}| \ll c$. We therefore consider that the electric term in the first square bracket in equation (10) is dominant and hence neglect the magnetic term. Further, under the same approximation the argument of the cosine function reduces to

$$\omega(t'-t)[1-\hat{\boldsymbol{k}}\cdot\frac{\boldsymbol{x}'-\boldsymbol{x}}{c\,(t'-t)}] \cong \omega(t'-t), \tag{11}$$

and (10) simplifies into

$$\boldsymbol{F}_s = -\frac{e^2}{2\pi^2}\int d^3k \sum_\lambda \boldsymbol{\epsilon}_{\boldsymbol{k}}^\lambda \int dt' \dot{\boldsymbol{x}}' \cdot \boldsymbol{\epsilon}_{\boldsymbol{k}}^\lambda \cos\omega_k(t'-t). \tag{12}$$

The angular integration over $\hat{\boldsymbol{k}}$ can be carried out by using

$$\sum_\lambda \int d\Omega_k \boldsymbol{\epsilon}_{\boldsymbol{k}}^\lambda (\dot{\boldsymbol{x}}' \cdot \boldsymbol{\epsilon}_{\boldsymbol{k}}^\lambda) = \frac{8\pi}{3}\dot{\boldsymbol{x}}',$$

so that

$$\boldsymbol{F}_s = -\frac{4e^2}{3\pi c^3}\int_0^t dt' \dot{\boldsymbol{x}}' \int d\omega\omega^2 \cos\omega(t'-t). \tag{13}$$

At this point, in usual treatments the integral over the frequency is extended up to infinity (see e.g. [16])

$$\boldsymbol{F}_s = -\frac{4e^2}{3\pi c^3}\int_0^t dt' \dot{\boldsymbol{x}}' \int_0^\infty d\omega\omega^2 \cos\omega(t'-t), \tag{14}$$

which gives after two integrations by parts

$$\boldsymbol{F}_s = \frac{4e^2}{3\pi c^3}\left(\frac{\pi}{2}\dddot{\boldsymbol{x}} - \ddot{\boldsymbol{x}}\int_0^\infty d\omega\right) = m\tau\dddot{\boldsymbol{x}} - \delta m_\infty \ddot{\boldsymbol{x}}, \tag{15}$$

with $\tau = 2e^2/3mc^3$ and

$$\delta m_\infty = \frac{2m\tau}{\pi}\int_0^\infty d\omega. \tag{16}$$

2.2 Causal Abraham-Lorentz equation

The approximate equation $\boldsymbol{F}_s = m\tau\dddot{\boldsymbol{x}} - \delta m_\infty \ddot{\boldsymbol{x}}$ furnishes a good description of the nonrelativistic motion of a point charge for many purposes. The downside of it, as is well known, is the appearance of acausality, manifested e.g. as preacceleration, linked to the $\dddot{\boldsymbol{x}}$ term, and of the divergent acquired mass δm_∞. These problems are normally avoided by approximating the radiation reaction in terms of the external force $\boldsymbol{F}$ (when there is one), $m\tau\ddot{\boldsymbol{x}} \simeq \tau(\nabla\boldsymbol{F})\cdot\dot{\boldsymbol{x}}$, and introducing a cutoff ω_c in the frequency integral of the mass correction,

$$\delta m_c = \frac{2m\tau}{\pi}\int_0^{\omega_c} d\omega = \frac{2m\tau\omega_c}{\pi}. \tag{17}$$

Strictly speaking, this procedure implies an inconsistency: whilst the nonrelativistic assumption implies that the effect of the high-frequency modes on the motion is considered small, the infinite mass correction is an (extreme) consequence of just these high-frequency modes. It is clear that these latter modes do affect the particle motion, by putting it to vibrate very rapidly. We shall therefore analyse with extra care the effect of the high-frequency modes on the dynamics of an otherwise slowly (nonrelativistically) moving particle. To avoid the inconsistency just mentioned, we pay close attention to the effect of the radiation field modes of high frequency on the particle dynamics, but assuming that the particle responds to the field only up to a certain frequency ω_c, after which it becomes transparent to it. With the introduction of the cutoff frequency ω_c, the mass correction is given by the finite expression (17).

Now we introduce the cutoff ω_c in equation (13), so that

$$\boldsymbol{F}_s = -\frac{4e^2}{3\pi c^3}\int_0^t dt'\dot{\boldsymbol{x}}' I(t'-t), \tag{18}$$

with

$$I(t'-t) = \int_0^{\omega_c} d\omega\omega^2 \cos\omega(t'-t). \tag{19}$$

Integration over the frequency gives

$$I(t'-t) = -\frac{\partial^2}{\partial t^2}\frac{\sin\omega_c(t'-t)}{t'-t}, \tag{20}$$

and equation (18) reads now

$$\boldsymbol{F}_s = \frac{4e^2}{3\pi c^3}\int_0^t dt'\dot{\boldsymbol{x}}'\frac{\partial^2}{\partial t^2}\frac{\sin\omega_c(t'-t)}{t'-t}. \tag{21}$$

With

$$S_c(t-t') = \frac{\sin\omega_c\,(t-t')}{\omega_c\,(t-t')}, \tag{22}$$

the self-force takes the form

$$\boldsymbol{F}_s = \frac{2m\tau\omega_c}{\pi}\int_0^t dt'\dot{\boldsymbol{x}}'\frac{\partial^2}{\partial t^2}S_c(t-t'). \tag{23}$$

By performing two integrations by parts and using equation (17), this result takes on the alternative form

$$\boldsymbol{F}_s = \delta m_c\frac{\partial^2}{\partial t^2}\int_0^t dt'\dot{\boldsymbol{x}}' S_c(t-t') - \delta m_c\ddot{\boldsymbol{x}}, \tag{24}$$

and the equation of motion that replaces the Abraham-Lorentz equation reads now alternatively

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F} + \frac{2m\tau\omega_c}{\pi}\int_0^t dt'\dot{\boldsymbol{x}}'\frac{\partial^2}{\partial t^2}S_c(t-t'), \tag{25}$$

or

$$(m+\delta m_c)\ddot{\boldsymbol{x}} = \boldsymbol{F} + \delta m_c\frac{\partial^2}{\partial t^2}\int_0^t dt'\dot{\boldsymbol{x}}'S_c(t-t'). \tag{26}$$

Note that the radiation force term in equations (21) and (25) is a *causal* function, with its instantaneous value dependent on the whole (past) history of the motion. However, none of the two extra terms in equation (26) obey causality separately: the mass correction δm_c seems to be present already at time $t=0$ (before the interaction), and the second term on the right-hand side, which generalizes the radiation reaction force $m\tau\dddot{\boldsymbol{x}}$, thus also violates causality and leads to preacceleration.

Notice that the function $S_c(t-t')$ changes very rapidly for high values of ω_c. Since

$$\left.\frac{\partial^2 S_c(t-t')}{\partial t^2}\right|_{t'=t} = -\frac{\omega_c^2}{3}, \tag{27}$$

the accelerations for small time intervals $t-t'$ of the order of ω_c^{-1} are so strong that the nonrelativistic treatment is clearly insufficient for a full description of the motion; while it may be appropriate to describe the slow motion, it can only give an approximate account of the fine variations around this slow motion.[1]

The previous discussion acquires its real importance by leading us to consider the presence of the cutoff ω_c as a physical requirement. This will become crucial for the rest of our analysis.

2.3 De Broglie's clock

An important property of the function $S_c(t-t')$ given by equation (22) is that it oscillates at the frequency ω_c, inducing oscillations of precisely this frequency on the particle. It is the presence of these high-frequency oscillations, which play the role of a nonrelativistic zitterbewegung, what generates an effective structure of the particle. In the particular case of the electron it is well known that its electromagnetic interaction induces on it an *effective* structure, with a radius of the order of the Compton wavelength [16], equivalent to the distance the particle travels with velocity c in a time interval $\Delta t \simeq 2\pi\omega_C^{-1}$ (the somewhat smaller value $(\lambda_C r_c)^{1/2}$, where r_c is the "classical" electron radius e^2/mc^2, is assigned to it in Ref. [17]).[2]

[1] Of course, relativistic treatments of the radiation reaction force exist, although they also present preacceleration and related problems (see e.g. [15]). The introduction of an effective structure associated with the cutoff, which may be understood as acquired by the particle as a result of its rapid oscillations (see below), helps to recover causality even in the nonrelativistic approximation (examples and references may be seen in [5].

[2] From the point of view of SED, the emergence of an effective structure of the electron is most natural. Consider the electron as a pointlike charged particle subject to the action of the random vacuum field. Its high-frequency wandering induced by the field during the short time required for the determination of its dimensions assigns to it a size of the order of the mean square root of the position fluctuations. Identifying this measure of the fluctuations with the zitterbewegung strengthens the selection λ_C.

The appearance of the Compton frequency $\omega_C = mc^2/\hbar$ should come as no surprise in the present context, in which two electromagnetic constants play an important role. These are the fine structure constant α, characterizing the interactions of the field with matter, and τ, characterizing the magnitude of the radiation reaction. Since $\alpha = e^2/\hbar c$ and $\tau = 2e^2/3mc^3$, it follows that $\omega_C = 2\alpha/3\tau$. From a more physical point of view, the fact that the cutoff frequency depends on the mass may be understood in terms of the particle's capacity to respond to the high-frequency modes, which is limited by its total energy, mc^2; at higher energies, higher processes occur that change the nature of the particle.

It is important to note that the function $S_c(t-t')$ decreases rapidly as $t-t'$ grows beyond Δt. This means that in the absence of an external field that maintains such oscillations, they will decay, because of the dissipative effect of the radiation reaction. According to the basic postulate of SED, however, the particle is permanently embedded in the ZPF, so that the high-frequency oscillations – and hence the effective structure – are continuously regenerated by the high-frequency modes of this field, with which it interacts resonantly.[3] With ω_c given by ω_C, these oscillations are in line with the old proposal made by de Broglie stating that an atomic particle of mass m carries with it a kind of clock that oscillates with the Compton frequency; however, in contrast to the present instance, in de Broglie's theory this behavior is assumed as a fundamental property of the particle, without any specification of the underlying cause that gives rise to it. His postulate, as is well known, led de Broglie to the existence of the de Broglie wavelength and the construction of the first version of quantum mechanics around 1925 (a detailed discussion of this and related events is given in [19]).

3 The de Broglie wave

The above discussion endows the de Broglie wave with a concrete physical meaning, as will be shown in what follows. For this purpose we consider a representative particle traveling with a (constant or slowly varying) velocity v_0 along the z direction and look at the ZPF modes with which the particle interacts resonantly, namely those with frequency ω_C in the co-moving reference system S_p. Let us confine our attention to just the couple of modes that travel along the z axis (in both directions).[4] The corresponding frequencies as seen from the laboratory are given by

$$\omega = \frac{\gamma\omega_C}{1 \pm \beta},$$

the sign depending on the direction of propagation of the mode, with $\gamma = 1/\sqrt{1-\beta^2}$ and $\beta = v_0/c$. We write the vector potential for the two modes of the ZPF as seen from the laboratory, in the form

$$\boldsymbol{A}^\lambda = A_+^\lambda(k)\hat{\epsilon}_+^\lambda e^{i\theta_+^\lambda} e^{i\gamma(1+\beta)(k_C z - \omega_C t)} + A_-^\lambda(k)\hat{\epsilon}_-^\lambda e^{i\theta_-^\lambda} e^{i\gamma(1-\beta)(-k_C z - \omega_C t)} + \text{c.c.} \quad (28)$$

[3]According to several authors (see e.g. [18]) the self-force induces oscillations on a particle with structure, that are self-sustained and depend on the size of the particle. This argument rests on the assumption of the particle being already endowed with a structure, in contrast with the present treatment, in which the structure is created and sustained by the particle's permanent interaction with the vacuum field.

[4]The field modes traveling in the x and y directions are symmetrically distributed about the z axis and are therefore considered to have a mean null effect on the dynamics.

for every circular polarization λ ($\lambda = \pm 1$).[5] Here $A_+^\lambda(k)$ and $A_-^\lambda(k)$ stand for the amplitudes of the waves that travel in the $+z$ and $-z$ direction, respectively; $\hat{\epsilon}^\lambda$ are the polarization vectors, $k_C = \omega_C/c$, and $\theta_\pm^\lambda$ are independent random phases. Since $\beta \ll 1$, we may take both ZPF mode amplitudes of equal size A_0, with $A_0 = \sqrt{\pi c^2 \hbar/\omega_C V}$, corresponding to an energy $\hbar\omega_C/2$ per mode. It is convenient to introduce the following shorthand definitions

$$\varphi_P = \gamma(\beta k_C z - \omega_C t), \quad \varphi_M = \gamma(k_C z - \beta\omega_C t); \tag{29}$$

$$\theta_P^\lambda = \tfrac{1}{2}(\theta_+^\lambda + \theta_-^\lambda), \quad \theta_M^\lambda = \tfrac{1}{2}(\theta_+^\lambda - \theta_-^\lambda); \tag{30}$$

$$\epsilon_+^\lambda = \hat{\epsilon}_x + i\hat{\epsilon}_y, \quad \epsilon_-^\lambda = \epsilon_+^{*\lambda} = \hat{\epsilon}_x - i\hat{\epsilon}_y, \tag{31}$$

$$\hat{\boldsymbol{k}}_\pm \times \hat{\epsilon}_\pm = -i\lambda\hat{\epsilon}_\pm^\lambda. \tag{32}$$

Then $\boldsymbol{A}^\lambda$ takes the form

$$\boldsymbol{A}^\lambda = 4A_0 \cos P^\lambda \left[\hat{\epsilon}_x \cos M^\lambda - \lambda\hat{\epsilon}_y \sin M^\lambda\right], \tag{33}$$

with

$$M^\lambda = \varphi_M + \theta_M^\lambda, \quad P^\lambda = \varphi_P + \theta_P^\lambda. \tag{34}$$

Let us now look separately at the two factors contained in $\boldsymbol{A}^\lambda$, which we call P- and M-waves, respectively. The P-wave represents a high-frequency carrier signal in time; it oscillates with Compton's frequency, impressing on the particle a nonrelativistic zitterbewegung (of frequency $\gamma\omega_C$ instead of the relativistic value $2\omega_C$). In space it has a modulation, with wave number given by

$$k_B \equiv \gamma\beta k_C = \frac{\gamma m v_0}{\hbar} = \frac{p}{\hbar}, \tag{35}$$

where $p = \gamma m v_0$. With $k_B = 2\pi/\lambda_B$, the wavelength of the P-wave reads

$$\lambda_B = \frac{h}{p}, \tag{36}$$

which is just the famous de Broglie formula. From the dispersion relation $\omega_z^2 = \gamma^2\omega_C^2 = \omega_C^2 + c^2 k_B^2$, we get for the group velocity

$$v_g = \frac{\partial\omega_z}{\partial k_B} = v_0, \tag{37}$$

which indicates that wave and particle travel with the same velocity, the wave 'guiding' (or accompanying) the particle, just as proposed in de Broglie's theory. (A more detailed discussion of these matters can be seen in Refs. [13,21].) The association of the ZPF modulation to the particle defines a complex wave-plus-particle entity that gives a specific meaning to the familiar structure, pervasive in the whole of quantum

[5] The field is decomposed into modes of circular polarization, in attention to the fact that the electron interacts separately with such modes, as is known from the theory of atomic transitions [20].

mechanics and referred to under various names, such as Eddington's 'wavicle' [22], Bunge's 'quanton' [23], or Maxwell's 'smearon' [24].

The M-wave, in its turn, oscillates slowly in time but contains very fine oscillations in z-space, of Compton's wavelength. This part of the wave is therefore responsible for the zitterbewegung in space that gives rise to the effective structure of the particle mentioned above. Its phase travels with the velocity given by $\beta\omega_C/k_C = v_0$, which means that as the particle moves in space, it is 'anchored' to some place inside the M-wave.

Attention must be paid to the θ_P and θ_M phases contained in the P- and M-waves, respectively, which in free space may have any value at random between 0 and 2π. For an ensemble of similar systems, its members are affected by different random values of these phases. Thus, in any case only a statistical treatment of the problem is meaningful.[6] It follows that the de Broglie wave is a statistical concept, so that its use implies going from an individual case over to an ensemble. How this is achieved will become clear in what follows.

4 Mean dynamics of particles confined in an infinite potential well

In this section, we proceed to analyse the behavior of an ensemble of representative particles enclosed in a finite region of space and subject to the action of the ZPF, considering that the interaction occurs dominantly (resonantly) with the modes of frequency ω_C. Specifically, we are interested in the stationary states of motion of electrons moving freely within two parallel, impenetrable walls at $z = 0$ (lying on the xy plane) and $z = a$. According to the above discussions, the interaction of a particle with modes of the field (of frequency ω_C) traveling along z generates (small) stochastic motions on the xy plane. Now, the magnetic component of the field on that plane couples to such small motions, giving rise to a magnetic component of the Lorentz force along the z axis, which adds an extra component to the original systematic velocity of the particle along this direction. Therefore, the particles move inside the well with a velocity that varies around its mean value v_0, the variations depending on the position variable z. The presence of the impenetrable walls is taken account for by imposing appropriate stationarity and symmetry restrictions on the mean local systematic motion, the result of all this being an expression for the distribution of particles inside the well and a restriction on the possible values of the mean velocity v_0.[7]

4.1 Velocity field inside the well

To study the behavior of the electron inside the well we consider that it interacts dominantly with waves of Compton's frequency ω_C in its own reference system S_p, all the remaining modes of the ZPF, with frequencies other than ω_C, constituting a noisy background that may be omitted in a first approximation. (As discussed in Refs. [5,13] and references therein, in the context of SED this noisy background is identified with the vacuum fluctuations responsible for the radiative corrections of QED.) Further, we assume that the particle + field system has reached a stationary state; this means that an equilibrium has been established between the energy radiated by the particle and

[6] The statistical character of quantum mechanics has been recently underlined in an important series of works that analyse the dynamics of measurement, see references [25,26].

[7] While here we deal with charged particles, a similar treatment is applicable in principle to any particle that interacts with the zero-point radiation field or, for that matter, with any vacuum field.

the one gained from the random field, and one may therefore neglect any contribution from radiation reaction *inside* the well.

Because the system has axial symmetry around the z axis, for the calculation of the Lorentz force on the particle we may single out the effective ZPF components A^λ with wave vector along the direction of motion given by equation (33), with P^λ and M^λ given in the nonrelativistic approximation ($\gamma \simeq 1$) respectively by

$$P^\lambda = k_B z - \omega_C t + \theta_P^\lambda, \qquad M^\lambda = k_C z - \beta\omega_C t + \theta_M^\lambda, \tag{38}$$

according to equations (29) and (34). From the expression for the Lorentz force exerted by the field represented by equation (33), we get for the x-component of the velocity

$$\frac{dv_x}{dt} = -\frac{e}{mc}\frac{\partial A_x}{\partial t} - \frac{e}{mc}v_z\frac{\partial A_x}{\partial z} = -\frac{e}{mc}\frac{dA_x}{dt} \tag{39}$$

and a similar result for dv_y/dt, both for each λ. Integrating these equations, we find that the solutions of the equations of motion for the transverse components are, taking into account that the systematic motion has no x or y component,

$$v_x = -\frac{e}{mc}A_x^\lambda, \quad v_y = -\frac{e}{mc}A_y^\lambda. \tag{40}$$

Inserting these into the equation of motion for the longitudinal component

$$m\frac{dv_z}{dt} = \frac{e}{c}\left(v_x\frac{\partial A_x^\lambda}{\partial z} + v_y\frac{\partial A_y^\lambda}{\partial z}\right) \tag{41}$$

gives, according to (33),

$$\frac{dv_z}{dt} = -\frac{4e^2A_0^2}{m^2c^2}\frac{\partial}{\partial z}\cos 2P^\lambda. \tag{42}$$

Notice that the dynamics of the particle is determined by the carrier wave P only, which oscillates rapidly in time and varies smoothly in space. As mentioned above, different members of the ensemble may be attached to different points of the finely oscillating (in space) M-wave; this means that the position of the particle is defined up to a radius of size λ_C. Analogously, we may consider that the extremely rapid time oscillations (of frequency ω_C) contained in the P-wave are not detected by the measuring instrument because of its comparatively slow response. This is in line with the usual nonrelativistic quantum-mechanical description, which is limited to the slow motion and does not include the fine oscillations associated with the zitterbewegung.[8] To mimic this in the present description we erase the rapid oscillation from P^λ by making the substitution $\cos 2P^\lambda \to \cos 2\mathcal{P} = \cos 2(k_B z + \varphi)$, where φ is a fixed phase, the same for all members of the ensemble. The slow motion is therefore described by the solution of

$$\frac{dv_z}{dt} = -\frac{4e^2A_0^2}{m^2c^2}\frac{d}{dz}\cos 2\mathcal{P}, \quad \mathcal{P} = k_B z + \varphi. \tag{43}$$

[8] Things are somewhat different in a relativistic treatment. In particular, the solution of the Dirac equation for the free particle exhibits the zitterbewegung as an additional contribution in the relativistic expressions for velocity and position. In neglecting here the zitterbewegung we are assuming that its contribution to the particle energy is constant.

Since there is no more explicit dependence on t, we have $dv_z/dt = v_z(dv_z/dz)$. Integration of (43) is thus straightforward; with v_0 the average speed of the particle within the well (see Eq. (50)), the solution is

$$v_z(z) = \pm v_0\sqrt{1 - b\cos(2k_B z + 2\varphi)}, \tag{44}$$

with $b = 8e^2A_0^2/m^2c^2v_0^2$, and $z \in (0, a)$.

The spatial confinement of the particles is taken into account by imposing the symmetry considerations that distinguish a distribution of trapped particles from an ensemble of freely moving particles. Specifically, in a stationary state within a well of width a the following conditions must hold:

$$|v_z(z)| = |v_z(a - z)|\,, \tag{45}$$

$$\Delta\vartheta = \vartheta(a) - \vartheta(0) = 2n\pi, \quad n = 1, 2, 3, \ldots \tag{46}$$

where $\vartheta(z) = 2\mathcal{P}$. The first condition results in

$$k_B a = \pi n' - 2\varphi, \quad n' = 1, 2, 3, \ldots \tag{47}$$

which together with equation (46) gives

$$2\varphi = \pi(n' - n) = \pi\eta, \quad \eta = n' - n, \tag{48}$$

so that equation (47) becomes

$$k_B a = \pi n, \quad n = 1, 2, 3, \ldots \tag{49}$$

It follows from equations (35) and (49) that the speed v_0 is restricted to the values

$$v_{0n} = \frac{\pi\hbar}{ma}n = nv_1, \quad v_1 \equiv \frac{\pi\hbar}{ma}, \tag{50}$$

which gives for de Broglie's wavelength the familiar condition

$$\lambda_{Bn} = h/mv_{0n} \tag{51}$$

and for the energy spectrum the well-known result

$$E_n = \frac{1}{2}mv_{0n}^2 = \frac{\pi^2\hbar^2}{2ma^2}n^2. \tag{52}$$

Equation (44) becomes thus

$$v_n(z) = \pm nv_1\sqrt{1 - (-1)^\eta b_n \cos\frac{2\pi n}{a}z}, \tag{53}$$

with

$$b_n = \frac{8e^2A_0^2}{m^2c^2v_1^2n^2} = \frac{b_1}{n^2}. \tag{54}$$

Equation (53) gives the mean local velocity as a function of z, which is what we need in order to find the distribution of particles along the z axis.

4.2 Probability distribution of the confined particles

We have been led by the considerations of stationarity and symmetry to an approximate statistical description of the motion of the particle inside the well. To determine the corresponding probability distribution from equation (53) we resort to the usual classical argument. We consider that the average number of particles $dn = \rho(z)dz$ in a small region dz is proportional to the mean time dt that the particles spend in the small space interval dz (with $dt \gg \Delta t \simeq 2\pi\omega_C^{-1}$ and $dz \gg \Delta z \simeq 2\pi k_C^{-1} = \lambda_C$). Then with $\rho(z)$ the density of particles and N the normalization constant,

$$\rho(z)dz = Ndt = N\frac{dz}{|v|}, \tag{55}$$

or

$$\rho(z) = \frac{N}{|v|}. \tag{56}$$

Inserting here $v_n(z)$ as given by equation (53) we obtain

$$\rho_n(z) = \frac{N}{nv_1\sqrt{1-(-1)^\eta b_n \cos\frac{2\pi n}{a}z}}. \tag{57}$$

With any odd value for the free parameter η, the positions of the maxima and minima of this expression coincide precisely with those of the corresponding quantum results. For easiness of comparison, we introduce the modified distribution $\rho_n^C(z)$ by substracting from (57) its minimum value $\rho_n(0)$,

$$\rho_n^C(z) = \rho_n(z) - \rho_n(0) = \frac{N}{nv_1}\left(\frac{1}{\sqrt{1+b_n\cos\frac{2\pi n}{a}z}} - \frac{1}{\sqrt{1+b_n}}\right). \tag{58}$$

Figure 1 illustrates the result for the cases $n = 1, 2$, as well as the corresponding densities $\rho_n^Q(z)$ (in green) predicted by quantum theory. The densities have been normalized to a maximum value of 1. Figure 2 for the excited state $n = 5$, shows that the agreement becomes better as the value of n increases. Indeed, a series expansion of $\rho_n^C(z)$ gives, to first order in $(2b_n/1 + b_n)\sin^2 k_n z$,

$$\rho_n^C(z) \propto \sin^2(\pi n\frac{z}{a}), \quad n \gg 1. \tag{59}$$

Although the agreement between the adjusted classical curve $\rho_n^C(z)$ and the quantum one is quite satisfactory, the differences between the quantum and the classical predictions cannot be sidestepped: not only was it necessary to substract $\rho_n(0)$ from $\rho_n(z)$, but the adjusted $\rho_n^C(z)$ differs from $\rho_n^Q(z)$ for any finite value of n.

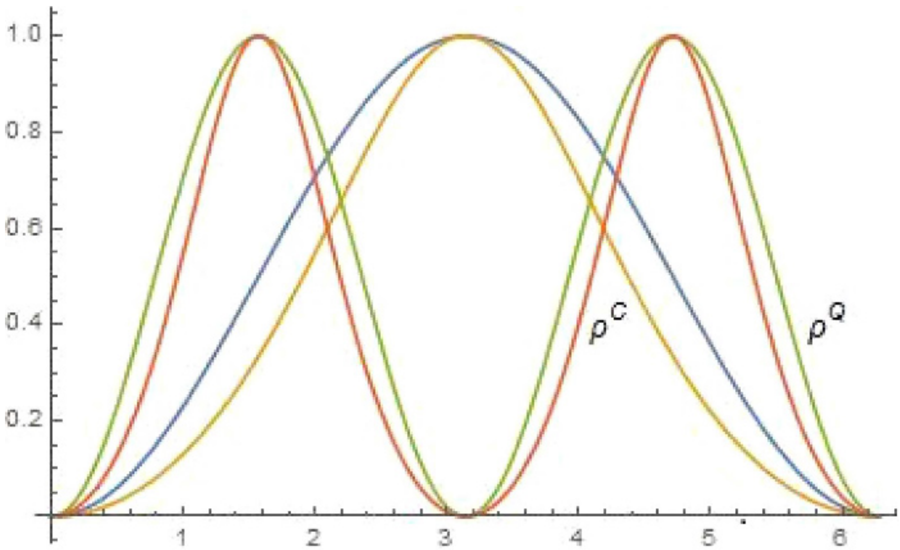

Fig. 1. Probability density for $n = 1, 2$; $b_1 = 0.8$.

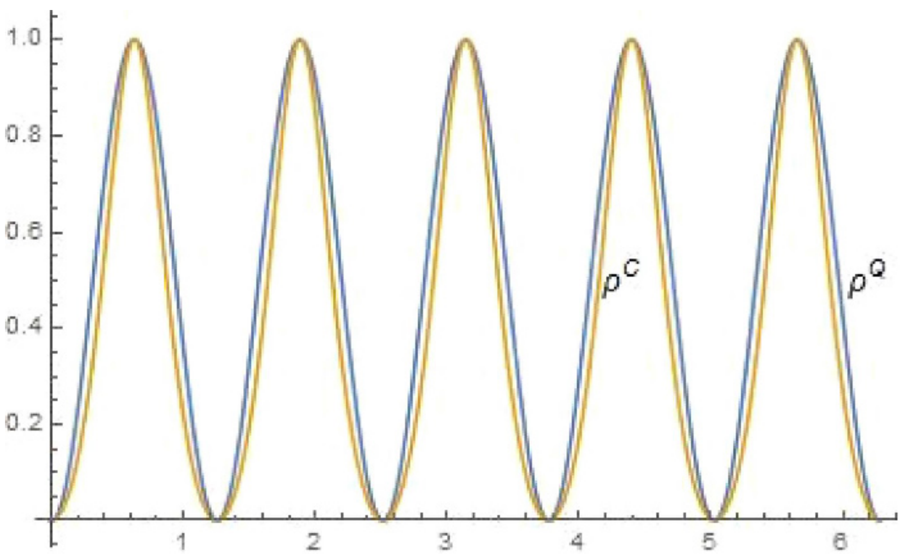

Fig. 2. Same as Figure 1, for $n = 5$.

5 Limitations of the classical approach

We recall that in certain instances – particularly linear problems – quantum results have been predicted with precision using the tools of CSED (see e.g. [3] for the Casimir effect and the van der Waals forces). However, more generally – and particularly with nonlinear problems – it happens that the predictions of CSED deviate from the quantum ones. Recent examples of relevance for the present discussion are provided in references [6–9], devoted to the ground state of the hydrogen atom. The numerical results reported in these important works adjust themselves reasonably well to the statistics of the orbit up to a point, after which the atom ionizes; according to references [7–9] this happens after about one million or more electron orbits.

This problem with CSED had been detected at an early stage by means of analytical calculations (e.g. [27–31]), but it was poorly understood, the then prevailing conviction among the SED community being that CSED and QM were but two different expressions of one and the same theory. The present case illustrates that not even in the linear case this is true, as is also discussed in Section 8.1.3 of reference [5], where it is shown that the *free* particle of CSED does not comply with detailed energy balance.

It is important therefore to recognize under which conditions SED and QM furnish equivalent results, despite their remarkable differences in approach. As discussed in some detail in reference [13] in connection with SED, assuming that the particle-field interaction causes the system to evolve asymptotically towards the quantum regime, in addition to energy balance this regime implies by necessity an ergodic behavior of the mechanical part of the system. As a consequence, the dynamical variables, originally functions in phase space, become replaced by operators (matrices), as is demonstrated in detail in Chapter 5 of reference [13]. Moreover, the transition to this regime does not leave the ZPF unaffected; in particular, the ergodic condition implies the establishment of specific correlations among the ZPF modes that sustain the stationary states [13]. This means that as a result, both matter and the (nearby) field become eventually affected in an essential way. For internal consistency one appeals to the Hilbert-space formalism, which turns out to be a powerful, synthetic tool to describe such state of affairs. Further, since SED contains the vacuum ab initio, one arrives at a theory that is in essence equivalent to nonrelativistic QED.[9]

The transition from the classical to the quantum regime represents a most delicate point of the theory, and perhaps the most difficult one to grasp, clearly still in want of a more detailed study and deeper understanding.

6 Concluding remarks

By taking an entirely classical approach to the problem of a charged particle in a box immersed in the ZPF, we have found that the ensuing accompanying de Broglie wave – which, correspondingly, is of electromagnetic nature, as anticipated in reference [33] – has the effect of producing an uneven distribution of matter inside the box, with alternating maxima and minima characterizing the stationary states, in consonance with the quantum case. The quantization of the mean speed of the particle (and hence of its associated kinetic energy) emerges as a consequence of periodicity and simple symmetry properties of the stationary state.

The quantum-like behavior obtained refers to the *slow* motion; the rapid oscillations that appear as a nonrelativistic zitterbewegung are ignored in this coarse approximate description, just as in nonrelativistic quantum mechanics. Moreover, we have not addressed here the complex dynamics of the evolution towards stationarity, nor did we prove that energy balance is attained in the asymptotic limit. We simply assumed that the system has reached a stationary state, and that the ZPF has not been affected by its interaction with the particle (except by fixing the value of the phase φ).

The main purpose of the present analysis has been the elucidation, by means of a simple example, of the extent to which a classical approach involving the ZPF is able to explain the origin of some core elements of the characteristic quantum behavior. This exercise has served the purpose, by proving that the silence of QM can be transcended to reveal a physics that remains concealed in the usual descriptions.

[9] In reference [13], the theory so constructed is shown to correctly predict the atomic lifetimes, the Lamb shift and other results that are proper of QED and fall beyond usual QM. More recently it has been possible to demonstrate that the spin of the electron is a further quantum property that emerges from the interaction with the ZPF [32].

However, another and no less important outcome of the exercise is that it confirms that CSED (which is just the original form of SED referred to in the introduction) and usual QM, are *different* theories, although in specific cases they may lead to similar or coincident results.

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